L^q DENSITIES FOR MEASURES ASSOCIATED WITH PARABOLIC IFS WITH OVERLAPS

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Abstract. We study parabolic iterated function systems (IFS) with overlaps on the real line and measures associated with them. A Borel probability measure μ on the coding space projects into a measure ν on the limit set of the IFS. We consider families of IFS satisfying a transversality condition. In [SSU2] sufficient conditions were found for the measure ν to be absolutely continuous for Lebesgue-a.e. parameter value. Here we investigate when ν has a density in $L^q(\mathbb{R})$ for $q > 1$. A necessary condition is that the q-dimension of μ (computed with respect to a certain metric associated with the IFS) is greater or equal to one. We prove that this is sharp for $1 < q \leq 2$ in the following sense: if μ is a Gibbs measure with a Hölder continuous potential, then ν has a density in $L^q(\mathbb{R})$ for Lebesgue-a.e. parameter value such that the q-dimension of μ is greater than one. This result is applied to a family of random continued fractions studied by R. Lyons.

1. Introduction

We continue to study parabolic iterated function systems (IFS) with overlaps on the real line. In our joint work with K. Simon [SSU1] we investigated the Hausdorff dimension of the limit set. The paper [SSU2] focused on the properties of invariant measures. An ergodic shift-invariant measure μ with positive entropy h_μ on the coding space induces an invariant (stationary) measure ν on the limit set of the iterated function system. The Hausdorff dimension of ν equals the ratio of entropy over Lyapunov exponent if the IFS has no "overlaps". [SSU2] investigated families of parabolic IFS which do have overlaps but satisfy a **transversality condition**. It was proved that for almost every (with respect to the Lebesgue measure) member of such a family, if the entropy exceeds the Lyapunov exponent, then the invariant measure is absolutely continuous, otherwise the above-mentioned formula for dimension still holds. We recall the set-up and the main result of [SSU2] in Section 2.

In this paper we explore the case when the projected measure ν is absolutely continuous and investigate when it has a density in $L^q(\mathbb{R})$. A similar question for linear IFS was studied in [PSo2], and we use some of the methods from [PSo2] here. It turns out that the crucial role is played by the **q-dimension** of the measure μ computed with respect to a natural

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metric associated with the IFS. If the q-dimension of μ is less than one, then ν cannot have an L^q-density, for any $q > 1$. Our main result is that, under some additional conditions, if $q \in (1, 2]$ and the q-dimension of μ is greater than one, then typically the projected measure does have a density in $L^q(\mathbb{R})$. The precise formulation is given in Section 3 (see Corollary 3.8). One of the conditions is transversality for the family of IFS and another one is that μ is a Gibbs measure with a Hölder continuous potential. Section 3 contains other results as well, which are not as sharp, but require only that μ be an atomless Borel probability measure (it need not even be invariant). In the case of a hyperbolic IFS these difficulties disappear, and we obtain a sharp result for a general measure (see Theorem 3.9).

In Section 4 our results are applied to an interesting family of measures considered by R. Lyons [L]. These measures (depending on a parameter) correspond to a class of random continued fractions. Lyons [L] found a threshold parameter above which the measures cannot have a density in $L^2(\mathbb{R})$. We prove that this threshold is sharp by showing that the measures are absolutely continuous with a density in $L^2(\mathbb{R})$ for almost every parameter value in some interval just below the threshold. We also prove that there is a similar threshold for the existence of L^q-density for any $q \in (1, 2]$.

2. Families of parabolic iterated function systems

Here we recall the set-up and the main result of [SSU2].

Let $X \subset \mathbb{R}$ be a closed interval and $\theta \in (0, 1]$. A $\mathcal{C}^{1+\theta}$ map $\phi : X \to X$ is hyperbolic if $0 < |\phi'(x)| < 1$ for all $x \in X$. We say that a $\mathcal{C}^{1+\theta}$ map $\phi : X \to X$ is **parabolic** if the following requirements are fulfilled:

- there is only one point $v \in X$ such that $\phi(v) = v$;
- $|\phi'(v)| = 1$ and $0 < |\phi'(x)| < 1$ for all $x \in X \setminus \{v\}$.
- There exists $C_1 \geq 1$ and $\beta = \beta(\phi) < \theta/(1 - \theta)$ ($= \infty$ if $\theta = 1$) such that

$$C_1^{-1} \leq \liminf_{x \to v} \frac{||\phi'(x)| - 1|}{|x - v|^\beta} \leq \limsup_{x \to v} \frac{||\phi'(x)| - 1|}{|x - v|^\beta} \leq C_1. \tag{2.1}$$

We will need the following basic lemma on iteration of a single parabolic function. For the proof see [U, Lemmas 2.2,2.3].

Lemma 2.1. *For every neighborhood V of v there exists a constant $L(V) \geq 1$ such that for all $x \in X \setminus V$ and all $n \geq 1$,*

$$L(V)^{-1} \leq |\phi^n(x) - v| \cdot (n + 1)^{\frac{1}{\beta}} \leq L(V); \tag{2.2}$$

$$L(V)^{-1} \leq |(\phi^n)'(x)| \cdot (n+1)^{\frac{\beta+1}{\beta}} \leq L(V). \tag{2.3}$$

Now, following [SSU1, SSU2], we define the class of parabolic IFS under investigation.

Definition 2.2. *Let $\Phi = \{\phi_1, \dots, \phi_k\}$ be a collection of $\mathcal{C}^{1+\theta}$ functions on a closed interval $X \subset \mathbb{R}$ such that ϕ_k is parabolic with the fixed point v and the other functions are hyperbolic. We write $\Phi \in \Gamma_X(\theta)$ if, in addition, $\phi_i(X) \subset \mathrm{Int}(X) \setminus \{v\}$ for all $i \leq k-1$.*

Let $\mathcal{A} = \{1, \dots, k\}$. We define the **natural projection map** $\pi_\Phi : \mathcal{A}^\infty \to \mathbb{R}$ by setting

$$\{\pi_\Phi(\omega)\} = \bigcap_{n \geq 1} \phi_{\omega|_n}(X)$$

where $\omega|_n = \omega_1 \dots \omega_n$ and $\phi_{\omega|_n} = \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n}$. If $\Phi \in \Gamma_X(\theta)$ then the map π_Φ is well-defined and continuous (see [SSU1, Lemma 5.6]). We have

$$\pi_\Phi(\omega) = \phi_{\omega|_n}(\pi_\Phi(\sigma^n\omega)) \quad \text{for all } \omega \in \mathcal{A}^\infty \text{ and } n \geq 1,$$

where σ is the left shift on $\mathcal{A}^\infty$. The **limit set**, or attractor, of the IFS Φ is defined by

$$J_\Phi = \pi_\Phi(\mathcal{A}^\infty).$$

It is easy to see that J_Φ is the unique non-empty compact set such that $J_\Phi = \bigcup_{i \leq k} \phi_i(J_\Phi)$. Let $U \subset \mathbb{R}^d$ be an open set. Consider a family of parabolic IFS

$$\Phi^{\mathbf{t}} = \{\phi_1^{\mathbf{t}}, \dots, \phi_{k-1}^{\mathbf{t}}, \phi_k\} \in \Gamma_X(\theta), \ \mathbf{t} \in \overline{U}. \tag{2.4}$$

Although the parabolic function *does not* depend on the parameter, it is sometimes convenient to write $\phi_k^{\mathbf{t}} \equiv \phi_k$ for $\mathbf{t} \in U$. Let $\pi_{\mathbf{t}} : \mathcal{A}^\infty \to \mathbb{R}$ be the natural projection associated with $\Phi^{\mathbf{t}}$ and denote $J_{\mathbf{t}} = J_{\Phi^{\mathbf{t}}}$. Two conditions which control the dependence on $\mathbf{t}$ will be needed.

CONTINUITY: the maps

$$\mathbf{t} \mapsto \phi_i^{\mathbf{t}} \text{ are continuous from } \overline{U} \text{ to } \mathcal{C}^{1+\theta}(X) \text{ for } i \leq k-1. \tag{2.5}$$

TRANSVERSALITY CONDITION: there exists a constant C_2 such that for all ω and τ in $\mathcal{A}^\infty$ with $\omega_1 \neq \tau_1$,

$$\mathcal{L}_d\{\mathbf{t} \in U : |\pi_{\mathbf{t}}(\omega) - \pi_{\mathbf{t}}(\tau)| \leq r\} \leq C_2 r \quad \text{for all } r > 0 \tag{2.6}$$

where $\mathcal{L}_d$ for the Lebesgue measure in $\mathbb{R}^d$.

A mild additional condition on the parabolic map ϕ_k will be needed in our theorems.

Definition 2.3. *Let us say that a parabolic function ϕ on X with the fixed point v is* **well-behaved** *on a connected open neighborhood V of v if $x = v$ is the only local extremum for $|\phi'(x)|$ on $V \cap X$.*

Clearly, any real-analytic parabolic function is well-behaved on some neighborhood of the parabolic point.

