

GEOMETRY AND ERGODIC THEORY OF NON-RECURRENT ELLIPTIC FUNCTIONS

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ABSTRACT. We explore the class of elliptic functions whose all critical points contained in the Julia set are non-recurrent and whose ω -limit sets form compact subsets of the complex plane. In particular, this class comprises hyperbolic, subhyperbolic and parabolic elliptic maps. Let h be the Hausdorff dimension of the Julia set of such elliptic function f . We construct an atomless h -conformal measure m and we show that the h -dimensional Hausdorff measure of the Julia set of f vanishes unless the Julia set is equal to the entire complex plane \mathcal{C} . The h -dimensional packing measure is always positive and it is finite if and only if there are no rationally indifferent periodic points. Furthermore, we prove the existence of a (unique up to a multiplicative constant) σ -finite f -invariant measure μ equivalent with m . The measure μ is then proved to be ergodic and conservative and we identify the set of those points whose all open neighborhoods have infinite measure μ . In particular we show that ∞ is not among them.

1. INTRODUCTION AND GENERAL PRELIMINARIES

1.1. Introduction.

First examples of elliptic (in fact \wp -Weierstrass) functions with detailed description of their Julia sets appeared in [11]. Our paper dealing with elliptic functions whose all critical points contained in the Julia set are non-recurrent and whose ω -limit sets form compact subsets of the complex plane, basically stems from [21], [22] and [12]. Any such elliptic function will be called non-recurrent. We study geometric properties of the Julia sets ultimately resulting in Theorem 4.1 which says that the h -dimensional Hausdorff measure of the Julia set of f vanishes unless the Julia set is equal to the entire complex plane \mathcal{C} . The h -dimensional packing measure is always positive and it is finite if and only if there are no rationally indifferent periodic points. We would like to emphasize that both Hausdorff and packing appearing in this theorem are taken with respect to the spherical metric on $\overline{\mathcal{C}}$. The fact of vanishing h -dimensional Hausdorff measure of the Julia set in the case when $h < 2$ (note that due to [12] $h > 1$) caused by the existence of poles, dramatically differentiate non-recurrent elliptic functions from the case of analogous class of rational functions (see [21]). Our main technical tool employed in this paper is the concept of semi-conformal, almost-conformal and conformal

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measures. We provide an elaborated proof of the existence, uniqueness and continuity of an h -conformal measure. Another important tool is provided by Proposition 2.21, where, expressed in an appropriate language, all non-singular points are shown to be conical. Although there are some overlaps with rational functions (see [21]), most of the proofs are substantially different, mainly because of the existence of poles in the Julia set.

Our second major theme in this paper is the dynamics of f with respect to the conformal measure m . As the first result in this direction we prove the existence of a conservative ergodic σ -finite measure μ equivalent to m . Developing this direction, we study points of finite and infinite condensation of the measure μ , the concepts introduced in [22]. After collecting some basic facts about these points we show in Subsection 5.2 that ∞ is always a point of finite condensation, perhaps the most interesting fact about the measure μ . In the next subsection we relate points of infinite condensation with the set $\Omega(f)$ of rationally indifferent periodic points, providing in particular some sufficient conditions ($\Omega(f) = \emptyset$) for the invariant measure μ to be finite. In the end of this section we deal with parabolic points themselves.

1.2. General Preliminaries.

Throughout the entire paper f^* , diam_s and $B_s(z, r)$ denote respectively the derivatives, diameters and open balls defined by means of the spherical metric whereas f' , diam and $B(z, r)$ are considered in the Euclidean sense.

Definition 1.1. *If $H : D \rightarrow \mathcal{C}$ is an analytic map, $z \in \mathcal{C}$, and $r > 0$, then by $\text{Comp}(z, H(z), H, r)$ we denote the connected component of $H^{-1}(B(H(z), r))$ that contains z .*

Suppose now that c is a critical point of an analytic map $H : D \rightarrow \overline{\mathcal{C}}$. Then there exists $R = R(H, c) > 0$ and $A = A(H, c) \geq 1$ such that

$$A^{-1}|z - c|^q \leq |H(z) - H(c)| \leq A|z - c|^q$$

and

$$A^{-1}|z - c|^{q-1} \leq |H'(z)| \leq A|z - c|^{q-1}$$

for every $z \in \text{Comp}(c, H(c), H, R)$ and that

$$H(\text{Comp}(c, H(c), H, R)) = B(H(c), R)$$

where $q = q(H, c)$ is the order of H at the critical point c . Moreover letting $R > 0$ to be sufficiently small we can require the two above inequalities to hold for every $z \in B(c, (AR)^{1/q})$ and the ball $B(c, (AR)^{1/q}) \cup \text{Comp}(c, H(c), H, R)$ to be expressed as a union of the point c and q open mutually disjoint sets such that H restricted to each of them is injective.

Koebe's Distortion Theorem, I (Euclidean version). There exists a function $k : [0, 1) \rightarrow [1, \infty)$ such that for any $z \in \mathcal{C}$, $r > 0$, $t \in [0, 1)$ and any univalent analytic function $H : B(z, r) \rightarrow \mathcal{C}$ we have that

$$\sup\{|H'(x)| : x \in B(z, tr)\} \leq k(t) \inf\{|H'(x)| : x \in B(z, tr)\}.$$

We put $K = k(1/2)$.

Koebe's Distortion Theorem, I (spherical version). Given a number $s > 0$ there exists a function $k_s : [0, 1) \rightarrow [1, \infty)$ such that for any $z \in \overline{\mathcal{C}}$, $r > 0, t \in [0, 1)$ and any univalent analytic function $H : B(z, r) \rightarrow \overline{\mathcal{C}}$ such that the complement $\overline{\mathcal{C}} \setminus H(B(z, r))$ contains a ball of radius s we have

$$\sup\{|H^*(x)| : x \in B(z, tr)\} \leq k_s(t) \inf\{|H^*(x)| : x \in B(z, tr)\}.$$

The following is straightforward consequence of these two Koebe's Distortion Theorems.

Lemma 1.2. *Suppose that $D \subset \mathcal{C}$ is an open set, $z \in D$ and $H : D \rightarrow \mathcal{C}$ is an analytic map which has an analytic inverse H_z^{-1} defined on $B(H(z), 2R)$ for some $R > 0$. Then for every $0 \leq r \leq R$*

$$B(z, K^{-1}r|H'(z)|^{-1}) \subset H_z^{-1}(B(H(z), r)) \subset B(z, Kr|H'(z)|^{-1}).$$

Lemma 1.3. *Suppose that $D \subset \overline{\mathcal{C}}$ is an open set, $z \in D$ and $H : D \rightarrow \overline{\mathcal{C}}$ is an analytic map which has an analytic inverse H_z^{-1} defined on $B(H(z), 2R)$ for some $R > 0$ avoiding a ball of some radius s . Then for every $0 \leq r \leq R$*

$$B(z, k_s^{-1}(1/2)r|H'(z)|^{-1}) \subset H_z^{-1}(B(H(z), r)) \subset B(z, k_s(1/2)r|H'(z)|^{-1}).$$

We shall also use the following more geometric versions of Koebe's Distortion Theorems involving moduli of annuli.

Koebe's Distortion Theorem, II (Euclidean version). There exists a function $w : (0, +\infty) \rightarrow [1, \infty)$ such that for any two open topological disks $Q_1 \subset Q_2$ with $\text{Mod}(Q_2 \setminus Q_1) \geq t$ and any univalent analytic function $H : Q_2 \rightarrow \overline{\mathcal{C}}$ such that the complement $\overline{\mathcal{C}} \setminus H(Q_2)$ contains a ball of radius s we have

$$\sup\{|H'(x)| : x \in Q_1\} \leq w(t) \inf\{|H'(x)| : x \in Q_1\}.$$

Koebe's Distortion Theorem, II (spherical version). Given a number $s > 0$ there exists a function $w_s : (0, +\infty) \rightarrow [1, \infty)$ such that for any two open topological disks $Q_1 \subset Q_2$ with $\text{Mod}(Q_2 \setminus Q_1) \geq t$ and any univalent analytic function $H : Q_2 \rightarrow \overline{\mathcal{C}}$ such that the complement $\overline{\mathcal{C}} \setminus H(Q_2)$ contains a ball of radius s we have

$$\sup\{|H'(x)| : x \in Q_1\} \leq w_s(t) \inf\{|H'(x)| : x \in Q_1\}.$$

Lemma 1.4. *Suppose that an analytic map $Q \circ H : D \rightarrow \overline{\mathcal{C}}$, a radius $R > 0$ and a point $z \in D$ are such that*

$$\text{Comp}(H(z), Q(H(z)), Q, 2R) \cap \text{Crit}(Q) = \emptyset \text{ and } \text{Comp}(z, Q \circ H(z), Q \circ H, R) \cap \text{Crit}(H) \neq \emptyset$$

If c belongs to the last intersection and

$$\text{diam}\left(\text{Comp}(z, Q \circ H(z), Q \circ H, R)\right) \leq (AR(H, c))^{1/q}$$

then

$$|z - c| \leq KA^2 |(Q \circ H)'(z)|^{-1} R.$$

2. THE DYNAMICS OF NON-RECURRENT ELLIPTIC FUNCTIONS

2.1. Preliminaries from Elliptic Functions. As we already indicated in the introduction, throughout the entire paper $f : \mathcal{C} \rightarrow \overline{\mathcal{C}}$ denotes a non-constant elliptic function. Every such function is doubly periodic and meromorphic. In particular there exist two vectors w_1, w_2 , $\text{Im}(\frac{w_1}{w_2}) \neq 0$, such that for every $z \in \mathcal{C}$ and $n, m \in \mathbb{Z}$,

$$f(z) = f(z + mw_1 + nw_2).$$

The set

$$\Lambda = \{mw_1 + nw_2 : m, n \in \mathbb{Z}\}$$

will be called the lattice of the elliptic function f . This object is independent of the choice of its generators w_1 and w_2 . Let

$$\mathcal{R} = \{t_1 w_1 + t_2 w_2 : 0 \leq t_1, t_2 \leq 1\},$$

be the basic fundamental parallelogram of f . It follows from periodicity of f that $f(\mathcal{C}) = f(\mathcal{R})$. Therefore $f(\mathcal{C})$ as a closed and open subset of the connected set $\overline{\mathcal{C}}$ is equal to $\overline{\mathcal{C}}$. This means that each elliptic function is surjective. It also follows from periodicity of f that

$$f^{-1}(\infty) = \bigcup_{m, n \in \mathbb{Z}} (\mathcal{R} \cap f^{-1}(\infty) + mw_1 + nw_2).$$

For every pole b of f let q_b denote its multiplicity. We define

$$q := \max\{q_b : b \in f^{-1}(\infty)\} = \max\{q_b : b \in f^{-1}(\infty) \cap \mathcal{R}\}.$$

Let

$$B_R = \{z \in \overline{\mathcal{C}} : |z| > R\}.$$

For every pole b of f by $B_b(R)$ we denote the connected component of $f^{-1}(B_R)$ containing b . If $R > 0$ is large enough, say $R \geq R_0$, then B_R contains no critical values of f , all sets $B_b(R)$ are simply connected, mutually disjoint and for $z \in B_b(R)$

$$f(z) = \frac{G_b(z)}{(z - b)^{q_b}} \tag{2.1}$$

where $G_b : B_b(R) \rightarrow \mathcal{C}$ is a holomorphic function taking values out of some neighbourhood of 0. If $U \subset B_R \setminus \{\infty\}$ is an open simply connected set, then all the holomorphic inverse

branches $f_{b,U,1}^{-1}, \dots, f_{b,U,q_b}^{-1}$ of f are well-defined on U and for every $1 \leq j \leq q_b$ and all $z \in U$ we have

$$|(f_{b,U,j}^{-1})'(z)| \asymp |z|^{-\frac{q_b+1}{q_b}}. \quad (2.2)$$

Therefore

$$|(f_{b,U,j}^{-1})^*(z)| \asymp |z|^{-\frac{q_b+1}{q_b}} \frac{1 + |z|^2}{1 + |(f_{b,U,j}^{-1})'(z)|^2} \asymp \frac{|z|^{\frac{q_b-1}{q_b}}}{1 + |b|^2} \asymp \frac{|z|^{\frac{q_b-1}{q_b}}}{|b|^2}, \quad (2.3)$$

where the last comparability sign we wrote assuming in addition that $|b|$ is large enough, say $|b| \geq R_1 > R_0$. Let M be an upper bound of the ratios of $|(f_{b,U,j}^{-1})^*(z)|$ and $|z|^{\frac{q_b-1}{q_b}}|b|^{-2}$ with b, U, j as above. A straightforward calculation based on (2.1) shows that there exists a constant $L \geq 1$ such that for all poles b and all $R \geq R_1$ we have

$$\begin{aligned} L^{-1}R^{-\frac{1}{q_b}} &\leq \text{diam}(B_b(R)) \leq LR^{-\frac{1}{q_b}}, \\ L^{-1}R^{-\frac{1}{q_b}}(1 + |b|^2)^{-1} &\leq \text{diam}_s(B_b(R)) \leq LR^{-\frac{1}{q_b}}(1 + |b|^2)^{-1}. \end{aligned} \quad (2.4)$$

We will frequently use the following fact proven in [12].

