Rigidity of Infinite one-dimensional Iterated Function Systems

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Abstract. In [MU] there has been introduced and developed the concept of infinite conformal iterated function systems. In this paper we consider 1-dimensional systems. We provide necessary and sufficient conditions for such systems to be bi-Lipschitz equivalent. We extend to such systems the concept of scaling functions and we pay special attention to the real-analytic systems.

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§**1. Preliminaries.** In [MU] we have provided the framework to study infinite conformal iterated function systems. Let us recall this notion assuming that X is a 1-dimensional interval. Let I be a countable index set with at least two elements and let $S = \{\phi_i : X \to X : i \in I\}$ be a collection of injective contractions from X into X for which there exists $0 < s < 1$ such that $\rho(\phi_i(x), \phi_i(y)) \leq s\rho(x, y)$ for every $i \in I$ and for every pair of points $x, y \in X$. Thus, the system S is uniformly contractive. Any such collection S of contractions is called an iterated function system. We are particularly interested in the properties of the limit set defined by such a system. We can define this set as the image of the coding space under a coding map as follows. Let $I^* = \bigcup_{n\geq 1} I^n$, the space of finite words, and for $\omega \in I^n$, $n \geq 1$, let $\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_n}$. If $\omega \in I^* \cup I^\infty$ and $n \geq 1$ does not exceed the length of ω, we denote by $\omega|_n$ the word $\omega_1\omega_2 \ldots \omega_n$. Since given $\omega \in I^\infty$, the diameters of the compact sets $\phi_{\omega|_n}(X)$, $n \geq 1$, converge to zero and since they form a descending family, the set

$$\bigcap_{n=0}^{\infty} \phi_{\omega|_n}(X)$$

is a singleton and therefore, denoting its only element by $\pi(\omega)$, defines the coding map $\pi : I^\infty \to X$. The main object of our interest will be the limit set

$$J = \pi(I^\infty) = \bigcup_{\omega\in I^\infty} \bigcap_{n=1}^{\infty} \phi_{\omega|n}(X),$$

Observe that J satisfies the natural invariance equality, $J = \bigcup_{i\in I} \phi_i(J)$. Notice that if I is finite, then J is compact and this property fails for infinite systems.
An iterated function system $S = \{\phi_i : X \to X : i \in I\}$, is said to satisfy the Open Set Condition if there exists a nonempty open set $U \subset X$ (in the topology of X) such that $\phi_i(U) \subset U$ for every $i \in I$ and $\phi_i(U) \cap \phi_j(U) = \emptyset$ for every pair $i, j \in I$, $i \neq j$.

An iterated function system S, satisfying the Open Set Condition is said to be conformal (c.i.f.s.) if the following conditions are satisfied.

(a) $U = \mathrm{Int}(X)$.

(b) There exists an open connected set $X \subset V \subset I\!R$ such that all maps ϕ_i, $i \in I$, extend to C^1 diffeomorphisms on V.

(c) Bounded Distortion Property(BDP). There exists $K \geq 1$ such that

$$|\phi'_\omega(y)| \leq K|\phi'_\omega(x)|$$

for every $\omega \in I^*$ and every pair of points $x, y \in V$, where $|\phi'_\omega(x)|$ means the norm of the derivative.

Notice that for simplicity and clarity of our exposition we assumed the open set U appearing in the open set condition to be $\mathrm{Int}(X)$. As was demonstrated in [MU], conformal iterated function systems naturally break into two main classes, irregular and regular. This dichotomy can be determined from either the existence of a zero of a natural pressure function or, equivalently, the existence of a conformal measure. The topological pressure function, P is defined as follows. For every integer $n \geq 1$ define

$$\psi_n(t) = \sum_{\omega \in I^n} ||\phi'_\omega||^t.$$

and

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log \psi_n(t).$$

For a conformal system S, we sometimes set $\psi_S = \psi_1 = \psi$. The finiteness parameter, θ_S, of the system S is defined by $\inf\{t : \psi(t) < \infty\} = \theta_S$. In [MU], it was shown that the topological pressure function $P(t)$ is non-increasing on $[0, \infty)$, strictly decreasing, continuous and convex on $[\theta, \infty)$ and $P(1) \leq 0$. Of course, $P(0) = \infty$ if and only if I is infinite. In [MU] (see Theorem 3.15) we have proved the following characterization of the Hausdorff dimension of the limit set J, which will be denoted by $\mathrm{HD}(J) = h_S$.

Theorem 1.1. *$HD(J) = \sup\{HD(J_F) : F \subset I$ is finite$\} = \inf\{t : P(t) \leq 0\}$. If $P(t) = 0$, then $t = HD(J)$.*

We called the system S regular provided that there is some t such that $P(t) = 0$. It follows from [MU] that t is unique. Also, the system is regular if and only if there is a t-conformal measure. Recall that a Borel probability measure m is said to be t-conformal provided $m(J) = 1$ and for every Borel set $A \subset X$ and every $i \in I$

$$m(\phi_i(A)) = \int_A |\phi'_i|^t \, dm$$

and

$$m(\phi_i(X) \cap \phi_j(X)) = 0,$$

for every pair $i, j \in I$, $i \neq j$. We recall also (see [MU, Theorem 3.8]) that there exists an invariant measure μ (in the sense that for every measurable set A, $\mu(\bigcup_{i\in I\!N} \phi_i(A)) = \mu(A)$ equivalent with m.

