RIGIDITY OF MULTI-DIMENSIONAL CONFORMAL ITERATED FUNCTION SYSTEMS

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ABSTRACT. The paper starts with an appropriate version of the bounded distortion theorem. We show that for a regular, satisfying the "Open Set Condition", iterated function system of countably many conformal contractions of an open connected subset of a Euclidean space $I\!R^d$ with $d \geq 3$, the Radon-Nikodym derivative $d\mu/dm$ has a real-analytic extension on an open neighbourhood of the limit set of this system, where m is the conformal measure and μ is the unique probability invariant measure equivalent with m. Next, we explore in this context the concept of essential affinity of iterated function systems providing its several necessary and sufficient conditions. We prove the following rigidity result. If $d \geq 3$ and h, a topological conjugacy between two not essentially affine systems F and G sends the conformal measure m_F to a measure equivalent with the conformal measure m_G, then h has a conformal extension on an open neighbourhood of the limit set of the system F. Finally in exactly the same way as in [MPU] we extend our rigidity result to the case of parabolic systems.

1. Introduction, Preliminaries

This paper extends the rigidity result of conformal iterated function systems in the complex plane (see [MPU]) to any dimension $d \geq 3$. It states the following (comp. Theorem 4.2). If h, a topological conjugacy between two not essentially affine systems F and G in a dimension $d \geq 3$, transports the conformal measure m_F to the equivalence class of the conformal measure m_G, then h has a conformal extension on an open neighbourhood of the limit set of the system F. The general approach undertaken in this paper is modeled on that in [MPU]. The ideas and technics involved in most of the proofs are partially or entirely different from and incomparable with their counterparts in [MPU]. One of the causes of this fact is the higher dimensional structure of the phase space and the lack of advanced technics of the complex function theory in the plane. On the other hand the special structure of conformal maps in dimensions $d \geq 3$ (see (1.1) makes the presented theory slightly more elegant and clearer. The first result of our paper, Theorem 1.1, provides the distortion tool. Then we show that for a regular, satisfying the "Open Set Condition", iterated function system of countably many conformal contractions of an open connected subset of a Euclidean space $I\!R^d$ with $d \geq 3$, the Radon-Nikodym derivative $d\mu/dm$ has a real-analytic extension on an open neighbourhood of the limit set of this system, where m is the conformal measure and μ is the unique probability invariant measure equivalent with m. Next, we explore in this context the concept of essential

The research of the second author was supported in part by the NSF Grant DMS 9801583. He also wishes to thank IHES, the University of Orleans and the Mathematical Institute of the Polish Academy of Science and the University of Warwick where a part of this paper was written for warm hospitality and excellent working conditions.

affinity of iterated function systems providing its several necessary and sufficient conditions. The main result, Theorem 1.4 (comp. Theorem 4.2) concerns rigidity. At the end of the paper we briefly discuss parabolic systems.

In [MU1] we have provided the framework to study infinite conformal iterated function systems. We shall recall first this notion and some of its basic properties. Let I be a countable index set with at least two elements and let $S = \{\phi_i : X \to X : i \in I\}$ be a collection of injective contractions from a compact metric space X into X for which there exists $0 < s < 1$ such that $\rho(\phi_i(x), \phi_i(y)) \leq s\rho(x, y)$ for every $i \in I$ and for every pair of points $x, y \in X$. Thus, the system S is uniformly contractive. Any such collection S of contractions is called an iterated function system. We are particularly interested in the properties of the limit set defined by such a system. We can define this set as the image of the coding space under a coding map as follows. Let I^n denote the space of words of length n, I^∞ the space of infinite sequences of symbols in I, $I^* = \bigcup_{n\geq 1} I^n$ and for $\omega \in I^n$, $n \geq 1$, let $\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_n}$. If $\omega \in I^* \cup I^\infty$ and $n \geq 1$ does not exceed the length of ω, we denote by $\omega|_n$ the word $\omega_1\omega_2 \ldots \omega_n$. Since given $\omega \in I^\infty$, the diameters of the compact sets $\phi_{\omega|_n}(X)$, $n \geq 1$, converge to zero and since they form a decreasing family, the set

$$\bigcap_{n=0}^{\infty} \phi_{\omega|_n}(X)$$

is a singleton and therefore, denoting its only element by $\pi(\omega)$, defines the coding map $\pi : I^\infty \to X$. The main object of our interest will be the limit set

$$J = \pi(I^\infty) = \bigcup_{\omega\in I^\infty} \bigcap_{n=1}^{\infty} \phi_{\omega|n}(X),$$

Observe that J satisfies the natural invariance equality, $J = \bigcup_{i\in I} \phi_i(J)$. Notice that if I is finite, then J is compact and this property fails for infinite systems.

An iterated function system S is said to be *conformal* if $X \subset I\!\!R^d$ for some $d \geq 1$, X is connected and the following conditions are satisfied.

- **(1a):** Open Set Condition (OSC). $\phi_i(\mathrm{Int}X) \cap \phi_j(\mathrm{Int}X) = \emptyset$ for every pair $i, j \in I$, $i \neq j$.
- **(1b):** There exists an open connected set V such that $X \subset V \subset I\!\!R^d$ such that all maps ϕ_i, $i \in I$, extend to C^1 conformal diffeomorphisms of V into V. (Note that for $d = 1$ this just means that all the maps ϕ_i, $i \in I$, are C^1 monotone diffeomorphisms, for $d \geq 2$ the words *conformal* mean holomorphic or anti-holomorphic, and for $d \geq 3$, the maps ϕ_i, $i \in I$ are Möbius transformations. The proof of the last statement can be found in [BP] and [Va] for example, where it is called Liouville's theorem)
- **(1c):** There exist $\gamma, l > 0$ such that for every $x \in \partial X \subset I\!\!R^d$ there exists an open cone $\mathrm{Con}(x, \gamma, l) \subset \mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure γ, and altitude l.
- **(1d):** Bounded Distortion Property(BDP). There exists $K \geq 1$ such that

$$|\phi'_\omega(y)| \leq K|\phi'_\omega(x)|$$

for every $\omega \in I^*$ and every pair of points $x, y \in V$, where $|\phi'_\omega(x)|$ means the norm of the derivative.

In fact throughout the whole paper we will need one more condition which (comp. [MU1]) can be considered as a strengthening of (BDP).

(1e): There are two constants $L \geq 1$ and $\alpha > 0$ such that

$$\Big||\phi'_i(y)| - |\phi'_i(x)|\Big| \leq L||\phi'_i|||y-x|^\alpha.$$

for every $i \in I$ and every pair of points $x, y \in V$.

As we have already mentioned in the item (1b), it is known (see [BP] and [Va]) that in every dimension $d \geq 3$ each C^1 conformal homeomorphism ϕ defined on an open connected subset of $I\!\!R^d$ extends to the entire space $I\!\!R^d$ and takes on the form

$$\phi = \lambda A \circ i_{a,r} + b, \tag{1.1}$$

where $0 < \lambda \in R$ is a positive scalar, A is a linear isometry in $I\!\!R^d$, $i_{a,r}$ is either the inversion with respect to some sphere centered at a point a and with radius r, or the identity map, and $b \in I\!\!R^d$. In the sequel λ will be called the scalar factor and $a = \phi^{-1}(\infty)$ - the center of inversion. If A is the identity map, ϕ will be called a conformal affine homeomorphism. From now on we assume that $d \geq 3$. We would first of all like to demonstrate that in this case the Bounded distortion Property (1d) and the property (1e) are satisfied automatically. Namely, we have the following.

Theorem 1.1. *If $\{\phi_i\}_{i\in I}$ is a collection of maps satisfying condition (1b), then the conditions (1d) and (1e) are also satisfied, perhaps with a smaller set V. The property (1e) takes on the following stronger form*

$$||\phi'_\omega(y) - |\phi'_\omega(x)|| \leq K||\phi'_\omega||||y-x|| \tag{1.2}$$

for all $\omega \in I^*$, all $x, y \in V$ and some K sufficiently large.