Theorem 2.1. *If $f : \mathcal{C} \rightarrow \overline{\mathcal{C}}$ is an arbitrary elliptic function, then*

$$\text{HD}(J(f)) > \frac{2q}{q+1} \geq 1,$$

where $q = \inf\{q_b : b \in \inf^{-1}(\infty)\} = \max\{q_b : b \in \mathcal{R} \cap f^{-1}(\infty)\}$.

2.2. Julia Sets and Non-Recurrent Elliptic Functions.

The Fatou set $F(f)$ of a meromorphic function $f : \mathcal{C} \rightarrow \overline{\mathcal{C}}$ is defined in exactly the same manner as for rational functions; $F(f)$ is the set of points $z \in \mathcal{C}$ such that all the iterates are defined and form a normal family on a neighborhood of z . The *Julia set* $J(f)$ is the complement of $F(f)$ in \mathcal{C} . Thus, $F(f)$ is open, $J(f)$ is closed, $F(f)$ is completely invariant while $f^{-1}(J(f)) \subset J(f)$ and $f(J(f)) = J(f) \cup \{\infty\}$. For a general description of the dynamics of meromorphic functions see e.g. [5]. We would however like to note that it easily follows from Montel's criterion of normality that if $f : \mathcal{C} \rightarrow \overline{\mathcal{C}}$ has at least one pole which is not an omitted value then

$$J(f) = \overline{\bigcup_{n \geq 0} f^{-n}(\infty)}.$$

Let $\text{Crit}(f)$ be the set of critical points of f i.e.

$$\text{Crit}(f) = \{z : f'(z) = 0\}.$$

Its image, $f(\text{Crit}(f))$, is called the set of critical values of f . Since $\mathcal{R} \cap \text{Crit}(f)$ is finite and since $f(\text{Crit}(f)) = f(\mathcal{R} \cap \text{Crit}(f))$, the set of critical values $f(\text{Crit}(f))$ is also finite. Let

$$I_\infty(f) = \{z \in \mathcal{C} : z \in \bigcup_{n \geq 0} f^{-n}(\infty) \text{ or } \lim_{n \rightarrow \infty} f^n(z) = \infty\}$$

be the set of points escaping to infinity under iterates of f . We say that the elliptic function $f : \mathcal{C} \rightarrow \overline{\mathcal{C}}$ is non-recurrent, if the following conditions are satisfied:

- (1) If $c \in \text{Crit}(f) \cap J(f)$, then the ω -limit set $\omega(c)$ is a compact subset of \mathcal{C} (i.e. $\infty \notin \omega(c)$) and $c \notin \omega(c)$
- (2) If $c \in \text{Crit}(f) \cap F(f)$ then either there exists an attracting periodic point w or a rationally indifferent periodic point w such that $\omega(c) \subset \{w, f(w), \dots, f^{p-1}(w)\}$, p is a period.

From now on, unless otherwise stated, we assume throughout the entire paper that the elliptic function $f : \mathcal{C} \rightarrow \overline{\mathcal{C}}$ is non-recurrent. If $t \geq 0$, then a measure m supported on $J(f)$ is said to be semi t -conformal for $f : \mathcal{C} \rightarrow \overline{\mathcal{C}}$, if

$$m(f(A)) \geq \int_A |f^*|^t dm \tag{2.5}$$

for every Borel set $A \subset J(f)$ such that $f|_A$ is injective and m is said to be t -conformal for $f : \mathcal{C} \rightarrow \overline{\mathcal{C}}$, if

$$m(f(A)) = \int_A |f^*|^t dm \tag{2.6}$$

for these sets A .

2.3. Local behavior around parabolic fixed points.

In this section $f : \mathcal{C} \rightarrow \overline{\mathcal{C}}$ is an arbitrary elliptic function of degree ≥ 2 . In particular the map f is not assumed yet to be non-recurrent. In what follows we basically summarize the results concerning local behavior around parabolic fixed points which have been proved in [1], [8], and [9]. Although they were formulated and proved in the context of parabolic rational maps that is assuming that the Julia set contains no critical points, nevertheless they and their proofs are of local character and, in particular, extend to the class of all elliptic functions. Through this section ω is a simple parabolic fixed point of f , that is $f(\omega) = \omega$ and $f'(\omega) = 1$.

First note that on a sufficiently small open neighbourhood V of ω a holomorphic inverse branch $f_\omega^{-1} : V \rightarrow \overline{\mathcal{C}}$ of f is well defined which sends ω to ω . Moreover, V can be taken so small that on V the transformation f_ω^{-1} expresses in the form

$$f_\omega^{-1}(z) = z - a(z - \omega)^{p+1} + a_2(z - \omega)^{p+2} + a_3(z - \omega)^{p+3} + \dots \tag{2.7}$$

where $a \neq 0$ and $p = p(\omega)$ is a positive integer.

$$f_\omega^{-1}(z) - \omega = z - \omega - a(z - \omega)^{p+1} + a_2(z - \omega)^{p+2} + a_3(z - \omega)^{p+3} + \dots$$

Consider the set $\{z : a(z - \omega)^p \in \mathbb{R} \text{ and } a(z - \omega)^p > 0\}$. This set is the union of p rays beginning in ω and forming angles which are integer multiples of $2\pi/p$. Denote these rays by

L_1, L_2, \dots, L_p . For $1 \leq j \leq p$, $0 < r \leq \infty$ and $0 \leq \alpha < 2\pi$ let $S_j(r, \alpha) \subset V$ be the set of those points z lying in the open ball $B(\omega, r)$ for which the angle between the rays L_j and the interval which joins the points ω and z does not exceed α . Using (2.7) an easy computation leads to the following

$$\begin{aligned} \forall \alpha > 0 \exists r_1(\alpha) > 0 \exists 0 < \alpha_0 \leq \alpha \forall 1 \leq j \leq p \\ f_\omega^{-1}(S_j(r_1(\alpha), \alpha_0)) \subset S_j(\infty, \alpha) \end{aligned} \quad (2.8)$$

and there are $\beta > 0$ and $\theta_1 > 0$ such that

$$|f_\omega^{-1}(z) - \omega| < |z - \omega| \quad \text{and} \quad |(f_\omega^{-1})'(z)| < 1 \quad (2.9)$$

for every $\omega \neq z \in S_1(\theta_1, \beta) \cup \dots \cup S_p(\theta_1, \beta)$. The following version of Fatou's flower theorem, (see [4], [17], comp. [1]) shows that the Julia set $J(f)$ approaches the fixed point ω tangentially to the lines L_1, L_2, \dots, L_p . This can be precisely formulated as follows.

Lemma 2.2. (*Fatou's flower theorem*) *For every $\alpha > 0$ there exists $r_2(\alpha) > 0$ such that*

$$J(f) \cap B(\omega, r_2(\alpha)) \subset S_1(r_2(\alpha), \alpha) \cup \dots \cup S_p(r_2(\alpha), \alpha).$$

Since the Julia set $J(f)$ is fully invariant ($f^{-1}(J(f)) = J(f)$ and $f(J(f)) = J(f) \cup \{\infty\}$), we conclude from this lemma and (2.9) that for every $0 < \theta_2 \leq \min\{\theta_1, r_2(\beta)\}$ we have

$$f_\omega^{-1}(J(f) \cap B(\omega, \theta_2)) \subset J(f) \cap B(\omega, \theta_2).$$

Thus all iterates $f_\omega^{-n} : J(f) \cap B(\omega, \theta_2) \rightarrow J(f) \cap B(\omega, \theta_2)$, $n = 0, 1, 2, \dots$ are well defined. From (2.8), Lemma 2.2, and (2.9) we obtain the following

$$\begin{aligned} \forall \alpha > 0 \exists r_3(\alpha) > 0 \forall 1 \leq j \leq p \\ f_\omega^{-1}(S_j(r_3(\alpha), \alpha) \cap J(f)) \subset S_j(r_3(\alpha), \alpha). \end{aligned} \quad (2.10)$$

Put

$$\theta = \theta(f, \omega) = \min\{\theta_2, r_2(\beta), r_3(\beta)\} \quad (2.11)$$

Then, it follows from (2.9), (2.8), and Lemma 2.2 that for every $z \in J(f) \cap B(\omega, \theta)$.

$$\lim_{n \rightarrow \infty} f_\omega^{-n}(z) = \omega \quad (2.12)$$

In fact it can be proved that this convergence is uniform on compact subsets of $B(\omega, \theta) \cap J(f) \setminus \{\omega\}$. See (2.13) for even stronger result. By precise computations one can prove the following.

Lemma 2.3. *For every $\tau > 0$ sufficiently small and every $z \in J(f) \cap B(\omega, \theta)$*

$$f_\omega^{-1}(B(z, \tau|z - \omega|)) \subset B(f_\omega^{-1}(z), \tau|f_\omega^{-1}(z) - \omega|).$$

This lemma immediately leads to the following.

Lemma 2.4. *For every $\tau > 0$ sufficiently small, every $z \in J(f) \cap B(\omega, \theta)$ and every $n \geq 0$ there exists a unique holomorphic inverse branch*

$$f_{\omega, z}^{-n} : B(z, 2\tau|z - \omega|) \rightarrow B(f_{\omega}^{-n}(z), 2\tau|f_{\omega}^{-n}(z) - \omega|)$$

of f^n which sends z to $f_{\omega}^{-n}(z)$.

The following three results (comp. Lemma 1 and Lemma 2 of [8] and Lemma 4.8 of [9]) can be proved in exactly the same way as in [8] and [9].

$$\lim_{n \rightarrow \infty} |f_{\omega, z}^{-n}(z) - \omega| n^{1/p} = (|a|p)^{-1/p} \quad \text{and} \quad n^{-\frac{p+1}{p}} \preceq |(f_{\omega, z}^{-n})'(z)|, |(f_{\omega, z}^{-n})^*(z)| \preceq n^{-\frac{p+1}{p}} \quad (2.13)$$

uniformly on compact subsets of $B(\omega, \theta) \cap J(f) \setminus \{\omega\}$.

Lemma 2.5. *Let m be a semi t -conformal measure for f . Then for every $R > 0$ there exists a constant $C = C(t, \omega, R) \geq 1$ such that for every $0 < r \leq R$*

$$\frac{m(B(\omega, r) \setminus \{\omega\})}{r^{\alpha_t(\omega)}}, \frac{m(B_s(\omega, r) \setminus \{\omega\})}{r^{\alpha_t(\omega)}} \leq C.$$

where $\alpha_t(\omega) = t + p(\omega)(t - 1)$. If m is t -conformal, then in addition

$$\frac{m(B(\omega, r) \setminus \{\omega\})}{r^{\alpha_t(\omega)}}, \frac{m(B_s(\omega, r) \setminus \{\omega\})}{r^{\alpha_t(\omega)}} \geq C^{-1}.$$

2.4. Basic properties of non-recurrent elliptic functions. .

In this section the elliptic function $f : \mathcal{C} \rightarrow \overline{\mathcal{C}}$ is assumed to be non-recurrent. A periodic point ω of f is called parabolic if there exists $q \geq 1$ such that $f^q(\omega) = \omega$ and $(f^q)'(\omega) = 1$. The set of all parabolic points will be denoted by $\Omega(f)$. Since the set of critical values of f is finite, it follows from Fatou's theorem that $\Omega(f)$ is also finite. In addition, $\Omega(f)$ is contained in the Julia set $J(f)$. The crucial tool for our approach in this paper similarly as in [21] is the following version of Mane's theorem proven in [13].