For $\phi \in \mathcal{C}^{1+\theta}(X)$ we write

$$\|\phi'\|_\theta = \sup\{|\phi'(x) - \phi'(y)| \cdot |x-y|^{-\theta} : \ x, y \in X\},$$

and $\|\Phi'\|_\theta := \max\{\|\phi_i'\|_\theta : \ i \in \mathcal{A}\}$ for an IFS $\Phi = \{\phi_i\}_{i\in\mathcal{A}}$. We denote by $\|\cdot\|$ the supremum norm on X. Given two IFS $\Phi = \{\phi_1, \dots, \phi_k\}$ and $\Psi = \{\psi_1, \dots, \psi_k\}$, we write

$$\|\Phi - \Psi\| = \max_{i \le k-1} \|\phi_i - \psi_i\| \quad \text{ and } \quad \|\Phi' - \Psi'\| = \max_{i\le k-1} \|\phi_i' - \psi_i'\|.$$

Following [SSU1, SSU2] we introduce additional notation useful for families of IFS. We write

$$\Phi \in \Gamma_X(\theta, V, \gamma, u, M)$$

for $\Phi \in \Gamma_X(\theta)$ if V is a connected open neighborhood of the parabolic point v such that

$$V \cap \bigcup_{i=1}^{k-1} \phi_i(X) = \emptyset, \tag{2.7}$$

$$\max\{\|(\phi_i)'\| : \ i \le k-1\} \le \gamma \in (0,1), \tag{2.8}$$

$$\min\{|\phi_i'(x)| : \ x \in X, \ i \le k\} \ge u \in (0,1), \tag{2.9}$$

and $\|\Phi'\|_\theta \le M$. By Definition 2.2, every $\Phi \in \Gamma_X(\theta)$ belongs to $\Gamma_X(\theta, V, \gamma, u, M)$ for some V, γ, u and M. Given a finite word $\omega \in \mathcal{A}^*$, let $h(\omega)$ denote the number of all hyperbolic letters (i.e. $\neq k$) appearing in ω. By (2.8),

$$\|\phi_w'\| \le \gamma^{h(w)} \quad \text{ for all } \ w \in \mathcal{A}^*. \tag{2.10}$$

We will need three technical lemmas from [SSU1, SSU2]. The first of them is a basic result on distortion for parabolic IFS; the second one says that if ϕ_k is well-behaved, then the derivative of a map ϕ_w cannot be too small near the parabolic point, and the third one compares the derivatives at a single point but for two distinct IFS.

Lemma 2.4. (see [SSU1, Lemma 5.8]). *There exists a constant $C_3 = C_3(X, \theta, V, \gamma, u, M) \ge 1$ such that for every $\Phi \in \Gamma_X(\theta, V, \gamma, u, M)$, all $\omega \in \mathcal{A}^\infty$ and all $n \ge 1$,*

$$C_3^{-1} \le \frac{|\phi'_{\omega|_n}(y)|}{|\phi'_{\omega|_n}(x)|} \le C_3 \quad \textit{for all} \ \ x, y \in X \setminus V. \tag{2.11}$$

Lemma 2.5. (see [SSU2, Lemma 4.3]). *There exists a constant $C_4 = C_4(X, \theta, V, \gamma, u, M) \geq 1$ such that for any parabolic IFS $\Phi \in \Gamma_X(\theta, V, \gamma, u, M)$ with the property that ϕ_k is well-behaved on V, the following holds: For all $\omega \in \mathcal{A}^\infty$ and all $n \in \mathbb{N}$,*

$$\frac{|\phi'_{\omega|_n}(y)|}{|\phi'_{\omega|_n}(x)|} \leq C_4 \qquad \text{for all } \ x \in X \quad \text{and} \quad y \in X \setminus V.$$

Lemma 2.6. (see [SSU1, Cor. 6.3]). *There exists a constant $C_5 = C_5(X, \theta, V, \gamma, u, M) > 0$ such that for any two IFS $\Phi = \{\phi_1, \dots, \phi_k\}$ and $\Psi = \{\psi_1, \dots, \psi_k\}$, in $\Gamma_X(\theta, V, \gamma, u, M)$, with $\phi_k = \psi_k$, for all $\omega \in \mathcal{A}^*$ and all $x \in X$,*

$$\frac{|\phi'_\omega(x)|}{|\psi'_\omega(x)|} \leq \exp\left(C_5 h(\omega)\left(\|\Phi - \Psi\|^\theta + \|\Phi' - \Psi'\|\right)\right) . \tag{2.12}$$

The following was the main result of [SSU2]. Recall that the Hausdorff dimension of a measure ν on $\mathbb{R}$ is defined by $\dim_{\mathrm{H}} \nu = \inf\{\dim_{\mathrm{H}}(F) : \ \nu(\mathbb{R} \setminus F) = 0\}$.

Theorem 2.7. (see [SSU2, Th. 2.3]). *Suppose that $\{\Phi^{\mathbf{t}}\}_{\mathbf{t}\in\overline{U}}$ is a family of parabolic IFS (2.4) satisfying (2.5) and (2.6), such that ϕ_k is well-behaved on some neighborhood of v. Let μ be a shift-invariant ergodic Borel probability measure on $\mathcal{A}^\infty$ with positive entropy h_μ and let $\nu_{\mathbf{t}} = \mu \circ \pi_{\mathbf{t}}^{-1}$. Then*

(i) *for Lebesgue-a.e. $\mathbf{t} \in U$,*

$$\dim_{\mathrm{H}} \nu_{\mathbf{t}} = \min\left\{\frac{h_\mu}{\chi_\mu(\Phi^{\mathbf{t}})}, 1\right\}$$

where $\chi_\mu(\Phi^{\mathbf{t}}) = -\int_{\mathcal{A}^\infty} \log|\phi'_{\omega_1}(\pi_{\mathbf{t}}(\sigma\omega))|\, d\mu(\omega)$ is the Lyapunov exponent of the IFS;

(ii) *the measure $\nu_{\mathbf{t}}$ is absolutely continuous for a.e. $\mathbf{t}$ in $\{\mathbf{t} \in U : \ \frac{h_\mu}{\chi_\mu(\Phi^{\mathbf{t}})} > 1\}$.*

It is therefore tempting to ask about the properties of the Radon-Nikodym derivative of the measure $\nu_{\mathbf{t}}$ with respect to the Lebesgue measure in the case when the former measure is absolutely continuous. The next section is devoted to this problem.

Notation. We write $B_\delta(\mathbf{t}_0)$ for the open ball of radius δ centered at $\mathbf{t}_0$ and $\mathcal{L}_d$ for the Lebesgue measure in $\mathbb{R}^d$. If μ is a measure we often write μF without parentheses. Recall that $\mathcal{A} = \{1, \dots, k\}$. For a finite word $w \in \mathcal{A}^n$ the corresponding cylinder set in $\mathcal{A}^\infty$ is denoted by $[w]$. For ω and τ in $\mathcal{A}^\infty$ we denote by $\omega \wedge \tau$ their common initial segment, so that $\omega, \tau \in [\omega \wedge \tau]$ and $\omega_{n+1} \neq \tau_{n+1}$ for $n = |\omega \wedge \tau|$. The symbol $\preceq$ means that the inequality holds up to an absolute multiplicative constant, and $\asymp$ means that both $\preceq$ and $\succeq$ are true.

3. L^q densities

In this section we explore the problem of when the projection measures $\nu_{\mathbf{t}}$ are absolutely continuous with L^q densities. It is related to the notion of q**-dimension** of a probability measure on a metric space.

Definition 3.1. *Suppose that $(\mathcal{X}, \varrho)$ is a metric space, μ is a Borel probability measure on $\mathcal{X}$, and $q > 1$. The q-dimension of μ with respect to the metric ϱ is defined by*

$$D_q(\mu) = \sup\{t \geq 0 : \ I_{t,q}(\mu) < \infty\} \tag{3.1}$$

where $I_{t,q}(\mu)$ is the (t,q)-energy of the measure μ, given by the formula

$$I_{t,q}(\mu) = \int \left(\int \frac{d\mu(y)}{\varrho(x,y)^t} \right)^{q-1} d\mu(x). \tag{3.2}$$

Remark. Hunt and Kaloshin [HK, Prop. 2.1] showed that $D_q(\mu)$ equals the lower L^q-dimension of μ defined as $\liminf_{r\to 0} (\log r)^{-1} \log \int (\mu B_r(x))^{q-1}\, d\mu$ (the upper L^q-dimension is obtained by taking $\limsup$). The upper and lower L^q-dimensions are usually denoted $D_q^-(\mu)$ and $D_q^+(\mu)$, and the L^q-dimension is said to exist if $D_q^-(\mu) = D_q^+(\mu)$. In this paper we suppress the "$-$" sign, since in what follows we only need the definition (3.1); this does not mean to imply that the L^q-dimension of μ exists. We note also that in the important case when $q = 2$, the L^q-dimension of a measure is called its correlation dimension.

The following elementary lemma and its corollary are known; we include a short proof for completeness.

Lemma 3.2. *Let X be a compact interval on the real line and consider the space $L^q(X)$ with respect to the Lebesgue measure. If $q > 1$ and $h \in L^q(X)$, then*

$$\int_X \left(\int_X \frac{h(y)}{|x-y|^s}\, dy \right)^{q-1} h(x)\, dx < \infty \qquad \textit{for all} \ \ s < 1. \tag{3.3}$$

Proof. Fix $s < 1$. Since $k(y) = |x-y|^{-s} \in L^1(X)$, the convolution $(h * k)(x) = \int_X \frac{h(y)}{|x-y|^s}\, dy$ is in $L^q(X)$, and hence $(h*k)^{q-1} \in L^{\frac{q}{q-1}}(X)$. But the latter space is the dual of $L^q(X)$, and (3.3) follows. $\square$

Corollary 3.3. *Let ν be a compactly supported Borel probability measure on the real line. If ν is absolutely continuous with a density in $L^q(\mathbb{R})$, then $D_q(\nu) \geq 1$, where the q-dimension is computed with respect to the Euclidean metric.*

This is immediate from Definition 3.1 and Lemma 3.2.