Proof. Let $U = B(X, \frac{1}{2}\text{dist}(X, \partial V))$. Then U is an open neighbourhood of X and U is connected since X is. Fix $\omega \in I^*$. In view of (1.1) there exist $\lambda_\omega > 0$, a linear isometry A_ω, an inversion (or the identity map) $i_\omega = i_{a_\omega, r_\omega}$ and a vector $b_\omega \in I\!\!R^d$ such that $\phi_\omega = \lambda_\omega A_\omega \circ i_\omega + b_\omega$. In case when i_ω is the identity map the statement of our theorem is obvious. So, we may assume that i_ω is an inversion. Then for every $z \in I\!\!R^d$

$$|\phi'_\omega(z)| = \frac{\lambda_\omega r_\omega^2}{||z - a_\omega||^2}.$$

Hence, for all $x, y \in I\!\!R^d$

$$\frac{|\phi'_\omega(y)|}{|\phi'_\omega(x)|} = \frac{||x - a_\omega||^2}{||y - a_\omega||^2}. \tag{1.3}$$

Since $\phi_\omega(V) \subset V$, $a_\omega \notin V$. Therefore for all $x, y \in U$

$$\begin{aligned}\frac{||x-a_\omega||}{||y-a_\omega||} &\le \frac{||x-y||+||y-a_\omega||}{||y-a_\omega||} = 1+\frac{||x-y||}{||y-a_\omega||} \\ &\le 1+\frac{\operatorname{diam}(U)}{\operatorname{dist}(U,\partial V)} \le 1+\frac{\operatorname{diam}(U)}{\operatorname{dist}(X,\partial U)}.\end{aligned} \tag{1.4}$$

Thus

$$\frac{|\phi_\omega'(y)|}{|\phi_\omega'(x)|} \le \left(1+\frac{\operatorname{diam}(U)}{\operatorname{dist}(X,\partial U)}\right)^2.$$

The proof of the first part of our theorem is complete. In order to prove the second part we may assume without loosing generality that $|\phi_\omega'(x)| \le |\phi_\omega'(y)|$. Using (1.3) and (1.4) we then get

$$\begin{aligned}||\phi_\omega'(y)|-|\phi_\omega'(x)|| &\le ||\phi_\omega'||\left(\frac{|\phi_\omega'(y)|}{|\phi_\omega'(x)|}-1\right) = ||\phi_\omega'||\left(\frac{||x-a_\omega||^2}{||y-a_\omega||^2}-1\right) \\ &= ||\phi_\omega'||\left(\frac{||x-a_\omega||}{||y-a_\omega||}-1\right)\left(\frac{||x-a_\omega||}{||y-a_\omega||}+1\right) \\ &\le ||\phi_\omega'||\left(2+\frac{\operatorname{diam}(U)}{\operatorname{dist}(X,\partial U)}\right)\frac{||x-y||}{||y-a_\omega||} \\ &\le \left(2+\frac{\operatorname{diam}(U)}{\operatorname{dist}(X,\partial U)}\right)\frac{1}{\operatorname{dist}(U,\partial V)}||\phi_\omega'||||y-x||\end{aligned}$$

The proof is complete with V replaced by U. ■

As was demonstrated in [MU1], conformal iterated function systems naturally break into two main classes, irregular and regular. This dichotomy can be determined from either the existence of a zero of a natural pressure function or, equivalently, the existence of a conformal measure. Let us explore this latter option since conformal measures and not topological pressure will be used in the sequel. So, recall that a Borel probability measure m is said to be t-conformal provided $m(J) = 1$ and for every Borel set $A \subset X$ and every $i \in I$

$$m(\phi_i(A)) = \int_A |\phi_i'|^t \, dm$$

and

$$m(\phi_i(X) \cap \phi_j(X)) = 0,$$

for every pair $i, j \in I$, $i \ne j$. It has been proved in [MU1] that if a t-conformal measure exists, then $= \delta$, the Hausdorff dimension of the limit set J_S of S and this measure is unique. The system S is called *regular* if a conformal measure exists and from now on we assume that the system S is regular unless stated otherwise. We define the associated Perron-Frobenius operator acting on $C(X)$ as follows

$$\mathcal{L}(f)(x) = \sum_{i\in I} |\phi_i'(x)|^\delta f(\phi_i(x)).$$

Notice that the norm of $\mathcal{L}$ is equal to $||\mathcal{L}(1\!\!1)|| \le \psi(\delta)$ and the nth iterate of $\mathcal{L}$ is given by the formula

$$\mathcal{L}^n(f)(x) = \sum_{|\omega|=n} |\phi'_\omega(x)|^\delta f(\phi_\omega(x)).$$

In [MU1] we have proved that for every $n \ge 1$

$$||\mathcal{L}^n|| = \sum_{|\omega|=n} ||\phi'_\omega||^\delta \le K^\delta \tag{1.5}$$

Theorem 1.2 below explains what we really need the Perron-Frobenius operator for in this paper. The conformal measure m is a fixed point of the operator conjugate to $\mathcal{L}$. We recall also (see [MU1, Theorem 3.8]) that there exists a Borel probability measure μ, invariant in the sense that for every measurable set A,

$$\mu(\bigcup_{i\in I} \phi_i(A)) = \mu(A),$$

and in addition μ is equivalent with m with the Radon-Nikodym derivative $d\mu/dm$ bounded away from zero and infinity. In Sections 4 and 2 we will need more knowledge about this derivative and in particular we will need to know how it is computed. The appropriate information is contained in the following (see [MU3]).

Theorem 1.2. *The Radon-Nikodym derivative $d\mu/dm$ has a version which continuously extends to a function $\rho : X \to (0, \infty)$ and which is a unique fixed point of the Perron-Frobenius operator $\mathcal{L}$ whose integral with respect to the conformal measure m is equal to 1. Moreover the iterates $\mathcal{L}^n(1\!\!1)$ converge uniformly on X to ρ.*

We call two iterated function systems $F = \{f_i : X \to X, i \in I\}$ and $G = \{g_i : Y \to Y, i \in I\}$ topologically conjugate if and only if there exists a homeomorphism $h : J_F \to J_G$ such that

$$h \circ f_i = g_i \circ h$$

for all $i \in I$. Then by induction we easily get that $h \circ f_\omega = g_\omega \circ h$ for every finite word ω. The Section 2 of the paper [HU] (see also Appendix 1 of [MPU]) contains the proof of the following.

Theorem 1.3. *Suppose that $F = \{f_i : X \to X, i \in I)$ and $G = \{g_i : Y \to Y, i \in I\}$ are two topologically conjugate conformal iterated function systems not necessarily regular. Then the following four conditions are equivalent.*

(1): $\exists C \ge 1\ \forall \omega \in I^*$

$$C^{-1} \le \frac{\mathrm{diam}(g_\omega(Y))}{\mathrm{diam}(f_\omega(X))} \le C.$$

(2): *$|g'_\omega(y_\omega)| = |f'_\omega(x_\omega)|$ for all $\omega \in I^*$, where x_ω and y_ω are the only fixed points of $f_\omega : X \to X$ and $g_\omega : Y \to Y$ respectively.*

(3): $\exists E \ge 1\ \forall \omega \in I^*$

$$E^{-1} \le \frac{||g'_\omega||}{||f'_\omega||} \le E.$$

(4): *For every finite subset T of I,* $\mathrm{HD}(J_{G,T}) = \mathrm{HD}(J_{F,T})$ *and the conformal measures* $m_{G,T}$ *and* $m_{F,T} \circ h^{-1}$ *are equivalent.*

Suppose additionally that both systems F and G are regular. Then the following condition is also equivalent to the four conditions above.

(5): $\mathrm{HD}(J_G) = \mathrm{HD}(J_F)$ *and the conformal measures* m_G *and* $m_F \circ h^{-1}$ *are equivalent.*

As we already mentioned, our main goal in this paper is to prove the rigidity theorem saying that the conditions (1)-(5) imply that the conjugacy has a conformal extension. For finite systems arising from inverse branches of a holomorphic expanding map on a mixing repeller contained in a complex plane $\mathbb{C}$ a sufficient condition for this implication is that the systems are non-linear (not essentially affine in the terminology of the present paper), (see [Su], [Pr]). This condition is also sufficient for infinite 1-dimensional real-analytic systems (see [HU]) and infinite 2-dimensional holomorphic systems (see [MPU]). Here we shall prove this rigidity for infinite C^1-conformal systems in the dimension $d \ge 3$.

The main result of this paper is the following.

Theorem 1.4. *If two Open Set Condition conformal regular iterated function systems* $\{f_i : X \to X : i \in I\}$ *and* $\{g_i : Y \to Y : i \in I\}$ *are not essentially affine and conjugate by a homeomorphism* $h : J_F \to J_G$, *then the following conditions are equivalent.*

(i): *The conjugacy between the systems* $\{f_i : X \to X : i \in I\}$ *and* $\{g_i : Y \to Y : i \in I\}$ *extends in a conformal fashion to an open neighbourhood of X.*

(ii): *The measures* m_G *and* $m_F \circ h^{-1}$ *are equivalent.*

The concept of essentially affine systems is introduced in Section 3 and the full extended version for of Theorem 1.4 is stated and proven in Section 4 as Theorem 4.2. In Section 2 we prove an important technical result which is interesting itself, namely the real analyticity of the Radon-Nikodym derivative $d\mu/dm$ of invariant measure μ with respect to conformal measure m. In Section 3 we deal with various conditions equivalent with essential affinity and, as we already mentioned, we prove in Section 4 our main result, Theorem 1.4 which is presented there in its extended version enumerated as Theorem 4.2. In Section 5 we extend the results of Section 4 to the case of parabolic iterated function systems.

We would like to conclude this introduction by emphasizing that although the undertaken to the subjects are by nature different, there exists however a close ideological analogy between our main result Theorem 1.4 and celebrated Mostow's rigidity theorem whose furthest going generalizations and and a fairly comlete list of relevant literature can be found in [Tu].

