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Examples of positive recurrent functions

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Abstract. In [Sa] Sarig has introduced and explored the concept of positively recurrent functions. In this paper we construct a natural wide class of such functions and we show that they have stronger properties than the general functions considered in [Sa].

§**1. Preliminaries.** In [Sa] Sarig has introduced and explored the concept of positively recurrent functions. In this paper, using the concept of an iterated function system, we construct a natural wide class of positively recurrent functions and we show that they have stronger properties than the general functions considered in [Sa]. In some parts our exposition is similar and follows the approach developed in [MU]. To begin with, let $I\!\!N$ be the set of positive integers and let $\Sigma = I\!\!N^\infty$ be the infinitely dimensional shift space equipped with the product topology. Let $\sigma : \Sigma \to \Sigma$ be the shift transformation (cutting out the first coordinate), $\sigma(\{x_n\}_{n=1}^\infty) = (\{x_n\}_{n=2}^\infty)$. Fix $\beta > 0$. If $\phi : \Sigma \to I\!\!R$ and $n \geq 1$, we set

$$V_n(\phi) = \sup\{|\phi(x) - \phi(y)| : x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n\}.$$

The function ϕ is said to be Hölder continuous of order β if and only if

$$V(\phi) = \sup_{n\geq 1}\{\mathrm{e}^{\beta n} V_n(\phi)\} < \infty.$$

We also assume that

$$\sup_{\omega\in\Sigma} \sum_{\tau\in\sigma^{-1}(\omega)} \mathrm{e}^{\phi(\tau)} < \infty. \tag{1.1}$$

This assumption allows us to introduce the Perron-Frobenius-Ruelle operator $\mathcal{L}_\phi : C_b(\Sigma) \to C_b(\Sigma)$,

$$\mathcal{L}_\phi(g)(\omega) = \sum_{\tau\in\sigma^{-1}(\omega)} \mathrm{e}^{\phi(\tau)} g(\tau)$$

acting on $C_b(\Sigma)$, the space of all bounded continuous real-valued functions on Σ equipped with the norm $||\cdot||_0$, where $||k||_0 = \sup_{x\in\Sigma}|k(x)|$. Moreover,

$$||\mathcal{L}_\phi||_0 \leq \mathcal{L}_\phi(1\!\!1) = \sup_{\omega\in\Sigma} \sum_{\tau\in\sigma^{-1}(\omega)} \mathrm{e}^{\phi(\tau)} < \infty.$$

We extend the standard definition of topological pressure by setting

$$\mathrm{P}(\phi) = \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{|\omega|=n} \sup_{\tau\in[\omega]} \exp\left(\sum_{j=0}^{n-1} \phi\circ\sigma^j(\tau)\right)\right), \tag{1.2}$$

where $[\omega] = \{\rho \in \Sigma : \rho_1 = \omega_1, \rho_2 = \omega_2, \ldots, \rho_{|\omega|} = \omega_{|\omega|}\}$. Notice that the limit exists since the partition functions

$$Z_n(\phi) = \sum_{|\omega|=n} \sup_{\tau\in[\omega]} \exp\left(\sum_{j=0}^{n-1} \phi\circ\sigma^j(\tau)\right)$$

form a subadditive sequence. Notice also that our definition of pressure formally differs from that provided by Sarig in [Sa] which reads that given $i \in I\!\!N$

$$\mathrm{P}(\phi) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\phi, i), \tag{1.3}$$

where

$$Z_n(\phi, i) = \sum \exp\left(\sum_{j=0}^{n-1} \phi \circ \sigma^j(\omega)\right)$$

and the summation is taken over all elements ω satisfying $\sigma^n(\omega) = \omega$ and $\omega_1 = i$. However in [Sa] Sarig proves Theorem 2 which says that $\mathrm{P}(\phi) = \sup\{\mathrm{P}(\phi|_Y)\}$, where the supremum is taken over all topologically mixing subshifts of finite type $Y \subset \Sigma$ and the same proof goes through with (1.3) replaced by (1.2). Thus we have the following.

Lemma 1.1. The definitions of topological pressures given by (1.2) and (1.3) coincide.

Following the definition 2 of [Sa] we call the function $\phi : \Sigma \to I\!R$ positive recurrent if for every $i \in I\!N$ there exists a constant M_i and an integer N_i such that for all $n \geq N_i$

$$M_a^{-1} \leq Z_n(\phi, i)\lambda^{-n} \leq M_i$$

for some $\lambda > 0$. As we already have said the main purpose of this paper is to provide a wide natural class of examples of positive recurrent potential which additionally satisfy much stronger properties than those claimed in Theorem 4 of [Sa]. In order to describe our setting let (X, d) be a compact metric space and let $\phi_i : X \to X$, $i \in I\!N$, be a family of uniform contractions, i.e. $d(\phi_i(x), \phi_i(y)) \leq s d(x, y)$ for all $i \in I\!N$, $x, y \in X$ and some $s < 1$. Given $\omega \in \Sigma$ consider the intersection $\bigcap_{n\geq 1} \phi_{\omega|_n}(X)$, where $\phi_{\omega|_n} = \phi_{\omega_1} \circ \ldots \circ \phi_{\omega_n}$. Since $\phi_{\omega|_n}(X)$, $n \geq 1$, form a descending family of compact sets, this intersection is non-empty and since the maps ϕ_i, $i \in I\!N$, are uniform contractions, it is a singleton. So, we have defined a projection map $\pi : \Sigma \to X$ given by the formula

$$\{\pi(\omega)\} = \bigcap_{n\geq 1} \phi_{\omega|_n}(X).$$

J, the range of π, is said to be the limit set of the iterated function system $\phi_i : X \to X$, $i \in I\!N$. Let now $\phi^{(i)} : X \to I\!R$, $i \in I\!N$, be a family of continuous functions such that

$$\sup_X \sum_{i \in I\!N} \mathrm{e}^{\phi^{(i)}(x)} < \infty. \tag{1.4}$$

We define a function $\phi : \Sigma \to I\!R$ by setting

$$\phi(\omega) = \phi^{(\omega_1)}(\pi(\sigma(\omega))). \tag{1.5}$$

It easily follows from (1.4) that $\mathrm{P}(\phi) < \infty$. In the next section we shall prove the following.

