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Rigidity of Tame Rational Functions

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Abstract. We introduce and establish some basic properties of the tame rational functions. The class of these functions contains all the rational functions with no recurrent critical points in their Julia sets. For tame non-exceptional functions we prove that the Lipschitz conjugacy, the same spectra of moduli of derivatives at periodic orbits and conformal conjugacy are mutually equivalent. We prove also the following rigidity result: If h is a Borel measurable invertible map which conjugates two tame functions f and g a.e. and if h transports conformal measure m_f to a measure equivalent to m_g then h extends from a set of full measure m_f to a conformal homeomorphism of neighbourhoods of respective Julia sets. This extends D. Sullivan's rigidity theorem for holomorphic expanding repellers. We provide also a few lines proof of E. Prado's theorem that two generalized polynomial-like maps at zero Teichmüller's distance are holomorphically conjugate.

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§**1. Tame maps.** Let $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ be a rational function of degree ≥ 2 and let $J(f)$ be its Julia set. We recall (see [DU1] for ex.) that given $t\geq 0$ a Borel probability measure m supported on $J(f)$ is called (Sullivan's) t-conformal if

$$m(f(A))=\int_A |f'|^t dm$$

for every Borel set $A\subset J(f)$ such that $f|_A:A\to f(A)$ is injective. By $\mathrm{Crit}(f)$ we denote the set of all critical points of the map $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ and by $\mathrm{Sing}(f)$, $\overline{\bigcup_{n=0}^\infty f^n(\mathrm{Crit}(f))}$, the closure of its forward orbit. Notice that $\mathrm{Sing}(f)$ includes all parabolic periodic points (other name: rationally indifferent, i.e. such that $(f^p)'(x)$ is a root of 1, where p is a period of x). Indeed, such points are in the forward limit set of critical points from outside of $J(f)$.

Finally, by $T(f)$ denote the set of all points in $J(f)$ for which

$$\limsup_{n\to\infty}\mathrm{dist}(f^n(z),\mathrm{Sing}(f))>0.$$

Definition 1.1 We say that a rational function $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ is tame if $\mathrm{Sing}(f)\cap J(f)$ is nowhere dense in $J(f)$ and there exists a t-conformal measure m (sometimes to be more specific denoted by m_f) such that $m(T(f))=1$.

Remark 1.2. The set $T(f)$ was already considered in [GPS] (called there the transverse limit set). In fact its origin goes back to [Ly].

Remark 1.3. Notice that if the singular set $\mathrm{Sing}(f)$ of an arbitrary rational function f is nowhere dense in $J(f)$, then $T(f)$ contains the set of all transitive points of f (i.e. points with dense forward orbit). Therefore for such a function to be tame it is sufficient to have a conformal measure m supported on the set of transitive points.

Remark 1.4. It is possible to define tame conformal repellers which generalize the class of tame rational functions. Namely a triple (X,U,f) is called tame conformal repeller if U is an open subset of $\overline{\mathbb{C}}$, X is a compact subset of U and $f:U\to\overline{\mathbb{C}}$ is an analytic map such that

(a) $f(X)=X$.

(b) $\bigcap_{n\geq 0} f^{-n}(U)=X$.

(c) The requirements of Definition 1.1 are satisfied with $J(f)$ replaced by X.

Actually to get a proper extension of tame rational functions one should weaken the items (a) and (b) by allowing the existence of parabolic periodic points in X as it has been done in [U4] and [DU2]. For these objects all the theorems proven in this paper remain valid (wherever it makes sense) replacing only in Theorem 1.9 the property "not critically finite with parabolic orbifold for which $J(f)=J(g)=\overline{\mathbb{C}}$" by "non-linear" and removing condition (1).

Clearly $T(f)$ is a subset of the set of conical points of f (for the definition of the latter see [U3], comp. [DMNU] and [Mc2]) and therefore as an immediate consequence of Theorem 1.2 in [DMNU] we get the following.

Theorem 1.5. If $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is tame, then there exists at most one value t for which a t-conformal measure exists and is supported on $T(f)$. Additionally, for such a t there exists exactly one t-conformal measure supported on $T(f)$ and this measure is ergodic.

Thus the measure $m = m_f$ and the exponent $t = t_f$ are determined uniquely. It is easy to see that m_f is nonatomic. We can say something more about the exponent $t = t_f$. Indeed, with considerations similar to those in [U4], we can prove the following.

Theorem 1.6. If a rational function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is tame, then the t_f-dimensional Hausdorff measure of $T(f)$ is finite. In particular $t_f \geq \mathrm{HD}(T(f))$ (HD abbreviates Hausdorff dimension).

Employing the method introduced by M. Martens in [Ma] and proceeding in the same way as in the proof of Proposition 4.2 in [U2], we can demonstrate the following.

Theorem 1.7. Up to a multiplicative constant there exists exactly one f-invariant σ-finite measure $\mu = \mu_f$ absolutely continuous with respect to m. Moreover, μ is equivalent with m, conservative and ergodic.

In fact it follows from [U2] that there exists a set X_0 of positive conformal measure such that $\sum_{k=0}^{\infty} m \circ f^{-k}(X_0) = \infty$ and up to a multiplicative constant the measure μ_f is given by the formula

$$\mu_f(F) = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} m \circ f^{-k}(F)}{\sum_{k=0}^{n} m \circ f^{-k}(X_0)}.$$

for every Borel subset F of $J(f)$. Notice that as X_0 one can take any open ball such the the ball centered at the same point but with radius twice as big is disjoint from $\mathrm{Sing}(f)$. So, X_0 and F are not related one to each other; in particular F does not have to be a subset of X_0. A straightforward computation using the above display and conformality of m shows then that for every $x \in J(f) \setminus \mathrm{Sing}(f)$ the limit below exists and for m_f-a.e. x

$$\frac{d\mu_f}{dm_f}(x) = \lim_{n\to\infty} \frac{1}{a_n} \sum_{k=0}^{n} \sum_{y \in f^{-k}(x)} |(f^k)'(y)|^{-t_f}, \tag{1.1}$$

where $a_n = \sum_{k=0}^{n} m \circ f^{-k}(X_0)$. Moreover the functions $\frac{d\mu_f}{dm_f}$ and $\frac{d\mu_f}{dm_f}^{-1}$ are bounded on every compact subset of $J(f) \setminus \mathrm{Sing}(f)$.

In [U1] and [U2] the second author explored the class of rational functions with no recurrent critical points in their Julia sets, abbr. NCP. This class comprised expanding (on Julia set), subexpanding and parabolic maps. Its properties most interesting for us at the moment are collected in the following.

Theorem 1.8. Each NCP rational function f is tame and $t_f = \mathrm{HD}(J(f))$. Morover the t_f-conformal measure is supported on the set of transitive points.

Other examples of tame rational functions are provided by Collet-Eckmann-Tsujii polynomials with real coefficients and all critical points in the real axis, see [Przy3], with the conformal measure m defined in [Przy3]. Indeed it was proved there that there exists an invariant ergodic probability measure μ equivalent to m. So since m is positive on open sets, it is supported on transitive points by Birkhoff Ergodic Theorem applied to μ. Hence by Remark 1.3 f is tame.

The main result of our paper is the following.

Theorem 1.9. Suppose that f and g are two tame rational maps. Let h be an invertible map from a full measure m_f subset of $J(f)$ onto a full measure m_g subset of $J(g)$, preserving the algebras of measurable sets for m_f and m_g and conjugating f to g, namely $h \circ f = g \circ h$. Then the following conditions (1)-(6) are equivalent.