Finally, we call two iterated function systems $\{f_i : X \to X, i \in I\!N\}$ and $\{g_i : Y \to Y, i \in I\!N\}$ topologically conjugate if and only if there exists a homeomorphism $h : X \to Y$ such that $h \circ f_i = g_i \circ h$ for all $i \in I\!N$. Then by induction we easily get that $h \circ f_\omega = g_\omega \circ h$ for every finite word ω.

§**2. General Systems.** The main result of this section is the following

Theorem 2.1. Suppose that $F = \{f_i : X \to X, i \in I\!N)$ and $G = (g_i : Y \to Y, i \in I\!N)$ are two topologically conjugate one-dimensional conformal iterated function systems. Then the following 4 conditions are equivalent.

(1) $\exists S \geq 1\ \forall \omega \in \mathbb{N}^*$

$$S^{-1} \leq \frac{\mathrm{diam}(g_\omega(Y))}{\mathrm{diam}(f_\omega(X))} \leq S.$$

(2) $|g'_\omega(y_\omega)| = |f'_\omega(x_\omega)|$ for all $\omega \in \mathbb{N}^*$, where x_ω and y_ω are the only fixed points of $f_\omega : X \to X$ and $g_\omega : Y \to Y$ respectively.

(3) $\exists E \geq 1\ \forall \omega \in \mathbb{N}^*$

$$E^{-1} \leq \frac{||g'_\omega||}{||f'_\omega||} \leq E.$$

(4) For every finite subset T of $\mathbb{N}$, $\mathrm{HD}(J_{G,T}) = \mathrm{HD}(J_{F,T})$ and the conformal measures $m_{G,T}$ and $m_{F,T} \circ h^{-1}$ are equivalent.

Suppose additionally that both systems F and G are regular. Then the following condition is also equivalent with the four conditions above.

(5) $\mathrm{HD}(J_G) = \mathrm{HD}(J_F)$ and the conformal measures m_G and $m_F \circ h^{-1}$ are equivalent.

Proof. Let us first demonstrate that conditions (2) and (3) are equivalent. Indeed, suppose that (2) is satisfied and let K_F and K_G denote the distortion constants of the systems F and G respectively. Then for all $\omega \in \mathbb{N}^*$, $||g'_\omega|| \leq K_G|g'_\omega(y_\omega)| = K_G|f'_\omega(x_\omega)| \leq K_G||f'_\omega||$ and similarly $||f'_\omega|| \leq K_F||g'_\omega||$. So suppose that (3) holds and (2) fails, that is that there exists $\omega \in \mathbb{N}^*$ such that $|g'_\omega(y_\omega)| \neq |f'_\omega(x_\omega)|$. Without loosing generality we may assume that $|g'_\omega(y_\omega)| < |f'_\omega(x_\omega)|$. For every $n \geq 1$ let ω^n be the concatenation of n words ω. Then $g_{\omega^n}(y_\omega) = g^n_\omega(y_\omega) = y_\omega$ and similarly $f_{\omega^n}(x_\omega) = x_\omega$. So, $x_{\omega^n} = x_\omega = \pi_F(\omega^\infty)$ and $y_{\omega^n} = y_\omega = \pi_G(\omega^\infty)$. Moreover $|g'_{\omega^n}(y_\omega)| = |g'_\omega(y_\omega)|^n$ and $|f'_{\omega^n}(x_\omega)| = |f'_\omega(x_\omega)|^n$. Hence

$$\lim_{n\to\infty} \frac{|g'_{\omega^n}(y_\omega)|}{|f'_{\omega^n}(x_\omega)|} = 0.$$

On the other hand, by (3) and the Bounded Distortion Property

$$\frac{|g'_{\omega^n}(y_\omega)|}{|f'_{\omega^n}(x_\omega)|} \geq \frac{K_G^{-1}||g'_{\omega^n}||}{||f'_{\omega^n}||} \geq E^{-1}K_G^{-1}$$

for all $n \geq 1$. This contradiction finishes the proof of equivalence of conditions (2) and (3). Since the equivalence of (1) and (3) is immediate, the proof of the equivalence of conditions (1)-(3) is finished. We shall now prove that (3) $\Rightarrow$ (5). Indeed, it follows from (3) that $E^{-1}\psi_{G,n}(t) \leq \psi_{F,n}(t) \leq E\psi_{G,n}(t)$ for all $t \geq 0$ and all $n \geq 1$. Hence $P_G(t) = P_F(t)$ and therefore by Theorem 1.1, $\mathrm{HD}(J_G) = \mathrm{HD}(J_F)$. Denote this common value by h. Although we keep the same symbol for the homeomprphism establishing conjugacy between the systems F and G, it will never cause misunderstandings.