Theorem 2.6. *Let $f : \mathcal{C} \rightarrow \overline{\mathcal{C}}$ be a non-recurrent elliptic function. If $X \subset J(f) \setminus \Omega(f)$ is a closed subset of \mathcal{C} , then for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x \in X$ and every $n \geq 0$, all connected components of $f^{-n}(B(x, \delta))$ have diameters $\leq \epsilon$.*

Corollary 2.7. *Let $f : \mathcal{C} \rightarrow \overline{\mathcal{C}}$ be a non-recurrent elliptic function. If $X \subset J(f) \cup \{\infty\} \setminus \Omega(f)$ is compact, then for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x \in X$ and every $n \geq 0$, all connected components of $f^{-n}(B_s(x, \delta))$ have Euclidean diameters $\leq \epsilon$.*

Proof. Apply Theorem 2.6 for the set $f^{-1}(\infty)$ and given $\epsilon > 0$. This gives us the corresponding number $\delta_1 > 0$. Taking now $\xi > 0$ so small that each connected component of $f^{-n}(B_s(\infty, \xi))$ is contained in $B(b, \delta_1)$ for some pole $b \in f^{-1}(\infty)$ consider the set $Y = X \setminus B_s(\infty, \xi)$. Since Y is a compact subset of \mathcal{C} , it follows from Theorem 2.6 that there exists $\delta_2 >$

0 such that for every $x \in Y$ and every $n \geq 0$ all the connected components of $f^{-n}(B_s(x, \delta))$ have Euclidean diameters $\leq \epsilon$. Consider a finite cover $\{B_s(x_1, \delta_2), \dots, B_s(x_k, \delta_2), B_s(\infty, \xi)\}$ of X , where $x_j \in Y$ for all $j = 1, 2, \dots, k$. Taking as δ half of the Lebesgue number of this cover finishes the proof. ■

Because of an extremal importance of this theorem and its corollary for our considerations, we provide in the Appendix the proof of Theorem 2.6 adapting to the context of elliptic functions original Mane's proof from [13] and some lemma from [19].

We put

$$\text{Crit}(J(f)) = \text{Crit}(f) \cap J(f),$$

$$\theta = \theta(f) = \min\left\{\min\{\theta(f^a, \omega) : \omega \in \Omega(f)\}, \frac{1}{2}\text{dist}(\Omega(f), \text{Crit}(f))\right\} > 0 \quad (2.14)$$

where $a \geq 1$ is so large that all parabolic points of f^a are simple and the numbers $\theta(f^a, \omega)$ are defined in (2.11). We also denote for every set $A \subset \mathcal{C}$

$$O_+(A) = \bigcup_{n \geq 0} f^n(A).$$

and

$$A = A(f) = \max\{A(f, c) : c \in \text{Crit}(f)\} \quad (2.15)$$

We call two points z and w equivalent and we write $z \sim w$ if $w - z \in \Lambda$, the lattice associated with the elliptic function f . Obviously $z \sim w$ implies that $O_+(z) = O_+(w)$ and $\omega(z) = \omega(w)$. Since the set $\text{Crit}(f) \cap \mathcal{R}$ is finite, we conclude that the sets $\omega(\text{Crit}(f)) = \bigcup_{c \in \text{Crit}(f) \cap \mathcal{R}} \omega(c)$ and $\overline{O_+(\text{Crit}(f))} = \bigcup_{c \in \text{Crit}(f) \cap \mathcal{R}} \overline{O_+(c)}$ are compact subsets of \mathcal{C} . A positive number $\beta < \theta/2$ is now chosen to be less than the following three numbers.

$$\begin{aligned} & \min\{\text{dist}(c, O_+(f(c))) : c \in \text{Crit}(f)\} \\ & \min\{(A(c)R(f, c))^{1/q(c)} : c \in \text{Crit}(f)\} \\ & \min\{|c - c'| : c, c' \in \text{Crit}(f) \text{ and } c \neq c'\}, \end{aligned}$$

where $q(c) = q(f, c)$ is the order of the critical point c of f . Notice that the first of these numbers is positive since $O_+(f(\text{Crit}(f)))$ is a compact subset of \mathcal{C} and $\text{Crit}(f)$ has no accumulation points in \mathcal{C} . Since f contains no recurrent critical points, it follows from Theorem 2.6 that there exists $0 < \gamma < 1/4$ such that if $n \geq 0$ is an integer, $z \in J(f)$ and $f^n(z) \notin B(\Omega(f), \theta)$, then

$$\text{diam}\left(\text{Comp}(z, f^n(z), f^n, 2\gamma)\right) < \beta. \quad (2.16)$$

From now on fix also $0 < \tau < \theta^{-1} \min\{\beta, 2\gamma\}$ so small as required in Lemma 2.4 for every $\omega \in \Omega(f)$ and so small that for every $z \in J(f)$

$$\text{diam}\left(\text{Comp}(z, f(z), f, \theta\tau)\right) < \min\{\beta, 2\gamma\}. \quad (2.17)$$

Lemma 2.8. *If $n \geq 0$ is an integer, $\eta > 0$, $z \in J(f)$ and for every $k \in \{0, 1, \dots, n\}$*

$$\text{diam}\left(\text{Comp}(f^k(z), f^n(z), f^{n-k}, \eta)\right) \leq \beta,$$

then each connected component $\text{Comp}(f^k(z), f^n(z), f^{n-k}, \eta)$ contains at most one critical point of f and the equivalence class of each critical point intersects at most one of these components.

Proof. The first part is obvious by the choice of β . In order to prove the second part suppose that

$$c_1 \in \text{Crit}(f) \cap \text{Comp}(f^{k_1}(z), f^n(z), f^{n-k_1}, \eta), \quad c_2 \in \text{Comp}(f^{k_2}(z), f^n(z), f^{n-k_2}, \eta)$$

and $c_1 \sim c_2$, where $0 \leq k_1 < k_2 \leq n$. But then

$$f^{k_2-k_1}(c_2) = f^{k_2-k_1}(c_1) \in \text{Comp}(f^{k_2}(z), f^n(z), f^{n-k_2}, \eta)$$

and therefore $|f^{k_2-k_1}(c_2) - c_1| < \beta$, contrary to the choice of β . ■

Lemma 2.9. *The set $\omega(\text{Crit}(J(f)))$ is nowhere dense in $J(f)$.*

Proof. Suppose that the interior (relative to $J(f)$) of $\omega(\text{Crit}(J(f)))$ is nonempty. Then there exists $c \in \text{Crit}(J(f))$ such that $\omega(c)$ has nonempty interior. But then there would exist $n \geq 0$ such that $f^n(\omega(c)) = J(f)$ and consequently $\omega(c) = J(f)$. This however is a contradiction as $c \notin \omega(c)$. ■

Let $\kappa = \left(\prod_{c \in \text{Crit}(f) \cap \mathcal{R}} q(c)\right)^{-1}$. We shall prove the following.

Lemma 2.10. *If $z \in J(f)$, $f^n(z) \notin B(\Omega(f), \theta)$, then*

$$\text{Mod}\left(\text{Comp}(z, f^n(z), f^n, 2\gamma) \setminus \text{Comp}(z, f^n(z), f^n, \gamma)\right) \geq \kappa \log 2 / \#\text{Crit}(f) \cap \mathcal{R}$$

Proof. In view of Lemma 2.8 there exists a geometric annulus $R \subset B(f^n(z), 2\gamma) \setminus B(f^n(z), \gamma)$ centered at $f^n(z)$ of modulus $\log 2 / \#\text{Crit}(f)$ such that $f^{-n}(R) \cap \text{Comp}(z, f^n(z), f^n, 2\gamma) \cap \text{Crit}(f^n) = \emptyset$. Since covering maps increase moduli of annuli at most by factors equal to their degrees, we conclude that

$$\begin{aligned} & \text{Mod}\left(\text{Comp}(z, f^n(z), f^n, 2\gamma) \setminus \text{Comp}(z, f^n(z), f^n, \gamma)\right) \\ & \geq \text{Mod}(R_n) \geq \left(\log 2 / \#\text{Crit}(f) \cap \mathcal{R}\right) / \prod_{c \in \text{Crit}(f) \cap \mathcal{R}} q(c) \\ & = \frac{\kappa \log 2}{\#\text{Crit}(f) \cap \mathcal{R}}, \end{aligned}$$

where $R_n \subset \text{Comp}(z, f^n(z), f^n, 2\gamma)$ is the connected component of $f^{-n}(B(f^n(z), 2\gamma))$ enclosing $\text{Comp}(z, f^n(z), f^n, \gamma)$. ■

As an immediate consequence of this lemma and Koebe's Distortion Theorem, II (Euclidean version) we get the following.

Lemma 2.11. *Suppose that $z \in J(f)$ and $f^n(z) \notin B(\Omega(f), \theta)$. If $0 \leq k \leq n$ and $f^k : \text{Comp}(z, f^n(z), f^n, 2\gamma) \rightarrow \text{Comp}(f^k(z), f^n(z), f^{n-k}, 2\gamma)$ is univalent, then*

$$\frac{|(f^k)'(y)|}{|(f^k)'(x)|} \leq \text{const}$$

for all $x, y \in \text{Comp}(z, f^n(z), f^n, \gamma)$, where const is a number depending only on $\#(\text{Crit}(f) \cap \mathcal{R})$ and κ .

For A, B , any two subsets of a metric space put

$$\text{dist}(A, B) = \inf\{\text{dist}(a, b) : a \in A, b \in B\}$$

and

$$\text{Dist}(A, B) = \sup\{\text{dist}(a, b) : a \in A, b \in B\}.$$

We shall prove the following.

Lemma 2.12. *Suppose that $z \in J(f)$ and $f^n(z) \notin B(\Omega(f), \theta)$. Suppose also that $Q^{(1)} \subset Q^{(2)} \subset B(f^n(z), \gamma)$ are connected sets. If $Q_n^{(2)}$ is a connected component of $f^{-n}(Q^{(2)})$ contained in $\text{Comp}(z, f^n(z), f^n, \gamma')$ and $Q_n^{(1)}$ is a connected component of $f^{-n}(Q^{(1)})$ contained in $Q_n^{(2)}$, then*

$$\frac{\text{diam}(Q_n^{(1)})}{\text{diam}(Q_n^{(2)})} \geq \frac{\text{diam}(Q^{(1)})}{\text{diam}(Q^{(2)})}.$$

Proof. Let $1 \leq n_1 \leq \dots \leq n_u \leq n$ be all the integers k between 1 and n such that

$$\text{Crit}(f) \cap \text{Comp}(f^{n-k}(z), f^n(z), f^k, 2\gamma) \neq \emptyset.$$

Fix $1 \leq i \leq u$. If $j \in [n_i, n_{i+1} - 1]$ (we set $n_{u+1} = n - 1$), then by Lemma 2.10 there exists a universal constant $T > 0$ such that

$$\frac{\text{diam}(Q_j^{(1)})}{\text{diam}(Q_j^{(2)})} \geq T \frac{\text{diam}(Q_{n_i}^{(1)})}{\text{diam}(Q_{n_i}^{(2)})} \quad (2.18)$$

Since, in view of Lemma 2.8, $u \leq \#(\text{Crit}(f) \cap \mathcal{R})$, in order to conclude the proof is therefore enough to show the existence of a universal constant $E > 0$ such that for every $1 \leq i \leq u - 1$.

$$\frac{\text{diam}(Q_{n_{i+1}}^{(1)})}{\text{diam}(Q_{n_{i+1}}^{(2)})} \geq E \frac{\text{diam}(Q_{n_i}^{(1)})}{\text{diam}(Q_{n_i}^{(2)})}.$$

And indeed, let c be the critical point contained in $\text{Comp}(f^{n-n_{i+1}}(z), f^n(z), f^{n_{i+1}}, 2\gamma)$ and let q denote its order. Since both sets $Q_{n_{i+1}}^{(2)}$ and $Q_{n_{i+1}}^{(1)}$ are connected, we get for $i = 1, 2$ that

$$\text{diam}(Q_{n_{i+1}-1}^{(i)}) \asymp \text{diam}(Q_{n_{i+1}}^{(i)}) \sup\{|f'(x)| : x \in Q_{n_{i+1}}^{(i)}\} \asymp \text{diam}(Q_{n_{i+1}}^{(i)}) \text{Dist}(c, Q_{n_{i+1}}^{(i)}).$$

Hence, using (2.18), we obtain

$$\begin{aligned} \frac{\text{diam}(Q_{n_{i+1}}^{(1)})}{\text{diam}(Q_{n_{i+1}}^{(2)})} &\asymp \frac{\text{diam}(Q_{n_{i+1}-1}^{(1)})}{\text{diam}(Q_{n_{i+1}-1}^{(2)})} \cdot \frac{\text{Dist}(c, Q_{n_{i+1}}^{(2)})}{\text{Dist}(c, Q_{n_{i+1}}^{(1)})} \geq \frac{\text{diam}(Q_{n_{i+1}-1}^{(1)})}{\text{diam}(Q_{n_{i+1}-1}^{(2)})} \\ &\geq T \frac{\text{diam}(Q_{n_i}^{(1)})}{\text{diam}(Q_{n_i}^{(2)})}. \end{aligned}$$

We are done. ■

2.5. Partial order in $\text{Crit}(J(f))$ and stratifications of closed forward-invariant subsets of $J(f)$.

Now we introduce in $\text{Crit}(J(f))$ a relation $<$ which, in view of Lemma 2.13 below, is an ordering relation, by putting

$$c_1 < c_2 \iff c_1 \in \omega(c_2). \quad (2.19)$$

Since $c_2 \sim c_3$ implies $\omega(c_2) = \omega(c_3)$, if $c_1 < c_2$, then if $c_1 < c_2$ and $c_2 \sim c_3$, then $c_1 < c_3$

Lemma 2.13. *If $c_1 < c_2$ and $c_2 < c_3$, then $c_1 < c_3$.*

Proof. Indeed, we have $c_1 \in \omega(c_2) \subset \omega(c_3)$. ■

Lemma 2.14. *There is no infinite, linear subset of the partially ordered set $(\text{Crit}(J(f)), <)$*

Proof. Indeed, suppose on the contrary that $c_1 < c_2 < \dots$ is an infinite, linearly ordered subset of $\text{Crit}(J(f))$. Since the number of equivalency classes of relation \sim is equal to $\#(\text{Crit}(J(f)) \cap \mathcal{R})$ which is finite, there exist two numbers $1 \leq i < j$ such that $\omega(c_i) = \omega(c_j)$. But this implies that $c_i \in \omega(c_j) = \omega(c_i)$ and we get a contradiction. The proof is finished. ■

The following observation is a reformulation of the condition that $J(f)$ contains no recurrent critical points.