(1) h extends to a Möbius conjugacy between $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ and $g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$.
(2) h extends to a conformal homeomorphism conjugating f and g on neighbourhoods of $J(f)$ and $J(g)$ in $\overline{\mathbb{C}}$.
(3) h extends to a real-analytic diffeomorphism conjugating f and g on neighbourhoods of $J(f)$ and $J(g)$ in $\overline{\mathbb{C}}$.
(4) h extends to a homeomorphism from $J(f)$ to $J(g)$ such that h and h^{-1} are Lipschitz continuous.
(5) h extends to a homeomorphism from $J(f)$ to $J(g)$ such that for every periodic point x of f, say of period p, $|(f^p)'(x)| = |(g^p)'(h(x))|$.
(6) The measure class of m_f is transported under h to the measure class of m_g.

Here for the implication (6) $\Rightarrow$ (2), f and g are assumed not to be critically finite with parabolic orbifold (see [Thu, Ch.13], [DH, §9] or [Zd] for the definition) for which $J(f) = J(g) = \overline{\mathbb{C}}$. We call such maps exceptional.

Remark 1.10 The implication (5) $\Rightarrow$ (2) was in fact proved by E. Prado in [Pra] for all rational maps. The only missing point, the non-linearity (see §3 for the definition) caused by parabolic points will turn out to be easy (see Theorem 3.5). Prado's proof of the only hard part (5) $\Rightarrow$ (2) was done by approximating $J(f)$ by forward invariant expanding repellers inside $J(f)$, where h extends conformally, see [Su], [Przy1]. Our proof of the implication (5) $\Rightarrow$ (2) is different and goes via (6). This requires the assumptions that f is tame. Thus the main new result in our paper is the implication (6) $\Rightarrow$ (2) which extends Sullivan's result for non-linear repellers, see [Su] and [Przy1]); we use the same scheme of proof. This implication is called rigidity.

Note that the implication (2) $\Rightarrow$ (1) is straightforward (see §5, Proposition 5.4). This holds even in a general, not just tame, situation. Any conformal conjugacy on neighbourhoods of Julia sets for any two rational functions extends to a conformal conjugacy

(Möbius map) to the whole sphere. In particular by Theorem 1.9 the measure theoretic conjugacy class of a tame map coincides with its conformal conjugacy class. In other words one cannot perturb a tame map inside the measure theoretic conjugacy class changing its conformal conjugacy class. Equivalence classes of measures classify tame maps.

Remark on notation. We write derivatives in the euclidean metric in $\mathbb{C}$ which is correct if we assume $J(f) \neq \overline{\mathbb{C}}$, because then we can change holomorphically coordinates on $\overline{\mathbb{C}}$ so that $\infty \notin J(f)$.

If $J(f) = \overline{\mathbb{C}}$ one should consider the spherical metric. Then Koebe's Distortion Theorem will be used for families of functions on a disc whose values omit a set of a diameter ε. The Koebe's constants depend on ε. Our functions will be branches of f^{-n}. If the disc is small enough, the images omit a periodic orbit of period at least 2, so have complements of a definite positive diameter.

To simplify notation we always assume in proofs that Julia sets are in $\mathbb{C}$,

§2. Analiticity of Jacobian.

Definition 2.1. A rational function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is said to be Julia real-analytic if its Julia set is contained in a finite union of pairwise disjoint real-analytic curves which will be denoted by $\Gamma = \Gamma_f$. Frequently in such a context we will alternatively speak about real analyticity of the Julia set $J(f)$.

Remark 2.2. **[Przy1]** If there exist $x \in J(f)$ a real-analytic arc γ and $\delta > 0$ such that $J(f) \cap B(x, \delta) \subset \gamma$ then f is Julia real-analytic. This follows from the topological exactness of f on $J(f)$, i.e. $(\forall U)$ open in $J(f)$ $(\exists n \geq 0)$ such that $f^n(U) = J(f)$. Indeed $J(f) \subset \Gamma := f^n(\gamma)$. The set Γ has no branch point in $J(f)$, otherwise Γ would have infinitely many branch points at $J(f)$ at preimages for iterates of $f^k, k = 0, 1, ...$

We shall prove the following.

Proposition 2.3. If $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a tame mapping, then the Radon-Nikodym derivative $\rho = d\mu/dm$ has a real-analytic real-valued extension on a neighbourhood of $J(f) \setminus \mathrm{Sing}(f)$ in $\mathbb{C}$. If f is Julia real-analytic, then ρ has a real-analytic extension on a neighbourhood of $J(f) \setminus \mathrm{Sing}(f)$ in Γ.

Proof. Suppose first that f is Julia real-analytic. We need to show that there exists a holomorphic complex-valued extension of ρ on a neighbourhood of $J(f) \setminus \mathrm{Sing}(f)$ in $\mathbb{C}$. Taking an appropriate atlas we may assume that $J(f)$ is contained in a real axis (if a closed curve is a component of Γ we can use Arg). Then for every $x \in J(f) \setminus \mathrm{Sing}(f)$ there exists $r > 0$ such that $B(x, r) \cap \mathrm{Sing}(f) = \emptyset$. For all $k \geq 1$ and all $y \in f^{-k}(x)$ let $\nu(k, y) = 1$ or -1 depending as f_y^{-k} preserves or reverses the orientation. So

$$|(f_y^{-k})'(z)| = \nu(k, y)((f_y^{-k})'(z))$$

for all $z \in J(f) \cap B(x, r)$. Here $f_y^{-k} : B(x, r) \to \mathbb{C}$ are the inverse branches of f^k sending x to y.

Consider the following sequence of complex analytic functions on $z \in B(x,r)$

$$g_n(z) = \frac{1}{a_n}\sum_{k=0}^{n}\sum_{y\in f^{-k}(x)}\Big(\nu(k,y)((f_y^{-k})'(z))\Big)^t.$$

There is no problem here with raising to the t-th power since $B(x,r)$, the domain of all $\nu(k,y)(f_y^{-k})'$ is simply connected. Since the latter functions are positive in $\mathbb{R}$, we can choose the branches of the t-th powers to be also positive in $\mathbb{R}$. By Koebe's Distortion Theorem for every $z \in B(x,r/2)$, every $k \geq 1$ and every $y \in f^{-k}(x)$ we have $|(f_y^{-k})'(z)| \leq K|(f_y^{-k})'(x)|$. Hence $|g_n(z)| \leq Kg_n(x)$. Since the sequence $g_n(x)$ converges, see (1.1), the functions $\{g_n|_{B(x,r/2)}\}_{n\geq 1}$ are uniformly bounded, so they form a normal family in the sense of Montel. Since $g_n(z)$ converges for all $z \in J\cap B(x,r/2)$, it follows that g_n converges to an analytic function g on $B(x,r/2)$ which by our construction is an extension of ρ.