Suppose now that both systems are regular (in fact assuming (3) regularity of one of these systems implies regularity of the other). Then for every $\omega \in \mathbb{N}^*$

$$(K_FE)^{-h} \leq \frac{K_F^{-h}||f'_\omega||^h}{||g'_\omega||^h} \leq \frac{m_F(f_\omega(J_F))}{m_G(g_\omega(J_G))} \leq \frac{||f'_\omega||^h}{K_G^{-h}||g'_\omega||^h} \leq (EK_G)^h.$$

So, the measures m_G and $m_F \circ h^{-1}$ are equivalent, and even more

$$(K_F E)^{-h} \le \frac{dm_G}{dm_F \circ h^{-1}} \le (EK_G)^h.$$

Let us show now that (5) $\Rightarrow$ (3). Indeed, if (5) is satisfied then the measure $\mu_F \circ h^{-1}$ is equivalent with μ_G. Since additionally $\mu_F \circ h^{-1}$ and μ_G are both ergodic (see Theorem 3.8 of [MU], they are equal. Hence

$$\begin{aligned}
||g'_\omega||^h &\asymp \int |g'_\omega|^h \, dm_G = m_G(g_\omega(J_G)) \asymp \mu_G(g_\omega(J_G)) \\
&= \mu_F \circ h^{-1} = \mu_F(f_\omega(J_F)) \asymp m_F(f_\omega(J_F)) = \int |f'_\omega|^h \, dm_F \\
&\asymp ||f'_\omega||^h,
\end{aligned}$$

and raising the first and the last term of this sequence of comparabilities to the power $1/h$, we finish the proof of the implication (5) $\Rightarrow$ (3).

The equivalence of (4) and conditions (1) - (3) is now a relatively simple corollary. Indeed, to prove that (3) implies (4) fix a finite subset T of $I\!N$. By (3) $E^{-1} \le ||f'_\omega||/||g'_\omega|| \le E$ for all $\omega \in T^*$, and as every finite system is regular, the equivalence of measures $m_{G,T}$ and $m_{F,T} \circ h^{-1}$ follows from the equivalence of conditions (3) and (5) applied to the systems $\{f_i : i \in T\}$ and $\{g_i : i \in T\}$. If in turn (4) holds and $\omega \in I\!N^*$, then $\omega \in T^*$, where T is the (finite) set of letters making up the word ω and the measures $m_{G,T}$ and $m_{F,T} \circ h^{-1}$ are equivalent. Hence, by the equivalence of (2) and (5) applied to the systems $\{f_i : i \in T\}$ and $\{g_i : i \in T\}$ we conclude that $|g'_\omega(y_\omega)| = |f'_\omega(x_\omega)|$. Thus (2) follows and the proof of Theorem 2.1 is finished. ♣

We say that a conformal system $\{\phi_i : X \to X : i \in I\!N\}$ is of bounded geometry if and only if there exists $C \ge 1$ such that for all $i, j \in I\!N$, $i \ne j$

$$\max\{\mathrm{diam}(\phi_i(X)), \mathrm{diam}(\phi_j(X))\} \le C\mathrm{dist}\big(\phi_i(X), \phi_j(X)\big).$$

The next theorem provides a sufficient and necessary condition for two systems of bounded geometry to be bi-Lipschitz equivalent.

Theorem 2.2. If both systems $\{f_i : X \to X : i \in I\!N\}$ and $\{g_i : Y \to Y : i \in I\!N\}$ are of bounded geometry, then the topological conjugacy $h : J_f \to J_g$ is bi-Lipschitz continuous if and only if the following two conditions are satisfied.

(a) $$Q^{-1} \le \frac{\mathrm{diam}(f_\omega(X))}{\mathrm{diam}(g_\omega(Y))} \le Q$$

for some $Q \ge 1$ and all $\omega \in I\!N^*$.

(b) $$C^{-1} \le \frac{\mathrm{dist}\big(g_i(Y), g_j(Y)\big)}{\mathrm{dist}\big(f_i(X), f_j(X)\big)} \le C$$

for some $C \ge 1$ and all $i, j \in I\!N$, $i \ne j$.

Proof. First notice that (a) and (b) remain true, with modified constants Q and C if necessary, if X is replaced by J_F and Y is replaced by J_G respectively. Suppose now that $x \in f_i(J_F)$ and $y \in f_j(J_F)$ with $i \neq j$. Then

$$\begin{aligned}
|h(y)-h(x)| &\le \mathrm{diam}(g_i(J_G)) + \mathrm{dist}(g_i(J_G), g_j(J_G)) + \mathrm{diam}(g_j(J_g)) \\
&\le Q\mathrm{diam}(f_i(J_F)) + C\mathrm{dist}(f_i(J_F), f_j(J_F)) + Q\mathrm{diam}(f_j(J_F)) \\
&\le 2QC\mathrm{dist}(f_i(J_F), f_j(J_F)) + C\mathrm{dist}(f_i(J_F), f_j(J_F)) \\
&\le (2Q+1)C\mathrm{dist}(f_i(J_F), f_j(J_F)) \\
&\le (2Q+1)C|y-x|
\end{aligned}$$

Suppose in turn that $x \neq y$ both belong to the same element $f_k(J_F)$. Then there exist $\omega \in I^*$ ($|\omega| \ge 1$) and $i \neq j \in I\!N$ such that $x, y \in f_\omega(J_F)$, $x \in f_{\omega i}(J_F)$ and $y \in f_{\omega j}(J_F)$. From what has been proved so far we know that $|g_\omega^{-1}(h(y)) - g_\omega^{-1}(h(x))| \le (2Q+1)C|f_\omega^{-1}(y) - f_\omega^{-1}(x)|$. Since $|y-x| \asymp ||f'_\omega|||f_\omega^{-1}(y) - f_\omega^{-1}(x)|$ and $|h(y)-h(x)| \asymp ||g'_\omega|||g_\omega^{-1}(h(y)) - |g_\omega^{-1}(h(x))|$, we get

$$|h(y)-h(x)| \preceq \frac{||g'_\omega||}{||f'_\omega||}|y-x|.$$

In the same way we show that h^{-1} is Lipshitz continuous which completes the proof of the first part of our theorem.