2. The Radon-Nikodym derivative ρ is real-analytic

From now on throughout the whole paper we assume that $d \ge 3$ and $\{\phi_i\}$ is an Open Set Condition conformal regular iterated function system.

Our main goal in this section is to prove the following.

Theorem 2.1. *The Radon-Nikodym derivative ρ has a real-analytic extension on an open connected neighbourhood U of X in V.*

Proof. In view of (1.1), there exist $\lambda_\omega > 0$, a linear isometry A_ω, an inversion (or the identity map) $i_\omega = i_{a_\omega, r_\omega}$ and a vector $b_\omega \in I\!\!R^d$ such that $\phi_\omega = \lambda_\omega A_\omega \circ i_\omega + b_\omega$. Then

$$|\phi_\omega'(z)| = \frac{\lambda_\omega r_\omega^2}{||z - a_\omega||^2} \text{ if} i_\omega \neq \mathrm{Id}$$

and

$$|\phi_\omega'(z)| = \lambda_\omega \text{ if} i_\omega = \mathrm{Id}$$

Since $\phi_\omega(V) \subset V$, we get $a_\omega \notin V$. Fix $\xi \in X$ and consider the function $\rho_\omega : \mathbb{C}^d \to \mathbb{C}$ given by the formula

$$\rho_\omega(z) = \frac{||\xi - a_\omega||^2}{\sum_{j=1}^d \Big(z_j - (a_\omega)_j\Big)^2} \text{ or } \rho_\omega(z) = 1 \text{ if } i_\omega = \mathrm{Id}.$$

We shall show that there exist a constant $B > 0$ and a neighbourhood $\tilde{U}$ of X in $\mathbb{C}^d$ such that

$$|\rho_\omega(z)| \le B \tag{2.1}$$

for every $\omega \in I^*$ and every $z \in \tilde{U}$. Indeed, otherwise there exist sequences $\omega^{(n)} \in I^*$ and $z^{(n)} \in \mathbb{C}^d$, $n \ge 1$, such that $\lim_{n\to\infty} \mathrm{dist}(z^{(n)}, X) = 0$ and $|\rho_{\omega^{(n)}}(z^{(n)})| \ge n$ for every $n \ge 1$. Passing to a subsequence we may assume that the limit $w = \lim_{n\to\infty} z^{(n)}$ exists. Then $w \in X \subset I\!\!R^d$ and for every $n \ge 1$ we have

$$\sum_{j=1}^d \Big(z_j^{(n)} - (a_{\omega^{(n)}})_j\Big)^2 = \sum_{j=1}^d \Big((z_j^{(n)} - w_j) + (w_j - (a_{\omega^{(n)}})_j)\Big)^2 =$$

$$= \sum_{j=1}^d \Big((z_j^{(n)} - w_j)^2 + 2\sum_{j=1}^d (z_j^{(n)} - w_j)(w_j - (a_{\omega^{(n)}})_j) + \sum_{j=1}^d (w_j - (a_{\omega^{(n)}})_j)^2. \tag{2.2}$$

Now, since $a_{\omega^{(n)}} \notin V$ and $w \in X$, we get

$$\sum_{j=1}^d (w_j - (a_{\omega^{(n)}})_j)^2 = ||w - a_{\omega^{(n)}}||^2 \ge \mathrm{dist}^2(X, \partial V) \tag{2.3}$$

Fix now $q \ge 1$ so large that for every $n \ge q$

$$\sum_{j=1}^d |z_j^{(n)} - w_j|^2 \le \frac{1}{4}\mathrm{dist}^2(X, \partial V) \tag{2.4}$$

and

$$\sum_{j=1}^d |z_j^{(n)} - w_j| \le \min\left\{\frac{1}{8d}\mathrm{dist}^2(X, \partial V), \frac{1}{8d}\right\}. \tag{2.5}$$

So, if $|w_j - (a_{\omega^{(n)}})_j| \le 1$, then by (2.5)

$$2|z_j^{(n)} - w_j||w_j - (a_{\omega^{(n)}})_j| \le \frac{1}{4d}\mathrm{dist}^2(X, \partial V) \tag{2.6}$$

and if $|w_j - (a_{\omega(n)})_j| \le 1$, then by the other part of(2.5)

$$2|z_j^{(n)} - w_j||w_j - (a_{\omega(n)})_j| \le 2|z_j^{(n)} - w_j||w_j - (a_{\omega(n)})_j|^2 \le \frac{1}{4d}|w_j - (a_{\omega(n)})_j|^2$$
$$\le ||w - a_{\omega(n)}||^2. \tag{2.7}$$

Applying now (2.3) along with (2.3), (2.4), (2.6), and (2.7), we get for every $n \ge q$

$$\left|\sum_{j=1}^d \left(z_j^{(n)} - (a_{\omega(n)})_j\right)^2\right| \ge \sum_{j=1}^d (w_j - (a_{\omega(n)})_j)^2 - \sum_{j=1}^d |z_j^{(n)} - w_j|^2 - 2\sum_{j=1}^d |z_j^{(n)} - w_j||w_j - (a_{\omega(n)})_j|$$
$$\ge ||w - a_{\omega(n)}||^2 - \frac{1}{4}||w - a_{\omega(n)}||^2 - d\frac{1}{4d}||w - a_{\omega(n)}||^2 = \frac{1}{2}||w - a_{\omega(n)}||^2.$$

And therefore, using also Theorem 1.1(property (1e)) and (1.3), we get

$$n \le |\rho_{\omega(n)}(z^{(n)})| = \frac{||\xi - a_{\omega(n)}||^2}{\sum_{j=1}^d \left(z_j^{(n)} - (a_{\omega(n)})_j\right)^2} \le 2\frac{||\xi - a_{\omega(n)}||^2}{||w - a_{\omega(n)}||^2} \le 2K.$$

This contradiction finishes the proof of (2.2). Decreasing $\tilde U$ if necessary, we may assume that this set is connected. Now, for every $n \ge 1$ define the function $b_n : \tilde U \to \mathbb{C}$ by setting

$$b_n(z) = \sum_{|\omega|=n} \rho_\omega^\delta(z)|\phi'_\omega(\xi)|^\delta.$$

Notice that the power $\rho_\omega^\delta(z)$ (strictly speaking any of its branches) is well defined and analytic since $\rho_\omega(z)$ is defined on the simply connected set $\mathbb{C}^d$. Since each term of this series is an analytic function and since, by (ref1.pf) and (2.1)

$$\sum_{|\omega|=n} |\rho_\omega^\delta(z)||\phi'_\omega(\xi)|^\delta \le B^h \sum_{|\omega|=n} ||\phi'_\omega||^\delta \le B^\delta K^\delta,$$

we conclude that all the functions $b_n : \tilde U \to \mathbb{C}$ are analytic and $||b_n||_\infty \le B^\delta K^\delta$ for every $n \ge 1$. Hence, inview of Montel's theorem, we can choose a subsequence $\{b_{n_k}\}_{k=1}^\infty$ converging on a connected neighbourhood $\tilde U_1$ of X (with closure $\tilde U_1$ contained in $\tilde U$) to an analytic function $b : \tilde U_1 \to \mathbb{C}$. Since for every $n \ge 1$ and every $z \in X$, $b_n(z) = \sum_{|\omega|=n} |\phi'_\omega(z)|^\delta = \mathcal{L}^n(\mathbb{1})$, it therefore follows from Theorem 1.2 that $b|_X = \rho = \frac{d\mu}{dm}$. Hence, putting $U = \mathrm{Pr}(\tilde U_1)$, where Pr: $\mathbb{C}^d \to \mathbb{R}^d$ is the orthogonal projection from $\mathbb{C}^d$ to $\mathbb{R}^d$, completes the proof. ■

For every $\omega \in I^*$ denote by $D_{\phi_\omega} = \frac{d\mu\circ\phi_\omega}{d\mu}$ the Jacobian of the map $\phi_\omega : J \to J$ with respect to the measure μ. As an immediate consequence of Theorem 2.1, the following computation

$$\frac{d\mu\circ\phi_\omega}{d\mu} = \frac{d\mu\circ\phi_\omega}{dm\circ\phi_\omega}\cdot\frac{dm\circ\phi_\omega}{dm}\cdot\frac{dm}{d\mu} = \left(\frac{d\mu}{dm}\circ\phi_\omega\right)\cdot|\phi'_\omega|^\delta\cdot\frac{dm}{d\mu}$$

and the observation that $|\phi'_\omega|^\delta$ is real-analytic on V, we get the following.

Corollary 2.2. *For every $i \in I$ the Jacobian D_{ϕ_i} has a real-analytic extension $\tilde D_{\phi_i}$ on the neighbourhood U of X produced in Theorem 2.1.*

3. Essentially Affine Systems

We begin this section with the following.