Let us pass now to the proof of the first part of this proposition. That is, we relax the Julia real analyticity assumption and we want to construct a real-analytic real-valued extension of ρ to a neighbourhood of $J(f)\setminus \mathrm{Sing}(f)$ in $\mathbb{C}$. Our strategy is to work in $\mathbb{C}^2$, to use an appropriate version of Montel's theorem and, in general, to proceed similarly as in the first part of the proof. So, fix $v \in J(f)\setminus\mathrm{Sing}(f)$ and take $r>0$ so small that $B(v,2r)\cap\mathrm{Sing}(f)=\emptyset$. Thus for every $k\geq 0$ and every $v_k \in f^{-k}(v)$ there exists $f_{v_k}^{-k}: B(v,2r)\to\mathbb{C}$, a holomorphic inverse branch of f^k defined on $B(v,2r)$ mapping v to v_k. Identify now $\mathbb{C}$, where our f acts, to $\mathbb{R}^2$ with coordinates x,y, the real and complex part of z. Embed this into $\mathbb{C}^2$ with x,y complex. Denote the above $\mathbb{C}=\mathbb{R}^2$ by $\mathbb{C}_0$. We may assume that $v=0$ in $\mathbb{C}_0$. Given $k\geq 0$ and $v_k\in f^{-k}(v)$ define the function $\rho_{v_k}: B_{\mathbb{C}_0}(0,2r)\to\mathbb{C}$ (the ball in $\mathbb{C}_0$) by setting

$$\rho_{v_k}(z) = \frac{(f_{v_k}^{-k})'(z)}{(f_{v_k}^{-k})'(0)},$$

Since $B_{\mathbb{C}_0}(0,2r)\subset\mathbb{C}_0$ is simply connected and ρ_{v_n} nowhere vanishes, all the branches of logarithm $\log\rho_{v_n}$ are well defined on $B_{\mathbb{C}_0}(0,2r)$. Choose this branch that maps 0 to 0 and denote it also by $\log\rho_{v_n}$. By Koebe's Distortion Theorem $|\rho_{v_k}|$ and $|\mathrm{Arg}\rho_{v_k}|$ are bouned on $B(0,r)$ by universal constants K_1, K_2 respectively. Hence $|\log\rho_{v_k}|\leq K=(\log K_1)+K_2$. We write

$$\log\rho_{v_k} = \sum_{m=0}^{\infty} a_m z^m$$

and note that by Cauchy's inequalities

$$|a_m| \leq K/r^m. \tag{2.1}$$

We can write for $z=x+iy$ in $\mathbb{C}_0$

$$\mathrm{Re}\log\rho_{v_k} = \mathrm{Re}\sum_{m=0}^{\infty} a_m(x+iy)^m = \sum_{p,q=0}^{\infty}\mathrm{Re}\Big(a_{p+q}\binom{p+q}{q}i^q\Big)x^py^q := \sum c_{p,q}x^py^q.$$

In view of (2.1) we can estimate $|c_{p,q}| \leq |a_{p+q}|2^{p+q} \leq Kr^{-(p+q)}2^{p+q}$. Hence $\mathrm{Re}\log\rho_{v_k}$ extends, by the same power series expansion $\sum c_{p,q}x^p y^q$, to the polydisc $\mathbb{D}_{\mathbb{C}^2}(0,r/2)$ and its absolute value is bounded there from above by K. Now for every $n \geq 0$ consider a real-analytic function b_n on $B_{\mathbb{C}_0}(0,2r)$ by setting

$$b_n(z) = \frac{1}{a_n}\sum_{k=0}^{n}\sum_{v_k\in f^{-k}(0)} |(f_{v_k}^{-k})'(z)|^t.$$

By (1.1) the sequence $b_n(0)$ is bounded from above by a constant L. Each function b_n extends to the function

$$B_n(z) = \frac{1}{a_n}\sum_{k=0}^{n}\sum_{v_k\in f^{-k}(0)} |(f_{v_k}^{-k})'(0)|^t \mathrm{e}^{t\mathrm{Re}\log\rho_{v_k}(z)}.$$

whose domain, similarly as the domains of the functions $\mathrm{Re}\log\rho_{v_k}$, contains the polydisc $\mathbb{D}_{\mathbb{C}^2}(0,r/2)$. Finally we get for all $n \geq 0$ and all $z \in \mathbb{D}_{\mathbb{C}^2}(0,r/4)$

$$\begin{aligned}|B_n(z)| &= \frac{1}{a_n}\sum_{k=0}^{n}\sum_{v_k\in f^{-k}(0)} |(f_{v_k}^{-k})'(0)|^t \mathrm{e}^{\mathrm{Re}(t\mathrm{Re}\log\rho_{v_k}(z))}\\ &\leq \frac{1}{a_n}\sum_{k=0}^{n}\sum_{v_k\in f^{-k}(0)} |(f_{v_k}^{-k})'(0)|^t \mathrm{e}^{t|\mathrm{Re}\log\rho_{v_k}(z)|}\\ &\leq \mathrm{e}^{Kt}\frac{1}{a_n}\sum_{k=0}^{n}\sum_{v_k\in f^{-k}(0)} |(f_{v_k}^{-k})'(0)|^t \leq \mathrm{e}^{Kt}L.\end{aligned}$$

Now by Cauchy's integral formula (in $\mathbb{D}_{\mathbb{C}^2}(0,r/4)$) for the second derivatives we prove that the family B_n is equicontinuous on, say, $\mathbb{D}_{\mathbb{C}^2}(0,r/5)$. Hence we can choose a uniformly convergent subsequence and the limit function G is complex analytic and extends ρ on $J(f)\cap B(0,r/5)$, by (1.1). Thus we have proved that ρ extends to a complex analytic function in a neighbourhood of every $v \in J(f)\setminus \mathrm{Sing}(f)$ in $\mathbb{C}^2$, i.e. real analytic in $\mathbb{C}_0$. These extensions coincide on the intersections of the neighbourhoods, otherwise $J(f)$ is real analytic and we are in the case considered at the beginning of the proof. ♣

Denote the Jacobian of the map $f : J(f) \to J(f)$ with respect to the measure μ by $D_\mu f$. As an immediate consequence of Proposition 2.2, the following computation

$$\frac{d\mu_f\circ f}{d\mu_f} = \frac{d\mu_f\circ f}{dm_f\circ f}\cdot\frac{dm_f\circ f}{dm_f}\cdot\frac{dm_f}{d\mu_f} = \frac{d\mu_f}{dm_f}\circ f\cdot|f'|^t\cdot\frac{dm_f}{d\mu_f}$$

and the observation that $|f'|^t$ is real-analytic on $\mathbb{C}$ we get the following.

Corollary 2.4. If $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is tame, then the Jacobian $D_\mu f$ has a real-analytic extension on a neighbourhood of $J(f)\setminus\mathrm{Sing}(f)$ in $\mathbb{C}$. If the map f is Julia real-analytic, then the Jacobian $D_\mu f$ has a real-analytic extension on a neighbourhood of $J(f)\setminus\mathrm{Sing}(f)$ in Γ.

From now on we will keep for these extensions the same symbol $D_\mu f$ and the same name: the Jacobian of f.

§**3. Non-linearity.** We begin this section with the following.

Definition 3.1. The tame rational function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is said to be non-linear if there exists a point $w \in J(f) \setminus \mathrm{Sing}(f)$ such that if f is Julia real-analytic, then $D_\mu f$ is invertible on a neighbourhood of w in Γ_f and if f is not Julia real-analytic, then the map $F(z) = ((D_\mu f)(z), (D_\mu f) \circ f^k(z))$ is invertible on a neighbourhood of w in $\mathbb{C}$ for some $k \geq 1$. Otherwise, the map f is said to be linear.

Theorem 3.2. The following five conditions are equivalent.

(a) The tame map f is linear.

(b) The Jacobian $D_{\mu_f} f$ is locally constant on $J(f) \setminus \mathrm{Sing}(f)$.

(c) The function $\log |f'|$ is cohomologous in the class of real-analytic functions to a locally constant function on $J(f) \setminus \mathrm{Sing}(f)$.