Now consider a parabolic iterated function system $\Phi \in \Gamma_X(\theta, V, \gamma, u, M)$. For $w \in \mathcal{A}^*$ denote

$$X_w := \phi_w(X) \quad \text{and} \quad \widetilde{X}_w := \phi_w(X \setminus V).$$

We equip the coding space $\mathcal{A}^\infty$ with two metrics d_1 and d_2 given by the formulas (for $\omega \neq \tau$):

$$d_1(\omega, \tau) = |X_{\omega\wedge\tau}| \quad \text{and} \quad d_2(\omega, \tau) = |\widetilde{X}_{\omega\wedge\tau}|,$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$. Notice that Lemma 2.4 implies the existence of a constant $C_6 > 0$ such that if $\kappa \in \{\omega, \tau\}$ satisfies $(\sigma^{|\omega\wedge\tau|}\kappa)_1 \neq k$, then

$$C_6^{-1}|\phi'_{\omega\wedge\tau}(\pi(\sigma^{|\omega\wedge\tau|}\kappa))| \leq |\widetilde{X}_{\omega\wedge\tau}| \leq C_6|\phi'_{\omega\wedge\tau}(\pi(\sigma^{|\omega\wedge\tau|}\kappa))|. \tag{3.4}$$

Therefore, the metric d_2 is equivalent to another metric given by the formula

$$\widetilde{d}_2(\omega, \tau) = \max\left\{|\phi'_{\omega\wedge\tau}(\pi(\sigma^{|\omega\wedge\tau|}\omega))|\,,\, |\phi'_{\omega\wedge\tau}(\pi(\sigma^{|\omega\wedge\tau|}\tau))|\right\}.$$

Clearly, the q-dimension depends only on the Lipschitz equivalence class of the metric; thus, the q-dimension corresponding to d_2 is independent of the set V. Let us denote by $I_{t,q}^{(i)}(\mu)$ and $D_q^{(i)}(\mu)$, $i = 1, 2$, respectively the (t, q)-energy and the q-dimension of μ computed with respect to the metric d_i. It is immediate from the definition that

$$D_q^{(1)}(\mu) \geq D_q^{(2)}(\mu).$$

Of course, the metrics d_i and q-dimensions $D_q^{(i)}(\mu)$, $i = 1, 2$, depend on the iterated function system under consideration, so dealing with families of IFS we will write $|X_w^{(\mathbf{t})}|$, $I_{t,q}^{(i,\mathbf{t})}(\mu)$ and $D_q^{(i,\mathbf{t})}(\mu)$ to indicate the parameter.

We begin with the proof of the following.

Theorem 3.4. *Let μ be a Borel probability measure on $\mathcal{A}^\infty$ and $q > 1$. If $\nu = \mu \circ \pi^{-1}$ is absolutely continuous with a density in $L^q(\mathbb{R})$, then $D_q^{(1)}(\mu) \geq 1$.*

Proof. Since $|\pi(\omega) - \pi(\tau)| \leq d_1(\omega, \tau)$ for all $\omega, \tau \in \mathcal{A}^\infty$, we have $I_{t,q}(\mu \circ \pi^{-1}) \geq I_{t,q}^{(1)}(\mu)$ for all $t > 0$, and hence $D_q(\mu \circ \pi^{-1}) \leq D_q^{(1)}(\mu)$. Now the statement follows from Corollary 3.3. $\square$

Now we are in a position to prove the first new result of this paper. Its proof combines the methods of [PSo2, Th.4.1] and [SSU2].

Theorem 3.5. *Suppose that $\{\Phi^{\mathbf{t}}\}_{\mathbf{t}\in\overline{U}}$ is a family of parabolic IFS in $\Gamma_X(\theta, V, \gamma, u, M)$ satisfying (2.4), (2.5) and (2.6), such that ϕ_k is well-behaved on some neighborhood of the parabolic point v. Suppose that μ is a Borel atomless probability measure on $\mathcal{A}^\infty$ and that $D_q^{(2,\mathbf{t}_0)}(\mu) > 1$*

for some $q \in (1, 2]$ *and* $\mathbf{t}_0 \in U$. *Then there exists* $\delta > 0$ *such that for* $\mathcal{L}_d$*-a.e.* $\mathbf{t} \in B_\delta(\mathbf{t}_0)$ *the measure* $\nu_{\mathbf{t}} = \mu \circ \pi_{\mathbf{t}}^{-1}$ *is absolutely continuous with a density in* $L^q(\mathbb{R})$.

Proof. Let $b = \log(1/\gamma)$. Fix $1 < s < D_q^{(2,\mathbf{t}_0)}(\mu)$ and $\varepsilon = b(s-1)$. In view of Lemma 2.6, there exists $\delta > 0$ so small that for all $w \in \mathcal{A}^*$ and all $x \in X$,

$$|\mathbf{t} - \mathbf{t}_0| \le \delta \implies \frac{|(\phi_w^{\mathbf{t}_0})'(x)|}{|(\phi_w^{\mathbf{t}})'(x)|} \le e^{\varepsilon h(w)}. \tag{3.5}$$

We are going to show that

$$\mathcal{I} = \int_{B_\delta(\mathbf{t}_0)} \int_{\mathbb{R}} \underline{D}(\nu_{\mathbf{t}}, x)^{q-1}\, d\nu_{\mathbf{t}}(x)\, d\mathbf{t} < \infty, \tag{3.6}$$

where

$$\underline{D}(\nu_{\mathbf{t}}, x) = \liminf_{r \searrow 0} \frac{\nu_{\mathbf{t}}[x-r, x+r]}{2r}$$

is the lower density of the measure $\nu_{\mathbf{t}}$ at the point x. Then by [Mat, 2.12], the measure $\nu_{\mathbf{t}}$ is absolutely continuous for a.e. $\mathbf{t} \in B_\delta(\mathbf{t}_0)$. For such $\mathbf{t}$ we have $\underline{D}(\nu_{\mathbf{t}}, x) = \frac{d\nu_{\mathbf{t}}}{dx}$, so (3.6) will imply that $\frac{d\nu_{\mathbf{t}}}{dx} \in L^q(\mathbb{R})$ for a.e. $\mathbf{t} \in B_\delta(\mathbf{t}_0)$.

First we apply Fatou's Lemma and then make a change of variable to get

$$\begin{aligned} \mathcal{I} &\le \liminf_{r \searrow 0} \int_{B_\delta(\mathbf{t}_0)} \int_{\mathbb{R}} \frac{\big(\nu_{\mathbf{t}}[x-r, x+r]\big)^{q-1}}{(2r)^{q-1}}\, d\nu_{\mathbf{t}}(x)\, d\mathbf{t} \\ &= \liminf_{r \searrow 0} \int_{B_\delta(\mathbf{t}_0)} \int_{\mathcal{A}^\infty} \frac{\big(\nu_{\mathbf{t}} B_r(\pi_{\mathbf{t}}(\omega))\big)^{q-1}}{(2r)^{q-1}}\, d\mu(\omega)\, d\mathbf{t}. \end{aligned} \tag{3.7}$$

Next we reverse the order of integration and use Hölder's inequality (recall that $1 < q \le 2$) to obtain

$$\mathcal{I} \preceq \liminf_{r \searrow 0} (2r)^{1-q} \int_{\mathcal{A}^\infty} \left(\int_{B_\delta(\mathbf{t}_0)} \nu_{\mathbf{t}} B_r(\pi_{\mathbf{t}}(\omega))\, d\mathbf{t} \right)^{q-1} d\mu(\omega). \tag{3.8}$$

Denoting by $\mathbf{1}_E$ the characteristic function of a set E, we have

$$\begin{aligned} \int_{B_\delta(\mathbf{t}_0)} \nu_{\mathbf{t}} B_r(\pi_{\mathbf{t}}(\omega))\, d\mathbf{t} &= \int_{B_\delta(\mathbf{t}_0)} \int_{\mathbb{R}} \mathbf{1}_{B_r(\pi_{\mathbf{t}}(\omega))}\, d\nu_{\mathbf{t}}\, d\mathbf{t} \\ &= \int_{B_\delta(\mathbf{t}_0)} \int_{\mathcal{A}^\infty} \mathbf{1}_{\{\tau \in \mathcal{A}^\infty : |\pi_{\mathbf{t}}(\omega) - \pi_{\mathbf{t}}(\tau)| \le r\}}\, d\mu(\tau)\, d\mathbf{t} \\ &= \int_{\mathcal{A}^\infty} \mathcal{L}_d\{\mathbf{t} \in B_\delta(\mathbf{t}_0) :\ |\pi_{\mathbf{t}}(\omega) - \pi_{\mathbf{t}}(\tau)| \le r\}\, d\mu(\tau) \\ &= \int_{\mathcal{A}^\infty \setminus \{\omega\}} \mathcal{L}_d\{\mathbf{t} \in B_\delta(\mathbf{t}_0) :\ |\pi_{\mathbf{t}}(\omega) - \pi_{\mathbf{t}}(\tau)| \le r\}\, d\mu(\tau). \end{aligned} \tag{3.9}$$