Theorem 1.2. Suppose that the function $\phi : \Sigma \to I\!R$ defined by (1.5) and satisfying (1.4) is Hölder continuous. Let $\mathcal{L}_\phi^*$ be the operator conjugate to $\mathcal{L}_\phi$. Then ϕ is positive recurrent with $\lambda = \mathrm{e}^{\mathrm{P}(\phi)}$. Moreover there exists $M > 0$ such that $M^{-1} \leq \lambda^{-n}\mathcal{L}_\phi^n(1\!\!1) \leq M$ for all $n \geq 1$. Suppose additionally that $\phi_i(X) \cap \phi_j(X) = \emptyset$ for all $i, j \in I\!N$, $i \neq j$. Then there are a probability measure ν on Σ and a positive continuous function $h : \Sigma \to I\!R$ such that $\mathcal{L}_\phi^*(\nu) = \lambda\nu$, $\mathcal{L}_\phi(h) = \lambda h$, $\nu(h) = 1$ and $\lambda^{-n}\mathcal{L}_\phi^n(g) \to (\int g d\nu)h$ uniformly on compacts for every uniformly continuous function g such that $||gh^{-1}||_0 < \infty$. Additionally (see Theorem 5 of [Sa]) there exists L a large class of functions such that for all $g \in L$, $\lambda^{-n}\mathcal{L}_\phi^n(g) \to (\int g d\nu)h$ exponentially fast on each initial cylinder of length 1.

§**2. Proof of Theorem 1.2.** Define first an auxiliary Perron-Frobenius operator $L_\phi : C(X) \to C(X)$ given by the formula

$$L_\phi(g)(x) = \sum_{i \in I\!N} \mathrm{e}^{\phi^{(i)}(x)} g(\phi_i(x)).$$

L_ϕ is continuous, positive and $||L_\phi||_0 \le \sup_X \sum_{i \in I\!N} \mathrm{e}^{\phi^{(i)}(x)} < \infty$. Let $L_\phi^* : C(X)^* \to C(X)^*$ be the conjugate operator and consider the map

$$\mu \mapsto \frac{L_\phi^*(\mu)}{L_\phi^*(\mu)(1\!\!1)}.$$

of the space of Borel probability measures on X into itself. This map is continuous in the weak-* topology of measures and therefore, in view of the Schauder-Tichonov theorem, it has a fixed point, say m_ϕ. Thus

$$L_\phi^*(m_\phi) = \lambda m_\phi \tag{2.1}$$

with $\lambda = L_\phi^*(m_\phi)(1\!\!1)$.

Given $n \ge 1$ and $\omega \in I\!N^n$, denote $\sum_{j=1}^n \phi^{(\omega_j)} \circ \phi_{\sigma^j\omega}$ by $S_\omega(\phi)$. Let us then prove the following.

Lemma 2.1. If $x, y \in J$ are such that $x = \pi(\tau), y = \pi(\rho)$ are such that $\tau|_k = \rho|_k$, then for all $n \ge 1$, all $\omega \in I^n$

$$|S_\omega(\phi)(x) - S_\omega(\phi)(y)| \le \frac{V(\phi)}{1 - \mathrm{e}^{-\beta}} \mathrm{e}^{-\beta(k+1)}$$

Proof. By Hölder continuity of ϕ we have for every $j = 1, \dots, n$,

$$\begin{aligned}\left|\sum_{j=1}^n \phi^{(\omega_j)}(\phi_{\sigma^j\omega}(x)) - \sum_{j=1}^n \phi^{(\omega_j)}(\phi_{\sigma^j\omega}(y))\right| &= \left|\sum_{j=0}^{n-1} \phi(\sigma^j\omega\tau) - \sum_{j=0}^{n-1} \phi(\sigma^j\omega\rho)\right| \\ &\le \sum_{j=0}^{n-1} |\phi(\sigma^j\omega\tau) - \phi(\sigma^j\omega\rho)| \le \sum_{j=0}^{n-1} V(\phi)\mathrm{e}^{-\beta(k+n-j)} \\ &\le \frac{V(\phi)\mathrm{e}^{-\beta}}{1 - \mathrm{e}^{-\beta}} \mathrm{e}^{-\beta k}\end{aligned}$$

The proof is finished. ♣

Set

$$Q = \exp\left(V(\phi)\frac{\mathrm{e}^{-\beta}}{1-\mathrm{e}^{-\beta}}\right).$$

We shall prove the following.

Lemma 2.2. The eigenvalue λ (see 2.1) of the dual Perron-Frobenius operator is equal to $\mathrm{e}^{\mathrm{P}(\phi)}$.