Lemma 2.15. *If $c \in \text{Crit}(J(f))$, then $\sim(c < c)$.*

Define now inductively a sequence $\{Cr_i(f)\}$ of subsets of $\text{Crit}(J(f))$ by setting $Cr_0(f) = \emptyset$ and

$$Cr_{i+1}(f) = \left\{ c \in \text{Crit}(J(f)) \setminus \bigcup_{j=0}^i Cr_j(f) : \text{if } c' < c, \text{ then } c' \in Cr_0(f) \cup \dots \cup Cr_i(f) \right\} \quad (2.20)$$

Lemma 2.16. *We have*

- (a) *If $c \in Cr_i(f)$ and $c' \sim c$, then $c' \in Cr_i(f)$.*

- (b) The sets $\{Cr_i(f)\}$ are mutually disjoint.
- (c) $\exists_{p \geq 1} \forall_{i \geq p+1} Cr_i(f) = \emptyset$
- (d) $Cr_0(f) \cup \dots \cup Cr_p(f) = \text{Crit}(J(f))$
- (e) $Cr_1(f) \neq \emptyset$

Proof. The item (a) follows immediately from the definition of the sets Cr_i and the fact that two equivalent points have the same ω -limit sets. By definition $Cr_{i+1}(f) \cap \bigcup_{j=1}^i Cr_j(f) = \emptyset$, so disjointness in (b) is clear. As the number of equivalence classes of the relation \sim is equal to $\#(\text{Crit}(J(f)) \cap \mathcal{R})$ which is finite, (a) and (b) imply (c). Take p to be minimal number satisfying (b) and suppose that $c \in \text{Crit}(J(f)) \setminus \bigcup_{j=1}^p Cr_j(f)$. Since $Cr_{p+1}(f) = \emptyset$, there exists $c' \notin \bigcup_{j=1}^p Cr_j(f)$ such that $c' < c$. Iterating this procedure we would obtain an infinite sequence of critical points $c_1 = c > c_2 = c' > c_3 > \dots$. But this contradicts Lemma 2.14 proving (d). Now part (e) follows from (c) and (2.20). ■

As an immediate consequence of the definition of the sequence $\{Cr_i(f)\}$ we get the following simple lemma.

Lemma 2.17. *If $c, c' \in Cr_i(f)$, then $\sim(c < c')$.*

For every point $z \in J(f)$ define the set

$$\text{Crit}(z) = \{c \in \text{Crit}(J(f)) : c \in \omega(z)\}$$

We shall prove the following.

Lemma 2.18. *If $z \in J(f) \setminus I_\infty(f)$, then either $z \in \bigcup_{n \geq 0} f^{-n}(\Omega(f))$ or $\omega(z) \setminus \{\infty\}$ is not contained in $\overline{O_+(f(\text{Crit}(z)))} \cup \Omega(f)$.*

Proof. Suppose that $z \notin \bigcup_{n \geq 0} f^{-n}(\Omega(f)) \cup I_\infty(f)$. Then by (2.12) the set $\omega(z) \setminus \{\infty\}$ is not contained in $\Omega(f)$. So, if we suppose that

$$\omega(z) \setminus \{\infty\} \subset \overline{O_+(f(\text{Crit}(z)))} \cup \Omega(f), \quad (2.21)$$

then, as $\omega(z) \setminus \{\infty\} \neq \emptyset$, we conclude that $\text{Crit}(z) \neq \emptyset$. Let $c_1 \in \text{Crit}(z)$. It means that $c_1 \in \omega(z)$ and as $c_1 \notin \Omega(f)$, it follows from (2.21) that there exists $c_2 \in \text{Crit}(z)$ such that either $c_1 \in \omega(c_2)$ or $c_1 = f^{n_1}(c_2)$ for some $n_1 \geq 1$. Iterating this procedure we obtain an infinite sequence $\{c_j\}_{j=1}^\infty$ such that for every $j \geq 1$ either $c_j \in \omega(c_{j+1})$ or $c_j = f^{n_j}(c_{j+1})$ for some $n_j \geq 1$. Consider an arbitrary block c_k, c_{k+1}, \dots, c_l such that $c_j = f^{n_j}(c_{j+1})$ for every $k \leq j \leq l-1$ and suppose that $l - (k-1) \geq \#(\text{Crit}(f) \cap \mathcal{R})$. Then there are two indexes $k \leq a < b \leq l$ such that $c_a \sim c_b$. Then

$$f^{n_a + n_{a+1} + \dots + n_{b-1}}(c_a) = f^{n_a + n_{a+1} + \dots + n_{b-1}}(c_b) = c_a$$

and consequently, as $n_a + n_{a+1} + \dots + n_{b-1} \geq b - a \geq 1$, c_a is a super-attracting periodic point of f . Since $c_a \in J(f)$, this is a contradiction, and in consequence the length of the

block c_k, c_{k+1}, \dots, c_l is bounded above by $\#(\text{Crit}(f) \cap \mathcal{R})$. Hence, there exists an infinite subsequence $\{n_k\}_{k \geq 1}$ such that $c_{n_k} \in \omega(c_{n_{k+1}})$ for every $k \geq 1$. But then $c_{n_k} \in \omega(c_{n_{k+1}})$ for every $k \geq 1$, or equivalently $c_{n_k} < c_{n_{k+1}}$ for every $k \geq 1$. This however contradicts Lemma 2.14 and we are done. ■

Define now for every $i = 0, 1, \dots, p$

$$S_i(f) = Cr_0(f) \cup \dots \cup Cr_i(f)$$

and for every $i = 0, 1, \dots, p-1$ consider $c' \in \bigcup_{c \in Cr_{i+1}(f)} \omega(c) \cap \text{Crit}(J(f))$. Then there exists $c \in Cr_{i+1}(f)$ such that $c' \in \omega(c)$ which equivalently means that $c' < c$. Thus, by (2.20) we get $c' \in S_i(f)$. So

$$\bigcup_{c \in Cr_{i+1}(f)} \omega(c) \cap (\text{Crit}(J(f)) \setminus S_i(f)) = \emptyset \quad (2.22)$$

Therefore, since the set $\bigcup_{c \in Cr_{i+1}(f)} \omega(c) \subset \mathcal{C}$ is compact and $\text{Crit}(J(f)) \setminus S_i(f)$ has no accumulation point in \mathcal{C} ,

$$\delta_i = \text{dist}\left(\bigcup_{c \in Cr_{i+1}(f)} \omega(c), \text{Crit}(J(f)) \setminus S_i(f)\right) > 0 \quad (2.23)$$

Set

$$\rho = \min\{\delta_i/2 : i = 0, 1, \dots, p-1\}.$$

Fix a closed forward-invariant subset F of $J(f)$ and for every $i = 0, 1, \dots, p$ define

$$F_i(f) = \{z \in F : \text{dist}(O_+(z), \text{Crit}(J(f)) \setminus S_i(f)) \geq \rho\}.$$

Let us now prove the following two lemmas concerning the sets $F_i(f)$.

Lemma 2.19. $F_0 \subset F_1 \subset \dots \subset F_p = F$.

Proof. Since $S_{i+1}(f) \supset S_i(f)$, the inclusions $F_i \subset F_{i+1}$ is obvious. Since $S_p(f) = \text{Crit}(J(f))$, it holds $F_p(f) = F$. We are done. ■

Let

$$\text{PC}(f) = \overline{O_+(\text{Crit}(J(f)))}$$

We shall prove the following.

Lemma 2.20. *There exists $l = l(f)$ such that for every $i = 0, 1, \dots, p-1$*

$$\bigcup_{c \in Cr_{i+1}(f)} \omega(c) \subset \overline{O_+(f^l(Cr_{i+1}(f)))} \subset \text{PC}(f)_i$$

Proof. The left-hand inclusion is obvious regardless whatever $l(f)$ is. In order to prove the right-hand one fix $i \in \{0, 1, \dots, p-1\}$. By the definition of ω -limit sets there exists $l_i \geq 1$ such that for every $c \in Cr_{i+1}(f)$ we have $\text{dist}(O_+(f^{l_i}(c)), \bigcup_{c \in Cr_{i+1}(f)} \omega(c)) < \delta_i/2$. Thus, by (2.23), $\text{dist}(\overline{O_+(f^{l_i}(c))}, \text{Crit}(J(f)) \setminus S_i(f)) > \delta_i/2$. Since $\rho \leq \delta_i/2$ and since for every $z \in \overline{O_+(f^{l_i}(c))}$ also $O_+(z) \subset \overline{O_+(f^{l_i}(c))}$, we therefore get $O_+(f^l(Cr_{i+1}(f))) \subset \text{PC}(f)_i$. So, putting $l(f) = \max\{l_i : i = 0, 1, \dots, p-1\}$ the proof is completed. ■

2.6. Holomorphic inverse branches. In this section we prove the existence of suitable holomorphic inverse branches-our basic tools in the next section. Set

$$\text{Sing}^-(f) = \bigcup_{n \geq 0} f^{-n}(\Omega(f) \cup \text{Crit}(J(f)) \cup f^{-1}(\infty)) \text{ and } I_-(f) = \bigcup_{n \geq 1} f^{-n}(\infty).$$

We start with the following.

Proposition 2.21. *If $z \in J(f) \setminus \text{Sing}^-(f)$, then there exist a positive number $\eta(z)$, an increasing sequence of positive integers $\{n_j\}_{j \geq 1}$, and a point $x = x(z) \in \omega(z) \setminus (\Omega(f) \cup \omega(\text{Crit}(z)))$ such that $x \neq \infty$ if $z \notin I_\infty(f)$, $\lim_{j \rightarrow \infty} f^{n_j}(z) = x$ and*

$$\text{Comp}(z, f^{n_j}(z), f^{n_j}, \eta(z)) \cap \text{Crit}(f^{n_j}) = \emptyset$$

for every $j \geq 0$.

Proof. Suppose first that $z \in I_\infty(f) \setminus \text{Sing}^-(f)$. Since $\overline{O_+(\text{Crit}(f))}$ is a compact subset of \mathcal{C} , we conclude that for all n large enough $\text{dist}(f^n(z), O_+(\text{Crit}(f))) \geq 1$. We are therefore done taking $x = \infty$ and $\eta(z) = 1$. So, suppose that $z \notin I_\infty(f)$. This means that $\omega(z) \setminus \{\infty\} \neq \emptyset$. Suppose that $\omega(z) \setminus \{\infty\}$ is unbounded. Since $\overline{O_+(\text{Crit}(f))}$ is a compact subset of \mathcal{C} , there thus exists $x \in \omega(z) \setminus \{\infty\}$ such that $\text{dist}(x, \overline{O_+(\text{Crit}(f))}) \geq 2$ and we are done fixing a sequence $\{n_j\}_{j=1}^\infty$ such $|f^{n_j}(z) - x| \leq 1$ and taking $\eta(z) = 1$. So, assume that $\omega(z) = F \cup \{\infty\}$ where $F \subset \mathcal{C}$ is a compact set. Then $F \cap f^{-1}(\infty) \neq \emptyset$ and fix $x \in F \cap f^{-1}(\infty)$. Again, since $\overline{O_+(\text{Crit}(f))}$ is a compact subset of \mathcal{C} and since $f(\overline{O_+(\text{Crit}(f))}) \subset \overline{O_+(\text{Crit}(f))}$, we see that $x \notin \overline{O_+(\text{Crit}(f))}$ and we are done taking $\eta(z) = \text{dist}(x, \overline{O_+(\text{Crit}(f))})$. So suppose finally that $\omega(z)$ is a compact subset of \mathcal{C} . In view of Lemma 2.18 there exists $x \in \omega(z) \setminus (\Omega(f) \cup \overline{O_+(f(\text{Crit}(z)))} \cup \{\infty\})$. The number $\eta = \text{dist}(x, \Omega(f) \cup \overline{O_+(f(\text{Crit}(z)))})/2$ is positive since $\omega(\text{Crit}(z))$ is a compact subset of \mathcal{C} and $\Omega(f)$ is finite. Then there exists an infinite increasing sequence $\{m_j\}_{j \geq 1}$ such that

$$\varinjlim_{j \rightarrow \infty} f^{m_j}(z) = x \tag{2.24}$$

and

$$B(f^{m_j}(z), \eta) \cap \bigcup_{n \geq 1} f^n(\text{Crit}(z)) = \emptyset. \tag{2.25}$$