Lemma 3.1. *Suppose that $\phi : I\!R^d \to I\!R^d$, $d \geq 3$, is a conformal diffeomorphism that has an attracting fixed point a ($\phi(a) = a$, $|\phi'(a)| < 1$). If M is an open connected C^1-submanifold of $I\!R^d$ such that $\phi(M) \subset M$ and $a \in M$, then M is either a subset of a ϕ-invariant affine subspace of the same dimension as M, or a subset of a ϕ-invariant geometric sphere of the same dimension as M*

Proof. Since a is an attracting fixed point of ϕ, there exists a radius $r > 0$ so small that $\phi_{-1}\big(\overline{R}^d \setminus B(a,r)\big) \subset \overline{R}^d \setminus B(a,r)$, where $\overline{R}^d$ is the Alexandrov compactification of $I\!R^d$ obtained by adding the point at infinity. Since $\overline{R}^d \setminus B(a,r)$ is a closd topological disk, in view of Brower's fixed point theorem there exists a fixed point b of ϕ^{-1} in $\overline{R}^d \setminus B(a,r)$. Hence b is also a fixed point of ϕ and $b \neq a$. Then the map

$$\psi = i_{b,1} \circ \phi \circ i_{b,1}$$

($i_{b,1}$ equals identity if $b = \infty$) fixes ∞ which means that this map is affine, and $w = i_{b,1}(a)$ is an attracting fixed point of ψ. In addition $\psi(\tilde{M}) \subset \tilde{M}$, where $\tilde{M} = i_{b,1}(M)$, $w \in \tilde{M}$, and $\psi : I\!R^d \to I\!R^d$, as an affince map, can be written in the form $\lambda A + c$, where $\lambda > 0$ and A is an orthogonal matrix. Since $\psi(\tilde{M}) \subset \tilde{M}$, and since ψ is a diffeomorphism, $\psi'(z)(T_z\tilde{M}) = T_{\psi(z)}\tilde{M}$. In particular $\psi'(w)E = E$, where $E = T_w\tilde{M}$. Without loosing generality we may assume that $\tilde{M}$ is contained in the basin of immediate attraction to w. We shall show that

$$T_z\tilde{M} = E$$

for every $z \in \tilde{M}$. And indeed, take an arbitrary point point $z \in \tilde{M}$. Since $\psi'(x) = \lambda A$ for all $x \in I\!R^d$ and since λA is conformal, we get for all $n \geq 0$ that

$$\angle(T_z\tilde{M}, E) = \angle(A^n(T_z\tilde{M}), A^nE) = \angle((\psi^n)'(z)T_z\tilde{M}), E) = \angle(T_{\psi^n(z)}\tilde{M}, E),$$

where $\angle$ denotes the angle between linear hyperspaces. Since $\lim_{n\to\infty} T_{\psi^n(z)}\tilde{M} = T_w\tilde{M} = E$, we conclude that $\angle(T_z\tilde{M}, E) = 0$, or equivalently $T_z\tilde{M} = E$. Since the only integral manifolds of a constant field of linear subspaces are affine subspaces, we conclude that $\tilde{M}$ is contained in an affine subspace. Since $\tilde{M}$ is its open subset, this affine subspace is ϕ-invariant. Since $M = i_{b,1}(\tilde{M})$, we are done. ■

We call the system $S = \{\phi_i\}_{i\in I}$ at most q-dimensional, $1 \leq q \leq d$, if there exists M_S, either a q-dimensional liear subspace of $I\!R^d$ or a q-dimensional geometric sphere contained in $I\!R^d$ such that $\overline{J} \subset M_S$ and $\phi_i(M_S) = M_S$ for all $i \in I$. We call the system $S = \{\phi_i\}_{i\in I}$ q-dimensional if q is the minimal number with this property.

Lemma 3.2. *If a non-empty open subset of $\overline{J}$ is contained in a q-dimensional real-analytic submanifold, then the system S is at most q-dimensional.*

Proof. The assumptions of the lemma state that there exists a point $x \in \overline{J}$, an open ball $B(x)$ centered at x and M, a p-dimensional open connected real-analytic submanifold M containing $\overline{J} \cap B(x)$, where $1 \le p \le q$ is the minimal integer with this property. Fix now an arbitrary auxiliary point $z \in \overline{J}$. Since $x \in \overline{J}$, there exists $\omega \in I^*$ such that $\phi_\omega(z) \in \overline{J} \cap B(x)$, moreover $\phi_\omega(V) \subset B(x)$. Then the set $\phi_\omega(V) \cap M$ contains $\phi_\omega(V) \cap \overline{J}$, an open neighbourhood of $\phi_\omega(z)$ in $\overline{J}$ and consists of countably many connected p-dimensional real-analytic submanifolds. Taking the length of ω large enough we may assume that this countable family is just a one manifold. Then $N = \phi_\omega^{-1}(\phi_\omega(V) \cap M)$ is a connected p-dimensional real-analytic submanifold (there are no branching points since ϕ_ω^{-1} is 1-to-1) containing $\overline{J}$ and contained in V). Fix an arbitrary $i \in I$. Since $\lim_{n\to\infty} \phi_i^n(N) = x_i$, the only attracting fixed point of ϕ, and since the connected component of $\phi_i^n(N) \cap N$ containing x_i is a manifold containing an open neighbourhood of x_i in the space $\overline{J}$, recalling the definition of p, we conclude that for every $n \ge 1$ large enough, $\phi_i^n(N) \subset N$. Hence, in view of Lemma 3.1 applied with $\phi = \phi_i^n$, we gain that N is an open subset of a p-dimensional set M_S, either an affine subset or a geometric sphere contained in $I\!\!R^d$ invariant under ϕ_i^n. Now, for every $j \in I$, $\phi_J(M_S) \cap M_S \ne \emptyset$ since $J \subset M_S$. Since in addition $\phi_J(M_S) \cap M_S$ is either an affine subset or a geometric sphere contained in $I\!\!R^d$, we conclude from the minimality of p that $\phi_J(M_S) = M_S$. We are done. ■

Definition 3.3. *We say that the system S is essentially affine if S is conjugate by a conformal homeomorphism with a system consisting of conformal affine contractions ($\lambda A + b$)) only.*

The main goal of this section is to prove the following.

Theorem 3.4. *Suppose that the system $S = \{\phi_i\}_{i\in I}$ is regular and denote the corresponding conformal measure by m. Then the following conditions are equivalent.*

- **(a):** *For each $i \in I$ the extended Jacobian $\tilde{D}_{\phi_i} : U \to I\!\!R$ is constant, where U is the neighbourhood of X produced in Corollary 2.3.*
- **(b):** *There exist a continuous function $u : X \to I\!\!R$ and constants $c_i \in I\!\!R$, $i \in I$, such that*

$$\log|\phi_i'| = u - u \circ \phi_i + c_i$$

  for all $i \in I$.
- **(c):** *There exist a continuous function $u : \overline{J} \to I\!\!R$ and constants $c_i \in I\!\!R$, $i \in I$, such that*

$$\log|\phi_i'| = u - u \circ \phi_i + c_i$$

  for all $i \in I$.
- **(d):** *The system S is essentially affine.*
- **(er):** *There exist a real-analytic function $\gamma : V \to Lis(d)$ such that*

$$\gamma \circ \phi_i \cdot \frac{\phi_i'}{|\phi_i'|} \cdot \gamma^{-1} = k_i \in LC(d)$$

for every $i \in I$, where $Lis(d)$ is the group of all linear isometries on $\mathbb{R}^d$ and $LC(d)$ is the group of all linear conformal (of the form λA) homeomorphisms of $\mathbb{R}^d$. The composition of linear maps we denote here and in the sequel either by $\cdot$ or we put no sign.

(ec): *The same as (er) but γ is required to be continuous only.*

(g): *If S is not q-dimensional, then the vectors*

$$(\nabla \tilde{D}\phi_i \circ \phi_{\omega^{(j)}(z)})_{j=1}^{q}$$

are linearly dependent for all $z \in \overline{J}$, all $i \in I$ and all sequnces $(\omega^{(j)})_{j=1}^{q} \in (I^*)^q$.

(f): *If S is q-dimensional, $1 \leq q \leq d$, then either S is essentially affine or there exists a field of linear subspaces in TM_S of dimension and co-dimension greater than or equal to 1 defined on a neighbourhood of $\overline{J}$ in M_S and invariant under the action of derivatives of all maps ϕ_i, $i \in I$.*

Proof. We shall prove the following implications (a)⇒(b)⇒(c)⇒(d)⇒(a), $(d) \Rightarrow (er) \Rightarrow (ec) \Rightarrow (d)$ and (a)⇒(g)⇒(f)⇒(d).