(d) The conformal structure on $J(f) \setminus \mathrm{Sing}(f)$ admits a conformal affine refinement so that f becomes affine, i.e. there exists an atlas $\{\phi_t : U_t \to \mathbb{C}\}$, $\bigcup_t U_t \supset J(f) \setminus \mathrm{Sing}(f)$, consisting of conformal injections such that the compositions $\phi_t \circ \phi_s^{-1}$ and $\phi_t \circ f \circ \phi_s^{-1}$ are affine.

An analogous theorem holds for conformal expanding repellers, [Su], [Przy1, Prop.7.1.2]. Proofs are similar.

It occurs that the only rational function which may be linear are the exceptional ones (criticaly finite maps with parabolic orbifold).

Indeed, for f expanding on $J(f)$ linearity implies $f(z) = z^d$ or f is a Tchebyshev polynomial, see [Zd]. In fact Zdunik assumed $\log |f'|$ cohomologous to a constant, but the same proof holds for locally constant.

For nonexpanding case this immediately follows form the following two theorems, the first one also following from [Zd] or E. Prado's paper [Pra].

Theorem 3.4. If the Julia set of a tame non-exceptional rational function contains a critical point, then f is non-linear.

In fact it follows from Prado's proof that there are non-linear invariant hyperbolic Cantor sets contained in $J(f) \setminus \mathrm{Sing}(f)$. Prado proved this for all non-exceptional rational functions with a critical point in Julia set, not only for the tame ones.

Adapting the general idea of Prado's argument we shall prove the following.

Theorem 3.5. If the Julia set of a tame rational function contains a periodic parabolic point and no critical points, then f is non-linear.

Proof. Passing to a sufficiently high iterate we may assume that each periodic parabolic point is fixed under f and that the derivative of f evaluated at any parabolic fixed point is equal to 1. Take one such a point and call it ω. Suppose on the contrary that f is linear. Since f is tame and topologically exact, there exists $n \geq 1$ such that $f^{-n}(\omega) \setminus \mathrm{Sing}(f) \neq \emptyset$. Let $x \in f^{-n}(\omega) \setminus \mathrm{Sing}(f)$ and let $\phi_x : U_x \to \mathbb{C}$ be a map from an affine atlas $\mathcal{A} = \{\phi_i : U_i \to \mathbb{C} : i \in I\}$ such that $x \in U_x$. Since $J(f)$ contains no critical point, shrinking U_x if necessary, we may assume that $f^n|_{U_x} : U_x \to \mathbb{C}$ is injective. Denote by $f_x^{-n} : f^n(U_x) \to U_x$ the inverse map $(f^n|_{U_x})^{-1}$ and consider the composition

$$\phi_x \circ f_x^{-n} \circ f \circ f^n \circ \phi_x^{-1}$$

defined on $\phi_x\big(U_x \cap f^{-(n+1)}(f^n(U_x))\big)$. Since $x \in U_x$ and $f^{n+1}(x) = f(f^n(x)) = f(\omega) = \omega \in f^n(U_x)$, $x \in U_x \cap f^{-(n+1)}(f^n(U_x))$, and therefore it makes sense to consider V_x, the connected component of $U_x \cap f^{-(n+1)}(f^n(U_x))$ containing x. Since $f^n(V_x)$ is an open neighbourhood of ω, using again the fact that $\mathrm{Sing}(f)$ is nowhere dense in $J(f)$, we find a map $\phi_\omega : U_\omega \to \mathbb{C}$ from $\mathcal{A}$ such that $f^n(V_x) \cap U_\omega \neq \emptyset$ and (shrinking U_ω if necessary) $f(U_\omega) \subset U_j$ for some $\phi_j : U_j \to \mathbb{C}$ from $\mathcal{A}$. Now notice that $\phi_x \circ f_x^{-n} \circ f \circ f^n \circ \phi_x^{-1}$ and $(\phi_x \circ f_x^{-n} \circ \phi_j^{-1}) \circ (\phi_j \circ f \circ \phi_\omega^{-1}) \circ (\phi_\omega \circ f^n \circ \phi_x^{-1})$ coincide on $\phi_x\big(V_x \cap f_x^{-n}(U_\omega)\big)$. Since the latter map is affine, the map $g = \phi_x \circ f_x^{-n} \circ f \circ f^n \circ \phi_x^{-1} : \phi_x(V_x) \to \phi_x(U_x)$ is also affine and extends uniquely to an affine map from $\mathbb{C}$ to $\mathbb{C}$. Since $g(\phi_x(x)) = \phi_x(x)$, $g'(\phi_x(x)) = 1$ and assuming without generality that $\phi_x(x) = 0$, we therefore conclude that g is an identity map. Hence $\phi_x \circ f_x^{-n} \circ f = \phi_x \circ f_x^{-n}$ on $f^n(V_x)$, and since $\phi_x \circ f_x^{-n}$ is injective, $f|_{f^n(V_x)} = \mathrm{Id}|_{f^n(V_x)}$. Consequently $f = \mathrm{Id}$ on $\overline{\mathbb{C}}$, a contradiction. ♣

Remark that again the same proof works for general rational functions, proving the existence of expanding non-linear subsets as in [Pra]. In parabolic case Zdunik's proof [Zd, Lemma 3] does not work because the locally constant function Φ cohomologous to $D_{\mu_f} f$ can be *a priori* equal to 0 in a neighbourhood of ω so $|f'(\omega)|^{t_f} = \exp \Phi(\omega) = 1$ and we do not get any contradiction. (If $\Phi(\omega) > 0$ the formula implies that ω is a source.)

Let us repeat that we have proved the following.

Theorem 3.6. All tame linear functions are exceptional (critically finite with parabolic orbifold).

Remark 3.7. In contrast to Theorem 3.5 there exist of course an aboundance of linear conformal expanding repellers but their domains are proper open subsets of $\overline{\mathbb{C}}$.

§**4. The chain of implications.** In this section we will prove the chain of implications (2) $\Rightarrow$ (3) $\Rightarrow \ldots \Rightarrow$ (6) of Theorem 1.9. The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are obvious. For the sake of completeness we provide now an easy proof of the implication

(4) $\Rightarrow$ (5). So, suppose that $x \in J(f)$ is a periodic point of period p and $|(f^p)'(x)| \neq |(g^p)'(h(x))|$. Without loosing generality we may assume that $|(f^p)'(x)| < |(g^p)'(h(x))|$. Fix $|(f^p)'(x)| < \mu < \lambda < |(g^p)'(h(x))|$. Let U be a neighbourhood of x such that both inverse branches $f_x^{-p} : U \to U$ of f^p and $g_{h(x)}^{-p} : h(U) \to h(U)$ of g^p sending respectively x to x and $h(x)$ to $h(x)$ are well defined. Taking U sufficiently small we may assume that $|f_x^{-pn}(z) - x| \geq \mu^{-n}$ and $|g_{h(w)}^{-pn}(w) - h(x)| \leq \lambda^{-n}$ for all $n \geq 1$, $z \in U$ and $w \in h(U)$. Hence

$$\frac{|f_x^{-pn}(z) - x|}{|h(f_x^{-pn}(z)) - h(x)|} \geq \frac{\mu^{-n}}{\lambda^{-n}} = \left(\frac{\lambda}{\mu}\right)^n \to \infty$$

if $n \to \infty$. So, the implication (4) $\Rightarrow$ (5) is proved.