So suppose now that h is bi-Lipshitz continuous. We shall show that conditions (a) and (b) are satisfied. Indeed, to prove (a) suppose that a and b are taken so that $|h(a)-h(b)| \ge \frac{1}{2}\mathrm{diam}(g_\omega(J_G))$. Then $\mathrm{diam}(g_\omega(J_G)) \le 2|h(a)-h(b)| \le 2L|a-b| \le 2L\mathrm{diam}(f_\omega(J_F))$, where L is a Lipshitz constant of h and h^{-1}. In the same way it can be shown that $\mathrm{diam}(f_\omega(J_F)) \le 2L\mathrm{diam}(g_\omega(J_G))$ which completes the proof of property (a). In order to prove the right-hand side of property (b) we proceed as follows. Fix $i, j \in I\!N$, $i \neq j$ and $a \neq b \in J_F$. Then

$$\begin{aligned}
\mathrm{dist}(g_i(Y), g_j(Y)) &\le \mathrm{dist}(g_i(J_G), g_j(J_G)) \le |g_i(h(a)) - g_j(h(b))| \le L|f_i(a) - f_j(b)| \\
&\le L\big(\mathrm{diam}(f_i(X)) + \mathrm{dist}(f_i(X), f_j(X)) + \mathrm{diam}(f_j(X))\big) \\
&\le L(2C+1)\mathrm{dist}(f_i(X), f_j(X)),
\end{aligned}$$

where the last inequality we wrote due to boundedness of geometry of the system $\{f_i : i \in I\!N\}$. The proof is finished. ♣

Remark 2.3. Notice that Theorem 2.1 nd Theorem 2.2 remain true without assuming that the phase space X is one-dimensional. We only need to know that the maps ϕ_i, ff_i and g_i are conformal and the assumption (2.7)of [MU] is satisfied.

Remark 2.4. Suppose now that the maps $i \mapsto \phi_i(X)$ are monotone, that is suppose that for all i and j, $i < j$ implies $\phi_i(X) < \phi_j(X)$. We claim that then the bounded geometry of the system is equivalent with the following weaker condition

$$\max\{\mathrm{diam}(\phi_i(X), \mathrm{diam}(\phi_{i+1}(X)\} \le C\mathrm{dist}(\phi_i(X), \phi_{i+1}(X)).$$

Indeed, if $i < j$, then

$$\begin{aligned}\max\{\operatorname{diam}(\phi_i(X), \operatorname{diam}(\phi_j(X)\} &\le \max_{i\le k\le j-1}\{\max\{\operatorname{diam}(\phi_k(X)), \operatorname{diam}(\phi_{k+1}(X))\}\} \\ &\le \max_{i\le k\le j-1}\{C\operatorname{dist}(\phi_k(X), \phi_{k+1}(X))\} \\ &\le C\operatorname{dist}(\phi_i(X), \phi_j(X)),\end{aligned}$$

where writing the last inequality we used the monotonicity of the map $i \mapsto \phi_i(X)$. The opposite implication is obvious.

Remark 2.5. If both maps $i \mapsto f_i(X)$ and $i \mapsto g_i(X)$ are monotone, then condition (2.b) from Theorem 2.2 can be replaced by the following.

$$\text{(c)} \qquad C^{-1} \le \frac{\operatorname{dist}(g_k(Y), g_{k+1}(Y))}{\operatorname{dist}(f_k(X), f_{k+1}(X))} \le C$$

for some constant $C \ge 1$ and all $k \in \mathbb{N}$. Indeed, assuming (2.3) this follows from the following computation.

$$\begin{aligned}\operatorname{dist}(g_i(Y), g_j(Y)) &= \sum_{k=i}^{j-1}\operatorname{dist}(g_k(Y), g_{k+1}(Y)) + \sum_{k=i+1}^{j-1}\operatorname{diam}(g_k(X)) \\ &\le \sum_{k=i}^{j-1} C\operatorname{dist}(f_k(X), f_{k+1}(X)) + Q\sum_{k=i+1}^{j-1}\operatorname{diam}(f_k(X)) \\ &\le \max\{C, Q\}\left(\sum_{k=i}^{j-1} C\operatorname{dist}(f_k(X), f_{k+1}(X)) + \sum_{k=i+1}^{j-1}\operatorname{diam}(f_k(X))\right) \\ &= \max\{C, Q\}\operatorname{dist}(f_i(X), f_j(X))\end{aligned}$$

§**3. Real-analytic systems**. We call a 1-dimensional system $\Phi = \{\phi_i : X \to X, i \in \mathbb{N}\}$ real analytic if and only if there exists a topological disk D such that all the maps ϕ_i extend in a conformal (so 1-to-1) fashion to D. Let m be the conformal measure associated to the system Φ and let μ be the only probability invariant measure equivalent with m (see [MU,Theorem 3.8], where this measure was denoted by μ^*). We call the system Φ non-linear (comp. [S1]) if and only if at least one of the Jacobians $J_{\phi_i} = \frac{d\mu\circ\phi_i}{d\mu}$ is not constant. We shall prove the following theorem which is stronger than both Theorem 2.1 and Theorem 2.2.