We could write the last equality since μ is atomless. Now take an arbitrary $\tau \in \mathcal{A}^\infty \setminus \{\omega\}$ and denote $\rho = \omega \wedge \tau$. Then we have for some $c \in [\pi_{\mathbf{t}}(\sigma^{|\rho|}\omega), \pi_{\mathbf{t}}(\sigma^{|\rho|}\tau)]$ using the Mean Value Theorem and (3.5):

$$\begin{aligned} |\pi_{\mathbf{t}}(\omega) - \pi_{\mathbf{t}}(\tau)| &= |(\phi_\rho^{\mathbf{t}})'(c)| \cdot |\pi_{\mathbf{t}}(\sigma^{|\rho|}\omega) - \pi_{\mathbf{t}}(\sigma^{|\rho|}\tau)| \\ &\geq |(\phi_\rho^{\mathbf{t}_0})'(c)| e^{-h(\rho)\varepsilon} \cdot |\pi_{\mathbf{t}}(\sigma^{|\rho|}\omega) - \pi_{\mathbf{t}}(\sigma^{|\rho|}\tau)| \,. \end{aligned}$$

Since $\omega_{|\rho|+1} \neq \tau_{|\rho|+1}$, at least for one element $\kappa \in \{\omega, \tau\}$ we have $\kappa_{|\rho|+1} \neq k$. Since $(\sigma^{|\rho|}\kappa)_1 \neq k$, we have $\pi(\sigma^{|\rho|}\kappa) \notin V$ by (2.7). Therefore, by Lemma 2.5 and (3.4),

$$\begin{aligned} |\pi_{\mathbf{t}}(\omega) - \pi_{\mathbf{t}}(\tau)| &\geq C_4^{-1} |(\phi_\rho^{\mathbf{t}_0})'(\pi(\sigma^{|\rho|}\kappa))| \cdot e^{-h(\rho)\varepsilon} |\pi_{\mathbf{t}}(\sigma^{|\rho|}\omega) - \pi_{\mathbf{t}}(\sigma^{|\rho|}\tau)| \\ &\geq (C_4 C_6)^{-1} |\widetilde{X}_\rho^{(\mathbf{t}_0)}| \cdot e^{-h(\rho)\varepsilon} |\pi_{\mathbf{t}}(\sigma^{|\rho|}\omega) - \pi_{\mathbf{t}}(\sigma^{|\rho|}\tau)|. \end{aligned} \tag{3.10}$$

It follows that

$$\begin{aligned} &\mathcal{L}_\delta\{\mathbf{t} \in B_\delta(\mathbf{t}_0) : \ |\pi_{\mathbf{t}}(\omega) - \pi_{\mathbf{t}}(\tau)| \leq r\} \\ &\leq \mathcal{L}_d \left\{ \mathbf{t} \in B_\delta(\mathbf{t}_0) : \ |\pi_{\mathbf{t}}(\sigma^{|\rho|}\omega) - \pi_{\mathbf{t}}(\sigma^{|\rho|}\tau)| \leq \frac{C_4 C_6 e^{h(\rho)\varepsilon} r}{|\widetilde{X}_\rho^{(\mathbf{t}_0)}|} \right\} \\ &\leq C_2 C_4 C_6 e^{h(\rho)\varepsilon} r \cdot |\widetilde{X}_\rho^{(\mathbf{t}_0)}|^{-1} \,, \end{aligned}$$

by the transversality condition (2.6). Substituting this into (3.9) we obtain from (3.8)

$$\begin{aligned} \mathcal{I} &\preceq \int_{\mathcal{A}^\infty} \left(\int_{\mathcal{A}^\infty \setminus \{\omega\}} e^{h(\rho)\varepsilon} |\widetilde{X}_\rho^{(\mathbf{t}_0)}|^{-1} \, d\mu(\tau) \right)^{q-1} d\mu(\omega) \\ &= \int_{\mathcal{A}^\infty} \left(\sum_{n=0}^{\infty} e^{h(\omega|_n)\varepsilon} |\widetilde{X}_{\omega|_n}^{(\mathbf{t}_0)}|^{-1} \, \mu\big([\omega|_n] \setminus [\omega|_{n+1}]\big) \right)^{q-1} d\mu(\omega). \end{aligned} \tag{3.11}$$

In view of (3.4) and (2.10), we can continue our estimate of $\mathcal{I}$ as follows, using that $\gamma = e^{-b}$ and $\varepsilon = b(s-1)$,

$$\begin{aligned} \mathcal{I} &\preceq \int_{\mathcal{A}^\infty} \left(\sum_{n=0}^{\infty} e^{h(\omega|_n)\varepsilon} |\widetilde{X}_{\omega|_n}^{(\mathbf{t}_0)}|^{s-1} \cdot |\widetilde{X}_{\omega|_n}^{(\mathbf{t}_0)}|^{-s} \, \mu\big([\omega|_n] \setminus [\omega|_{n+1}]\big) \right)^{q-1} d\mu(\omega) \\ &\preceq \int_{\mathcal{A}^\infty} \left(\sum_{n=0}^{\infty} e^{(\varepsilon-(s-1)b)h(\omega|_n)} |\widetilde{X}_{\omega|_n}^{(\mathbf{t}_0)}|^{-s} \, \mu\big([\omega|_n] \setminus [\omega|_{n+1}]\big) \right)^{q-1} d\mu(\omega) \\ &\preceq \int_{\mathcal{A}^\infty} \left(\sum_{n=0}^{\infty} |\widetilde{X}_{\omega|_n}^{(\mathbf{t}_0)}|^{-s} \, \mu\big([\omega|_n] \setminus [\omega|_{n+1}]\big) \right)^{q-1} d\mu(\omega) \\ &= \int_{\mathcal{A}^\infty} \left(\frac{d\mu(\tau)}{|\widetilde{X}_{\omega\wedge\tau}^{(\mathbf{t}_0)}|^s} \right)^{q-1} d\mu(\omega) = I_{s,q}^{(2,\mathbf{t}_0)}(\mu) < \infty, \end{aligned}$$

since $D_q^{(2,\mathbf{t}_0)}(\mu) > s$ by the choice of s. The proof is complete. $\square$

Next we prove a result which uses the q-dimension calculated with respect to the metric d_1. Recall that $\beta > 0$ appears in the definition of a parabolic map (2.1).

Theorem 3.6. *Suppose that $\{\Phi^{\mathbf{t}}\}_{\mathbf{t}\in\overline{U}}$ is a family of parabolic IFS in $\Gamma_X(\theta, V, \gamma, u, M)$ satisfying (2.4), (2.5) and (2.6), such that ϕ_k is well-behaved on some neighborhood of the parabolic point v. Suppose that μ is a Borel atomless probability measure on $\mathcal{A}^\infty$ such that $D_q^{(1,\mathbf{t}_0)}(\mu) > 1+\beta$ for some $q \in (1,2]$ and $\mathbf{t}_0 \in U$. Then there exists $\delta > 0$ such that for $\mathcal{L}_d$-a.e. $\mathbf{t} \in B_\delta(\mathbf{t}_0)$ the measure $\nu_{\mathbf{t}} = \mu \circ \pi_{\mathbf{t}}^{-1}$ is absolutely continuous with a density in $L^q(\mathbb{R})$.*

Proof. As in the proof of Theorem 3.5, we let $b = \log(1/\gamma)$. Fix s so that $\beta+1 < s < D_q^{(1,\mathbf{t}_0)}(\mu)$ and let $\varepsilon = b(s-1)$. Then we repeat the proof of Theorem 3.5 until (3.11) almost word by word. However, we use only the first line of (3.10), keeping $|(\phi_\rho^{\mathbf{t}_0})'(\pi(\sigma^{|\rho|}\kappa))|$ instead of $|\widetilde{X}_\rho^{(\mathbf{t}_0)}|$ and obtain

$$\mathcal{I} \preceq \int_{\mathcal{A}^\infty}\left(\int_{\mathcal{A}^\infty\setminus\{\omega\}} e^{h(\rho)\varepsilon}|(\phi_\rho^{\mathbf{t}_0})'(\pi(\sigma^{|\rho|}\kappa))|^{-1}d\mu(\tau)\right)^{q-1} d\mu(\omega)|\phi_\rho'(\pi(\sigma^{|\rho|}\kappa))|.$$

Write $\rho = \alpha k^{p(\rho)}$, where the last letter of α is different from k. Since $(\sigma^{|\rho|}\kappa)_1 \neq k$, we have $\pi(\sigma^{|\rho|}\kappa) \notin V$. Therefore, applying Lemma 2.1 and Lemma 2.4, we get

$$|X_\rho^{(\mathbf{t}_0)}| \asymp \|(\phi_\alpha^{\mathbf{t}_0})'\|(p(\rho)+1)^{-\frac{1}{\beta}} \quad \text{and} \quad |(\phi_\rho^{\mathbf{t}_0})'(\pi(\sigma^{|\rho|}\kappa))| \asymp \|(\phi_\alpha^{\mathbf{t}_0})'\|(p(\rho)+1)^{-\frac{\beta+1}{\beta}}.$$