Given a set $A \subset \mathbb{C}$ and $r > 0$ let

$$\overline{B}(A, r) = \{z \in \mathbb{C} : \text{dist}(z, A) \leq r\}$$

be the closed ball centered at A and of radius r. In order to show that (5) $\Rightarrow$ (6) we need first the following version of the closing lemma (or shadowing lemma).

Lemma 4.1. Fix $s > 0$. Then for all $0 < \rho_2 < s$ there exist $\rho_1 > 0$ and an integer $n_1 \geq 1$ such that for every $n \geq n_1$ if $f^n(x) \in J(f) \setminus B(\text{Sing}(f), s)$ and if $f^n(x) \in B(x, \rho_1)$, then there exists $y \in J(f)$ such that $f^n(y) = y$ and $|f^j(y) - f^j(x)| \leq \rho_2$ for all $0 \leq j \leq n - n_1$ and $|y - f^n(x)| < \rho_2$.

Proof. It easily follows from the normal family argument that

$$\lim_{n\to\infty} \sup\{\text{diam}(P_n)\} = 0,$$

where P_n range over all connected components of $f^{-n}(\overline{B}(z, \rho_2))$, $z \in J(f) \setminus B(\text{Sing}(f), s)$. Take n_1 so large that $\text{diam}(P_n) < \rho_2/2$ for all $n \geq n_1$. Take $\rho_1 < \rho_2/2$. Let B_n, $n \geq n_1$, be the connected component of $f^{-n}(\overline{B}(f^n(x), \rho_2)$ containing x. Let $f_x^{-n} : \overline{B}(f^n(x), \rho_2) \to B_n$ be the holomorphic inverse branch of f^n sending $f^n(x)$ to x. We then have

$$f_x^{-n}\left(\overline{B}(f^n(x), \rho_2)\right) \subset \overline{B}(x, \rho_2/2) \subset \overline{B}\left(f^n(x), \rho_1 + \frac{\rho_2}{2}\right) \subset \overline{B}(f^n(x), \rho_2).$$

Hence by the Brouwer fixed point theorem there exists $y \in \overline{B}(f^n(x), \rho_2)$ such that $f_x^{-n}(y) = y$ which implies that $f^n(y) = y$. Finally note that $|f_x^{-j}(f^n(x)) - f_x^{-j}(y)| = |f_x^{-j}(f^n(x)) - f_x^{-j}(f^n(y))| \leq \rho_2/2 \leq \rho_2$ for all $j \geq n_1$. ♣

By topological exactness of f the set of transitive points is dense in $J(f)$. Choose one such a point, say x. For every $z \in J(f)$ define

$$\eta(z) = \log|g'(h(z))| - \log|f'(z)|$$

and for every $n \geq 1$ set

$$u(f^n(x)) = \sum_{j=0}^{n-1} \eta(f^j(x)). \tag{4.1}$$

Similarly as the preceding result the next one has its origins in [Bo]. In Bowen's expanding case Hölder continuity of the function $\log|f'|$ on $J(f)\setminus B(\mathrm{Sing}(f),s)$ is sufficient. Here instead, we use the Koebe's Distortion Theorem to control the distortion.

Lemma 4.2. Suppose that condition 3 of Theorem 1.9 holds. Then for every $s>0$ the function u restricted to the set $(J(f)\setminus B(\mathrm{Sing}(f),s))\cap\{f^n(x):n\ge 0\}$ is uniformly continuous.

Proof. Fix $0<\rho_2<s$ and choose ρ_1 and n_1 according to Lemma 4.41. Consider two points $f^n(x)$ and $f^m(x)$ in $J(f)\setminus B(\mathrm{Sing}(f),s)$ such that $n\ge m$ and $|f^n(x)-f^m(x)|<\rho_1$. Then in view of Lemma 4.1 there exists a point $y\in J(f)$ such that $f^{n-m}(f^m(y))=f^m(y)$, $|f^{m+j}(x)-f^{m+j}(y)|<\rho_2$ for all $j=0,1,n-m-n_1$, and $|f^n(x)-f^n(y)|<\rho_2$. Since by the assumption $\sum_{j=m}^{n-1}\eta(f^j(y))=0$, we therefore get

$$
\begin{aligned}
u(f^n(x))-u(f^m(x)) &= \sum_{j=m}^{n-1}\eta(f^j(x))=\sum_{j=m}^{n-1}\big(\eta(f^j(x))-\eta(f^j(y))\big)\\
&= \sum_{j=m}^{n-1}\big(\log|g'(h(f^j(x)))|-\log|g'(h(f^j(y)))|\big)-\big(\log|(f'(f^j(x))|-\log|f'(f^j(y))|\big)\\
&= \log\left|\frac{(g^{n-m})'(h(g^m(x)))}{(g^{n-m})'(h(g^m(y)))}\right|-\log\left|\frac{(f^{n-m})'(f^m(x))}{(f^{n-m})'(f^m(y))}\right|.
\end{aligned}
$$

We want $u(f^n(x))$ and $u(f^m(x))$ to be close one to the other if $f^n(x)$ and $f^m(x)$ are. For this it suffices to know that each term $\log\left|\frac{(g^{n-m})'(h(g^m(x)))}{(g^{n-m})'(h(g^m(y)))}\right|$ and $\log\left|\frac{(f^{n-m})'(f^m(x))}{(f^{n-m})'(f^m(y))}\right|$ is small. But for the latter term this follows from the Koebe's Distortion Theorem since $|f^n(x)-f^n(y)|<\rho_2$ and the inverse branch $f^{-(n-m)}_{f^m(x)}$ sending $f^n(x)$ to $f^m(x)$ is defined on $B(f^n(x),s)$. A similar argument works for the former term. The proof is finished. ♣

Consequently the function u extends continuously to each set $J(f)\setminus B(\mathrm{Sing}(f),s)$, $s>0$, and therefore to the set $J(f)\setminus\mathrm{Sing}(f)$.

Lemma 4.3. The functions $\log|f'(z)|$ and $\log|g'(h(z))|$ are cohomologous in the class of continuous functions on $(J(f)\setminus\mathrm{Sing}(f))\cap f^{-1}(J(f)\setminus\mathrm{Sing}(f))$. More precisely there exists continuous $u:J(f)\setminus\mathrm{Sing}(f)\to I\!\!R$ such that

$$\log|g'(h(z))|-\log|f'(z)|=u(f(z))-u(z)$$

for all $z\in(J(f)\setminus\mathrm{Sing}(f))\cap f^{-1}(J(f)\setminus\mathrm{Sing}(f))$.

Proof. Subtracting (4.1) from (4.1) written for $n-1$ we get $\eta(f^n(x))=u(f^n(x))-u(f^{n-1}(x))$. Since the set $\{f^n(x):n\ge 1\}$ is dense in $(J(f)\setminus\mathrm{Sing}(f))\cap f^{-1}(J(f)\setminus\mathrm{Sing}(f))$ and all the functions η, u, $u\circ f$ are continuous in this set, the lemma is proved. ♣

Lemma 4.4. Let μ and ν be Borel probability measures on Y, a bounded subset of a Euclidean space. Suppose that there are a constant $M>0$ and for every point $x\in Y$ a decreasing to zero sequence $\{r_j(x) : j \geq 1\}$ of positive radii such that for all $j \geq 1$ and all $x \in Y$

$$\mu(B(x, r_j(x)) \leq M\nu(B(x, r_j(x)).$$

Then the measure μ is absolutely continuous with respect to ν and the Radon-Nikodym derivative $d\mu/d\nu \leq CM$, where C is a universal constant depending only on the dimension of the Euclidean space under consideration.