Theorem 3.1. If both systems $\{f_i : X \to X : i \in \mathbb{N}\}$ and $\{g_i : Y \to Y : i \in \mathbb{N}\}$ are real-analytic and non-linear, then the following conditions are equivalent.

(a) The conjugacy between the systems $\{f_i : X \to X : i \in \mathbb{N}\}$ and $\{g_i : Y \to Y : i \in \mathbb{N}\}$ is real analytic.

(b) The conjugacy between the systems $\{f_i : X \to X : i \in I\!N\}$ and $\{g_i : Y \to Y : i \in I\!N\}$ is Lipschitz continuous.

(c) $|g'_\omega(y_\omega)| = |f'_\omega(x_\omega)|$ for all $\omega \in I\!N^*$, where x_ω and y_ω are the only fixed points of $f_\omega : X \to X$ and $g_\omega : Y \to Y$ respectively.

(d) $\exists S \geq 1\ \forall \omega \in I\!N^*$

$$S^{-1} \leq \frac{\operatorname{diam}(g_\omega(Y))}{\operatorname{diam}(f_\omega(X))} \leq S.$$

(e) $\exists E \geq 1\ \forall \omega \in I\!N^*$

$$E^{-1} \leq \frac{||g'_\omega||}{||f'_\omega||} \leq E.$$

(f) For every finite subset T of $I\!N$, $\mathrm{HD}(J_{G,T}) = \mathrm{HD}(J_{F,T})$ and the measures $m_{G,T}$ and $m_{F,T} \circ h^{-1}$ are equivalent.

(g) For every finite subset T of $I\!N$, the measures $m_{G,T}$ and $m_{F,T} \circ h^{-1}$ are equivalent.

Suppose additionally that both systems F and G are regular. Then the following two conditions are also equivalent with the conditions above.

(h) $\mathrm{HD}(J_G) = \mathrm{HD}(J_F)$ and the measures m_G and $m_F \circ h^{-1}$ are equivalent.

(i) The measures m_G and $m_F \circ h^{-1}$ are equivalent.

Proof. The implication $(a) \Rightarrow (b)$ is obvious.That $(b) \Rightarrow (c)$ results from the fact that (b) implies condition (1) of Theorem 2.1 which in view of this theorem is equivalent with condition (2) of Theorem 2.1 which finally is the same as condition (c) of Theorem 3.1 The implications $c \Rightarrow (d) \Rightarrow (e) \Rightarrow (h)$ have been proved in Theorem 2.1. The implication $(h) \Rightarrow (i)$ is again obvious. Let us now prove that $(i) \Rightarrow (a)$. As the first step we shall show that if a regular system $\{\phi_i : i \in I\!N\}$ is real-analytic, then the Jacobians J_{ϕ_ω} of all the maps ϕ_ω, $\omega \in I\!N^*$ with respect to the invariant measure μ are also real analytic. Since $\frac{d(m\circ\phi_\omega)}{dm} = |\phi'_\omega|^h$ and since ϕ'_ω is a real valued, either positive or negative, real analytic function, the function $|\phi'_\omega|^h$ is also real analytic. Consequently, to check that

$$\frac{d\mu\circ\phi_\omega}{d\mu} = \frac{d\mu\circ\phi_\omega}{dm\circ\phi_\omega}\cdot\frac{dm\circ\phi_\omega}{dm}\cdot\frac{dm}{d\mu} = \frac{d\mu}{dm}\circ\phi_\omega\cdot\frac{dm\circ\phi_\omega}{dm}\cdot\frac{dm}{d\mu}$$

is real-analytic it suffices to check that $\frac{d\mu}{dm}$ is real-analytic. Let $D \subset \overline{\mathbb{C}}$ be the open topological disk claimed in the definition of real analytic systems. Since for each $\omega \in I\!N^*$, $|\phi'_\omega||_X =^+_- \phi'_\omega|_X$, all the derivatives extend (complex) analytically to the corresponding maps $\nu(\omega)\phi'_\omega$, where $\nu(\omega) \in \{1, -1\}$. Given $n \geq 1$ consider the series of (complex) analytic functions $\mathcal{L}^n(1) = \sum_{|\omega|=n}(\nu(\omega)\phi'_\omega)^h$, where $(\nu(\omega)\phi'_\omega)^h$ are well-defined since D is simply connected. Fix $x_0 \in X$. By the Koebe Distortion Theorem and (3.3) of [MU] we can write for all $n \geq 1$ and all $x \in D$

$$\left|\sum_{|\omega|=n}(\nu(\omega)\phi'_\omega)^h\right| \leq \sum_{|\omega|=n}|\phi'_\omega(x)|^h \leq K^h\sum_{|\omega|=n}|\phi'_\omega(x_0)|^h = K^h\mathcal{L}^n(1)(x_0) \leq K^{2h}.$$