Proof. Iterating (2.1) we get

$$\begin{aligned}\lambda^n &= \lambda^n m_\phi(1\!\!1) = L_\phi^{*n}(1\!\!1) = \int_X L_\phi^n(1\!\!1)dm_\phi \\ &= \int_X \sum_{|\omega|=n} \exp(S_\omega(\phi)(x)) \le \sum_{|\omega|=n} ||\exp(S_\omega(\phi))||_0.\end{aligned}$$

So,

$$\log\lambda \le \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{|\omega|=n}||\exp(S_\omega(\phi))||_0\right) = \mathrm{P}(\phi).$$

Fix now $\omega \in I^n$ and take a point x_ω where the function $S_\omega(\phi)$ takes on its maximum. In view of Lemma 2.1, for every $x \in X$ we have

$$\sum_{|\omega|=n}\exp(S_\omega(\phi)(x)) \ge Q^{-1}\sum_{|\omega|=n}\exp(S_\omega(\phi)(x_\omega)) = Q^{-1}\sum_{|\omega|=n}||\exp(S_\omega(\phi))||_0.$$

Hence, iterating (2.1) as before,

$$\lambda^n = \int_X \sum_{|\omega|=n}\exp(S_\omega(\phi))dm_\phi \ge Q^{-1}\sum_{|\omega|=n}||\exp(S_\omega(\phi))||_0.$$

So, $\log\lambda \ge \lim_{n\to\infty}\frac{1}{n}\log\big(\sum_{|\omega|=n}||\exp(S_\omega(\phi))||_0\big) = \mathrm{P}(\phi)$. The proof is finished. ♣

Let L_0 and $\mathcal{L}_0$ denote the corresponding normalized Perron-Frobenius operators, i.e. $L_0 = \mathrm{e}^{-\mathrm{P}(\phi)}L_\phi$ and $\mathcal{L}_0 = \mathrm{e}^{-\mathrm{P}(\phi)}\mathcal{L}_\phi$. We shall prove the following.

Proposition 2.3. $m_\phi(J) = 1$.

Proof. Since by (2.1)

$$L_0^*(m_\phi) = m_\phi \tag{2.2}$$

and consequently $L_0^{*n}(m_\phi) = m_\phi$ for all $n \ge 0$, we have

$$\int_X \sum_{|\omega|=n}\exp\big(S_\omega(\phi) - \mathrm{P}(\phi)n\big)\cdot(f\circ\phi_\omega)dm_\phi = \int_X f dm_\phi \tag{2.3}$$

for all $n \geq 0$ and all continuous functions $f : X \to I\!R$. Since this equality extends to all bounded measurable functions f, we get

$$(2.4)\quad m_\phi(A) = \sum_{\tau \in I^n} \int \exp\big(S_\tau(\phi) - \mathrm{P}(\phi)n\big) \cdot 1\!\!1_{\phi_\omega(A)} \circ \phi_\tau \, dm_\phi \geq \int_A \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)n\big) dm_\phi$$

for all $n \geq 0$, all $\omega \in I^n$, and all Borel sets $A \subset X$. Now, for each $n \geq 1$ set $X_n = \bigcup_{|\omega|=n} \phi_\omega(X)$. Then $1\!\!1_{X_n} \circ \phi_\omega = 1\!\!1$ for all $\omega \in I\!N^n$. Thus apllying (2.3) to the function $f = 1\!\!1_{X_n}$ and later to the function $f = 1\!\!1$, we obtain

$$\begin{aligned} m_\phi(X_n) &= \int_X \sum_{|\omega|=n} \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)n\big) \cdot (1\!\!1_{X_n} \circ \phi_\omega) dm_\phi \\ &= \int_X \sum_{|\omega|=n} \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)n\big) dm_\phi = \int 1\!\!1 dm_\phi = 1. \end{aligned}$$

Hence $m_\phi(J) = m_\phi\big(\bigcap_{n \geq 1} X_n\big) = 1$. The proof is complete. ♣

Theorem 2.4. For all $n \geq 1$

$$Q^{-1} \leq L_0^n(1\!\!1) \leq Q.$$

Proof. Given $n \geq 1$ by (2.3) there exits $x_n \in X$ such that $L_0^n(1\!\!1)(x_n) \leq 1$. It then follows from Lemma 2.1 that for every $x \in X$, $L_0^n(1\!\!1) \leq Q$. Similarly by (2.3) there exists $y_n \in X$ such that $L_0^n(1\!\!1) \geq 1$. It then follows from Lemma 2.1 that for every $x \in X$, $L_0^n(1\!\!1) \geq Q^{-1}$. The proof is finished. ♣

So far we have worked downstairs in the compact space X. It is now time to lift our considerations up to the shift space Σ.

Lemma 2.5. There exists a unique Borel probability measure $\tilde{m}_\phi$ on $I\!N^\infty$ such that $\tilde{m}_\phi([\omega]) = \int \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)n\big) dm_\phi$ for all $\omega \in I\!N^*$.

Proof. In view of (2.4) $\int \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)n\big) dm_\phi = 1$ for all $n \geq 1$ and therefore one can define a Borel probability measure m_n on C_n, the algebra generated by the cylinder sets of the form $[\omega]$, $\omega \in I\!N^n$, putting $m_n([\omega]) = \int \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)n\big) dm_\phi$. Hence, applying (2.4) again we get for all $\omega \in I\!N^n$.

$$\begin{aligned} m_{n+1}(\omega) &= \sum_{i \in I\!N} m_{n+1}([\omega i]) = \sum_{i \in I\!N} \int \exp\big(S_{\omega i}(\phi) - \mathrm{P}(\phi)n\big) dm_\phi \\ &= \int \sum_{i \in I\!N} \exp\left(\sum_{j=1}^{n} \phi^{(\omega_j)} \circ \phi_{\sigma^j(\omega i)} - \mathrm{P}(\phi)n + \phi^{(i)} - \mathrm{P}(\phi) \right) dm_\phi \\ &= \int \sum_{i \in I\!N} \exp\big(S_\omega \circ \phi_i - \mathrm{P}(\phi)n\big) \exp\big(\phi^{(i)} - \mathrm{P}(\phi)\big) dm_\phi \\ &= \int L_0\big(\exp(S_\omega(\phi) - \mathrm{P}(\phi))\big) dm_\phi = \int \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)\big) dm_\phi = m_n([\omega]) \end{aligned}$$

and therefore in view of Kolmogorov extension theorem there exists a unique probability measure $\tilde{m}_\phi$ on $I\!N^\infty$ such that $\tilde{m}_\phi([\omega]) = \tilde{m}_{|\omega|}([\omega])$ for all $\omega \in I\!N^*$. The proof is complete. ♣