(a)⇒(b). Since for every $i \in I$, $\tilde{D}_{\phi_i} = (\rho \circ \phi_i) \cdot |\phi_i'|^{\delta} \cdot \rho^{-1}$, we have

$$\log(|\tilde{D}_{\phi_i}|) = \log(|\rho| \circ \phi_i) + \delta \log|\phi_i'| - \log|\rho|.$$

Thus to finish the proof of the implication (a)⇒(b) it suffices to set $c_i = \frac{1}{\delta}\log(\tilde{D}_{\phi_i})$ and $u = \frac{1}{\delta}\log|\rho|$.

The implication (b)⇒(c) is obvious.

(c)⇒(d). If all the maps ϕ_i, $i \in I$, are linear, there is nothing to prove. So, we may assume that there exists $j \in I$ such that the map ϕ_j is not affine. For every $n \geq 1$ let $a^{(n)}$ be the inversion center of ϕ_j^n. We shall prove that the sequnce $\{a^{(n)}\}_{n=1}^{\infty}$ does not converge to ∞. Indeed, suppose on the contrary that this sequence converges to inffinity. Since $a^{(n)} = \phi_{j^n}^{-1}(\infty) = (\phi_j^{-1})^n(\infty)$, we therefore get

$$\infty = \lim_{n\to\infty} a^{(n-1)} = \lim_{n\to\infty} \phi_j((\phi_j^{-1})^n) = \phi_j(\lim_{n\to\infty}(\phi_j^{-1})^n) = \phi_j(\lim_{n\to\infty} a^{(n)}) = \phi_j(\infty)$$

which means that ϕ_j is affine. This contradiction shows that there exists a subsequence $\{k_n\}_{n=1}^{\infty}$ such that $a^{(k_n)} \to a$ for some $a \in \mathbb{R}^d$. Fix $v \in J$, the unique fixed point of $\phi_j : V \to V$. Iterating equation (c) n times, we get for every $z \in \overline{J}$ that

$$\begin{aligned} u(z) - u(v) &= \log|\phi_{j^n}'(z)| - \log|\phi_{j^n}'(v)| + u(\phi_{j^n}(z)) - u(\phi_{j^n}(v)) \\ &= -2\log||z - a^{(n)}|| + 2\log||v - a^{(n)}|| + u(\phi_{j^n}(z)) - u(v). \end{aligned}$$

Since $\phi_{j^n}(z)$ converges to v and since the function u is continuous, passing to the limit along the subsequence $\{k_n\}_{n=1}^{\infty}$, we get $u(z) = u(v) - 2\log||z-a|| + 2\log||v-a||$. Define the conformal map $G : \mathbb{R}^d \to \mathbb{R}^d$ by setting

$$G(z) = \mathrm{e}^{u(v)}||v-a||^2 i_{a,1}.$$

Then $\log|G'(z)| = u(v) + 2\log||v-a|| - 2\log||z-a|| = u(z)$. Therefore, using (c) again, we get for every $i \in I$ that $\log|\phi_i'| = \log|G'| - \log|G' \circ \phi_i| + c_i$, or equivalently that

$$|(G \circ \phi_i \circ G^{-1})(z)| = \mathrm{e}^{c_i} \tag{3.1}$$

for all $z \in G(\overline{J})$. Suppose that $G \circ \phi_i \circ G^{-1}$ is not affine and let w and $\lambda >$ denote respectively its inversion center and the scalar coefficient. Then (3.1) takes on the form $\lambda||z-w||^{-2} = \mathrm{e}^{c_i}$ on $G(\overline{J})$ or

$$||z-w||^2 = \lambda \mathrm{e}^{-c_i} \text{ on } G(\overline{J}).$$

So,$G(\overline{J})$ is contained in the sphere $S(w, \sqrt{\lambda \mathrm{e}^{-c_i}})$ centered at w and of radius $\sqrt{\lambda \mathrm{e}^{-c_i}}$. Since for every $n \geq 0$, $G \circ \phi_i^n \circ G^{-1}(G(\overline{J})) = G \circ \phi_i^n(\overline{J}) \subset G(\overline{J})$, we conclude that all the descending sets $G \circ \phi_i^n \circ G^{-1}(G(\overline{J}))$ are contained in the sphere $S(a, \sqrt{\lambda \mathrm{e}^{-c_i}})$. Let $H \subset S(w, \sqrt{\lambda \mathrm{e}^{-c_i}})$ be a minimal sphere (in the sense of inclusion) containing at least one of the sets $G \circ \phi_i^n \circ G^{-1}(G(\overline{J}))$, $n \geq 0$. Thus there exists $k \geq 0$ such that $G \circ \phi_i^k \circ G^{-1}(G(\overline{J}))$. Then

$$G \circ \phi_i^{k+1} \circ G^{-1}(G(\overline{J})) \subset H \cap (G \circ \phi_i \circ G^{-1}(H)). \tag{3.2}$$

Since $G \circ \phi_i \circ G^{-1}(H)$ is either a sphere or an affine subspace of $\mathbb{R}^d$ and since $G \circ \phi_i^{k+1} \circ G^{-1}(G(\overline{J}))$ contains at least three points (is uncountable in fact), the intersection $H \cap (G \circ \phi_i \circ G^{-1}(H))$ is a sphere (at least 1-dimensional) again. Therefore by the minimality of H and by (3.2) we conclude that $H \cap (G \circ \phi_i \circ G^{-1}(H)) \supset H$, which means that $G \circ \phi_i \circ G^{-1}(H) \supset H$. Therefore, since $\dim(G \circ \phi_i \circ G^{-1}(H)) = \dim(H)$, we conclude that

$$G \circ \phi_i \circ G^{-1}(H) = H \tag{3.3}$$

Let x_i be the unique fixed point of the map $\phi_i : V \to V$. Since $G \circ \phi_i^n \circ G^{-1}(z) \to G(x_i)$ uniformly on $G(V) \subset G(\overline{J})$, it follows from (3.1) that $c_i > 0$. So, for every $z \in S(w, \sqrt{\lambda \mathrm{e}^{-c_i}}) \supset H$,

$$|(G \circ \phi_i \circ G^{-1})'(z)| = \lambda_i||z-w||^{-2} = \lambda_i \lambda_i^{-1} \mathrm{e}^{c_i} = \mathrm{e}^{c_i}.$$

This implies that $G \circ \phi_i \circ G^{-1}$ is a uniform contraction on H and therefore $G \circ \phi_i^n \circ G^{-1}(z) \to G(x_i)$ uniformly on H. This however contradicts (3.3) and finishes the proof of the implication (c)⇒(d).

(d)⇒(a). Let $G : \mathbb{R}^d \to \mathbb{R}^d$ be a conformal homeomorphism providing conjugacy of S with a system consisting only of conformal affine contractions. Then for every $i \in I$

$$g_i = |(G \circ \phi_i \circ G^{-1})'(z)|$$

is a number independent of $z \in G(V)$. By the chain rule we have for every $z \in V$ the following

$$\mathcal{L}^n(\mathbb{1})(z) = \sum_{|\omega|=n} |\phi_\omega(z)'|^\delta = \sum_{|\omega|=n} |G'(z)|^\delta |G'(\phi_\omega(z))|^{-\delta} \prod_{i=1}^n g_{\omega_i}^\delta.$$

Fix now $j \in I$. Then for every $n \geq 1$ and all $z \in V$ we get

$$\frac{\mathcal{L}^n(1\!\!1)(\phi_j(z))}{\mathcal{L}^n(1\!\!1)(z)}|\phi_j'(z)|^\delta = \frac{\sum_{|\omega|=n} |G'(\phi_j(z))|^\delta |G'(\phi_\omega(\phi_j(z)))|^{-\delta} \prod_{i=1}^n g_{\omega_i}^\delta}{\sum_{|\omega|=n} |G'(z)|^\delta |G'(\phi_\omega(z))|^{-\delta} \prod_{i=1}^n g_{\omega_i}^\delta} g_j^\delta |G'(z)|^\delta |G'(\phi_j(z))|^{-\delta}$$
$$= \frac{\sum_{|\omega|=n} \prod_{i=1}^n |G'(\phi_\omega(z))|^\delta}{\sum_{|\omega|=n} \prod_{i=1}^n |G'(\phi_\omega(\phi_j(z))|^\delta} g_j^\delta.$$

Since $||\phi_\omega(x) - x_\omega|| \leq const\, s^{|\omega|}$, where x_ω is the only fixed point of $\phi_\omega : V \to V$, we conclude that

$$\frac{\mathcal{L}^n(1\!\!1)(\phi_j(z))}{\mathcal{L}^n(1\!\!1)(z)}|\phi_j'(z)|^\delta \to g_j^\delta$$

uniformly on V. Hence applying Theorem 1.2, we conclude that

$$\tilde{D}_{\phi_j} = \rho(\phi_j(z))\rho(z)|\phi_j'(z)|^\delta = \lim_{n\to\infty} \frac{\mathcal{L}^n(1\!\!1)(\phi_j(z))}{\mathcal{L}^n(1\!\!1)(z)}|\phi_j'(z)|^\delta = g_j^\delta$$

on X. Since $\tilde{D}_{\phi_j}$ is real-analytic on U, we conclude that $\tilde{D}_{\phi_j} = g_j^\delta$ on U. The proof of the implication (d)$\Rightarrow$(a) is complete.