Hence, the maps $\mathcal{L}^n(1) : D \to \mathbb{C}$ form a normal family in the sense of Montel. Since $\mathcal{L}^n(1)|_X$ converges to $\rho = \frac{d\mu}{dm}$, we conclude more, that $\mathcal{L}^n(1)|_D$ converges to an analytic extension of ρ on D. We will keep the same notation ρ for this extension. So, we have proved that all the Jacobians $J_{\phi_\omega} = \frac{d\mu\circ\phi_\omega}{d\mu}$ are real-analytic, and in fact extends analytically onto D. Now suppose that condition (i) of Theorem 3.1 is satisfied. Then $\mu_F = \mu_G \circ h$ meaning that $J_h = \frac{d\mu_G\circ h}{d\mu_F} = 1$. Since $h \circ f_\omega = g_\omega \circ h$, the chain rule implies that $J_h \circ f_\omega \cdot J_{f_\omega} = J_{g_\omega} \circ h \cdot J_h$ and consequently

$$J_{f_\omega} = J_{g_\omega} \circ h.$$

Let now g_i be the contraction produced by non-linearity. Notice then that J_{g_i} has only finitely many extremal points, since otherwise the equation $J'_{g_i} = 0$ would have an accumulation point in Y which in turn would imply that J_{g_i} would be constant on Y, contrary to non-linearity of the system G. Hence $J_{g_i}^{-1} \circ J_{f_i}$ is well-defined and $1-to-1$ on an open set $V \subset X$, and $h = J_{g_i}^{-1} \circ J_{f_i}$ on $V \cap J_F$. Consider now $\omega \in I\!N^*$ such that $f_\omega(X) \subset V$. Then the map $g_\omega^{-1} \circ (J_{g_i}^{-1} \circ J_{f_i}) \circ f_\omega : X \to X$ extends h and is real analytic. So, we have proved that the conditions (a), (b), (c), (d), (e), (h), and (i) are equivalent To complete the proof we need to demonstrate that (f) and (g) are equivalent with conditions (a)-(e). Indeed, the implications $(e) \Rightarrow (f)$ and $(g) \Rightarrow (c)$ are proved in exactly the same way as respectively implication $(3) \Rightarrow (4)$ and $(4) \Rightarrow (2)$ of Theorem 2.1. Since the implication $(f) \Rightarrow (g)$ is obvious, the proof is finished. ♣

§**4. Scaling functions**. From now on we assume that all our systems satisfies condition (a) of Lemma 2.2 of [MU]. This condition reads as follows:

There are two constants $L \geq 1$ and $\alpha > 0$ such that

$$\big||\phi_i'(y)| - |\phi_i'(x)|\big| \leq L||(\phi_i')^{-1}||^{-1}|y-x|^\alpha, \tag{4.1}$$

for every $i \in I$ and every pair of points $x, y \in V$. As a byproduct of the demonstration that $(b) \Rightarrow (c)$ (p.112 of [MU]) we have shown that for all $\omega \in I\!N^*$, say $\omega \in I\!N^n$ and all $x, y \in X$

$$|\log|\phi_\omega'(y)| - \log|\phi_\omega'(x)| \leq \sum_{j=1}^n ||(\phi_{\omega_j}')^{-1}|| \cdot |\phi_{\omega_j}'(y_{n-j})| - |\phi_{\omega_j}'(x_{n-j})|,$$

where $z_k = \phi_{\omega_{n-k+1}} \circ \ldots \phi_{\omega_n}(z)$. In view of (4.1) this estimate continuous as follows

$$\begin{aligned}|\log|\phi_\omega'(y)| - \log|\phi_\omega'(x)|| &\leq \sum_{j=1}^n L|y_{n-j} - x_{n-j}|^\alpha \\ &\leq \sum_{j=0}^{n-1} Ls^{j\alpha}|y-x|^\alpha \\ &= \frac{L}{1-s^\alpha}|y-x|^\alpha \qquad (4.2)\end{aligned}$$

or equivalently

$$\exp\left(\frac{-L}{1-s^\alpha}|y-x|^\alpha\right) \le \frac{|\phi'_\omega(y)|}{|\phi'_\omega(x)|} \le \exp\left(\frac{L}{1-s^\alpha}|y-x|^\alpha\right) \tag{4.3}$$

Now, since for every t sufficiently small $|\mathrm{e}^t - 1| \le 2t$, we get

$$||\phi'_\omega(y)|-|\phi'_\omega(x)|| = \left|\frac{|\phi'_\omega(y)|}{|\phi'_\omega(x)|} - 1\right| |\phi'_\omega(x)| \le \frac{2L}{1-s^\alpha}|y-x|^\alpha|\phi'_\omega(x)| \le \frac{2Ls^n}{1-s^\alpha}|y-x|^\alpha \tag{4.4}$$

In order to define scaling functions we will need the following basic lemma.