Now we are ready to prove that the function ϕ is positive recurrent. Let us first notice that

$$\begin{aligned}\mathcal{L}_\phi(1\!\!1)(\omega) &= \sum_{\tau\in\sigma^{-1}(\omega)} \mathrm{e}^{\phi(\tau)} = \sum_{\tau\in\sigma^{-1}(\omega)} \exp\big(\phi^{(\tau_1)}(\pi(\sigma(\tau)))\big) \\ &= \sum_{\tau\in\sigma^{-1}(\omega)} \exp\big(\phi^{(\tau_1)}(\pi(\omega))\big) = \sum_{i\in I\!N} \mathrm{e}^{\phi^{(i)}(\pi(\omega))} = L_\phi(1\!\!1)(\pi(\omega)).\end{aligned}$$

Since $L_0 = \mathrm{e}^{-\mathrm{P}(\phi)} L_\phi$, it then follows from Theorem 2.4 that as M we can take Q. In order to demonstrate that the function ϕ is positive recurrent we first show that

$$\frac{Z_n(\phi,i)}{\mathcal{L}_\phi^n(1\!\!1)(\omega)} \le M_i$$

for all $n \ge 1$, $\omega \in \Sigma$, and some constant $M_i > 0$. So fix $\omega \in \Sigma$. We shall define an injection j from $\{\rho \in \Sigma : \sigma^n(\rho) = \rho \text{ and } \rho_1 = i\}$ into $\sigma^{-n}(\omega)$ as follows: $j(\rho) = \rho_1\rho_2\ldots\rho_n\omega$. Now, by Lemma 2.1

$$\left|\sum_{j=0}^{n-1}\phi(\sigma^j(\rho)) - \sum_{j=0}^{n-1}\phi(\sigma^j(j(\rho)))\right| \le \log Q$$

and therefore $Z_n(\phi,i) \le Q\mathcal{L}_\phi^n(1\!\!1)(\omega)$. Thus by Theorem 2.4 and the definition of the operators L_0 and $\mathcal{L}_0$, $Z_n(\phi,i) \le M_i\lambda^n$, where $M_i = Q^2$. Now we shall prove that $Z_n(\phi,i) \ge M_a'\lambda^n$ for some constant M_i' and all $n \ge 1$. We demonstrate first that for all $n \ge 1$ and all $i \in \Sigma$

$$\mathcal{L}_0(1\!\!1_{[i]}) \ge \tilde{m}_\phi([i]).$$

Indeed, since $\int \mathcal{L}_0(1\!\!1_{[i]})d\tilde{m}_\phi = \int 1\!\!1_{[i]})d\tilde{m}_\phi = \tilde{m}_\phi([i]) > 0$, there exists $\tau \in \Sigma$ such that $\mathcal{L}_0(1\!\!1_{[i]})(\tau) \ge \tilde{m}_\phi([i])$. It the follows from Lemma 2.1 that for every $\omega \in \Sigma$

$$\begin{aligned}\mathcal{L}_0^n(1\!\!1_{[i]})(\omega) &= \sum_{\rho\in\sigma^{-n}(\omega)} \exp\left(\sum_{j=0}^{n-1}\phi\circ\sigma^j(\rho)1\!\!1_{[i]}(\rho)\right) \\ &\ge Q^{-1}\sum_{\rho\in\sigma^{-n}(\tau)} \exp\left(\sum_{j=0}^{n-1}\phi\circ\sigma^j(\rho)1\!\!1_{[i]}(\rho)\right) = Q^{-1}\mathcal{L}_0(1\!\!1_{[i]})(\tau) \\ &\ge \tilde{m}_\phi([i]).\end{aligned}$$

Hence $\mathcal{L}_\phi^n(1\!\!1_{[i]})(\omega) \ge \lambda^n\tilde{m}_\phi([i])$. So, in order to conclude the proof that ϕ is positively recurrent it suffices now to show that

$$\frac{Z_n(\phi,i)}{\mathcal{L}_\phi^n(1\!\!1_{[i]})(\omega)} \ge M_i''$$

for all $n \geq 1$, all $\omega \in \Sigma$ and some constant $M_i'' > 0$. Indeed, we shall define an injection k from $\sigma^{-n}(\omega) \cap [i]$ to $\{\rho : \Sigma : \sigma^n(\rho) = \rho \text{ and } \rho_1 = i\}$ by taking as $k(\tau)$ the infinite concatenation of the first n words of τ. Then by Lemma 2.1,

$$\left| \sum_{j=0}^{n-1} \phi(\sigma^j(\tau)) - \sum_{j=0}^{n-1} \phi(\sigma^j(k(\tau))) \right| \leq \log Q$$

and therefore

$$\begin{aligned}
\mathcal{L}_\phi^n(1\!\!1_{[i]})(\omega) &= \sum_{\rho \in \sigma^{-n}(\omega)} \exp\left(\sum_{j=0}^{n-1} \phi \circ \sigma^j(\rho) 1\!\!1_{[i]}(\rho) \right) = \sum_{\rho \in \sigma^{-n}(\omega) \cap [i]} \exp\left(\sum_{j=0}^{n-1} \phi \circ \sigma^j(\rho) \right) \\
&\leq \sum_{\rho \in \sigma^{-n}(\omega) \cap [i]} \exp\left(\sum_{j=0}^{n-1} \phi \circ \sigma^j(k(\rho)) + \log Q \right) \\
&\leq Q \sum \exp\left(\sum_{j=0}^{n-1} \phi \circ \sigma^j(\rho) \right) = Q Z_n(\phi, i),
\end{aligned}$$

where the last summation is taken over all elements ω satisfying $\sigma^n(\omega) = \omega$ and $\omega_1 = i$. So, the proof of the positive recurrence of ϕ is complete taking Q^{-1} as M_i''. Now we pass to proving the existence of the measure ν and the function h. We begin with the following two facts.