Proof. Consider a Borel set $E \subset Y$ and fix $\varepsilon > 0$. Since $\lim_{j\to\infty} r_j(x) = 0$ and since ν is regular, for every $x \in E$ there exists a radius $r(x)$ being of the form $r_j(x)$ such that $\nu(\bigcup_{x\in E} B(x, r(x)) \setminus E) < \varepsilon$. Now by the Besicovič theorem (see [Gu]) we can choose a countable subcover $\{B(x_i, r(x_i))\}_{i=1}^{\infty}$ from the cover $\{B(x, r(x))\}_{x\in E}$ of E, of multiplicity bounded by some constant $C \geq 1$, independent of the cover. Therefore we obtain

$$\begin{aligned}
\mu(E) &\leq \sum_{i=1}^{\infty} \mu(B(x_i, r(x_i))) \leq M \sum_{i=1}^{\infty} \nu(B(x_i, r(x_i))) \\
&\leq MC\nu\big(\bigcup_{i=1}^{\infty} B(x_i, r(x_i))\big) \\
&\leq MC(\varepsilon + \nu(E)).
\end{aligned}$$

Letting $\varepsilon \searrow 0$ we obtain $\mu(E) \leq MC\nu(E)$. So μ is absolutely continuous with respect to ν with the Radon–Nikodym derivative bounded by MC. ♣

Proof of the implication (5) $\Rightarrow$ (6)**.** Given $n \geq 1$ let

$$T_n(f) = \{z \in J(f) : \limsup_{j\to\infty} \operatorname{dist}(f^j(z), \operatorname{Sing}(f) > 2/n\} \subset T(f).$$

Since $h(\operatorname{Crit}(f) \cap J(f)) = \operatorname{Crit}(g) \cap J(g)$, since, in view of (5), indifferent periodic points of f are mapped to indifferent periodic points of g, and since a topological conjugacy on Julia sets cannot send a rationally indifferent periodic point to an irrationally indifferent periodic point of g and vice versa ("snail"argument), we conclude that $h(\operatorname{Sing}(f)) = \operatorname{Sing}(g)$. (By the way irrationally indifferent periodic points in $J(f)$ also belong to $\operatorname{Sing}(f)$, see [GPS].) Therefore, since $\operatorname{Sing}(f)$ and $\operatorname{Sing}(g)$ are respectively f and g invariant and since $h^{-1} : J(g) \to J(g)$ is uniformly continuous, there exists $k_n \geq 1$ such that $\operatorname{dist}(x, \operatorname{Sing}(f)) \geq 2/n$ implies $\operatorname{dist}(h(x), \operatorname{Sing}(g)) \geq 2/k_n$. In particular $h(T_n(f)) \subset T_{k_n}(g)$. Fix now $z \in T_n(f)$. Then there exists an infinite sequence $n_j = n_j(z)$ such that $\operatorname{dist}(f^{n_j}(z), \operatorname{Sing}(f)) \geq 2/n$ for all $j \geq 1$. Applying now Lemma 4.3 and Lemma 4.2 we see that there exists a constant $Q_n \geq 1$ such that

$$Q_n^{-1} \leq \frac{|(g^{n_j})'h((z))|}{|(f^{n_j})'(z)|} \leq Q_n. \tag{4.2}$$

In view of uniform continuity of h there exists $\gamma_n \leq 1/n$ such that $h(J(f) \cap B(x, \gamma_n) \subset B(h(x), 1/k_n)$ for all $x \in J(f)$. By the choice of the sequence $n_j = n_j(z)$ for every

$j \geq 1$ there exists an inverse branch $f_z^{-n_j} : B(f_z^{n_j}(z), 2/n) \to \overline{\mathbb{C}}$ sending $f^{n_j}(z)$ to z. Set $r_j(z) = \frac{1}{4}|(f_z^{-n_j})'(f^{n_j}(z))|\gamma_n$. Then by the Koebe Distortion Theorem

$$B(z, r_j(z)) \supset f_z^{-n_j}(B(f^{n_j}(z), K^{-1}\gamma_n/4),$$

where K is a universal constant, and therefore

$$\begin{aligned} m_f(B(z, r_j(z))) &\geq K^{-t_f}|(f_z^{-n_j})'(f^{n_j}(z)|^{t_f} m(B(f^{n_j}(z), K^{-1}\gamma_n/4) \\ &\geq M_n K^{-t_f}|(f_z^{-n_j})'(f^{n_j}(z)|^{t_f}, \end{aligned} \tag{4.3}$$

where $M_n = \inf\{m_f(B(x, K^{-1}\gamma_n/4) : x \in J(f)\} > 0$. Similarly by the $\frac{1}{4}$-Koebe Distortion Theorem $B(z, r_j(z)) \subset f_z^{-n_j}(B(f^{n_j}(z), \gamma_n)$. Hence

$$\begin{aligned} h\big(J(f) \cap B(z, r_j(z))\big) &\subset h\big(f_z^{-n_j}(B(f^{n_j}(z), \gamma_n) \cap J(f))\big) \\ &= g_{h(z)}^{-n_j}\big(h(B(f^{n_j}(z), \gamma_n) \cap J(f))\big) \subset g_{h(z)}^{-n_j}\big(B(h(f^{n_j}(z)), 1/k_n))\big) \\ &= g_{h(z)}^{-n_j}\big(B(g^{n_j}(h(z)), 1/k_n))\big) \end{aligned}$$

Therefore using (4.2), (4.3) and the Koebe distortion theorem, we get

$$\begin{aligned} m_g \circ h(B(z, r_j(z))) &\leq m_g\big(g_{h(z)}^{-n_j}\big(B(g^{n_j}(h(z)), 1/k_n))\big)\big) \\ &\leq K^{t_g}|(g_{h(z)}^{-n_j})'(g^{n_j}(h(z)))|^{t_g} m_g\big(B(g^{n_j}(h(z)), 1/k_n))\big)\big) \\ &\leq K^{t_g}|(g^{n_j})'(h(z))|^{-t_g} \leq K^{t_g} Q_n^{t_g}|(f^{n_j})'(z)|^{-t_f+(t_f-t_g)} \\ &\leq M_n^{-1} K^{t_f+t_g} Q_n^{t_g} m_f(B(z, r_j(z))|(f^{n_j})'(z)|^{t_f-t_g} \end{aligned}$$

If $t_f - t_g < 0$ then it would follow from Lemma 4.4 and the fact that $\lim_{n\to\infty}|(f^{n_j})'(z)| = \infty$ that $m_g \circ h(T_n(f)) = 0$ for every $n \geq 1$. Since $\bigcup_{n\geq 1} T_n(f) = T(f)$, it would imply that $m_g(h(T(f))) = 0$. But since $h(\text{Sing}(f)) = \text{Sing}(g)$, using uniform continuity of h and h^{-1}, we conclude that $h(T(f)) = T(g)$. Hence $m_g(T(g)) = 0$ which would contradict the definition of tame maps. Thus for every $n \geq 1$ and every $z \in T_n(f)$

$$m_g \circ h(B(z, r_j(z))) \leq M_n^{-1} K^{2t_f} Q_n^{t_g} m_f(B(z, r_j(z))).$$

Therefore, applying Lemma 4.4 we conclude that $m_g \circ h|_{T_n(f)}$ is absolutely continuous with respect to $m_f|_{T_n(f)}$ for every $n \geq 1$. Since $\bigcup_{n\geq 1} T_n(f) = T(f)$, this implies that $m_g \circ h|_{T(f)}$ is absolutely continuous with respect to $m_f|_{T(f)}$. Since $m_g \circ h(\text{Sing}(f)) = m_g(\text{Sing}(g)) = 0$ and $m_f(\text{Sing}(f)) = 0$, we obtain that $m_g \circ h$ is absolutely continuous with respect to m_f. By symmetry $m_f \circ h^{-1}$ is absolutely continuous with respect to m_g and consequently the measures $m_g \circ h$ and m_f are equivalent. The proof of the implication (5) $\Rightarrow$ (6) is finished. ♣

§5. Closing the chain of implications: rigidity.