Hence

$$\begin{aligned}
\mathcal{I} &\preceq \int_{\mathcal{A}^\infty}\left(\int_{\mathcal{A}^\infty\setminus\{\omega\}} e^{h(\rho)\varepsilon}\|(\phi_\alpha^{\mathbf{t}_0})'\|^{-1}(p(\rho)+1)^{\frac{\beta+1}{\beta}}d\mu(\tau)\right)^{q-1} d\mu(\omega)\\
&= \int_{\mathcal{A}^\infty}\left(\int_{\mathcal{A}^\infty\setminus\{\omega\}} e^{h(\rho)\varepsilon}\left(\|(\phi_\alpha^{\mathbf{t}_0})'\|(p(\rho)+1)^{-\frac{1}{\beta}}\right)^{-s}\|(\phi_\alpha^{\mathbf{t}_0})'\|^{s-1}(p(\rho)+1)^{\frac{\beta+1}{\beta}-\frac{s}{\beta}}d\mu(\tau)\right)^{q-1} d\mu(\omega)\\
&\preceq \int_{\mathcal{A}^\infty}\left(\int_{\mathcal{A}^\infty\setminus\{\omega\}} e^{h(\rho)\varepsilon-bh(\rho)(s-1)}(p(\rho)+1)^{\frac{\beta+1-s}{\beta}}|X_\rho^{(\mathbf{t}_0)}|^{-s}\,d\mu(\tau)\right)^{q-1} d\mu(\omega)\\
&\le \int_{\mathcal{A}^\infty}\left(\int_{\mathcal{A}^\infty\setminus\{\omega\}} |X_\rho^{(\mathbf{t}_0)}|^{-s}d\mu(\tau)\right)^{q-1} d\mu(\omega) \le I_{s,q}^{(1,\mathbf{t}_0)}(\mu) < \infty,
\end{aligned}$$

since $D_q^{(1,\mathbf{t}_0)} > s$ by the choice of s. Above, when passing from the second to the third displayed line we used that $\|(\phi_\alpha^{\mathbf{t}_0})'\| \le \gamma^{h(\alpha)} = \gamma^{h(\rho)} = e^{-bh(\rho)}$. After that we used that $\varepsilon = b(s-1)$ and $\beta+1-s < 0$. The proof is complete. $\square$

Next we show that for certain shift-invariant measures Theorem 3.5 is in a sense optimal. We call a Borel probability measure μ on $\mathcal{A}^\infty$ **well-mixing** if $\mu[i] < 1$ for every $i \in \mathcal{A}$ and if there exist constants $C_7 > 0$ and $\ell \ge 1$ such that if $\alpha, \eta, \zeta \in \mathcal{A}^*$, and $|\eta| \ge \ell$, then

$$\mu[\alpha\eta\zeta] \le C_7\mu[\alpha]\mu[\zeta].$$

Every Bernoulli measure is obviously well-mixing and, more generally, every Gibbs (equilibrium) state of a Hölder continuous potential is well-mixing, see [Bo]. An easy induction shows that if μ is well-mixing then there exist constants $C_8 > 0$ and $0 < \lambda < 1$ such that for every $n \geq 0$, every $\alpha \in \mathcal{A}^*$ and every $\eta \in \mathcal{A}^n$

$$\mu[\alpha\eta] \leq C_8 \lambda^n \mu[\alpha]. \tag{3.12}$$

Proposition 3.7. *If μ is a well-mixing measure on $\mathcal{A}^\infty$, then $D_q^{(1)}(\mu) = D_q^{(2)}(\mu)$ for every $q > 1$.*

Proof. Since $D_q^{(1)}(\mu) \geq D_q^{(2)}(\mu)$, we are only left to show that $D_q^{(1)}(\mu) \leq D_q^{(2)}(\mu)$. Fix $\omega, \tau \in \mathcal{A}^\infty$, $\omega \neq \tau$, and write $\rho := \omega \wedge \tau = \alpha k^{p(\rho)}$, where the last letter of $\alpha = \alpha(\rho)$ is different from k. By (2.3), $|\phi_\rho'(\pi(\sigma^{|\rho|}\kappa))| \asymp \|\phi_\alpha'\|(p(\rho)+1)^{-\frac{\beta+1}{\beta}}$, where $\kappa \in \{\omega, \tau\}$ is that element for which $(\sigma^{|\rho|}\kappa)_1 \neq k$. Therefore, using (3.4), we can estimate for any $s > 0$:

$$\begin{aligned} I_{s,q}^{(2)}(\mu) &= \int_{\mathcal{A}^\infty} \left(\frac{d\mu(\tau)}{d_2(\omega,\tau)^s} \right)^{q-1} d\mu(\omega) \\ &\preceq \int_{\mathcal{A}^\infty} \left(\frac{d\mu(\tau)}{|\phi_\rho'(\pi(\sigma^{|\rho|}\kappa))|^s} \right)^{q-1} d\mu(\omega) \\ &\preceq \int_{\mathcal{A}^\infty} \left(\int_{\mathcal{A}^\infty} \|\phi_\alpha'\|^{-s} (p(\rho)+1)^{\frac{\beta+1}{\beta}s} \, d\mu(\tau) \right)^{q-1} d\mu(\omega) \\ &\preceq \int_{\mathcal{A}^\infty} \left(\int_{\mathcal{A}^\infty} |X_\alpha|^{-s} (p(\rho)+1)^{\frac{\beta+1}{\beta}s} \, d\mu(\tau) \right)^{q-1} d\mu(\omega) \end{aligned} \tag{3.13}$$

Given $\omega \in \mathcal{A}^\infty$ we define the sequence $\{k_n(\omega)\}_{n\geq 0}$ to be the increasing enumeration of all elements $i \geq 1$ such that $\omega_i \neq k$, with the convention $k_0(\omega) = 0$. Then fix an arbitrary $\ell \geq 1$ such that $C_8\lambda^\ell < 1$ and define

$$l_n(\omega) = \min\{k_n(\omega) + \ell, k_{n+1}(\omega)\} \tag{3.14}$$

and

$$C_9 = C_8/(1 - C_8\lambda^\ell). \tag{3.15}$$

Note that both $\omega \mapsto k_n(\omega)$ and $\omega \mapsto l_n(\omega)$ are Borel measurable functions. Further,

$$p(\omega|_{k_n(\omega)+i}) = p(\omega|_{k_n(\omega)} k^i) = i \qquad \text{for} \quad 0 \leq i \leq k_{n+1}(\omega) - k_n(\omega) - 1,$$

so we can continue (3.13) as follows

$$I_{s,q}^{(2)}(\mu) \preceq \int_{\mathcal{A}^\infty} \left(\sum_{n\geq 0} |X_{\omega|_{k_n(\omega)}}|^{-s} \sum_{i=0}^{k_{n+1}(\omega)-k_n(\omega)-1} (i+1)^{\frac{\beta+1}{\beta}s} \mu\big([\omega_{k_n(\omega)+i}] \setminus [\omega_{k_n(\omega)+i+1}]\big) \right)^{q-1} d\mu(\omega). \tag{3.16}$$

We claim that

$$\mu[\omega|_{k+j\ell}] - \mu[\omega|_{k+(j+1)\ell}] \le C_9\lambda^j \ (\mu[\omega|_k] - \mu[\omega|_{k+\ell}]) \tag{3.17}$$

for every $k \ge 1$ and every $j \ge 0$. And indeed, applying (3.12), we get

$$\begin{aligned}\mu[\omega|_{k+j\ell}] + C_9\lambda^j\mu[\omega|_{k+\ell}] &\le \mu[\omega|_k]C_8\lambda^{j\ell} + C_9\lambda^j\mu[\omega|_k]C_8\lambda^\ell \\ &= \mu[\omega|_k]C_8\lambda^j(\lambda^{j(\ell-1)} + C_9\lambda^\ell) \\ &\le \mu[\omega|_k]C_8\lambda^j(1 + C_9\lambda^\ell) = C_9\lambda^j\mu[\omega|_k],\end{aligned}$$

which obviously implies (3.17). In the last displayed line we used (3.15).

Now fix $n \ge 0$. There are two possibilities for $l_n(\omega)$ in (3.14). First suppose that $l_n(\omega) = k_n(\omega) + \ell$. We are going to estimate the inner sum in (3.16) by writing $\sum_i \le \sum_{i=0}^\infty = \sum_{j=0}^\infty \sum_{i=j\ell}^{(j+1)\ell}$. Thus, in view of (3.17),

$$\begin{aligned}
&\sum_{i=0}^{k_{n+1}(\omega)-k_n(\omega)-1} (i+1)^{\frac{\beta+1}{\beta}s}\mu\big([\omega_{k_n(\omega)+i}] \setminus [\omega_{k_n(\omega)+i+1}]\big) \\
\le\ & \sum_{i=0}^{\infty}(i+1)^{\frac{\beta+1}{\beta}s}\big(\mu[\omega_{k_n(\omega)+i}] - \mu[\omega_{k_n(\omega)+i+1}]\big) \\
\le\ & \sum_{j=0}^{\infty}((j+1)\ell+1)^{\frac{\beta+1}{\beta}s}\big(\mu[\omega_{k_n(\omega)+j\ell}] - \mu[\omega_{k_n(\omega)+(j+1)\ell}]\big) \\
\preceq\ & \left(\sum_{j=0}^{\infty}(j+1)^{\frac{\beta+1}{\beta}s}\lambda^j\right)\big(\mu[\omega|_{k_n(\omega)}] - \mu[\omega|_{k_n(\omega)+\ell}]\big) \\
\preceq\ & \mu[\omega|_{k_n(\omega)}] - \mu[\omega|_{l_n(\omega)}].
\end{aligned} \tag{3.18}$$

The second possibility is that $l_n(\omega) = k_{n+1}(\omega)$. Then $k_{n+1}(\omega) - k_n(\omega) \le \ell$ and we get

$$\begin{aligned}
&\sum_{i=0}^{k_{n+1}(\omega)-k_n(\omega)-1} (i+1)^{\frac{\beta+1}{\beta}s}\mu\big([\omega_{k_n(\omega)+i}] \setminus [\omega_{k_n(\omega)+i+1}]\big) \\
\le\ & \big(k_{n+1}(\omega) - k_n(\omega)\big)^{1+\frac{\beta+1}{\beta}s} \sum_{i=0}^{k_{n+1}(\omega)-k_n(\omega)-1} \big(\mu[\omega_{k_n(\omega)+i}] - \mu[\omega|_{k_n(\omega)+i+1}]\big) \\
\le\ & \ell^{1+\frac{\beta+1}{\beta}s}\big(\mu[\omega|_{k_n(\omega)}] - \mu[\omega|_{k_{n+1}(\omega)}]\big) \\
\preceq\ & \mu[\omega|_{k_n(\omega)}] - \mu[\omega|_{l_n(\omega)}].
\end{aligned}$$