(d)$\Rightarrow$(er). Define

$$\gamma = \frac{G'}{|G'|}.$$

Since for every $i \in I$, $G \circ \phi_i \circ G^{-1}$ is affine, we conclude that $G' \circ \phi_i \circ G^{-1} \cdot \phi_i' \circ G^{-1} \cdot (G')^{-1} \circ G^{-1} = k_i \in LC(d)$. Hence

$$\left(\left(\frac{G'}{|G'|}\right) \circ \phi_i \cdot \left(\frac{\phi_i'}{|\phi_i'|}\right) \cdot \left(\frac{G'}{|G'|}\right)^{-1}\right) \circ G^{-1} = \frac{k_i}{|k_i|}$$

and it suffices to take $\gamma = \frac{G'}{|G'|}$. Thus the proof of the implication (d)$\Rightarrow$(er) is complete.

The implication (er)$\Rightarrow$(ec) is obvious.

(ec)$\Rightarrow$(d). If all the maps ϕ_i, $i \in I$ are affine, there is nothing to prove. So, assume that there is $j \in I$ such that ϕ_j is not affine. Then no iterate ϕ_{j^n} is affine and let $a^{(n)}$ denote the inversion center of ϕ_{j^n}. Fix $v \in J$. By (ec) the following holds for every $z \in V$ and every $n \geq 1$,

$$(\gamma(v))^{-1}\gamma(z) = \left(\frac{\phi_{j^n}'(v)}{|\phi_{j^n}'(v)|}\right)^{-1} \cdot \Big(\gamma \circ \phi_{j^n}(v)\Big)^{-1} k_j^n k_j^{-n} \Big(\gamma \circ \phi_{j^n}(z)\Big) \left(\frac{\phi_{j^n}'(z)}{|\phi_{j^n}'(z)|}\right)$$
$$= \left(\frac{\phi_{j^n}'(v)}{|\phi_{j^n}'(v)|}\right)^{-1} \cdot \Big(\gamma \circ \phi_{j^n}(v)\Big)^{-1} \Big(\gamma \circ \phi_{j^n}(z)\Big) \left(\frac{\phi_{j^n}'(z)}{|\phi_{j^n}'(z)|}\right)$$
$$= (T_n(v))^{-1}\Big(\gamma \circ \phi_{j^n}(v)\Big)^{-1}\Big(\gamma \circ \phi_{j^n}(z)\Big)T_n(z),$$

where $T_n(w) = \mathrm{Id} - 2Q(w - a^{(n)})$ and in the canonical coordinates Q is given by the matrix

$$Q(x) = \frac{x_i x_j}{||x||^2}.$$

We shall now prove that the sequnce $\{a^{(n)}\}_{n=1}^\infty$ does not converge to ∞. Indeed, suppose on the contrary that $\lim_{n\to\infty} a^{(n)} = \infty$. Since $a^{(n)} = \phi_{j^n}^{-1}(\infty) = (\phi_j^{-1})^n(\infty)$, we therefore get

$$\infty = \lim_{n\to\infty} a^{(n-1)} = \lim_{n\to\infty} \phi_j((\phi_j^{-1})^n) = \phi_j(\lim_{n\to\infty}(\phi_j^{-1})^n) = \phi_j(\lim_{n\to\infty} a^{(n)}) = \phi_j(\infty)$$

which means that ϕ_j is affine. This contradiction shows that there exists a subsequence $\{k_n\}_{n=1}^\infty$ such that $a^{(k_n)} \to a$ for some $a \in I\!R^d$. Then for every $n \geq 1$,

$$(\gamma(v))^{-1}\gamma(z) = (T_{k_n}(v))^{-1}\Big(\gamma \circ \phi_{j^{n_k}}(v)\Big)^{-1}\Big(\gamma \circ \phi_{j^{n_k}}(z)\Big)T_{k_n}(z),$$

and taking limit when $n \to \infty$, we obtain $(\gamma(v))^{-1}\gamma(z) = \Big(\mathrm{Id} - 2Q(v-a)\Big)^{-1}\Big(\mathrm{Id} - 2Q(z-a)\Big)$ or, equivalently,

$$\gamma(z) = \gamma(v)\Big(\mathrm{Id} - 2Q(v-a)\Big)^{-1}\Big(\mathrm{Id} - 2Q(z-a)\Big).$$

Define

$$G = \gamma(v)\Big(\mathrm{Id} - 2Q(v-a)\Big)^{-1} \circ i_{a,1}.$$

Then

$$G'(z) = \gamma(v)\Big(\mathrm{Id} - 2Q(v-a)\Big)^{-1}\frac{1}{||z-a||^2}\Big(\mathrm{Id} - 2Q(z-a)\Big).$$

Hence

$$\frac{G'(z)}{|G'(z)|} = \gamma(v)\Big(\mathrm{Id} - 2Q(v-a)\Big)^{-1}\Big(\mathrm{Id} - 2Q(z-a)\Big) = \gamma(z)$$

Therefore for every $i \in I$, (ec) takes on the form

$$\frac{G' \circ \phi_i(z)}{|G' \circ \phi_i(z)|} \cdot \frac{\phi_i'(z)}{|\phi_i'(z)|} \cdot \left(\frac{G'(z)}{|G'(z)|}\right) = k_i.$$

Suppose that $G \circ \phi_i \circ G^{-1}$ is not affine. Then for y, the inversion center of $G \circ \phi_i \circ G^{-1}$, we get in canonical coordinates that

$$\delta_{mn} - 2\frac{(z_m - y_m)(z_n - y_n)}{||z-y||^2} = (k_i)_{mn}$$

for all $z \in V$ and all $m, n \in \{1, 2, \ldots, d\}$, where δ denotes the Kronecker symbol here. But this is impossible and we conclude that $G \circ \phi_i \circ G^{-1}$ is affine. The proof of the implication (ec)$\Rightarrow$(d) is complete.

The implication (a)$\Rightarrow$(g) follows from the implication (a)$\Rightarrow$(d).

(g)$\Rightarrow$(f). Conjugating the system S by a conformal diffeomorphism we may assume that $M_S = I\!R^q$. Given $i \in I$ and $(\omega^{(j)})_{j=1}^q \in (I^*)^q$ let

$$A = (i, \omega^{(1)}, \ldots, \omega^{(q)})$$

and let $H_A : I\!\!R^q \to I\!\!R^q$ be the map defined by the formula

$$H_A(z) = \Big(\tilde{D}_{\phi_i} \circ \phi_{\omega^{(1)}}(z), \dots, \tilde{D}_{\phi_i} \circ \phi_{\omega^{(q)}}(z)\Big).$$

Suppose first that for every $i \in I$ there exists A such that $H'_A = 0$ on $\overline{J}$. Since S is not $(q-1)$-dimensional, this implies that $H'_A = 0$ on a neighbourhood of $\overline{J}$ in $I\!\!R^q$. But then $\tilde{D}_{\phi_i}$ is constant on an open subset of $I\!\!R^q$ having a non-empty intersection with $\overline{J}$. Since by Corollary 2.2, $\tilde{D}_{\phi_i}$ is real-analytic, it is therefore constant on the appropriate set U_q produced in this corollary. Hence, in view of already proven implication (a)⇒(d), the system $\{\phi_i : I\!\!R^q \to I\!\!R^q$ is conjugate by a conformal diffeomorphism $\rho : R^q \to I\!\!R^q$ with an affine system. Since ρ extends to a conformal diffeomorphism from $I\!\!R^d$ to $I\!\!R^d$ and since an extension of an affine map in $I\!\!R^q$ to an affince map in $I\!\!R^d$ is also affine (if $q \le 1$ we need to be certain that these extensions are of the form $\lambda A + b$), we are done in this case.