Now we claim that there exists $\eta(z)$ such that for every $j \geq 1$ large enough

$$\text{Comp}(z, f^{m_j}(z), f^{m_j}, \eta(z)) \cap \text{Crit}(f^{m_j}) = \emptyset. \quad (2.26)$$

Otherwise we would find an increasing to infinity subsequence $\{m_{j_i}\}$ of $\{m_j\}$ and a decreasing to zero sequence of positive numbers η_i such that $\eta_i < \eta$ and

$$\text{Comp}(z, f^{m_{j_i}}(z), f^{m_{j_i}}, \eta_i) \cap \text{Crit}(f^{m_{j_i}}) \neq \emptyset$$

Let $\tilde{c}_i \in \text{Comp}(z, f^{m_{j_i}}(z), f^{m_{j_i}}, \eta_i) \cap \text{Crit}(f^{m_{j_i}})$. Then there exists $c_i \in \text{Crit}(f)$ such that $f^{p_i}(\tilde{c}_i) = c_i$ for some $0 \leq p_i \leq m_{j_i} - 1$. Since the set $f^{-1}(x)$ is at a positive distance from $\Omega(f)$ and since $\eta_i \rightarrow 0$, it follows from Theorem 2.6 that $\lim_{i \rightarrow \infty} \tilde{c}_i = z$. Since $z \notin \bigcup_{n \geq 0} f^{-n}(\text{Crit}(f))$, it implies that $\lim_{i \rightarrow \infty} p_i = \infty$. But then using Lemma 2.6 again and the formula $f^{p_i}(\tilde{c}_i) = c_i$ we conclude that the set of all accumulation points of the sequence $\{c_i\}$ is contained in $\omega(z)$. Hence, passing to a subsequence, we may assume that the limit $c = \lim_{i \rightarrow \infty} c_i$ exists. But since $c \in \omega(z)$, since $\omega(z)$ is a compact subset of \mathcal{C} and since ∞ is the only accumulation point of $\text{Crit}(f)$, we conclude that the sequence c_i is eventually constant. Thus, dropping some finite number of initial terms, we may assume that this sequence is constant. This means that $c_i = c$ for all $i = 1, 2, \dots$. Since $c = f^{p_i}(\tilde{c}_i)$, we get

$$|f^{m_{j_i}}(z) - f^{m_{j_i}-p_i}(c)| = |f^{m_{j_i}}(z) - f^{m_{j_i}}(\tilde{c}_i)| < \eta_i.$$

Since $\lim_{i \rightarrow \infty} \eta_i = 0$ and since $\omega(z)$ is a compact subset of \mathcal{C} , we conclude that $\lim_{i \rightarrow \infty} |f^{m_{j_i}}(z) - f^{m_{j_i}-p_i}(c)| = 0$. Since $c \in \text{Crit}(z)$, in view of (2.25) this implies that $m_{j_i} - p_i \leq 0$ for all i large enough. So, we get a contradiction as $0 \leq p_i \leq m_{j_i} - 1$ and (2.26) is proved. We are done. ■

Since if $z \in J(f) \setminus (\text{Sing}^-(f) \cup I_\infty(f))$, the limit points of the normal family

$$f_z^{-n_j} : B(x(z), \eta(z)/2) \rightarrow \overline{\mathcal{C}}$$

consist only of constant functions. Therefore we get the following.

Corollary 2.22. *If $z \in J(f) \setminus (\text{Sing}^-(f) \cup I_\infty(f))$ and the sequence $\{n_j\}_{j=1}^\infty$ is taken from Proposition 2.21, then*

$$\limsup_{n \rightarrow \infty} |(f^n)^*(z)| = \limsup_{n \rightarrow \infty} |(f^n)'(z)| = \lim_{n \rightarrow \infty} |(f^{n_j})'(z)| = +\infty.$$

In addition, if we assume only that $z \in J(f) \setminus \text{Sing}^-(f)$, then

$$\limsup_{n \rightarrow \infty} |(f^n)'(z)| = \infty.$$

3. CONFORMAL MEASURES

In this section we deal in detail with the existence, uniqueness and some properties of conformal measures. Let HD denote the Hausdorff dimension, H^t and l_2 denote respectively

t -dimensional Hausdorff measure and 2-dimensional Lebesgue measure, both considered with respect to the spherical metric on $\overline{\mathcal{C}}$. Throughout this section and the entire paper we set

$$h = \text{HD}(J(f)).$$

We begin with the following.

Lemma 3.1. *If m is a t -conformal measure for $f : J(f) \rightarrow J(f) \cup \{\infty\}$, then $t \geq \text{HD}(J(f))$ and $\text{H}^t|_{J(f)}$ is absolutely continuous with respect to m .*

Proof. Fix $z \in J(f) \setminus (\text{Sing}^-(f) \cup I_\infty(f))$. Let $\eta(z) > 0$, $x \in \omega(z) \setminus \{\infty\}$ and the sequence $\{n_j\}_{j \geq 1}$ be taken from Proposition 2.21. It then follows from this proposition and Koebe's Distortion Theorem, I(spherical version) that

$$f_z^{-n_j}(B(f^{n_j}(z), \eta(z)/2)) \asymp B(z, |(f^{n_j})^*(z)|^{-1}\eta(z)/2).$$

Applying again this Koebe's Distortion Theorem and conformality of the measure m , we get for all $j \geq 1$ large enough

$$\begin{aligned} m(B(z, |(f^{n_j})'(z)|^{-1}\eta(z)/2)) &\asymp |(f^{n_j})^*(z)|^{-t} m(B(f^{n_j}(z), \eta(z)/2)) \\ &\succeq |(f^{n_j})^*(z)|^{-t} m(B(x, \eta(z)/4)) \\ &\asymp |(f^{n_j})'(z)|^{-t} m(B(x, \eta(z)/4)) \\ &= (2\eta(z)^{-1})^t m(B(x, \eta(z)/4)) \left(|(f^{n_j})^*(z)|^{-1}\eta(z)/2 \right)^t, \end{aligned}$$

where the second comparability sign depends on $|z|$ and holds for all $j \geq 1$ large enough so that $f^{n_j}(z)$ is sufficiently close to x . In particular

$$\limsup_{r \rightarrow 0} \frac{m(B(z, r))}{r^t} \geq R(z) > 0,$$

where $R(z) = (2\eta(z)^{-1})^t m(B(x, \eta(z)/4))$. Therefore, putting

$$X_k = \{z \in J(f) \setminus \text{Sing}^-(f) : |z| \leq k \text{ and } R(z) \geq 1/k\}$$

we have $\bigcup_{k \geq 1} X_k = J(f) \setminus (\text{Sing}^-(f) \cup I_\infty(f))$ and in view of Theorem 4.3(1) (which is of purely geometric character independent of our considerations here), $d\text{H}^t/dm \leq b(2)k$ on X_k . In particular $\text{H}^t \ll m$ on $J(f) \setminus (\text{Sing}^-(f) \cup I_\infty(f))$. Hence $\text{HD}(J(f) \setminus (\text{Sing}^-(f) \cup I_\infty(f))) \leq t$. By Theorem 1 and Theorem 2 in [12]), $\text{HD}(J(f)) > \text{HD}(I_\infty(f))$. Thus $\text{HD}(J(f)) = \text{HD}(J(f) \setminus (\text{Sing}^-(f) \cup I_\infty(f))) \leq t$ and $\text{H}^t \ll m$ on $J(f)$. ■

We will need in the sequel the following result which is interesting itself.

Lemma 3.2. *If m is a t -conformal measure for $f : \mathcal{C} \rightarrow \overline{\mathcal{C}}$, then $m(I_\infty(f) \setminus I_-(f)) = 0$. Even more, there exists $R > 0$ such that*

$$m(\{z : \liminf_{n \rightarrow \infty} |f^n(z)| > R\}) = 0.$$

Proof. Let b be a pole of $f : \mathcal{C} \rightarrow \overline{\mathcal{C}}$. We shall obtain first an upper estimate on $m(B_b(R))$ similar to the second inequality in (2.4). And indeed, covering $B_R \setminus \{\infty\}$ by two simply connected domains

$$B_R^+ = \{z \in B_R \setminus \{\infty\} : \operatorname{Im} z > 0\} \quad \text{and} \quad B_R^1 = \{z \in B_R \setminus \{\infty\} : \operatorname{Im} z < 1\}$$

we obtain

$$m(B_b(R) \setminus \{b\}) \leq \sum_{j=1}^{q_b} \int_{B_R^+} |(f_{b, B_R^+, j}^{-1})^*|^t dm + \sum_{j=1}^{q_b} \int_{B_R^1} |(f_{b, B_R^-, j}^{-1})^*|^t dm.$$

Using now (2.3), we obtain

$$\begin{aligned} \int_{B_R^+} |(f_{b, B_R^+, j}^{-1})^*|^t dm &\asymp \int_{B_R^+} \left(\frac{1}{1 + |b|^2} |z|^{\frac{q_b-1}{q_b}} \right)^t dm(z) = \frac{1}{(1 + |b|^2)^t} \int_{B_R^+} |z|^{\frac{q_b-1}{q_b} t} dm(z) \\ &\leq (1 + |b|^2)^{-t} \int_{B_R^+} |z|^{\frac{q-1}{q} t} dm(z). \end{aligned}$$

Looking at the first line of this formula with a pole b of maximal multiplicity, we see that the integral $\int_{B_R^+} |z|^{\frac{q-1}{q} t} dm(z)$ is finite and even more:

$$\lim_{R \rightarrow \infty} \int_{B_R^+} |z|^{\frac{q-1}{q} t} dm(z) = 0. \quad (3.1)$$

Similarly is finite the integral $\int_{B_R^1} |z|^{\frac{q-1}{q} t} dm(z)$ and it also converges to 0 as $R \rightarrow \infty$. Putting

$$\Sigma_R = \max \left\{ \int_{B_R^+} |z|^{\frac{q-1}{q} t} dm(z), \int_{B_R^1} |z|^{\frac{q-1}{q} t} dm(z) \right\}$$

we therefore conclude that

$$m(B_b(R) \setminus \{b\}) \leq 2q \Sigma_R (1 + |b|^2)^{-t} \leq 2q \Sigma_R |b|^{-2t}. \quad (3.2)$$

Now the argument goes essentially in the same way as in [12]. We present it here for the sake of completeness. We take $R_2 \geq R_1$ defined in Section 2.1 so large that

$$LR^{-\frac{1}{q_b}} < R_0 \quad (3.3)$$

for all poles $b \in B_{R_2}$ and all $R \geq R_2$. Given two poles $b_1, b_2 \in B_{2R_2}$ we denote by $f_{b_2, b_1, j}^{-1} : B(b_1, R_0) \rightarrow \mathcal{C}$ all the holomorphic inverse branches $f_{b_2, B(b_1, R_0), j}^{-1}$. It follows from (2.4) and (3.3) that

$$f_{b_2, b_1, j}^{-1}(B(b_1, R_0)) \subset B_{b_2}(2R_2 - R_0) \subset B_{b_2}(R_2) \subset B(b_2, R_0) \quad (3.4)$$

Set

$$I_R(f) = \{z \in \mathcal{C} : \forall_{n \geq 0} |f^n(z)| > R\}.$$

Since the series $\sum_{b \in f^{-1}(\infty) \setminus \{0\}} |b|^{-s}$ converges for all $s > 2$ and since by Lemma 3.1 and Theorem 3 from [12], $t \geq h > \frac{2q}{q+1}$ there exists $R_3 \geq R_2$ such that

$$qM^t \sum_{b \in B_{R_3} \cap f^{-1}(\infty)} |b|^{-\frac{q+1}{q}t} \leq 1/2. \quad (3.5)$$

Consider $R \geq 4R_3$. Put

$$I = f^{-1}(\infty) \cap B_{(R/2)}$$

Since $R/2 + R_0 \leq R/2 + R_3 < R/2 + R/2 = R$, it follows from (3.4), (2.4) and (3.3) that for every $l \geq 1$ the family W_l defined as

$$\left\{ f_{b_l, b_{l-1}, j_l}^{-1} \circ f_{b_{l-1}, b_{l-2}, j_{l-1}}^{-1} \circ \dots \circ f_{b_2, b_1, j_2}^{-1} \circ f_{b_1, b_0, j_1}^{-1} \left(B_{b_0}(R/2) \setminus f^{-1}(\infty) \right) \right\},$$

where $b_i \in I : 1 \leq j_i \leq q_{b_i}, i = 0, 1, \dots, l$, is well-defined and covers $I_R(f)$. Applying (2.3) and (2.4) we may now estimate as follows.