Lemma 2.6. The measures m_ϕ and $\tilde{m}_\phi \circ \pi^{-1}$ are equal.

Proof. Let $A \subset J$ be an arbitrary closed subset of J and for every $n \geq 1$ let $A_n = \{\omega \in I\!\!N^n : \phi_\omega(X) \cap A \neq \emptyset\}$. In view of (2.3) applied to the characteristic function $1\!\!1_A$ we have for all $n \geq 1$

$$\begin{aligned}
m_\phi(A) &= \sum_{\omega \in I\!\!N^n} \int \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)|\omega|\big)(1\!\!1_A \circ \phi_\omega)\, dm_\phi \\
&= \sum_{\omega \in A_n} \int \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)|\omega|\big)(1\!\!1_A \circ \phi_\omega)\, dm_\phi \\
&\leq \sum_{\omega \in A_n} \int \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)|\omega|\big)\, dm_\phi = \sum_{\omega \in A_n} \tilde{m}_\phi([\omega]) = \tilde{m}_\phi\Big(\bigcup_{\omega \in A_n} [\omega]\Big)
\end{aligned}$$

Since the family of sets $\{\bigcup_{\omega \in A_n} [\omega] : n \geq 1\}$ is descending and $\bigcap_{n \geq 1} \bigcup_{\omega \in A_n} [\omega] = \pi^{-1}(A)$ we therefore get $m_\phi(A) \leq \lim_{n \to \infty} \tilde{m}_\phi\big(\bigcup_{\omega \in A_n}[\omega]\big) = \tilde{m}_\phi(\pi^{-1}(A))$. Since the limit set J is a metric space , using the Baire classification of Borel sets we easily see that this inequality extends to the family of all Borel subsets of J. Since both measures m_ϕ and $\tilde{m}_\phi \circ \pi^{-1}$ are probabilistic we get $m_\phi = \tilde{m}_\phi \circ \pi^{-1}$. The proof is finished. ♣

Theorem 2.7. There exists a unique ergodic σ-invariant probability measure $\tilde{\mu}_\phi$ absolutely continuous with respect to $\tilde{m}_\phi$. Moreover $\tilde{\mu}_\phi$ is equivalent with $\tilde{m}_\phi$ and $Q^{-1} \leq d\tilde{\mu}_\phi / d\tilde{m}_\phi \leq Q$.

Proof. Let L be a Banach limit defined on the Banach space of all bounded sequences of real numbers. Straightforward computations and an application of Kolmogorov's extension theorem show that the function $\tilde{\mu}_\phi([\omega]) = L\big((\tilde{m}_\phi(\sigma^{-n}([\omega])))_{n\geq 0}\big)$ defined on $I\!N^*$, extends to a σ-invariant probability measure on $I\!N\infty$. Keep for it the same symbol $\tilde{\mu}_\phi$. Notice that, using Lemma 2.5, for each $\omega \in I\!N^*$ and each $n \geq 0$ we have

$$\begin{aligned}
\tilde{m}_\phi(\sigma^{-n}([\omega])) &= \sum_{\tau\in I\!N^n} \tilde{m}_\phi([\tau\omega]) = \sum_{\tau\in I\!N^n} \int \exp\big(S_{\tau\omega}(\phi) - \mathrm{P}(\phi)|\tau\omega|\big)\, dm \\
&\geq \sum_{\tau\in I\!N^n} Q^{-1} || \exp\big(S_\tau(\phi) - \mathrm{P}(\phi)|\tau|\big)||_0 \exp\big(S_\omega(\phi - \mathrm{P}(\phi)|\omega|\big)\, dm \\
&= Q^{-1} \int \exp\big(S_\omega(\phi - \mathrm{P}(\phi)|\omega|\big)\, dm \sum_{\tau\in I\!N^n} || \exp\big(S_\tau(\phi - \mathrm{P}(\phi)|\tau|\big)||_0 \\
&\geq Q^{-1}\tilde{m}_\phi([\omega])\tilde{m}_\phi(I\!N^\infty) = Q^{-1}\tilde{m}_\phi([\omega])
\end{aligned}$$

and

$$\begin{aligned}
\tilde{m}_\phi(\sigma^{-n}([\omega])) &= \sum_{\tau\in I\!N^n} \tilde{m}_\phi([\tau\omega]) = \sum_{\tau\in I\!N^n} \int \exp\big(S_{\tau\omega}(\phi - \mathrm{P}(\phi)|\tau\omega|\big)\, dm_\phi \\
&\leq \sum_{\tau\in I\!N^n} || \exp\big(S_\tau(\phi - \mathrm{P}(\phi)|\tau|\big)||_0 \int \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)|\omega|\big)\, dm_\phi \\
&= \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)|\omega|\big)\, dm \sum_{\tau\in I\!N^n} || \exp\big(S_\tau(\phi) - \mathrm{P}(\phi)|\tau|\big)||_0 \\
&\leq Q\tilde{m}_\phi([\omega]).
\end{aligned}$$