The main result of this section is the following.

Theorem 5.1 (Rigidity Theorem). Suppose that f and g are two tame rational functions not critically finite with parabolic orbifold. Suppose they are conjugate in measure-theoretic sense by an isomorphism $h : J(f) \to J(g)$ mapping m_f to a measure equivalent to m_g. Then h extends from a set of full measure m_f to a conformal homeomorphism from a neighbourhood of $J(f)$ to a neighbourhood of $J(g)$.

To prove this Theorem we proceed as in [Su], [Przy1]. We do not write down all details, they can be found in [Przy1]. We start with

Lemma 5.2. Under the above assumptions there exists an open subset of $J(f)$ where h extends from a set of full measure m_f to a biLipschitz homeomorphism. Moreover h extends from a set of full measure in $J(f)\setminus\mathrm{Sing}(f)$ to a locally biLipschitz homeomorphism from $J(f) \setminus \mathrm{Sing}(f)$ to $J(g) \setminus \mathrm{Sing}(g)$.

Proof. Assume $t_f \le t_g$. In view of Theorem 3.6 f and g are non-linear. Since g is non-linear there exists $w \in J(g) \setminus \mathrm{Sing}(g)$ and a neighbourhood $W \subset \mathbb{C}$ of w such that G is invertible on W, where $G(z) = (D_{\mu_g} g)(z)$ in the Julia real-analytic case and $G(z) = ((D_{\mu_g} g)(z), (D_{\mu_g} g) \circ g^k(z))$, for some $k \ge 1$, otherwise. By ergodicity of m_f and m_g and since $m_f(\mathrm{Sing}(f)) = m_g(\mathrm{Sing}(g)) = 0$, there exists $Y \subset J(g)$ of positive measure m_g and $\delta > 0$ such that $Y \subset W \setminus B(\mathrm{Sing}(g), \delta)$ and $h^{-1}(Y) \subset J(f) \setminus B(\mathrm{Sing}(f), \delta)$ and for every $y \in Y$ there exists $n_j = n_j(y) \to \infty$ such that $g^{n_j}(y) \in Y$.

Let $x \in T(f)$ be a density point with respect to m_f of the set $h^{-1}(Y)$ and $x \in h^{-1}(Y)$. Write $F(z) := (D_{\mu_f} f)(z)$ if g is real-analytic and $F(z) = ((D_{\mu_f} f)(z), D_{\mu_f} f \circ f^k(z))$ otherwise. The measures $\mu_f \circ h^{-1}$ and μ_g coincide (up to multiplication by a constant) because they are both equivalent and ergodic. Hence we obtain the crucial equality

$$(D_{\mu_f} f)(f^n(z)) = (D_{\mu_g} g)(g^n \circ h(z)) \tag{5.1}$$

for every $n = 0, 1, ...$ Therefore

$$h = G^{-1} \circ F$$

on $A_r := B(x, r) \cap h^{-1}(Y)$, where $\frac{m_f(A_r)}{m_f(B(x,r))} \to 1$ as $r \to 0$.

By $t_f \le t_g$, due to (5.1) and

$$(D_{\mu_f} f^n)(x) = \frac{d\mu_f}{dm_f}(f^n(x))|(f^n)'(x)|^{t_f} \frac{d\mu_f}{dm_f}(x)^{-1}$$

$$(D_{\mu_g} g^n)(h(x)) = \frac{d\mu_g}{dm_g}(g^n h(x))|(g^n)'(h(x))|^{t_g} \frac{d\mu_g}{dm_g}(h(x))^{-1}$$

we obtain

$$(5.2) \qquad |(f^{n_j})'(x)| \geq C_Y^4 |(g^{n_j})'(h(x))|,$$

for $C_Y := \min\{\inf_{h^{-1}(Y)} \frac{d\mu_f}{dm_f}, \inf_{h^{-1}(Y)} \frac{d\mu_f}{dm_f}^{-1}, \inf_Y \frac{d\mu_g}{dm_g}, \inf_Y \frac{d\mu_g}{dm_g}^{-1}\}$.

Since $x \in h^{-1}(Y)$ for an arbitrary $\delta' > 0$ there exists $x' \in J(f)$ such that $B' := B(x', \delta') \subset \mathbb{C} \setminus B(\mathrm{Sing}(f), \delta)$ and there exist branches ϕ_j of f^{-n_j} on it, satisfying $\phi_j(B') \ni x$ (this may require passing to a subsequence of n_j). Then by (5.2) and Koebe's Distortion Theorem for δ' small enough $G^{-1}F\phi_j(B') \subset g_{h(x)}^{-n_j}(B(g^{n_j}(h(x)), \delta)$, where the branch $g_{h(x)}^{-n_j}$ is the branch of g^{n_j} mapping $g^{n_j}(h(x))$ to $h(x)$. Hence the functions $h = g^{n_j} h \phi_j$ are uniformly Lipschitz continuous on sets $f^{n_j}(A_{r_j})$, where r_j are such that f^{n_j} are subsets of B' of diameters $\geq \mathrm{Const}\delta'$. Hence h is Lipschitz continuous on subsets of a disc B'' in B' whose m_f-measures converge to $m_f(B'')$, with the same Lipschitz constant. Therefore h is Lipschitz continuous on a dense subset of B''. We obtain also $t_f = t_g$, because $t_f < t_g$ would lead to a subset of full measure in B" mapped by h to a point. Therefore we can exchange the roles of f and g in the above proof. Finally h is Lipschitz on a full measure subset of one disc B implies that h is Lipschitz on a subset of full measure on a disc centered at an arbitrary point $x \in J(f) \setminus \mathrm{Sing}(f)$. Just choose a backward trajectory from x to B. This in particular allows to find B such that h and h^{-1} are Lipschitz on B and $h(B)$ respectively (on subsets of full measure) perhaps with larger Lipschitz constants. Since m_f is a measure of full support on $J(f)$, we are done. ♣

Lemma 5.3. If the conjugating mapping h in Theorem 5.1 is a homeomorphism from $J(f) \setminus \mathrm{Sing}(f)$ to $J(g) \setminus \mathrm{Sing}(g)$ then the Julia real analytic of f implies Julia real analycity of g. In case they are both Julia real analytic, h extends to a real analytic diffeomorphism mapping a neighbourhood of $J(f) \setminus \mathrm{Sing}(f)$ to a neighbourhood of $J(g) \setminus \mathrm{Sing}(g)$ in respective real analytic curves, hence complex analytic in neighbourhoods in $\mathbb{C}$. If f and g are not Julia real-analytic, h extends to a real-analytic diffeomorphism in neighbourhoods in $\mathbb{C}$.