In both cases the estimate ends with the same expression which we can substitute into (3.16) to obtain

$$\begin{aligned} I^{(2)}_{s,q}(\mu) &\preceq \int_{\mathcal{A}^\infty} \left(\sum_{n\geq 0} |X_{\omega|_{k_n(\omega)}}|^{-s} \big(\mu([\omega|_{k_n(\omega)}]) - \mu([\omega|_{l_n(\omega)}])\big) \right)^{q-1} d\mu(\omega) \\ &\leq \int_{\mathcal{A}^\infty} \left(\sum_{n\geq 0} \int_{[\omega|_{k_n(\omega)}]\setminus[\omega|_{l_n(\omega)}]} \frac{d\mu(\tau)}{|X_{\omega\wedge\tau}|^s} \right)^{q-1} d\mu(\omega) \\ &\leq \int_{\mathcal{A}^\infty} \left(\int_{\mathcal{A}^\infty} \frac{d\mu(\tau)}{|X_{\omega\wedge\tau}|^s} \right)^{q-1} d\mu(\omega) = I^{(1)}_{s,q}(\mu). \end{aligned}$$

This implies that $D^{(1)}_q(\mu) \geq D^{(2)}_q(\mu)$, and the proof of the proposition is complete. $\square$

Corollary 3.8. *Suppose that $\{\Phi^{\mathbf{t}}\}_{\mathbf{t}\in\overline{U}}$ is a family of parabolic IFS (2.4) satisfying (2.5) and (2.6), such that ϕ_k is well-behaved on some neighborhood of v. Suppose that μ is a well-mixing atomless Borel probability measure on $\mathcal{A}^\infty$ (in particular, a Bernoulli measure or, more generally, a Gibbs state of a Hölder continuous potential), such that $D^{(1,\mathbf{t}_0)}_q(\mu) > 1$ for some $q \in (1,2]$ and $\mathbf{t}_0 \in U$ Then there exists $\delta > 0$ such that for $\mathcal{L}_d$-a.e. $\mathbf{t} \in B_\delta(\mathbf{t}_0)$, the measure $\nu_{\mathbf{t}} = \mu\circ\pi_t^{-1}$ is absolutely continuous with a density in $L^q(\mathbb{R})$.*

Proof. This is an immediate consequence of Theorem 3.5 and Proposition 3.7. $\square$

Since for hyperbolic systems the two metrics d_1 and d_2 are equal, the proof of Theorem 3.5 (simplified by the observation that in hyperbolic case $h(\rho) = |\rho|$) demonstrates also the following.

Theorem 3.9. *Let $U \subset \mathbb{R}^d$ be an open set. Suppose that $\Phi^{\mathbf{t}} = \{\phi^{\mathbf{t}}_1, \dots, \phi^{\mathbf{t}}_k\}_{\mathbf{t}\in\overline{U}}$ is a family of hyperbolic IFS such that the mappings $\mathbf{t} \mapsto \phi^{\mathbf{t}}_i$ are continuous from $\overline{U}$ to $\mathcal{C}^{1+\theta}(X)$ for all $i \leq k$ and the transversality condition holds. Suppose that μ is a Borel atomless probability measure on $\mathcal{A}^\infty$ such that $D^{(1,\mathbf{t}_0)}_q(\mu) > 1$ for some $q \in (1,2]$ and $\mathbf{t}_0 \in U$. Then there exists $\delta > 0$ such that for $\mathcal{L}_d$-a.e. $\mathbf{t} \in B_\delta(\mathbf{t}_0)$ the measure $\mu\circ\pi_t^{-1}$ is absolutely continuous with a density in $L^q(\mathbb{R})$.*

4. Example: a class of random continued fractions

Here we apply our results to the family $\Phi^\alpha := \{\frac{x+\alpha}{x+\alpha+1}, \frac{x}{x+1}\}$. The interesting interval of parameters is $0 < \alpha < 0.5$, when the limit set of Φ^α is the interval $[0, \frac{1}{2}(-\alpha + \sqrt{\alpha^2+4\alpha})]$ and the IFS has an overlap, see [L]. The function $\phi_2(x) = \frac{x}{x+1}$ has a parabolic fixed point at

$x = 0$. It is easy to see that $\Phi^\alpha \in \Gamma_{[0,1]}(1)$ for all $\alpha > 0$ and the family $\{\Phi^\alpha\}_{\alpha>0}$ satisfies the continuity condition (2.5).

Let $\mu = (\frac{1}{2}, \frac{1}{2})^{\mathbb{N}}$ be the Bernoulli measure on the coding space $\mathcal{A}^\infty$ and let ν_α be its projection on the real line corresponding to the IFS Φ^α. The connection with continued fractions is as follows: denote

$$[a_1, a_2, a_3, \ldots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$$

Then ν_α is the distribution of the random continued fraction $[1, Y_1, 1, Y_2, 1, Y_3, \ldots]$ where Y_i are independent and take the values in $\{0, \alpha\}$ with probabilities $(\frac{1}{2}, \frac{1}{2})$. Lyons [L, Thm. 1.1] proved that there is a "critical value" $\alpha_c \in (0.2688, 0.2689)$ such that ν_α is singular with respect to the Lebesgue measure and is concentrated on a set of Hausdorff dimension less than 1 for all $\alpha > \alpha_c$. He also proved in [L, Prop. 3.1] that if ν_α has a density in $L^2(\mathbb{R})$, then $\alpha \le \frac{\sqrt{6}}{2} - 1$. In our joint work with K. Simon [SSU2], we showed that ν_α is absolutely continuous for a.e. $\alpha \in (0.215, \alpha_c)$. Our proof was based on Theorem 2.7 and on the following lemma.

Lemma 4.1. (see [SSU2, Lemma 6.2]). *The family $\{\Phi^\alpha\}$ satisfies the transversality condition (2.6) for $\alpha \in (0.215, 0.5)$.*

Now we can state the new result.

Theorem 4.2. *For any $1 < q \le 2$ there exists $\alpha_q \in [\frac{\sqrt{6}}{2} - 1, \alpha_c]$ such that*

(i) *If ν_α is absolutely continuous with a density in $L^q(\mathbb{R})$ then $\alpha \le \alpha_q$;*

(ii) *For Lebesgue-a.e. $\alpha \in (0.215, \alpha_q)$, the measure ν_α is absolutely continuous with a density in $L^q(\mathbb{R})$;*

(iii) $\alpha_2 = \frac{\sqrt{6}}{2} - 1 = 0.22474\ldots$

Remarks. 1. Although we restrict ourselves to the specific family $\{\Phi^\alpha\}$, many of the arguments below apply in a more general setting.

2. Since $L^{q_1}[0,1] \supset L^{q_2}[0,1]$ for $q_1 < q_2$, it is immediate from parts (i) and (ii) that the function $q \mapsto \alpha_q$ is non-increasing. We believe that it is actually strictly decreasing but we have not verified this. It also seems plausible that $\lim_{q\to 1} \alpha_q = \alpha_c$.

3. Besides the explicit value of α_2, following [L] one can show that $\alpha_{3/2} = 3\sqrt{2} - 4 = 0.24264\ldots$ We do not know any other explicit values for α_q. Lyons [L, Prop. 3.2] also gave a

necessary condition for ν_α to have a density in $\bigcap_{q<\infty} L^q(\mathbb{R})$, but our methods do not extend to $q > 2$.

The proof of the theorem will require some preparation. First we need the following simple lemma; its proof essentially repeats the argument from [L, Prop. 3.1]. Whenever we write summation over w it is understood that $w \in \mathcal{A}^*$. Recall that $X_w = \phi_w(X)$.

Lemma 4.3. *Let $\{\phi_1, \phi_2\}$ be an IFS on the interval $X \subset \mathbb{R}$ (parabolic or hyperbolic) and let ν be a Borel probability measure satisfying $\nu = \frac{1}{2}(\nu \circ \phi_1^{-1} + \nu \circ \phi_2^{-1})$. Suppose that ν is absolutely continuous with a density $\frac{d\nu}{dx} \in L^q(\mathbb{R})$ for some $q > 1$. Then*

$$\sup_n \; 2^{-nq} \sum_{|w|=n} |X_w|^{-(q-1)} < \infty. \tag{4.1}$$

Proof. For each $n \in \mathbb{N}$ we have that $\nu = \sum_{|w|=n} 2^{-n}(\nu \circ \phi_w^{-1})$. Thus the density $h(x) := \frac{d\nu}{dx}$ satisfies $h(x) = \sum_{|w|=n} 2^{-n} h_w(x)$ where $h_w = h \circ \phi_w^{-1}$ has support contained in X_w. Then

$$\|h\|_q^q = \int_X \Big(\sum_{|w|=n} 2^{-n} h_w(x) \Big)^q \, dx \geq \int_X \sum_{|w|=n} 2^{-nq} h_w^q(x) \, dx = \sum_{|w|=n} 2^{-nq} \int_X h_w^q(x) \, dx. \tag{4.2}$$

By the Hölder's inequality,

$$1 = \int_{X_w} h_w(x) \, dx \leq \Big(\int_{X_w} h_w^q(x) \, dx \Big)^{1/q} |X_w|^{\frac{q-1}{q}},$$

hence $\int_{X_w} h_w^q(x) \, dx \geq |X_w|^{-(q-1)}$. Substituting this into (4.2) yields (4.1). $\square$

Proof of Theorem 4.2. Consider the family $\{\Phi^\alpha\}$ and let

$$\alpha_q = \inf \Big\{ \alpha > 0 : \; \sup_n \; 2^{-nq} \sum_{|w|=n} |X_w^{(\alpha)}|^{-(q-1)} = \infty \Big\}. \tag{4.3}$$

Observe that $|X_w^{(\alpha)}| = \phi_w^\alpha(1) - \phi_w^\alpha(0)$.