Therefore $Q^{-1}\tilde{m}_\phi([\omega]) \leq \tilde{\mu}_\phi([\omega]) \leq Q\tilde{m}_\phi([\omega])$ and these inequalities extend to all Borel subsets of $I\!N^*$. Thus, to complete the proof of our theorem we only need to show ergodicity of $\tilde{\mu}_\phi$ or equivalently of $\tilde{m}_\phi$. Toward this end take a Borel set $A \in I\!N^\infty$ with $\tilde{m}_\phi(A) > 0$. Since the nested family of sets $\{[\tau] : \tau \in I\!N^*\}$ generates the Borel σ-algebra on $I\!N^\infty$, for every $n \geq 0$ and every $\omega \in I\!N^n$ we can find a subfamily Z of $I\!N^*$ consisting of mutually incomparable words and such that $A \subset \bigcup\{[\tau] : \tau \in Z\}$ and $\sum_{\tau\in Z} \tilde{m}_\phi([\omega\tau]) \leq 2\tilde{m}_\phi(\omega A)$, where $\omega A = \{\omega\rho : \rho \in A\}$. Then

$$\begin{aligned}
\tilde{m}_\phi\big(\sigma^{-n}(A) \cap [\omega]\big) &= \tilde{m}_\phi(\omega A) \geq \frac{1}{2}\sum_{\tau\in Z} \tilde{m}_\phi([\omega\tau]) = \frac{1}{2}\sum_{\tau\in Z} \int \exp\big(S_{\omega\tau}(\phi - \mathrm{P}(\phi)|\omega\tau|\big)\, dm_\phi \\
&\geq \frac{1}{2}Q^{-1} \exp\big(S_\omega(\phi - \mathrm{P}(\phi)|\omega|\big)||_0 \sum_{\tau\in Z} \int \exp\big(S_\tau(\phi - \mathrm{P}(\phi)|\tau|\big)\, dm_\phi \\
&\geq \frac{1}{2}Q^{-1} \int \exp\big(S_\omega(\phi - \mathrm{P}(\phi)|\omega|\big)\, dm \sum_{\tau\in Z} \tilde{m}_\phi([\tau]) \\
&\geq \frac{1}{2}Q^{-1}\tilde{m}_\phi([\omega])\tilde{m}_\phi\big((\bigcup\{[\tau] : \tau \in Z\}\big) \geq \frac{1}{2}Q^{-1}\tilde{m}_\phi(A)\tilde{m}_\phi([\omega]).
\end{aligned}$$

Therefore $\tilde{m}_\phi\big(\sigma^{-n}(I\!N^\infty\backslash A)\cap[\omega]\big) = \tilde{m}_\phi\big([\omega]\backslash\sigma^{-n}(A)\cap[\omega]\big) = \tilde{m}_\phi([\omega]) - \tilde{m}_\phi\big(\sigma^{-n}(A)\cap[\omega]\big) \le \big(1-(2Q)^{-1}\tilde{m}_\phi(A)\big)\tilde{m}_\phi([\omega])$. Hence for every Borel set $A \subset I\!N^\infty$ with $\tilde{m}_\phi(A) < 1$, for every $n \ge 0$, and for every $\omega \in I\!N^n$ we get

$$\tilde{m}_\phi(\sigma^{-n}(A) \cap [\omega]\big) \le \big(1 - (2Q)^{-1}(1 - \tilde{m}_\phi(A))\big)\tilde{m}_\phi([\omega]). \tag{2.5}$$

In order to conclude the proof of ergodicity of σ suppose that $\sigma^{-1}(A) = A$ and $0 < \tilde{m}_\phi(A) < 1$. Put $\gamma = 1 - (2Q)^{-1}(1 - \tilde{m}_\phi(A))$. Note that $0 < \gamma < 1$. In view of (2.5), for every $\omega \in I\!N^*$ we get $\tilde{m}_\phi(A \cap [\omega]) = \tilde{m}_\phi(\sigma^{-|\omega|}(A) \cap [\omega]\big) \le \gamma\tilde{m}_\phi([\omega])$. Take now $\eta > 1$ so small that $\gamma\eta < 1$ and choose a subfamily R of $I\!N^*$ consisting of mutually incomparable words and such that $A \subset \bigcup\{[\omega] : \omega \in R\}$ and $\tilde{m}_\phi\big(\bigcup\{[\omega] : \omega \in R\}\big) \le \eta\tilde{m}_\phi(A)$. Then $\tilde{m}_\phi(A) \le \sum_{\omega\in R}\tilde{m}_\phi(A \cap [\omega]) \le \sum_{\omega\in R}\gamma\tilde{m}_\phi([\omega]) = \gamma\tilde{m}_\phi\big(\bigcup\{[\omega] : \omega \in R\}\big) \le \gamma\eta\tilde{m}_\phi(A) < \tilde{m}_\phi(A)$. This contradiction finishes the proof. ♣

Set $\nu = \tilde{m}_\phi$. Clearly our assumption $\phi_i(X) \cap \phi_j(X) = \emptyset$ for $i, j \in I\!N$, $i \ne j$ implies that $\pi : \Sigma \to J$ is a homeomorphism; in particuluar, in view of Lemma 2.6, it establishes a measure preserving isomorphism between measure spaces (Σ, ν) and (J, m_ϕ). To check that $\mathcal{L}_\phi^*(\nu) = \lambda\nu$ take $g \in C_b(\Sigma)$ and compute

$$\begin{aligned}
\int g d\mathcal{L}_0^*(\nu) &= \int \mathcal{L}_0(g) d\nu = \int \mathcal{L}_0(g)(\pi^{-1}(x)) d\nu \circ \pi^{-1}(x) = \int \mathcal{L}_0(g)(\pi^{-1}(x)) dm_\phi \\
&= \int \sum_{\tau\in\sigma^{-1}(\pi^{-1}(x))} \exp(\phi(\tau) - \mathrm{P}(\phi)) dm_\phi \\
&= \int \sum_{i\in I\!N} \exp\big(\phi^{(i)}(x) - \mathrm{P}(\phi)\big) g \circ \pi^{-1}(\phi_i(x)) dm_\phi(x) \\
&= \int L_0(g \circ \pi^{-1}) dm_\phi = \int g \circ \pi^{-1} dm_\phi = \int g d\nu.
\end{aligned}$$