$$\begin{aligned} m(I_R(f)) &\leq \\ &\leq \sum_{b_l \in I} \sum_{j_l=1}^{q_{b_l}} \dots \sum_{b_1 \in I} \sum_{j_1=1}^{q_{b_1}} \sum_{b_0 \in I} m \left(f_{b_l, b_{l-1}, j_l}^{-1} \circ f_{b_{l-1}, b_{l-2}, j_{l-1}}^{-1} \circ \dots \circ f_{b_2, b_1, j_2}^{-1} \circ f_{b_1, b_0, j_1}^{-1} \left(B_{b_0}(R/2) \right) \right) \\ &\leq \sum_{b_l \in I} \sum_{j_l=1}^{q_{b_l}} \dots \sum_{b_1 \in I} \sum_{j_1=1}^{q_{b_1}} \sum_{b_0 \in I} \left\| \left(f_{b_l, b_{l-1}, j_l}^{-1} \circ f_{b_{l-1}, b_{l-2}, j_{l-1}}^{-1} \circ \dots \circ f_{b_2, b_1, j_2}^{-1} \circ f_{b_1, b_0, j_1}^{-1} \right)^* \Big|_{B_{b_0}(R/2)} \right\|_{\infty}^t m \left(B_{b_0}(R/2) \right) \\ &\leq \sum_{b_l \in I} \sum_{j_l=1}^{q_{b_l}} \dots \sum_{b_1 \in I} \sum_{j_1=1}^{q_{b_1}} \sum_{b_0 \in I} M^{lt} \left(\frac{|b_{l-1}|^{\frac{q_{b_l}-1}{q_{b_l}}}}{|b_l|^2} \right)^t \cdot \left(\frac{|b_{l-2}|^{\frac{q_{b_{l-1}}-1}{q_{b_{l-1}}}}}{|b_{l-1}|^2} \right)^t \dots \left(\frac{|b_0|^{\frac{q_{b_1}-1}{q_{b_1}}}}{|b_1|^2} \right)^t (2q\Sigma_R)^t \frac{1}{|b_0|^{2t}} \\ &= (2q\Sigma_R)^t M^{lt} \sum_{b_l \in I} \sum_{j_l=1}^{q_{b_l}} \dots \sum_{b_1 \in I} \sum_{j_1=1}^{q_{b_1}} \sum_{b_0 \in I} |b_l|^{-2t} \left(|b_{l-1}|^{-\frac{q+1}{q}t} \dots |b_0|^{-\frac{q+1}{q}t} \right) \\ &\leq (2q\Sigma_R)^t M^{lt} \sum_{b_l \in I} \sum_{j_l=1}^{q_{b_l}} \dots \sum_{b_1 \in I} \sum_{j_1=1}^{q_{b_1}} \sum_{b_0 \in I} \left(|b_l|^{-\frac{q+1}{q}t} |b_{l-1}|^{-\frac{q+1}{q}t} \dots |b_0|^{-\frac{q+1}{q}t} \right) \\ &\leq (2q\Sigma_R)^t M^{lt} \left(\sum_{b \in I} |b|^{-\frac{q+1}{q}t} \right)^l q^l \\ &\leq (2q\Sigma_R)^t \left(qM^t \sum_{b \in B_{R_3} \cap f^{-1}(\infty)} |b|^{-\frac{q+1}{q}t} \right)^l \end{aligned}$$

Applying (3.5) we therefore get $m(I_R(f)) \leq (2q\Sigma_R)^t 2^{-l}$. Letting $l \rightarrow \infty$ we therefore get $m(I_R(f)) = 0$. Since $m \circ f^{-1} \ll m$ and since $\{z : \liminf_{n \rightarrow \infty} |f^n(z)| > R\} = \bigcup_{j=0}^{\infty} f^{-j}(I_R(f))$, we conclude that $m(\{z : \liminf_{n \rightarrow \infty} |f^n(z)| > R\}) = 0$. We are done. \blacksquare

Developing the general scheme from [7] we shall now prove in several steps the existence of an h -conformal measure. In order to begin we call $Y \subset \{\infty\} \cup \Omega(f) \cup \bigcup_{n \geq 1} f^n(\text{Crit}(J(f)))$ a crossing set if Y is finite and the following two conditions are satisfied.

- (y1) $\infty \in Y$.
- (y2) $Y \cap \{f^n(x) : n \geq 1\}$ is a singleton for all $x \in \text{Crit}(J(f))$.
- (y3) $Y \cap \text{Crit}(f) = \emptyset$.
- (y4) $\Omega(f) \subset Y$.

Since $f(\text{Crit}(f))$ is finite, crossing sets do exist. Let $V \subset \overline{\mathcal{C}}$ be an open neighbourhood of Y . We define

$$K(V) = \{z \in J(f) : f^n(z) \notin V \ \forall (n \geq 0)\}.$$

Obviously $f(K(V)) \subset K(V)$ and since $f : \mathcal{C} \rightarrow \overline{\mathcal{C}}$ is continuous and V is open, we see that $K(V)$ is a closed subset of \mathcal{C} . Since in addition $K(V) \subset \overline{\mathcal{C}} \setminus V$, we conclude that $K(V)$ is a compact subset of \mathcal{C} . Fix $w \in K(V)$ and $t \geq 0$. For all $n \geq 1$ consider the sets

$$E_n = \left(f|_{K(V)}\right)^{-n}(w)$$

and the number

$$c(f) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{x \in E_n} |(f^n)^*(x)|^{-t}.$$

Since the continuous map $f|_{K(V)} : K(V) \rightarrow K(V)$ has no critical points, all the sets $K(V)$ are (n, δ) -separated, where

$$\delta = \inf_{y \in K(V)} \{\min\{|z - x| : x, z \in (f|_{K(V)})^{-1}(y) \text{ and } x \neq z\}\} > 0.$$

Therefore

$$c(f) \leq P\left(f|_{K(V)}, -t \log |f^*|\right), \quad (3.6)$$

where the right-hand side of this inequality is the topological pressure of the potential $-t \log |f^*|$ with respect to the dynamical system $f|_{K(V)} : K(V) \rightarrow K(V)$. Denote this pressure simply by $P(f, V)$. We call a Borel set $A \subset \mathcal{C}$ special if $|_A$ is injective. Lemma 3.1 and 3.2 from [7] (comp. [6]) enlarged by the reasoning started from the second paragraph of the proof of Lemma 5.3 in [7] can be now formulated together as follows.

Lemma 3.3. *For every $t \geq 0$ there exists a Borel probability measure $m_{V,t}$ supported on $K(V)$ such that*

- (a) $m_{V,t}(f(A)) \geq \int_A e^{c(f)} |f^*|^t dm_{V,t}$ for every special set $A \subset \mathcal{C}$ and
- (b) $m_{V,t}(f(A)) = \int_A e^{c(f)} |f^*|^t dm_{V,t}$ for every special set $A \subset \mathcal{C} \setminus \overline{V}$.

We will need the following technical lemma.

Lemma 3.4. *The function $t \mapsto c(f)$ is continuous, $c(0) > 0$ and $c^{-1}(0) \cap (0, h] \neq \emptyset$ if V has a sufficiently small diameter.*

Proof. Continuity of the function $c(f)$ follows from the fact that $0 < \inf_{K(V)}\{|f^*|\} \leq \sup_{K(V)}\{|f^*|\} < \infty$. Since periodic points of f are dense in $J(f)$, $K(V) \neq \emptyset$ for all V sufficiently small. Also if V is sufficiently small and $w \in K(V)$, then $\#E_n \geq 2^n$ and consequently $c(0) \geq \log 2 > 0$. Since $c(0) > 0$ and since the function $c(f)$ is continuous, in order to prove the last claim of our lemma, it suffices to show that $c(f) \leq 0$ for all $t \geq h$. So, suppose on the contrary that $c(f) > 0$ for some $t \geq h$. It follows from (3.6) that

$$P(f, V) > 0. \quad (3.7)$$

Since the proof of Lemma 4.1 and Corollary 4.2 from [7] go word by word in our context, we conclude that the Lyapunov exponent $\chi_\mu = \int \log |f^*| d\mu \geq 0$ for every Borel probability f -invariant measure μ supported on $K(V)$. It follows from (3.7) and the variational principle for topological pressure that there exists a Borel probability f -invariant measure μ supported on $K(V)$ such that $h_\mu(f) - t\chi_\mu > 0$. Since $\chi_\mu \geq 0$, this implies that $h_\mu(f) > 0$ and due to Ruelle's inequality $\chi_\mu > 0$. Hence, applying Przytycki's-Manne volume lemma (see [18], comp. [14]), we can write

$$t < \frac{h_\mu(f)}{\chi_\mu} = \text{HD}(\mu) \leq h$$

and this contradiction finishes the proof. ■

Let

$$s(V) = \min\{c^{-1}(0) \cap (0, h]\} > 0.$$

Combining Lemma 3.3 and Lemma 3.4 we get the following.

Lemma 3.5. *There exists a Borel probability measure m_V supported on $K(V)$ such that*

- (a) $m_V(f(A)) \geq \int_A |f^*|^{s(V)} dm_V$ for every special set $A \subset \mathcal{C}$ and
- (b) $m_V(f(A)) = \int_A |f^*|^{s(V)} dm_V$ for every special set $A \subset \mathcal{C} \setminus \overline{V}$.

Since the sequence $n \mapsto s(B_s(Y, 1/n))$ is monotonically non-decreasing, proceeding similarly as in the proof of Lemma 5.4 from [7] (note that in the place where Lemma 3.3 from [7] is invoked, only the first inequality in (d) is needed; in particular $m_Y(\infty) = 0$, where m_Y is an arbitrary weak accumulation point of the sequence $m_{B_s(Y, 1/n)}$ we obtained the following.

Lemma 3.6. *For every $s(Y)$, an accumulation point of the sequence $s(B_s(Y, 1/n))$, $s(Y) \in (0, h]$ and there exists a Borel probability measure m_Y (an appropriate weak accumulation point of the sequence $\{m_{B_s(Y, 1/n)}\}_{n \geq 1}$) supported on $J(f)$ such that*

- (a) $m_Y(f(A)) \geq \int_A |f^*|^{s(Y)} dm_Y$ for every special set $A \subset \mathcal{C}$ and
- (b) $m_Y(f(A)) = \int_A |f^*|^{s(Y)} dm_Y$ for every special set $A \subset \mathcal{C} \setminus Y$.

The next fact proven in this section is provided by the following.

Lemma 3.7. *For every crossing set Y , $m = m_Y$ is an $s(Y)$ -conformal measure for $f : J(f) \rightarrow J(f) \cup \{\infty\}$, $s(Y) = h$, and all atoms of m are contained in $I_-(f) \cup \bigcup_{n \geq 0} f^{-n}(\text{Crit}(J(f)))$.*

Proof. Since we already know that $m(\infty) = 0$ and since $Y \cap (\text{Sing}^-(f) \cup I_\infty(f)) \subset \Omega(f) \cup \{\infty\}$, it follows from Lemma 3.6(b) and Corollary 2.22 that

$$m(Y \setminus \Omega(f)) = 0. \quad (3.8)$$

We shall show now that $m(\Omega(f)) = 0$. And indeed, fix $\omega \in \Omega(f)$. Take $a \geq 1$ so large that $f^a(\omega) = \omega$ and $(f^a)'(\omega) = 1$. It then follows from (2.13) that there exist a compact set $F_\omega \subset B(\omega, \theta) \setminus \{\omega\}$ and a constant $C \geq 1$ such that for every $k \geq 1$

$$C^{-1}k^{-\frac{p(\omega)+1}{p(\omega)}} \leq |(f_\omega^{-ak})^*(z)| \leq Ck^{-\frac{p(\omega)+1}{p(\omega)}} \quad (3.9)$$

and for every $n \geq 1$ there exists $k_n \geq 1$ such that

$$B(\omega, 1/n) \subset \bigcup_{j=k_n}^{\infty} f_\omega^{-aj}(F_\omega) \quad \text{and} \quad \lim_{n \rightarrow \infty} k_n = \infty. \quad (3.10)$$