Lemma 4.1. If $\{\phi_n : X \to X : n \ge 1\}$ is a one-dimensional conformal iterated function system satisfying condition (4.1), then for every closed subinterval K of X and $\omega \in I\!N^\infty$ the following limit exists

$$\lim_{n\to\infty} \frac{|\phi_{\omega_n\omega_{n-1}\ldots\omega_0}(K)|}{|\phi_{\omega_n\omega_{n-1}\ldots\omega_0}(X)|} := S(\omega, K)$$

and the convergence is uniform with respect to K, n and ω.

Proof. We shall show that the above sequence satisfies an appropriate Cauchy condition, So, fix $k < n$. We then have

$$\begin{aligned}\frac{|\phi_{\omega_n\ldots\omega_k\ldots\omega_0}(K)|}{|\phi_{\omega_n\ldots\omega_k\ldots\omega_0}(X)|}\Big/\frac{|\phi_{\omega_k\ldots\omega_0(K)}|}{|\phi_{\omega_k\ldots\omega_0}(X)|} &= \frac{|\phi_{\omega_n\ldots\omega_{k+1}}(\phi_{\omega_k\ldots\omega_0}(K))|}{|\phi_{\omega_k\ldots\omega_0}(K)|}\Big/\frac{|\phi_{\omega_n\ldots\omega_{k+1}}(\phi_{\omega_k\ldots\omega_0}(X))|}{|\phi_{\omega_k\ldots\omega_0}(X)|} \\ &= \frac{|\phi'_{\omega_n\ldots\omega_{k+1}}(x_n)|}{|\phi'_{\omega_n\ldots\omega_{k+1}}(y_n)|}\end{aligned} \tag{4.5}$$

for some $x_n \in \phi_{\omega_k\ldots\omega_0}(K)$ and $y_n \in \phi_{\omega_k\ldots\omega_0}(X)$, where the last equality sign we wrote due to the Mean Value Theorem. Denote now $|\phi_{\omega_j\ldots\omega_0}(K)|/|\phi_{\omega_j\ldots\omega_0}(X)|$ by a_j. In view of (4.5) and (4.2) we get

$$|\log a_n - \log a_k| \le \frac{L}{1-s^\alpha}|x_n - y_n|^\alpha \le \frac{L}{1-s^\alpha}|\phi_{\omega_k\ldots\omega_0}(X)|^\alpha \le \frac{L}{1-s^\alpha}s^{k\alpha}.$$

Thus the sequence $\{\log a_n\}_{n=1}^\infty$ is a Cauchy sequence, and consequently $\{a_n\}_{n=1}^\infty$ itself is a Cauchy sequence too, The proof is finished. ♣

Let $\tilde{I\!N}^\infty$ denote the set of infinite sequences of the form $\ldots\omega_n\omega_{n-1}\ldots\omega_1\omega_0$ and let $\tilde{I\!N}_*$ denote the set of all finite words of the form $\omega_n\omega_{n-1}\ldots\omega_1\omega_0$. Lemma 4.1 allows us to introduce the scaling function (comp. also [S2] and [PT]). In this section we will explore this notion. The weaker scaling function S^w is defined on the space $\tilde{I\!N}^\infty \times I\!N$, takes values in $(0,1)$, and is given by the formula

$$S^w(\{\omega_n\}_{n=0}^\infty, i) = \lim_{n\to\infty}\frac{|\phi_{\omega_n\omega_{n-1}\ldots\omega_0}(\phi_i(X))|}{|\phi_{\omega_n\omega_{n-1}\ldots\omega_0}(X)|},$$

where the limit exists due to Lemma 4.1.

The stronger scaling function S^s is defined similarly but on the larger space $\tilde{N} \times (I\!N \cup \mathcal{C})$, where $\mathcal{C}$ denotes the set of all connected components of $X \setminus \bigcup_{i=1}^{\infty} \phi_i(X)$. Frequently, given $\omega \in I\!N^*$ we will consider the function $S^s(\omega) : (I\!N \cup \mathcal{C}) \to (0,1)$ given by the formula $S^s(\omega)(Z) = S^s(\omega, Z)$, and similarly we define the function $S^w(\omega)$. The following theorem is an immediate consequence of Lemma 4.2

Theorem 4.2. Both scaling functions $S^w : \tilde{I\!N}^{\infty} \times I\!N$ and $S^s : \tilde{N} \times (I\!N \cup \mathcal{C})$ are continuous.

We now pass to consider two systems $F = \{f_i : i \in I\!N\}$ and $G = \{g_i : i \in I\!N\}$. Our first theorem about them reads as follows.

Theorem 4.3. If two topologically conjugate 1-dimensional i.f.s. F and G have the same weak scaling functions and condition (b) of Theorem 2.2 is satisfied, then the topological conjugacy is Lipschitz continuous. Conversely, if the topological conjugacy $h : J_F \to J_G$ extends in a diffeomorphic fashion onto X, then J_F and J_G have the same strong scaling functions.