Proof. Suppose that f is Julia real-analytic. Then $h = G^{-1} \circ F$ in an open set $W' := h^{-1}(W)$, see the beginning of Proof of Lemma 5.2. So h is real-analytic in this set and if $J(f) \cap W'$ is contained in Γ the union of real-analytic curves. Then $J(g) \cap W$ is contained in $h(\Gamma \cap W')$, also the union of real analytic curves. So, using topological exactness of $g : J(g) \to J(g)$ we conclude that g is Julia real-analytic. The same formulas $h = G^{-1} \circ F$ and $h^{-1} = F^{-1} \circ G$ prove real analyticity of h and h^{-1}. ♣

Proof of Theorem 5.1. Let $x \in \mathrm{Sing}(f)$. There exists $y \in J(f) \setminus \mathrm{Sing}(f)$ and a positive integer n such that $f^n(y) = x$. By Lemmas 5.2 and 5.3 there exists $\delta > 0$ such that h extends real-analytically (or complex analytically for Julia real analytic f) to the component B' of $f^{-n}(B(x, \delta))$ containing y. Moreover we can take δ so small that f^n has no critical points in B' maybe except the point y. Hence $h = g^n \circ h \circ f^{-n}$ on a dense subset of $B(x, \delta) \cap J(f)$ extends to $B(x, \delta)$ by the same formula, to a real analytic (complex analytic) diffeomorphism. Formally the extension depends on the branch of f^{-n} (i.e. the resulting h is multivalued) but two extensions must coincide on a real analytic set containing $J(f) \cap B(x,\delta)$. If they did not coincide in the entire $B(x,\delta)$ then f would be Julia real-analytic. But in this case the extensions h are complex analytic so they coincide. The singularity x (in the nontrivial case where $\deg f^n > 1$ on B') is therefore removable. Thus the proof is finished for Julia real-analytic case: h extends to a biholomorphic map on a neighbourhood of $J(f)$, real-analytic from Γ_f to Γ_g.

If f and g are not Julia real-analytic we first notice that $\tilde{h}$, the real analytic extension of h to a neighbourhood of $J(f) \setminus \mathrm{Sing}(f)$ is in fact conformal. The proof is the same as in [Przy1, Lemma 7.2.7]. The idea of the proof is that otherwise (in case $\tilde{h}$ preserves orientation) there would exist an invariant line field, with arguments $\alpha = \frac{1}{2}\mathrm{Arg}(\frac{dh}{d\bar{z}}/\frac{dh}{dz}) \bmod \pi$. This is possible only if f is linear. Indeed locally we can consider functions β conjugate to α and $\exp(\beta + i\alpha)$ will be charts ϕ_t satisfying the condition (d) in Theorem 3.2. Again as in Julia real-analytic case we extend $\tilde{h}$ conformally to $B(x,\delta) \setminus \{x\}$ for any $x \in \mathrm{Sing}(f)$ and remove the singularity x. ♣

The implication ((6) $\Rightarrow$ (2)) has been proved except for f and g critically finite with parabolic orbifold. Such maps for $J(f)$ and $J(g)$ not being the whole $\overline{\mathbb{C}}$ must be of the form $z \mapsto z^n$, $|n| \geq 2$, or Tchebyshev polynomials (up to conformal changes of coordinates). For such maps m_f and m_g are length measures and the conjugacy is real analytic because the densities of invariant measures are real-analytic, see for example [SS]. So, in order to conclude the proof of Theorem 1.9 we only need to show that (2) $\Rightarrow$ (1). This follows from the following proposition which is interesting itself.

Proposition 5.4. For any rational functions f, g any conformal conjugacy on neighbourhoods of Julia sets extends to a Möbius map conjugating f and g on $\overline{\mathbb{C}}$.

Proof. For every $x \in \overline{\mathbb{C}}$ except at most two points there exists y close to $J(f)$ so that h is defined in a neighbourhood of y and there exists $n \geq 0$ such that $f^n(y) = x$. Assume additionally that $\{f^j(x), j = 0, ..., n-1\} \cap \mathrm{Crit}(f) = \emptyset$. So we can define conformal $h := g^n h \phi_n$ on a neighbourhood of x, where ϕ is the branch of f^{-n} mapping x to y. This does not depend on the choice of the branch (i.e. the choice of y and n) because the formula extends the formula for x close to $J(f)$ where it does not depend on the branch. A countable number of singularities at $f^n(\mathrm{Crit}(f))$ are removable. Similarly extend h^{-1}. The compositions which are identities on neighbourhoods of $J(f)$ or $J(g)$ are therefore identities on $\overline{\mathbb{C}}$, so the extensions are invertible on $\overline{\mathbb{C}}$ ♣

In Proof of Theorem 5.1 a proof of the following general fact is contained.

Lemma 5.5. Let f, g be two arbitrary rational functions. If a homeomorphism $h : J(f) \to J(g)$ conjugates f to g and extends conformally to an open disc intersecting $J(f)$ then h extends to a conformal homeomorphism from a neighbourhood of $J(f)$ to a neighbourhood of $J(g)$.

The proof goes through as above via $g^n h f^{-n}$. In fact it is sufficient to assume that f and g are defined only on neighbourhoods of their invariant sets, here $J(f), J(g)$. Lemma 5.5 gives a very simple proof of the following.

Theorem 5.6 (E. Prado [Pra]). If f, g are generalized polynomial-like maps at zero Teichmüller distance, namely there exists a sequence $h_n : U_n \to V_n$ of K_n-quasiconformal conjugacies between neighbourhoods U_n and V_n of Julia sets and $K_n \to 1$, then there is a biholomorphic conjugacy between domains of f and g.

Proof. Fix a periodic source $x \in J(f)$ of period, say p and a disk $B(x, \delta)$ with $\delta > 0$ so small that all $k = 0, 1, ..., p-1$ the components of $f^{-k}(B(x, \delta))$ intersecting the periodic orbit of x are disjoint from $\mathrm{Crit}(f)$ and their diameters shrink (exponentially fast) to 0. Let $\delta_n > 0$ be such that $B(x, 2\delta_n) \subset U_n$. Write $B_n := B(x, \delta_n)$. Then there exist sequences k_j, n_j such that $f^{k_j}(B_{n_j}) \subset B(x, \delta/2)$ and the diameters of are $f^{k_j}(B_{n_j})$ are comparable to $\delta/2$. Since $\sup K_n < \infty$ there exists $B' = B(x, \delta') \subset f^{k_j}(B_{n_j})$ for all j. Hence

$$H_j := g^{k_j} \circ h_{n_j} \circ \phi_j$$

for ϕ_j the branches of f^{-k_j} fixing x, are all well defined on B' and K_{n_j}-quasiconformal. They all coincide on $J(f) \cap B'$ so a limit, which is conformal by $K_n \to 0$, exists. There is no problem with the domain of definition of g^{k_j} above and also all H_j are uniformly bounded because $g^k h_{n_j} \phi_j((J(f) \cap B')$ for all $k \leq k_j$ have small diameters and all $g^k h_{n_j} \phi_j(B')$ are uniformly distorted. Finally h extends holomorphically from B' to a neighbourhood of $J(f)$ by Lemma 5.5. ♣

Remark 5.7. In the literature there exist a large number of papers devoted to various kinds of rigidity theorems. Let us list here only those which seem to be the closest to what is contained in this paper: [Co], [Mc1], [Pra], [Przy1], [SS], [Su] and [U5].

Remark 5.8. It follows from [FU] that if the Julia set of a Julia real-analytic rational function is connected, then, in fact, it is equal to a geometric circle or an interval.

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