Lemma 4.4. (a) *The function $\alpha \mapsto \phi_w^\alpha(x)$ is non-decreasing for all $x \in [0,1]$ and for all $w \in \mathcal{A}^*$.*

(b) *The function $\alpha \mapsto |X_w^{(\alpha)}|$ is non-increasing for all $w \in \mathcal{A}^*$.*

Proof. (a) This is an easy induction in $|w|$; we use that $\frac{x+\alpha}{x+\alpha+1}$ is increasing in α and all $\phi_w^\alpha(x)$ are increasing in x.

(b) One checks that if $0 \leq A < B$, $0 \leq C < D$, $A \leq C$, $B \leq D$, and $B - A \geq D - C$, then for any $\alpha < \widetilde{\alpha}$ and $i = 1, 2$, we have $\phi_i^\alpha(B) - \phi_i^\alpha(A) \geq \phi_i^\alpha(D) - \phi_i^\alpha(C)$ and

$\phi_i^\alpha(D) - \phi_i^\alpha(C) \geq \phi_i^{\widetilde\alpha}(D) - \phi_i^{\widetilde\alpha}(C)$. Then the desired statement follows by induction, using $A = \phi_w^\alpha(0)$, $B = \phi_w^\alpha(1)$, $C = \phi_w^{\widetilde\alpha}(0)$, and $D = \phi_w^{\widetilde\alpha}(1)$. □

It follows from Lemma 4.4(b), (4.3) and Lemma 4.3 that ν_α cannot have a density in $L^q(\mathbb{R})$ for $\alpha > \alpha_q$, verifying the property (i) of Theorem 4.2.

Fix any $\widetilde\alpha < \alpha_q$; we have

$$\sup_n \; 2^{-nq} \sum_{|w|=n} |X_w^{(\widetilde\alpha)}|^{-(q-1)} < \infty. \tag{4.4}$$

The following lemma is the key step in the proof of the property (ii).

Lemma 4.5. *Suppose that $\widetilde\alpha$ and $q \in (1,2]$ are such that (4.4) holds. Then for any $\alpha < \widetilde\alpha$ there exist $t > 1$ and $\xi \in (0,1)$ such that*

$$2^{-nq} \sum_{|w|=n} |X_w^{(\alpha)}|^{-t(q-1)} \leq const \cdot \xi^n. \tag{4.5}$$

Before proving the lemma let us show how the property (ii) of Theorem 4.2 is deduced. With the same notation as in Lemma 4.5 we have, using (3.2) and the fact that $(q-1) \in (0,1]$,

$$\begin{aligned} I_{t,q}^{(1,\alpha)}(\mu) &= \int_{\mathcal{A}^\infty} \left(\sum_{n=0}^\infty 2^{-n} |X_{\omega|_n}^{(\alpha)}|^{-t} \right)^{q-1} d\mu(\omega) \\ &\leq \int_{\mathcal{A}^\infty} \sum_{n=0}^\infty 2^{-n(q-1)} |X_{\omega|_n}^{(\alpha)}|^{-t(q-1)} d\mu(\omega) \\ &= \sum_{n=0}^\infty 2^{-nq} \sum_{|w|=n} |X_w^{(\alpha)}|^{-t(q-1)} < \infty, \end{aligned} \tag{4.6}$$

by (4.5). This shows that $D_q^{(1,\alpha)}(\mu) \geq t > 1$, so by Corollary 3.8, in view of Lemma 4.1, ν_α has a density in $L^q(\mathbb{R})$ for Lebesgue-a.e. $\alpha \in (0.215, \widetilde\alpha)$. Since $\widetilde\alpha$ can be arbitrarily close to α_q, the property (ii) of Theorem 4.2 follows. □

Proof of Lemma 4.5. We have $\alpha < \widetilde\alpha$ and $q \in (1,2]$ fixed. Recall that $h(w)$ is the number of hyperbolic letters (just the number of 1's in our case) in the word w. Split the sum in (4.5) as follows:

$$\sum_{|w|=n} |X_w^{(\alpha)}|^{-t(q-1)} = \sum_{\substack{|w|=n \\ h(w) \leq n/K}} |X_w^{(\alpha)}|^{-t(q-1)} + \sum_{\substack{|w|=n \\ h(w) > n/K}} |X_w^{(\alpha)}|^{-t(q-1)} \tag{4.7}$$

Sublemma 4.6. *For any $\zeta > 1$ there exist $K > 1$ such that*

$$\sum_{\substack{|w|=n \\ h(w)\leq n/K}} |X_w^{(\alpha)}|^{-2} \leq const \cdot \zeta^n \quad \text{for all} \quad n \in \mathbb{N}.$$

Proof. It is clearly enough to restrict ourselves to n sufficiently large. Write

$$w = 1^{k_1} 2^{\ell_1} 1^{k_2} 2^{\ell_2} \dots 1^{k_m} 2^{\ell_m},$$

where all k_i are positive except possibly k_1 and all ℓ_i are positive except possibly ℓ_m. Let $\widetilde{w} = w|_{|w|-\ell_m}$. Then it follows from (2.2) that

$$|X_w^{(\alpha)}| \geq (\ell_m + 1)^{-1/\beta} L(V)^{-1} \min_{x\in[0,1]} |(\phi_{\widetilde{w}}^{\alpha})'(x)|.$$

Therefore, estimating from below the derivative of powers of parabolic elements by (2.3) and the derivative of hyperbolic elements by (2.9) we obtain

$$\begin{aligned} |X_w^{(\alpha)}| &\geq (\ell_m + 1)^{-1/\beta} L(V)^{-m} u^{\sum_{i=1}^m k_i} \prod_{i=1}^{m-1} (\ell_i + 1)^{-(\beta+1)/\beta} \\ &\geq L(V)^{-m} u^{h(w)} \prod_{i=1}^{m} (\ell_i + 1)^{-(\beta+1)/\beta}, \end{aligned} \tag{4.8}$$

since $\sum_{i=1}^m k_i = h(w)$. Here u can be easily chosen independent of α (in fact, $(\phi_i^{\alpha})'(x) \geq (2+\alpha)^{-2} \geq 1/9$ for all $x \in [0,1]$ and $\alpha \leq 1$). Assume that $h(w) \leq n/K$ where K will be chosen later. Observe that $m - 1 \leq \sum_{i=1}^m k_i \leq n/K$ hence $\frac{n}{m} \geq (1 - \frac{1}{m})K$. We can estimate using the Geometric-Arithmetic Means Inequality:

$$\prod_{i=1}^m (\ell_i + 1) \leq \left(\frac{1}{m} \sum_{i=1}^m (\ell_i + 1) \right)^m = \left(1 + \frac{1}{m} \sum_{i=1}^m \ell_i \right)^m \leq (1 + n/m)^m.$$

It follows that

$$\prod_{i=1}^m (\ell_i + 1)^{-(\beta+1)/\beta} \geq (1 + n/m)^{-m(\beta+1)/\beta}.$$

We claim that the last expression is greater than $\zeta^{-n/4}$ if K is sufficiently large. Indeed, this is equivalent to

$$\frac{m(\beta+1)}{\beta} \log(1 + n/m) \leq (n/4) \log(\zeta),$$

or

$$\log(1 + n/m) \leq \frac{n}{m} \cdot \frac{\beta}{4(\beta+1)} \log(\zeta). \tag{4.9}$$

If $m = 1$, then (4.9) holds for n sufficiently large, and if $m \geq 2$, then $\frac{n}{m} \geq (1 - \frac{1}{m})K \geq \frac{K}{2}$, and it is clear that (4.9) holds for K sufficiently large. We have proved that there exist K_0 and N_0 such that

$$\prod_{i=1}^{m} (\ell_i + 1)^{-(\beta+1)/\beta} \geq \zeta^{-n/4} \quad \text{for all} \quad n \geq N_0$$

provided that $K \geq K_0$. Combined with (4.8) and using that $m \leq 1 + \frac{n}{K}$ this implies

$$\begin{aligned} \sum_{\substack{|w|=n \\ h(w)\leq n/K}} |X_w^{(\alpha)}|^{-2} &\leq \# \{w \in \mathcal{A}^n : \ h(w) \leq n/K\} \cdot L(V)^{2m} u^{-h(w)} \zeta^{n/2} \\ &\leq \# \{w \in \mathcal{A}^n : \ h(w) \leq n/K\} \cdot L(V)^{2+2n/K} u^{-2n/K} \zeta^{n/2}. \end{aligned} \tag{4.10}$$

Observe that $\#\{w \in \mathcal{A}^n : \ h(w) = p\} = \begin{pmatrix} n \\ p \end{pmatrix}$, and it is easy to deduce from Stirling's formula that

$$\#\{w \in \mathcal{A}^n : \ h(w) \leq n/K\} \leq \text{const} \cdot \zeta^{n/4} \quad \text{for all} \quad n \in \mathbb{N},$$

provided that K is sufficiently large. Since also $L(V)^{2n/K} u^{-2n/K} \leq \zeta^{n/4}$ for K sufficiently large, the sublemma follows from (4.10). $\square$