Thus $\mathcal{L}_0(\nu) = \nu$ and by the definition of $\mathcal{L}_0$ and $\mathcal{L}_0^*$, $\mathcal{L}_\phi^*(\nu) = \lambda\nu$. The fact that $\mathcal{L}_\phi(h) = \lambda h$ follows immediately from the definition of the operator $\mathcal{L}_0$ and Theorem 2.7, where $h = d\tilde{\mu}_\phi/d\tilde{m}_\phi$. The last two parts of Theorem 1.2 are consequences of Theorem 4 and Theorem 5 of [Sa].

§**3. Equilibrium states.** In this section we further investigate the σ-invariant measure $\tilde{\mu}_\phi$ introduced in Theorem 2.7. We begin with the following technical result.

Lemma 3.1. The following 3 conditions are equivalent

(a) $\int -\phi d\tilde{\mu}_\phi < \infty$.

(b) $\sum_{i\in I\!N} \inf(-\phi|_{[i]}) \exp(\inf \phi|_{[i]}) < \infty$.

(c) $\mathrm{H}_{\tilde{\mu}_\phi}(\alpha) < \infty$, where $\alpha = \{[i] : i \in I\!N\}$ is the partition of Σ into initial cylinders of length 1.

Proof. Suppose that $\int -\phi d\tilde{\mu}_\phi < \infty$. It means that $\sum_{i\in I\!N}\int_{[i]} -\phi d\tilde{\mu}_\phi < \infty$ and consequently

$$\begin{aligned}
\infty &> \sum_{i\in I\!N} \inf(-\phi|_{[i]}) \int_{[i]} d\tilde{\mu}_\phi = \sum_{i\in I\!N} \inf(-\phi|_{[i]}) \int_{[i]} h d\tilde{m}_\phi \\
&\geq Q^{-1} \sum_{i\in I\!N} \inf(-\phi|_{[i]}) \tilde{m}_\phi([i]) = Q^{-1} \sum_{i\in I\!N} \inf(-\phi|_{[i]}) \int_X \exp(\phi^{(i)}(x) - \mathrm{P}(\phi)) dm_\phi(x) \\
&= Q^{-1} \mathrm{e}^{-\mathrm{P}(\phi)} \sum_{i\in I\!N} \inf(-\phi|_{[i]}) \int_X \exp(\phi^{(i)}(x)) dm_\phi(x)
\end{aligned}$$

Thus

$$\begin{aligned}
\infty &> \sum_{i\in I\!N} \inf(-\phi|_{[i]}) \int_X \exp(\phi^{(i)}(x)) dm_\phi(x) \geq \sum_{i\in I\!N} \inf(-\phi|_{[i]}) \exp(\inf_X(\phi^{(i)}) \\
&= \sum_{i\in I\!N} \inf(-\phi|_{[i]}) \exp(\inf \phi|_{[i]})
\end{aligned}$$

Now suppose that $\sum_{i\in I\!N} \inf(-\phi|_{[i]}) \exp(\inf \phi|_{[i]}) < \infty$. We shall show that $\mathrm{H}_{\tilde{\mu}_\phi}(\alpha) < \infty$. So,

$$\mathrm{H}_{\tilde{\mu}_\phi}(\alpha) = \sum_{i\in I\!N} -\tilde{\mu}_\phi([i]) \log \tilde{\mu}_\phi([i]) \leq \sum_{i\in I\!N} -Q\tilde{m}_\phi([i]) \big(\log \tilde{m}_\phi([i]) - \log Q\big).$$

But $\sum_{i\in I\!N} -Q\tilde{m}_\phi([i])(-\log Q) = Q \log Q$, so it suffices to show that

$$\sum_{i\in I\!N} -\tilde{m}_\phi([i]) \log \tilde{m}_\phi([i]) < \infty.$$

But

$$\begin{aligned}
\sum_{i\in I\!N} -\tilde{m}_\phi([i]) \log \tilde{m}_\phi([i]) &= \sum_{i\in I\!N} -\tilde{m}_\phi([i]) \log \left(\int_X \exp\big(\phi^{(i)} - \mathrm{P}(\phi)\big) \right) dm_\phi \\
&\leq \sum_{i\in I\!N} -\tilde{m}_\phi([i]) (\inf_X \phi^{(i)} - \mathrm{P}(\phi)).
\end{aligned}$$

But $\sum_{i\in I\!N} \tilde{m}_\phi([i]) \mathrm{P}(\phi) = \mathrm{P}(\phi)$, so it suffices to show that $\sum_{i\in I\!N} -\tilde{m}_\phi([i]) \inf_X \phi^{(i)} < \infty$. And indeed, using Lemma 2.1 we get

$$\sum_{i\in I\!N} -\tilde{m}_\phi([i]) \inf_X \phi^{(i)} = \sum_{i\in I\!N} \tilde{m}_\phi([i]) \sup_X(-\phi^{(i)}) \leq \sum_{i\in I\!N} \tilde{m}_\phi([i]) \big(\inf_X(-\phi^{(i)}) + \log Q\big).$$