It follows from Lemma 3.6(b), (3.9) and the fact that the family $\{f_\omega^{-an}(F_\omega)\}_{n \geq 1}$ is of bounded multiplicity, that

$$\sum_{n \geq 1} n^{-\frac{p(\omega)+1}{p(\omega)}s(Y)} < \infty.$$

In particular $\frac{p(\omega)+1}{p(\omega)}s(Y) > 1$. Denote $m|_{B_s(Y, 1/n)}$ by m_n and $s(B_s(Y, 1/n))$ by s_n . Since $\lim_{n \rightarrow \infty} s_n = s(Y)$, we see that for every $n \geq 1$ large enough, say $n \geq n_0$,

$$\frac{p(\omega)+1}{p(\omega)}s_n > 1 + \sigma.$$

for some $\sigma > 0$. It therefore follows from Lemma 3.6(a), (3.10) and (3.9) that for all $n \geq n_0$ and all $l \geq 1$

$$\begin{aligned} m_n(B(\omega, 1/l)) &\leq \sum_{j=k_l}^{\infty} m_n(f_\omega^{-aj}(F_\omega)) \leq C^{\frac{p(\omega)+1}{p(\omega)}s_n} \sum_{j=k_l}^{\infty} j^{-\frac{p(\omega)+1}{p(\omega)}s_n} \\ &\leq C^{\frac{p(\omega)+1}{p(\omega)}s(Y)} \sum_{j=k_l}^{\infty} j^{-(1+\sigma)}. \end{aligned}$$

Consequently

$$m(B(\omega, 1/l)) \leq C^{\frac{p(\omega)+1}{p(\omega)}s(Y)} \sum_{j=k_l}^{\infty} j^{-(1+\sigma)}.$$

Since $\lim_{l \rightarrow \infty} k_l = \infty$, we infer

$$m(\Omega(f)) = 0.$$

Combining this and (3.8), we see that $m(Y) = 0$. Since $f(\Omega(f)) = \Omega(f)$, in order to prove $s(Y)$ -conformality of the measure m , it therefore suffices to show that $m(f(Y \setminus \Omega(f))) =$

0. But if $y \in Y \setminus (\Omega(f) \cup \{\infty\})$, then due to our definition of Y , $y \notin \text{Sing}^-(f)$ and the formula $m(f(y)) = 0$ immediately follows from Corollary 2.22, the formula $m(f^n(f(y))) \geq |(f^n)^*(y)|^{s(Y)} m(f(y))$ and the stated in Lemma 3.6 fact that $s(Y) > 0$. Thus the $s(Y)$ -conformality of m is proven and in addition all the atoms of m must be contained in $J(f) \setminus \Omega$. In view of Lemma 3.6 and Lemma 3.1, $s(Y) = h$. Applying now Lemma 3.2 and Corollary 2.22 we see that all atoms of m must be contained in $I_-(f) \cup \bigcup_{n \geq 0} f^{-n}(\text{Crit}(J(f)))$. The proof is complete. ■

4. HAUSDORFF AND PACKING MEASURES

Let Π^h denote the packing measure considered with respect to the spherical metric on $\overline{\mathcal{C}}$. We shall prove in this section that the conformal measure m is atomless and the following main result.

Theorem 4.1. *Let $f : \mathcal{C} \rightarrow \overline{\mathcal{C}}$ be a non-recurrent elliptic function. If $h = \text{HD}(J(f)) = 2$, then $J(f) = \mathcal{C}$. So suppose that $h < 2$. Then*

- (a) $\text{H}^h(J(f)) = 0$.
- (b) $\Pi^h(J(f)) > 0$.
- (c) $\Pi^h(J(f)) = \infty$ if and only if $\Omega(f) \neq \emptyset$.

As an immediate consequence of this theorem we get the following.

Corollary 4.2. *If $\Omega(f) = \emptyset$, then the Euclidean h -dimensional packing measure Π_e^h is finite on each bounded subset of $J(f)$.*

4.1. Preliminaries from Geometric Measure Theory. In this section we collect some facts from the geometric measure theory as well as we list without proofs some more technical facts taken from Section 2, Section 3 and Section 4 of [21]. Given a subset A of a metric space (X, d) , a countable family $\{B(x_i, r_i)\}_{i=1}^\infty$ of open balls centered at the set A is said to be a packing of A if and only if for any pair $i \neq j$

$$d(x_i, x_j) > r_i + r_j.$$

Given $t \geq 0$, the t -dimensional outer Hausdorff measure $\text{H}^t(A)$ of the set A is defined as

$$\text{H}^t(A) = \sup_{\epsilon > 0} \inf \left\{ \sum_{i=1}^\infty r_i^t \right\}$$

where infimum is taken over all covers $\{B(x_i, r_i)\}_{i=1}^\infty$ of the set A by open balls centered at A with radii which do not exceed ϵ .

The t -dimensional outer packing measure $\Pi^t(A)$ of the set A is defined as

$$\Pi^t(A) = \inf_{\cup A_i = A} \left\{ \sum_i \Pi_*^t(A_i) \right\}$$

(A_i are arbitrary subsets of A), where

$$\Pi_*^t(A) = \sup_{\epsilon > 0} \sup \left\{ \sum_{i=1}^{\infty} r_i^t \right\}.$$

Here the second supremum is taken over all packings $\{B(x_i, r_i)\}_{i=1}^{\infty}$ of the set A by open balls centered at A with radii which do not exceed ϵ . These two outer measures define countable additive measures on Borel σ -algebra of X .

The definition of the Hausdorff dimension $\text{HD}(A)$ of A is the following

$$\text{HD}(A) = \inf\{t : \text{H}^t(A) = 0\} = \sup\{t : \text{H}^t(A) = \infty\}.$$

Let ν be a Borel probability measure on X which is positive on open sets. Define the function $\rho = \rho_t(\nu) : X \times (0, \infty) \rightarrow (0, \infty)$ by

$$\rho(x, r) = \frac{\nu(B(x, r))}{r^t}$$

The following two theorems (see [DU5]) are for our aims the key facts from geometric measure theory. Their proofs are an easy consequence of Besicovič covering theorem (see [G]).

Theorem 4.3. *Let $X = \mathbb{R}^d$ for some $d \geq 1$. Then there exists a constant $b(n)$ depending only on n with the following properties. If A is a Borel subset of \mathbb{R}^d and $C > 0$ is a positive constant such that*

- (1) *for all (but countably many) $x \in A$*

$$\limsup_{r \rightarrow 0} \rho(x, r) \geq C^{-1},$$

then for every Borel subset $E \subset A$ we have $\text{H}^t(E) \leq b(n)C\nu(E)$ and, in particular, $\text{H}^t(A) < \infty$.

or

- (2) *for all $x \in A$*

$$\limsup_{r \rightarrow 0} \rho(x, r) \leq C^{-1},$$

then for every Borel subset $E \subset A$ we have $\text{H}^t(E) \geq C\nu(E)$.

Theorem 4.4. *Let $X = \mathbb{R}^d$ for some $d \geq 1$. Then there exists a constant $b(n)$ depending only on n with the following properties. If A is a Borel subset of \mathbb{R}^d and $C > 0$ is a positive constant such that*

(1) for all $x \in A$

$$\liminf_{r \rightarrow 0} \rho(x, r) \leq C^{-1},$$

then for every Borel subset $E \subset A$ we have $\Pi^t(E) \geq Cb(n)^{-1}\nu(E)$,

or

(2) for all $x \in A$

$$\liminf_{r \rightarrow 0} \rho(x, r) \geq C^{-1},$$

then $\Pi^t(E) \leq C\nu(E)$ and, consequently, $\Pi^t(A) < \infty$.

(1') If ν is non-atomic then (1) holds under the weaker assumption that the hypothesis of part (1) is satisfied on the complement of a countable set.

Assume now that ν is a Borel measure on \mathcal{C} finite on bounded sets. These two theorems motivated us in [21] to introduce the following notions.

Definition 4.5. Given $r > 0$ and $L > 0$ a point $x \in \mathcal{C}$ is said to be (r, L) -*t.upper estimable* if $\rho(x, r) \leq L$ and is said to be (r, L) -*t.lower estimable* if $\rho(x, r) \geq L$. We will frequently abbreviate the notation writing (r, L) -*u.e.* for (r, L) -*t.upper estimable* and (r, L) -*l.e.* for (r, L) -*t.lower estimable*. We also say that the point x is *t-upper estimable* (*t-lower estimable*) if it is (r, L) -*t.upper estimable* ((r, L) -*t.lower estimable*) for some $L > 0$ and all $r > 0$ sufficiently small.

We will also need the following more technical notion.

Definition 4.6. Given $r > 0$, $\sigma > 0$ and $L > 0$ the point $x \in X$ is said to be (r, σ, L) -*t.strongly lower estimable*, or shorter (r, σ, L) -*s.l.e.* if $\nu(B(y, \sigma r)) \geq Lr^t$ for every $y \in B(x, r)$.

We collect now from [21] the technical facts about the notions defined above.

Lemma 4.7. If z is (r, σ, L) -*s.l.e.*, then every point $x \in B(z, r/2)$ is $(r/2, 2\sigma, 2^t L)$ -*s.l.e.*.

Lemma 4.8. If x is (r, σ, L) -*s.l.e.*, then for every $0 < u \leq 1$ it is $(ur, \sigma/u, Lu^{-t})$ -*s.l.e.*.

Lemma 4.9. If ν is positive on nonempty open sets, then for every $r > 0$ there exists $E(r) \geq 1$ such that every point $x \in X$ is $(r, E(r))$ -*u.e.* and $(r, E(r)^{-1})$ -*l.e.*.

Passing to conformal maps we consider now the situation where $H : U_1 \rightarrow U_2$ is an analytic map of open subsets U_1, U_2 of the complex plane \mathcal{C} . We say that given $t \geq 0$, the Borel measure ν finite on bounded sets of \mathcal{C} is a Euclidean semi t -conformal if and only if

$$\nu(H(A)) \geq \int_A |H'|^t d\nu$$

for every Borel subset A of U_1 such that $H|_A$ is one-to-one and is call t -conformal if the “ \geq ” sign can be replaced by an “ $=$ ” sign.

Lemma 4.10. *Let ν be a Euclidean semi t -conformal measure. Suppose that $D \subset \mathcal{C}$ is an open set, $z \in D$ and $H : D \rightarrow \mathcal{C}$ is an analytic map which has an analytic inverse H_z^{-1} defined on $B(H(z), 2R)$ for some $R > 0$. Then for every $0 \leq r \leq R$*

$$K^{-t}\nu(B(z, K^{-1}r|H'(z)|^{-1})) \leq |H'(z)|^{-t}\nu((B(H(z), r))).$$

If, in addition, ν is t -conformal, then also

$$|H'(z)|^{-t}\nu((B(H(z), r))) \leq K^t\nu(B(z, Kr|H'(z)|^{-1})).$$

Lemma 4.11. *Suppose that ν is a Euclidean t -conformal measure. If the point $H(z)$ is (r, σ, L) -s.l.e., where $r \leq R/2$ and $\sigma \leq 1$, then the point z is $(K^{-1}|H'(z)|^{-1}r, K^2\sigma, L)$ -s.l.e..*

Lemma 4.12. *Suppose that ν is a Euclidean t -conformal measure. Let c be a critical point of an analytic map $H : D \rightarrow \mathcal{C}$. If $0 < r \leq R(H, c)$ and $H(c)$ is (r, L) -l.e., then c is $((Ar)^{1/q}, A^{-2t}L)$ -l.e..*

Lemma 4.13. *Let c be a critical point of an analytic map $H : D \rightarrow \mathcal{C}$. Let ν be a Euclidean semi t -conformal measure such that $\nu(c) = 0$. If $0 < r \leq R(H, c)$ and $H(c)$ is (s, L) -u.e. for all $0 < s \leq r$, then c is $((A^{-1}r)^{1/q}, q(2A^2)^t(2^{t/q} - 1)^{-1}L)$ -u.e..*

Note that the proof of this lemma is the same as the proof of Lemma 3.4 in [21]. The only modification is that the equality sign in the first line of the first displayed formula of this proof is to be replaced by the “ \geq ” sign.