Sublemma 4.7. *Suppose that $0 < \alpha < \widetilde{\alpha}$. Then there exist $\eta > 1$ and $C > 0$ such that*

$$|X_w^{(\alpha)}| \geq C\eta^{h(w)} |X_w^{(\widetilde{\alpha})}|.$$

Proof. Suppose that $w = \rho 2^\ell$ where ρ ends with the hyperbolic letter 1 (or is empty). Then

$$|X_w^{(\alpha)}| \asymp (\ell + 1)^{-1/\beta} |(\phi_\rho^\alpha)'(1)| \quad \text{and} \quad |X_w^{(\widetilde{\alpha})}| \asymp (\ell + 1)^{-1/\beta} |(\phi_\rho^{\widetilde{\alpha}})'(1)|.$$

Therefore, omitting the $|\cdot|$ signs (since all the elements of the IFS are increasing), we obtain

$$\frac{|X_w^{(\alpha)}|}{|X_w^{(\widetilde{\alpha})}|} \asymp \frac{(\phi_\rho^\alpha)'(1)}{(\phi_\rho^{\widetilde{\alpha}})'(1)} = \prod_{i=1}^{n-\ell} \frac{(\phi_{w_i}^\alpha)'(\phi_{\sigma^i\rho}^\alpha(1))}{(\phi_{w_i}^{\widetilde{\alpha}})'(\phi_{\sigma^i\rho}^{\widetilde{\alpha}}(1))}. \tag{4.11}$$

We have $\phi_{\sigma^i\rho}^{\widetilde{\alpha}}(1) \geq \phi_{\sigma^i\rho}^\alpha(1)$ by Lemma 4.4(a), and $(\phi_{w_i}^{\widetilde{\alpha}})'(\cdot)$ is decreasing, hence

$$\frac{(\phi_{w_i}^\alpha)'(\phi_{\sigma^i\rho}^\alpha(1))}{(\phi_{w_i}^{\widetilde{\alpha}})'(\phi_{\sigma^i\rho}^{\widetilde{\alpha}}(1))} \geq \frac{(\phi_{w_i}^\alpha)'(\phi_{\sigma^i\rho}^\alpha(1))}{(\phi_{w_i}^{\widetilde{\alpha}})'(\phi_{\sigma^i\rho}^\alpha(1))} \geq \frac{(\phi_{w_i}^\alpha)'(1)}{(\phi_{w_i}^{\widetilde{\alpha}})'(1)}. \tag{4.12}$$

The last inequality holds since $\frac{(\phi_1^\alpha)'(x)}{(\phi_1^{\widetilde{\alpha}})'(x)} = (\frac{x+\widetilde{\alpha}}{x+\alpha})^2$ is decreasing in x and $\frac{(\phi_2^\alpha)'(x)}{(\phi_2^{\widetilde{\alpha}})'(x)} \equiv 1$. Now observe that the last expression in (4.12) is equal to 1 if $w_i = 2$ (the parabolic letter), and is equal to $\frac{2+\widetilde{\alpha}}{2+\alpha} > 1$ if $w_i = 1$. In view of (4.11), this implies the sublemma, with $\eta = \frac{2+\widetilde{\alpha}}{2+\alpha}$. $\square$

Conclusion of the proof of Lemma 4.5. Fix $\zeta \in (1,2)$ and find $K > 1$ from Sublemma 4.6. For any $t \in (1,2)$ and $q \in (1,2]$ we have, using that $|X_w^{(\alpha)}| \le 1$:

$$2^{-nq} \sum_{\substack{|w|=n \\ h(w)\le n/K}} |X_w^{(\alpha)}|^{-t(q-1)} \le 2^{-n} \sum_{\substack{|w|=n \\ h(w)\le n/K}} |X_w^{(\alpha)}|^{-2} \le \text{const} \cdot (\zeta/2)^n \tag{4.13}$$

by Sublemma 4.6, so it remains to estimate the second sum in (4.7). By Sublemma 4.7,

$$\begin{aligned} \sum_{\substack{|w|=n \\ h(w)>n/K}} |X_w^{(\alpha)}|^{-t(q-1)} &\preceq \sum_{\substack{|w|=n \\ h(w)>n/K}} |X_w^{(\widetilde{\alpha})}|^{-t(q-1)} \eta^{-h(w)t(q-1)} \\ &\le \sum_{|w|=n} |X_w^{(\widetilde{\alpha})}|^{-t(q-1)} \eta^{-nt(q-1)/K} \\ &= \sum_{|w|=n} |X_w^{(\widetilde{\alpha})}|^{-(q-1)} |X_w^{(\widetilde{\alpha})}|^{-(t-1)(q-1)} \eta^{-nt(q-1)/K}. \end{aligned} \tag{4.14}$$

Recall that $|X_w^{(\widetilde{\alpha})}| \ge \inf_{x\in[0,1]} |(\phi_w^{\widetilde{\alpha}})'(x)| \ge u^{|w|}$ by (2.9). Then we can continue the estimate (4.14) as follows:

$$\sum_{\substack{|w|=n \\ h(w)>n/K}} |X_w^{(\alpha)}|^{-t(q-1)} \preceq \sum_{|w|=n} |X_w^{(\widetilde{\alpha})}|^{-(q-1)} u^{-n(t-1)(q-1)} \eta^{-nt(q-1)/K}. \tag{4.15}$$

So far, we only assumed that $t > 1$; now we specify the condition on t needed for the proof. Suppose that

$$0 < t - 1 \le \frac{\log \eta}{2K \log(1/u)}.$$

Then $u^{-n(t-1)(q-1)} \le \eta^{nt(q-1)/2K}$, and (4.15) can be continued as follows:

$$\sum_{\substack{|w|=n \\ h(w)>n/K}} |X_w^{(\alpha)}|^{-t(q-1)} \preceq \sum_{|w|=n} |X_w^{(\widetilde{\alpha})}|^{-(q-1)} \eta^{-nt(q-1)/2K}.$$

Since $\eta > 1$ and $t > 1$, this, together with (4.13), (4.4) and (4.7), implies (4.5), where we can take $\xi = \max\{\frac{\zeta}{2}, \eta^{-(q-1)/2K}\}$. This concludes the proof of Lemma 4.5, and so the proof of Theorem 4.2(ii) is complete as well. $\square$

Proof of Theorem 4.2(iii). This is essentially proven by Lyons [L], but we repeat some of the steps using our notation for the reader's convenience. For $w \in \mathcal{A}^n$ let

$$M_n^{(\alpha)}(w) := \begin{pmatrix} a_n^{(\alpha)}(w) & b_n^{(\alpha)}(w) \\ c_n^{(\alpha)}(w) & d_n^{(\alpha)}(w) \end{pmatrix} := \prod_{i=1}^{n} \begin{pmatrix} 1 & w_i\alpha \\ 1 & 1 + w_i\alpha \end{pmatrix}. \tag{4.16}$$

Then it is easy to verify by induction that

$$\phi_w^\alpha(x) = \frac{a_n^{(\alpha)}(w)x + b_n^{(\alpha)}(w)}{c_n^{(\alpha)}(w)x + d_n^{(\alpha)}(w)}.$$

Therefore,

$$|X_w^{(\alpha)}| = \phi_w^\alpha(1) - \phi_w^\alpha(0) = [d_n^{(\alpha)}(w)(c_n^{(\alpha)}(w) + d_n^{(\alpha)}(w))]^{-1},$$

since the determinant in (4.16) equals 1. Recall that $\alpha_2 = \inf\{\alpha : \sup_n \mathcal{S}_n^{(\alpha)} = \infty\}$ where

$$\mathcal{S}_n^{(\alpha)} := 2^{-2n} \sum_{|w|=n} |X_w^{(\alpha)}|^{-1} = 2^{-2n} \sum_{|w|=n} d_n^{(\alpha)}(w)(c_n^{(\alpha)}(w) + d_n^{(\alpha)}(w)).$$

Consider the product measure $\mu_n = (\frac{1}{2}, \frac{1}{2})^n$ on $\mathcal{A}^n$; then

$$\mathcal{S}_n^{(\alpha)} = 2^{-n}\mathbb{E}[d_n^{(\alpha)} c_n^{(\alpha)} + (d_n^{(\alpha)})^2],$$

where $\mathbb{E}$ is the expectation with respect to μ_n. Consider the tensor product $M_n^{(\alpha)} \otimes M_n^{(\alpha)}$ and note that

$$\begin{pmatrix} d_n^{(\alpha)} c_n^{(\alpha)} & (d_n^{(\alpha)})^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} (M_n^{(\alpha)} \otimes M_n^{(\alpha)}) \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lyons [L] observed that, by independence, $\mathbb{E}[M_n^{(\alpha)} \otimes M_n^{(\alpha)}] = (R^{(\alpha)})^n$, where

$$R^{(\alpha)} := \mathbb{E}\left[\begin{pmatrix} 1 & w_1\alpha \\ 1 & 1 + w_1\alpha \end{pmatrix} \otimes \begin{pmatrix} 1 & w_1\alpha \\ 1 & 1 + w_1\alpha \end{pmatrix}\right].$$

An easy computation, see [L], yields that $R^{(\alpha)}$ has the largest eigenvalue less than 2 if and only if $\alpha < \frac{\sqrt{6}}{2} - 1$. In view of the above, it follows that $\alpha_2 = \frac{\sqrt{6}}{2} - 1$, keeping in mind that the Perron-Frobenius eigenvector is strictly positive. □

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