So, suppose that there exists $i \in I$ such that for every A with the first element equal to i there exists $x \in \overline{J}$ such that $H'_A(x) \ne 0$. Choose $w \in \overline{J}$ and $A = (i, \omega^{(1)}, \dots, \omega^{(q)})$ such that $\dim \mathrm{Ker} H'_A(w)$ is minimal, say equal to $p \le q-1$. By the assumptions of (g), $\dim \mathrm{Ker} H'_A(w) \ge 1$. So

$$1 \le \dim \mathrm{Ker} H'_A = p \le q-1$$

on W, a neighbourhood of w in $I\!\!R^q$. By the definition of the limit set J for every $z \in \overline{V}$ there exists $\tau \in I^*$ such that $\phi_\tau(z) \in W$. Then define

$$l(z) = (\phi_\tau^{-1})'_{\phi_\tau(z)}(\mathrm{Ker} H'_A(\phi_\tau(z))),$$

where, changing temporarily notation, $(\phi_\tau^{-1})'_{\phi_\tau(z)}$ denotes the derivative of the map ϕ_τ^{-1} evaluated at the point $\phi_\tau(z)$. We want to show first that we define in this manner a line field on $\overline{V}$. So, we need to show that if $\phi_\tau(z), \phi_\eta(z) \in W$, then

$$(\phi_\tau^{-1})'_{\phi_\tau(z)}(l(\phi_\tau(z))) = (\phi_\eta^{-1})'_{\phi_\eta(z)}(l(\phi_\eta(z))). \tag{3.4}$$

Suppose on the contrary that (3.4) fails with some z, τ, η as required above. Then there exists a point $x \in W$ and $\gamma \in I^*$ (in fact for every $x \in W$ there exists γ) such that $\phi_\gamma(x)$ is so close to z that

$$(\phi_\tau^{-1})'_{\phi_\tau(\phi_\gamma(x))}(l(\phi_\tau(\phi_\gamma(x)))) \ne (\phi_\eta^{-1})'_{\phi_\eta(\phi_\gamma(x))}(l(\phi_\eta(\phi_\gamma(x)))).$$

Hence

$$(\phi_{\tau\gamma}^{-1})'_{\phi_{\tau\gamma}(x)} l(\phi_{\tau\gamma}(x)) \ne (\phi_{\eta\gamma}^{-1})'_{\phi_{\eta\gamma}(x)} l(\phi_{\eta\gamma}(x)).$$

So, either

$$(\phi_{\tau\gamma}^{-1})'_{\phi_{\tau\gamma}(x)} l(\phi_{\tau\gamma}(x)) \ne \mathrm{Ker} H'_A(x)$$

or

$$(\phi_{\eta\gamma}^{-1})'_{\phi_{\eta\gamma}(x)} l(\phi_{\eta\gamma}(x)) \ne \mathrm{Ker} H'_A(x).$$

Without loosing generality we may assume that the first inequality holds. Since $(H_A \circ \phi_{\tau\gamma})'(x) = H'_A(\phi_{\tau\gamma}(x))\phi'_{\tau\gamma}(x)$, we get $\mathrm{Ker}\Big(H_A \circ \phi_{\tau\gamma})'(x)\Big) = \phi'_{\tau\gamma}(x)^{-1}(\mathrm{Ker} H'_A(\phi_{\tau\gamma}(x)))$ and therefore

$$\mathrm{Ker}\Big(H_A \circ \phi_{\tau\gamma})'(x)\Big) \ne \mathrm{Ker} H'_A(x). \tag{3.5}$$

If now $\phi_\gamma(x)$ is sufficiently close to z, then $\phi_{\tau\gamma}(x)$ is so close to $\phi_\tau(z)$ that $\phi_{\tau\gamma}(x) \in W$. Then

$$\dim\Big(\mathrm{Ker}H'_A(\phi_{\tau\gamma}(x))\Big) = p = \dim(\mathrm{Ker}H'_A(x)). \tag{3.6}$$

Consider now linearly independent vectors $\Big(\nabla\tilde{D}\phi_i \circ \phi_{\omega^{(k_1)}}(x), \dots, \nabla\tilde{D}\phi_i \circ \phi_{\omega^{(k_t)}}(x)\Big)$, $t = q - \dim(\mathrm{Ker}H'_A(x))$. If $v \in \mathrm{Ker}H'_A(x)$, then $< \nabla\tilde{D}\phi_i \circ \phi_{\omega^{(k_j)}}(x), v >= 0$ for all $j = 1, 2, \dots, t$. Suppose that each vector $\nabla\tilde{D}\phi_i \circ \phi_{\omega^{(j)}\tau\gamma}(x)$, $j = 1, \dots, q$, is a linear combination of the vectors $\Big(\nabla\tilde{D}\phi_i \circ \phi_{\omega^{(k_1)}}(x), \dots, \nabla\tilde{D}\phi_i \circ \phi_{\omega^{(k_t)}}(x)\Big)$, $t = q - \dim(\mathrm{Ker}H'_A(x))$. Then $< \nabla\tilde{D}\phi_i \circ \phi_{\omega^{(j)}\tau\gamma}(x), v >= 0$ for all $j = 1, \dots, q$ and all $v \in \mathrm{Ker}H'_A(x)$. Hence $\mathrm{Ker}\Big((H_A \circ \phi_{\tau\gamma})'\Big)(x) \supset \mathrm{Ker}H'_A(x)$. Thus using (3.6) we conclude that $\mathrm{Ker}\Big((H_A \circ \phi_{\tau\gamma})'\Big)(x) = \mathrm{Ker}H'_A(x)$. This contradicts (3.5) and shows that there exists $1 \le u \le q$ such that the vectors $\Big(\nabla\tilde{D}\phi_i \circ \phi_{\omega^{(k_j)}}(x)\Big)_{j=1}^t$ together with the vector $\nabla\tilde{D}\phi_i \circ \phi_{\omega^{(u)}\tau\gamma}(x)$ form a linearly independent set. Hence, if $B = (i, \omega^{(u)}\tau\gamma, \omega^{(k_1)}, \dots, \omega^{(k_t)}, i, \dots, i)$ (($q - (t+1)$)i's at the end), then the rank of $H'_B(x)$ is greater than or equal to $t + 1$. Thus $\mathrm{Ker}H'_B(x) = q - \mathrm{rank}(H'_B(x)) \le q - (t+1) = q - q + \dim(\mathrm{Ker}H'_A(x)) - 1 = p - 1$ which is a contradiction with the definition of p and finishes the proof of the implication (g)⇒(f).

(f)⇒(d). In order to prove this implication suppose that there exists a field of linear subspaces E_x in TM_S of dimension and co-dimension greater than or equal to 1 defined on a neighbourhood of $\overline{J}$ in M_S and invariant under the action of derivatives of all maps ϕ_i, $i \in I$. Conjugating our system by a conformal diffeomorphism, we may assume that $M_S = I\!\!R^q$. Fix an element $j \in I$. In the course of the proof of Lemma 3.1 we have shown that besides one attracting fixed point $x_j \in X$, the map ϕ_j has a different fixed point $y_j \in I\!\!R^d$. Conjugate the system S by the inversion $i_{y_j,1}$ (equal to identity if $y_j = \infty$) and denote the resulting system by S_1. Put $\psi_i = i_{y_j,1} \circ \phi_i \circ i_{y_j,1}$ for all $i \in I$. The field $F_x = i'_{y_j,1}(E_x)$ is defined on a neighbourhood W of J_{S_1} and it is S_1-invariant. Since $\psi_j : I\!\!R^q \to I\!\!R^q$ is linear, in view of the appropriate part of the proof of Lemma 3.1, the field $\{F_x\}_{x\in W}$ is constant, say equal to F. So, the field of affine subspaces $\{x + F\}_{x\in W}$, as the unique field of integral manifolds of the S_1-invariant field $\{F\}$ of linear subspaces, is S_1-invariant, which means that $\psi(x + F) = \psi_i(x) + F$ for every $i \in I$ and every $x \in W$. So, if ψ_i is not affine for some $i \in I$, then $x + F$ must contain $\psi_i^{-1}(\infty)$, the center of inversion of ψ_i for every $x \in W$. Since W is open in $I\!\!R^q$ and since $\dim(F) \le q - 1$, this is impossible and proves that ψ_i is affine. The implication (f)⇒(d) is thus proven. ■

4. **Rigidity.**

We begin this section with the following.

Proposition 4.1. *Suppose that $F = \{f_i : X \to X\}_{i\in I}$ and $G = \{g_i : Y \to Y\}_{i\in I}$ are two not essentially affine topologically conjugate systems. If the measures m_G and $m_F \circ h^{-1}$ are equivalent, then the systems F and G are of the same dimension.*

Proof. Suppose on the contrary that the dimensions of F and G are not equal. Without loosing generality we may assume that $p = \dim F < q = \dim G$. Since G is not essentially affine, it follows from Theorem 3.4 that there exist $y \in J_G$, $i \in I$, a sequence $(\omega^{(j)})_{j=1}^q \in (I^*)^q$ and a neighbourhood $W_G \subset M_G$ of y such that the map

$$\mathcal{G} = \left(\tilde{D}g_i \circ g_{\omega^{(1)}}, \dots, \tilde{D}g_i \circ g_{\omega^{(q)}}\right)$$

is invertible on W_G. Since the measures m_G and $m_F \circ h^{-1}$ are equivalent, after an appropriate normalization $\mu_F = \mu_G \circ h$ which means that $D_h = \frac{d\mu_G \circ h}{d\mu_F} = 1$. Since $h \circ f_\tau = g_\tau \circ h$ for all $\tau \in I^*$ and since $D_h = 1$,

$$\mathcal{G} \circ h = \mathcal{F}$$

on J_F, where $\mathcal{F} = \left(\tilde{D}f_i \circ f_{\omega^{(1)}}, \dots, \tilde{D}f_i \circ f_{\omega^{(q)}}\right)$. Write $x = h^{-1}(y)$. Then $h = \mathcal{G}^{-1} \circ \mathcal{F}$ on $W_F \cap J_F$ for some open neighbourhood W_F of x in M_F such that $\mathcal{F}(W_F) \subset \mathcal{G}(W_G)$. Since by Corollary 2.2, the maps $\mathcal{F}$ and $\mathcal{G}^{-1}$ are real-analytic, the image $\mathcal{G}^{-1} \circ \mathcal{F}(W_F)$ for an adequate W_F small enough, is a real-analytic submanifold of dimension $\leq q$ and $\mathcal{G}^{-1} \circ \mathcal{F}(W_F) \cap J_G$ contains an open neighbourhood of y in J_G. So, invoking Lemma 3.2, we conclude that G is at most p-dimensional. This contradiction finishes the proof. ■

The main result of this paper is contained in the following.