Since $\sum_{i\in I\!N} \tilde{m}_\phi([i]) \log Q = \log Q$, it is enough to show that $\sum_{i\in I\!N} \tilde{m}_\phi([i]) \inf_X(-\phi^{(i)}) < \infty$. And indeed,

$$\sum_{i\in I\!N} \tilde{m}_\phi([i]) \inf_X(-\phi^{(i)}) = \sum_{i\in I\!N} \int \exp(\phi^{(i)} - \mathrm{P}(\phi)) dm_\phi \inf_X(-\phi^{(i)})$$

But in view of (1.4) $\phi^{(i)}$ are negative everywhere for all i large enough, say $i \geq k$. Then using Lemma 2.1 again we get

$$\sum_{i\geq k} \tilde{m}_\phi([i]) \inf_X(-\phi^{(i)}) \leq \mathrm{e}^{-\mathrm{P}(\phi)} Q \sum_{i\geq k} \exp(\inf_X(\phi^{(i)})) \inf_X(-\phi^{(i)})$$

which is finite due to our assumption. Hence, $\sum_{i\in I\!N} \tilde{m}_\phi([i]) \inf_X(-\phi^{(i)}) < \infty$. Finally suppose that $\mathrm{H}_{\tilde{\mu}_\phi}(\alpha) < \infty$. We need to show that $\int -\phi d\tilde{\mu}_\phi < \infty$. We have

$$\infty > \mathrm{H}_{\tilde{\mu}_\phi}(\alpha) = \sum_{i\in I\!N} -\tilde{m}_\phi([i]) \log\big(\tilde{m}_\phi([i])\big) \leq \sum_{i\in I\!N} -\tilde{m}_\phi([i])\big(\inf(\phi|_{[i]} - \mathrm{P}(\phi) - \log Q)\big).$$

Hence $\sum_{i\in I\!N} -\tilde{m}_\phi([i]) \inf(\phi|_{[i]}) < \infty$ and therefore

$$\int -\phi d\tilde{\mu}_\phi = \sum_{i\in I\!N} \int_{[i]} -\phi d\tilde{\mu}_\phi \leq \sum_{i\in I\!N} \sup(-\phi|_{[i]}) \tilde{m}_\phi([i]) = \sum_{i\in I\!N} -\inf(\phi|_{[i]}) \tilde{m}_\phi([i]) < \infty.$$

The proof is complete. ♣

By Theorem 3 of [Sa] we know that $\sup\{\mathrm{h}_\mu(\sigma) + \int \phi d\mu\} = \mathrm{P}(\phi)$, where the supremum is taken over all σ-invariant probability measures such that $\int -\phi d\mu < \infty$. We call a σ-invariant probability measure μ an equilibrium state of the potential ϕ if $\mathrm{h}_\mu(\sigma) + \int \phi d\mu = \mathrm{P}(\phi)$. We shall prove the following.

Theorem 3.2. If $\sum_{i\in I\!N} \inf(-\phi|_{[i]}) \exp(\inf \phi|_{[i]}) < \infty$, then $\tilde{\mu}_\phi$ is an equilibrium state of the potential ϕ satisfying $\int -\phi d\tilde{\mu}_\phi < \infty$.

Proof. It follows from Lemma 3.1 that $\int -\phi d\tilde{\mu}_\phi < \infty$. To show that $\tilde{\mu}_\phi$ is an equilibrium state of the potential ϕ consider $\alpha = \{[i] : i \in I\!N\}$, the partition of Σ into initial cylinders of length one. By Lemma 3.1, $\mathrm{H}_{\tilde{\mu}_\phi}(\alpha) < \infty$. Applying the Breiman-Shanon-McMillan theorem and the Birkhoff ergodic theorem we therefore get for $\tilde{\mu}_\phi$-a.e. $\omega \in \Sigma$

$$\begin{aligned}
\mathrm{h}_{\mu_\phi}(\sigma) \geq \mathrm{h}_{\mu_\phi}(\sigma, \alpha) &= \lim_{n\to\infty} \frac{-1}{n} \log([\omega|_n]) \\
&= \lim_{n\to\infty} \frac{-1}{n} \log\left(\int \exp\big(S_\omega(\phi)(x) d\mu_\phi - \mathrm{P}(\phi)n\big)\right) \\
&= \lim_{n\to\infty} \frac{-1}{n} \log\left(\int \exp\Big(\sum_{j=0}^{n-1} \phi(\sigma^j(\omega|_n\tau)) d\mu_\phi(\tau) - \mathrm{P}(\phi)n\Big)\right) \\
&\geq \limsup_{n\to\infty} \frac{-1}{n} \log\left(\int \exp\Big(\sum_{j=0}^{n-1} \phi(\sigma^j(\omega)) + \log Q - \mathrm{P}(\phi)n\Big)\right) \\
&= \lim_{n\to\infty} \frac{-1}{n} \sum_{j=0}^{n-1} \phi(\sigma^j(\omega)) + \mathrm{P}(\phi) = -\int \phi d\tilde{\mu}_\phi + \mathrm{P}(\phi).
\end{aligned}$$

Hence $\mathrm{h}_{\mu_\phi}(\sigma) + \int \phi d\tilde{\mu}_\phi \geq \mathrm{P}(\phi)$, which in view of the variational principle (see Theorem 3 in [Sa]) implies that $\tilde{\mu}_\phi$ is an equilibrium state for the potential ϕ. The proof is finished. ♣

References

[MU] R.D. Mauldin, M. Urbański, Dimensions and measures in infinite iterated function systems, Proc. London Math. Soc. (3) 73(1996), 105-154.

[Sa] O. M. Sarig, Theormodynamic formalism for countable Markov shifts, Preprint 1997.