Proof. Let us first prove the second part of this theorem. Indeed, let us keep the same notation h for its diffeomorphic extension to X and let D be an arbitrary closed subinterval of X. For $\omega \in \tilde{N}^{\infty}$ we can write

$$\frac{S(\omega, D)}{S(\omega, h(D))} = \lim_{n\to\infty} \frac{|f_{\omega_n \ldots \omega_0}(D)|}{|f_{\omega_n \ldots \omega_0}(X)|} \Big/ \frac{|g_{\omega_n \ldots \omega_0}(h(D))|}{|g_{\omega_n \ldots \omega_0}(Y)|} = \lim_{n\to\infty} \frac{|f_{\omega_n \ldots \omega_0}(D)|}{|g_{\omega_n \ldots \omega_0}(h(D))|} \Big/ \frac{|f_{\omega_n \ldots \omega_0}(X)|}{|g_{\omega_n \ldots \omega_0}(Y)|}.$$

Now, by the Mean Value Theorem there exist a_n and b_n respectively in $f_{\omega_n \ldots \omega_0}(D)$ and in $f_{\omega_n \ldots \omega_0}(X)$ such that

$$\frac{S(\omega, D)}{S(\omega, h(D))} = \lim_{n\to\infty} \frac{|f_{\omega_n \ldots \omega_0}(D)|}{|h(f_{\omega_n \ldots \omega_0}(D))|} \Big/ \frac{|f_{\omega_n \ldots \omega_0}(X)|}{|h(f_{\omega_n \ldots \omega_0}(X))|} = \lim_{n\to\infty} \frac{h'(b_n)}{h'(a_n)}.$$

Since h' is uniformly continuous with no zeros and since $|b_n - a_n| \to 0$ the last limit is equal to 1 which finishes the proof of the second part of our theorem.

In order to show the first part of this theorem it suffices to show that condition (a) of Theorem 2.2 is satisfied. So, let $\tau = \tau_0 \ldots \tau_{q-1}$ be an arbitrary word. Our aim is to show that $|(g_\tau)'(h(x_\tau))| = |(f_\tau)'(x_\tau)|$, where x_τ is the only fixed point of the map $f_\tau : X \to X$. First notice that for every n

$$\frac{|g_{\tau^{n+1}\tau_0}(Y)|}{|g_{\tau^n\tau_0}(Y)|} = \frac{|g_{\tau^{n+1}\tau_0}(Y)|}{|g_{\tau^{n+1}}(Y)|} \cdot \frac{|g_{\tau^{n+1}}(Y)|}{|g_{\tau^n\tau_0\ldots\tau_{q-2}}(Y)|} \cdot \frac{|g_{\tau^n\tau_0\ldots\tau_{q-2}}(Y)|}{|g_{\tau^n\tau_0\ldots\tau_{q-3}}(Y)|} \cdot \ldots \cdot \frac{|g_{\tau^n\tau_0\tau_1}(Y)|}{|g_{\tau^n\tau_0}(Y)|}.$$

Hence

$$\lim_{n\to\infty} \frac{|g_{\tau^{n+1}\tau_0}(Y)|}{|g_{\tau^n\tau_0}(Y)|} = S^w_{\tau^\infty}(\tau_0) S^w_{\tau^\infty\tau_0\ldots\tau_{q-2}}(\tau_{q-1}) S^w_{\tau^\infty\tau_0\ldots\tau_{q-3}}(\tau_{q-2}) \ldots S^w_{\tau^\infty\tau_0}(\tau_1) \tag{4.6}$$

and similarly

$$(4.7)\qquad \lim_{n\to\infty}\frac{f_{\tau^{n+1}\tau_0}(X)|}{|f_{\tau^n\tau_0}(X)|}=S^w_{\tau^\infty}(\tau_0)S^w_{\tau^\infty\tau_0\ldots\tau_{q-2}}(\tau_{q-1})S^w_{\tau^\infty\tau_0\ldots\tau_{q-3}}(\tau_{q-2})\ldots S^w_{\tau^\infty\tau_0}(\tau_1).$$

Since $g_{\tau^{n+1}\tau_0}(Y)=g_\tau(g_{\tau_n\tau_0}(Y)$ and since $f_{\tau^{n+1}\tau_0}(X)=f_\tau(f_{\tau_n\tau_0}(X))$, it follows from the Mean Value theorem that there exists $x_n\in f_{\tau_n\tau_0}(X)$ and $y_n\in g_{\tau_n\tau_0}(Y)$ such that $|g_{\tau^{n+1}\tau_0}(Y)|=|g'_\tau(y_n)|\cdot|g_{\tau_n\tau_0}(Y)|$ and $|f_{\tau^{n+1}\tau_0}(X)|=|f'_\tau(y_n)|\cdot|f_{\tau_n\tau_0}(X)|$. Thus in view of our assumptions and (4.6) and (4.7) we get

$$\lim_{n\to\infty}\frac{|g'_\tau(y_n)|}{|f'_\tau(x_n)|}=\lim_{n\to\infty}\frac{|g_{\tau^{n+1}\tau_0}(Y)|}{|g_{\tau^n\tau_0}(Y)|}/\frac{f_{\tau^{n+1}\tau_0}(X)|}{|f_{\tau^n\tau_0}(X)|}=1.$$

Now, a straightforward computation shows that $y_n\to y_\tau$ and $x_n\to x_\tau$, where y_τ and x_τ are fixed points of g_τ and f_τ respectively. Hence $|g'_\tau(y_\tau)|=|f'_\tau(x_\tau)|$ and equivalence of this condition with condition (1) of Theorem 2.1 finishes the proof. ♣

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