Theorem 4.2. *If two Open Set Condition conformal regular iterated function systems $\{f_i : X \to X : i \in I\}$ and $\{g_i : Y \to Y : i \in I\}$ are not essentially affine and conjugate by a homeomorphism $h : J_F \to J_G$, then the following conditions are equivalent.*

- **(a):** *The conjugacy between the systems $\{f_i : X \to X : i \in I\}$ and $\{g_i : Y \to Y : i \in I\}$ extends in a conformal fashion to an open neighbourhood of X.*
- **(b):** *The conjugacy between the systems $\{f_i : X \to X : i \in I\}$ and $\{g_i : Y \to Y : i \in I\}$ extends in a real-analytic fashion to an open neighbourhood of X.*
- **(c):** *The conjugacy between the systems $\{f_i : X \to X : i \in I\}$ and $\{g_i : Y \to Y : i \in I\}$ is bi-Lipschitz continuous.*
- **(d):** $|g'_\omega(y_\omega)| = |f'_\omega(x_\omega)|$ *for all $\omega \in I^*$, where x_ω and y_ω are the only fixed points of $f_\omega : X \to X$ and $g_\omega : Y \to Y$ respectively.*
- **(e):** $\exists S \geq 1\ \forall \omega \in I^*$
$$S^{-1} \leq \frac{\mathrm{diam}(g_\omega(Y))}{\mathrm{diam}(f_\omega(X))} \leq S.$$
- **(f):** $\exists E \geq 1\ \forall \omega \in I^*$
$$E^{-1} \leq \frac{||g'_\omega||}{||f'_\omega||} \leq E.$$
- **(g):** $\mathrm{HD}(J_G) = \mathrm{HD}(J_F)$ *and the measures m_G and $m_F \circ h^{-1}$ are equivalent.*
- **(h):** *The measures m_G and $m_F \circ h^{-1}$ are equivalent.*

Proof. The implications (a)⇒(b) and (b)⇒(c) are obvious. That (c)⇒(d) results from the fact that (c) implies condition (1) of Theorem 1.3 which in view of this theorem is equivalent with condition (2) of Theorem 1.3 which finally is the same as condition (d) of

Theorem 4.2. The implications (d)$\Rightarrow$(e)$\Rightarrow$(f)$\Rightarrow$(g) have been proved in Theorem 1.3. The implication (g)$\Rightarrow$(h) is again obvious. We are left to prove that (h)$\Rightarrow$(a). We shall first prove that (h)$\Rightarrow$(b). So, suppose that (h) holds. Then, after an appropriate normalization $\mu_F = \mu_G \circ h$ which means that $D_h = \frac{d\mu_G \circ h}{d\mu_F} = 1$. Since G is not essentially affine, it follows from Theorem 3.4 that there exist $y \in J_G$, $i \in I$, a sequence $(\omega^{(j)})_{j=1}^q \in (I^*)^q$ and a neighbourhood $W_G \subset M_G$ of y such that the map

$$\mathcal{G} = \left(\tilde{D}g_i \circ g_{\omega^{(1)}}, \dots, \tilde{D}g_i \circ g_{\omega^{(q)}}\right)$$

is invertible on W_G. Since $h \circ f_\tau = g_\tau \circ h$ for all $\tau \in I^*$ and since $D_h = 1$, we have

$$\mathcal{G} \circ h = \mathcal{F}$$

on J_F, where $\mathcal{F} = \left(\tilde{D}f_i \circ f_{\omega^{(1)}}, \dots, \tilde{D}f_i \circ f_{\omega^{(q)}}\right)$. Fix W_F, an open neighbourhood of $x = h^{-1}(y)$ in M_F so small that $\mathcal{F}(W_1) \subset \mathcal{G}(W_2)$. Hence $\mathcal{G}^{-1} \circ \mathcal{F}$ is well-defined on W_1 and $\mathcal{G}^{-1} \circ \mathcal{F}|_{W_1 \cap J_F} = h$. Consider now $\omega \in I^*$ such that $f_\omega(J_F) \subset W_1$. Since

$$\mathcal{G}^{-1} \circ \mathcal{F}(f_\omega(J_F)) = h \circ f_\omega(J_F) = g_\omega \circ h(J_F) = g_\omega(J_G) \subset g_\omega(V_G),$$

since $g_\omega(V_G)$ is open, since f_ω and $\mathcal{G}^{-1} \circ \mathcal{F}$ are continuous, there exists an open neighbourhood $V_F' \subset V_F$ of X such that $f_\omega(V_F') \subset W_1$ and $\mathcal{G}^{-1} \circ \mathcal{F}(f_\omega(V_F')) \subset g_\omega(V_G)$. Hence, the map

$$g_\omega^{-1} \circ (\mathcal{G}^{-1} \circ \mathcal{F}) \circ f_\omega : V_F' \to \mathbb{C}$$

is well-defined, by Corollary 2.2 is real-analytic, and $g_\omega^{-1} \circ (\mathcal{G}^{-1} \circ \mathcal{F}) \circ f_\omega|_{J_F} = h$. Thus, the property (b) is proved.

The last step of the proof of Theorem 4.2, that is the implication (b)$\Rightarrow$(a) can be carried out using ideas from the proof of Lemma 7.2.7 in [Pr] as follows. Let H be this real-analytic extension of h on a neighbourhood of W_F of J_F in M_F. We may assume W_F to be so small that H' is a linear isomorphism at every point of W_F. Define the function $\psi : W_F \to \mathbb{R}$ by the formula

$$\psi(z) = \frac{||H'(z)||}{||(H'(z))^{-1}||}$$

Suppose that $\psi(\xi) = 1$ for some point $\xi \in W_F$. Since for every $\omega \in I^*$

$$\psi(f_\omega(\xi)) = \frac{||H'(f_\omega(\xi))||}{||(H'(f_\omega(\xi)))^{-1}||} = \frac{||g_\omega'(H(\xi)) \cdot H'(\xi) \cdot (f_\omega'(\xi))^{-1}||}{||\left(g_\omega'(H(\xi)) \cdot H'(\xi) \cdot (f_\omega'(\xi))^{-1}\right)^{-1}||} = \frac{||H'(\xi)||}{||(H'(\xi))^{-1}||} = \psi(\xi)$$

and since $\overline{\{f_\omega(\xi) : \omega \in I^*\}} \supset \overline{J}_F$, we conclude that $\psi = 1$ identically on $\overline{J}_F$. Since ψ is real-analytic and since F is q-dimensional, using Lemma 3.2 we conclude that $\psi = 1$ on an open neighbourhood of $\overline{J}_F$. But this means that H is conformal. So, we may assume that $\psi(z) \neq 1$ for every $z \in W_F$. Define the field $\{E_z\}_{z \in W_F}$ on W_F as follows.

$$E_z = \left\{w \in \mathbb{R}^q : \frac{||H'(z)w||}{||w||} = ||H'(z)||\right\} \cup \{0\}.$$

For every $z \in W_F$, the set E_z is a linear subspace of $\mathbb{R}^q$ of dimension ≥ 1. Its codimension is ≥ 1 since $\psi(z) \neq 1$. Obviously E_z depends continuously on z. Since the maps $f_i : \mathbb{R}^q \to \mathbb{R}^q$

are conformal, $f_i'(z)(E_z) = E_{f_i(z)}$ and it thereofore follows from Theorem 3.4 that the system F is essentially affine. This contradiction finishes the proof. ∎

Finally we want to recall that in [MU2] we have introduced the class of parabolic iterated function systems. For a brief exposition of this material done in the way suitable for the needs of rigidity see [MPU]. In exactly the same way as it has been done in [MPU] we can show that the canonically associate hyperbolic iterated function system is not essentially affine and we can prov the following.

Theorem 4.3. *If both topologically conjugate systems $F = \{f_i : X \to X, i \in I\}$ and $G = \{g_i : Y \to Y, i \in I\}$ are regular and at least one of them is parabolic, then the conditions listed in Theorem 4.2 are mutually equivalent where in the items (d),(e),(f) the words ω are required to be hyperbolic.*

Acknowledgment. The author wishes to thank Feliks Przytycki for providing him with the idea of the proof of Lemma 3.1.

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