Lemma 4.14. *Suppose that ν is a Euclidean t -conformal measure. Let c be a critical point of an analytic map $H : D \rightarrow \overline{\mathcal{C}}$. If $0 < r \leq \frac{1}{3}R(H, c)$, $0 < \sigma \leq 1$ and $H(c)$ is (r, σ, L) -s.l.e, then c is $((A^{-1}r)^{1/q}, \tilde{\sigma}, \tilde{L})$ -s.l.e, where $\tilde{\sigma} = (2^{q+1}KA^2\sigma)^{1/q}$ and $\tilde{L} = L \min\{K^{-t}, (A^2\sigma)^{\frac{1-q}{q}t}\}$.*

Notice now that if m is a semi t -conformal measure for $f : J(f) \rightarrow J(f) \cup \{\infty\}$, then the measure $m_e = (1 + |z|^2)^t m$ is Euclidean semi t -conformal, i.e.

$$m_e(f(A)) \geq \int_A |f'|^t dm_e$$

for every Borel set $A \subset J(f)$ such that $f|_A$ is 1-to-1. If m is t -conformal, then so is m_e in the obvious sense. The measure m_e is called the Euclidean version of m . Obviously m_e is equivalent to m and is finite on bounded subsets of \mathcal{C} . From now on throughout the entire paper we fix a crossing set Y and we consider an open neighbourhood $V \subset \overline{\mathcal{C}}$ of Y such that $\text{Crit}(f) \cap \overline{V} = \emptyset$ and the closure of V is disjoint from at least one fundamental parallelogram of f . A semi t -conformal measure m is said to be almost t -conformal if

$$m(f(A)) = \int_A |f'|^t dm$$

for every Borel set $A \subset J(f)$ such that $f|_A$ is 1-to-1 and $A \cap \overline{V} = \emptyset$. Hence for every Borel set A such that $f|_A$ is 1-to-1 and $A \cap \overline{V} = \emptyset$ and for every $w \in \Lambda$, we have

$$\int_A |f'|^t dm_e = m_e(f(A)) = m_e(f(A+w)) \geq \int_{A+w} |f'|^t dm_e$$

and the last inequality sign becomes an equality either if in addition $(A+w) \cap \overline{V} = \emptyset$ or if m is a t -conformal measure and we assume only that $f|_A$ is 1-to-1. Since f' is periodic with respect to the lattice Λ , all the above statements and assumptions lead to the following.

Lemma 4.15. *For every $w \in \Lambda$, every Borel set $A \subset \mathcal{C}$ such that $A \cap \overline{V} = \emptyset$ and every almost t -conformal measure m*

$$m_e(A+w) \leq m_e(A).$$

If either in addition $(A+w) \cap \overline{V} = \emptyset$ or if m is h -conformal and we assume only that $f|_A$ is 1-to-1, then this inequality becomes an equality. For every $r > 0$ there exists $M(r) \in (0, \infty)$ independent of any almost t -conformal measure m such that

$$m_e(F) \leq M(r). \tag{4.1}$$

for every Borel set $F \subset \mathcal{C}$ with the diameter $\leq r$. If in addition m is h -conformal, then for every $R > 0$ there exist constants $Q(R)$ and $Q_h(R)$ such that

$$m_e(B_e(x, r)) \geq Q(R)r^2 \geq Q_h(R)r^h \tag{4.2}$$

for all $x \in J(f)$ and all $r \geq R$.

The following lemma is proven in the same way as the corresponding lemma from Section 4 of [21].

Lemma 4.16. *Suppose that m_e is a Euclidean t -conformal measure. Then for every $R > 0$ and every $0 < \sigma \leq 1$ there exists $L = L(\omega, R, \sigma) > 0$ such that for every $0 < r \leq R$ every point $\omega \in \Omega(f)$ is (r, σ, L) - $\alpha_t(\omega)$.s.l.e. with respect to the measure m_e .*

4.2. Conformal Measure and Holomorphic Inverse Branches.

In this subsection we prove two technical propositions modeled on Proposition 6.3 and Proposition 6.4 from [21]. Let m be an almost t -conformal measure and let m_e be its Euclidean version. The upper estimability and strongly lower estimability will be considered in this section with respect to the measure m_e . When we speak about lower estimability we assume more, that the measure m is t -conformal. Since the number of parabolic points is finite, passing to an appropriate iteration, we assume in this and the next section without losing generality that all parabolic points of f are simple. Fix a forward f -invariant compact subset F of \mathcal{C} . Put

$$\|f'\|_F = \sup\{|f'(z)| : z \in F\}.$$

Recall that θ was defined in (2.11) and that $\tau > 0$ is so small as required in Lemma 2.3.

Proposition 4.17. *Fix a forward f -invariant compact subset F of \mathcal{C} . Let $z \in F$, $\lambda > 0$ and let $0 < r \leq \tau\theta\|f'\|_F^{-1}\lambda^{-1}$ be a real number. Suppose that at least one of the following two conditions is satisfied:*

$$z \in F \setminus \bigcup_{n \geq 0} f^{-n}(\text{Crit}(J(f)))$$

or

$$z \in F \quad \text{and} \quad r > \tau\theta\|f'\|_F^{-1}\lambda^{-1} \inf\{|(f^n)'(z)|^{-1} : n = 1, 2, \dots\}.$$

Then there exists an integer $u = u(\lambda, r, z) \geq 0$ such that $r|(f^u)'(z)| \leq \lambda^{-1}\theta\tau$ and the following four conditions are satisfied

$$\text{diam}\left(\text{Comp}(f^j(z), f^u(z), f^{u-j}, r|(f^u)'(z)|)\right) \leq \beta \tag{4.3}$$

for every $j = 0, 1, \dots, u$. For every $\eta > 0$ there exists a continuous function $t \mapsto B_t = B_t(\lambda, \eta) > 0$, $t \in [0, \infty)$, (independent of z , n , and r) and such that if $f^u(z) \in B(\omega, \theta)$ for some $\omega \in \Omega(f)$, then

$$f^u(z) \text{ is } (\eta r|(f^u)'(z)|, B_t) - \alpha_t(\omega).u.e. \tag{4.4}$$

and there exists a function $W_t = W_t(\lambda, \eta) : (0, 1] \rightarrow (0, 1]$ (independent of z , n , and r) such that if $f^u(z) \in B(\omega, \theta)$ for some $\omega \in \Omega(f)$, then for every $\sigma \in (0, 1]$

$$f^u(z) \text{ is } (\eta r|(f^u)'(z)|, \sigma, W_t(\sigma)) - \alpha_t(\omega).s.l.e. \tag{4.5}$$

If $f^u(z) \notin B(\Omega(f), \theta)$, then formulas (4.4) and (4.5) are also true with $\alpha_t(\omega)$ replaced by t . (4.6)

Proof. Suppose first that $\sup\{\lambda r|(f^j)'(z)| : j \geq 1\} > \theta\tau\|f'\|_F^{-1}$ and let $n = n(\lambda, z, r) \geq 0$ be a minimal integer such that

$$\lambda r|(f^n)'(z)| > \theta\tau\|f'\|_F^{-1}. \quad (4.7)$$

Then $n \geq 1$ (due to the assumption imposed on r) and also

$$\lambda r|(f^n)'(z)| \leq \theta\tau \quad (4.8)$$

If $f^n(z) \notin B(\Omega(f), \theta)$ set $u = u(\lambda, r, z) = n$. The items (4.4), (4.5) and (4.6) are obvious in view of our assumptions imposed on F .

So suppose that $f^n(z) \in B(\Omega(f), \theta)$, say $f^n(z) \in B(\omega, \theta)$, $\omega \in \Omega(f)$. Let $0 \leq k = k(\lambda, z, r) \leq n$ be the smallest integer such that $f^j(z) \in B(\Omega(f), \theta)$ for every $j = k, k+1, \dots, n$. Consider all the numbers

$$r_i = |f^i(z) - \omega| |(f^i)'(z)|^{-1}$$

where $i = k, k+1, \dots, n$. By (4.7) we have

$$r_n = |f^n(z) - \omega| |(f^n)'(z)|^{-1} \leq \theta\|f'\|_F \theta^{-1} \tau^{-1} \lambda r = \|f'\|_F \tau^{-1} \lambda r$$

and therefore there exists a minimal $k \leq u = u(\lambda, r, z) \leq n$ such that $r_u \leq \|f'\|_F \tau^{-1} \lambda r$. In other words

$$|f^u(z) - \omega| \leq \|f'\|_F \tau^{-1} \lambda r |(f^u)'(z)| \leq \|f'\|_F \tau^{-1} \eta^{-1} \lambda \eta r |(f^u)'(z)| \quad (4.9)$$

If $\sup\{\lambda r|(f^j)'(z)| : j \geq 1\} \leq \theta\tau\|f'\|_F^{-1}$, then it follows from Corollary 2.22 that $z \in \bigcup_{j \geq 0} f^{-j}(\Omega(f))$. Define then $u(\lambda, z, r) = k(\lambda, z, r)$ to be the minimal integer $j \geq 0$ such that $f^j(z) \in \Omega(f)$ and put $\omega = f^u(z)$. Notice that in this case formulas (4.8) and (4.9) are also satisfied. Our further considerations are valid in both cases. First note that by (4.9) we have

$$B(f^u(z), \eta r |(f^u)'(z)|) \subset B(\omega, (1 + \|f'\|_F \tau^{-1} \eta^{-1} \lambda) \eta r |(f^u)'(z)|) \quad (4.10)$$

and in view of Lemma 2.5 and (4.8)

$$\begin{aligned} m_e(B(f^u(z), \eta r |(f^u)'(z)|)) &\leq \\ &\leq C(\omega, (1 + \|f'\|_F \tau^{-1} \eta^{-1} \lambda) \theta \tau \eta \lambda^{-1}) (1 + \|f'\|_F \tau^{-1} \eta^{-1} \lambda)^{\alpha_t(\omega)} (\eta r |(f^u)'(z)|)^{\alpha_t(\omega)} \end{aligned}$$

So, item (4.4) is proved. Also applying (4.9), Lemma 4.16, Lemma 4.7 and (4.8) we see that the point $f^u(z)$ is

$$\left(\|f'\|_F \tau^{-1} \lambda r |(f^u)'(z)|, \sigma \tau \|f'\|_F^{-1} \eta \lambda^{-1}, 2^{\alpha_t(\omega)} L(\omega, 2\|f'\|_F \theta, \sigma \tau (2\|f'\|_F)^{-1} \eta \lambda^{-1}) \right)\text{-s.l.e.}$$

So, if $\|f'\|_F \tau^{-1} \lambda \geq \eta$, then by Lemma 4.8, $f^u(z)$ is

$$\left(\eta r |(f^u)'(z)|, \sigma, (2\|f'\|_F \tau^{-1} \lambda \eta^{-1})^{\alpha_t(\omega)} L(\omega, 2\|f'\|_F \theta, \sigma \tau (2\|f'\|_F)^{-1} \eta \lambda^{-1}) \right)\text{-s.l.e}$$

If instead $\|f'\|_F \tau^{-1} \lambda \leq \eta$, then again it follows from (4.9), Lemma 4.16, Lemma 4.7 and (4.8) that the point $f^u(z)$ is $\left(\eta r |(f^u)'(z)|, \sigma, 2^{\alpha_t(\omega)} L(\omega, 2\theta \tau \lambda^{-1} \eta, \sigma/2) \right)\text{-s.l.e.}$. So, part (4.5) is also proved.

In order to prove (4.3) suppose first that $u = k$. In particular this is the case if $z \in \bigcup_{j \geq 0} f^{-j}(\Omega(f))$. Then

$$\text{Comp}(f^{k-1}(z), f^k(z), f, r|(f^u)'(z)|) \subset \text{Comp}(f^{k-1}(z), f^k(z), f, \theta\tau)$$

and by the choice of k and (2.9) we have $f^{k-1}(z) \notin B(\Omega(f), \theta)$. Therefore (4.3) follows from the choice of τ (see (2.17)) and (2.16).

If $u > k$ (so the first case holds), then $r_{u-1} > \|f'\|_F \tau^{-1} \lambda r$ and by (2.16) we get

$$r_u = \frac{|f^u(z) - \omega|}{|f^{u-1}(z) - \omega|} |f'(f^{u-1}(z))|^{-1} r_{u-1} \geq \|f\|^{-1} r_{u-1} \geq \tau^{-1} \lambda r.$$

So, $\lambda r |(f^u)'(z)| \leq \tau |f^u(z) - \omega|$ and applying Lemma 2.4 and (2.9) $u - k$ times we conclude that for every $k \leq j \leq u$

$$\text{diam}\left(\text{Comp}(f^j(z), f^u(z), f^{u-j}, \lambda r |(f^u)'(z)|)\right) \leq \theta\tau < \beta$$

And now for $j = k - 1, k - 2, \dots, 1, 0$, the same argument applies as in the case $u = k$. ■

Proposition 4.18. *Fix a forward f -invariant compact subset F of \mathcal{C} . Let ϵ and λ be both positive numbers such that $\epsilon < \lambda \min\{1, \tau^{-1}, \theta^{-1} \tau^{-1} \gamma\}$. If $0 < r < \tau \theta \|f'\|_F^{-1} \lambda^{-1}$ and $z \in F \setminus \text{Crit}(J(f))$, then there exists an integer $s = s(\lambda, \epsilon, r, z) \geq 1$ with the following three properties.*

$$|(f^s)'(z)| \neq 0. \tag{4.11}$$

If $u = u(\lambda, r, z)$ is well-defined, then $s \leq u(\lambda, r, z)$. If either u is not defined or $s < u$, then there exists a critical point $c \in \text{Crit}(f)$ such that

$$|f^s(z) - c| \leq \epsilon r |(f^s)'(z)|. \tag{4.12}$$

In any case

$$\text{Comp}\left(z, f^s(z), f^s, (KA^2)^{-1} 2^{-\#\text{Crit}(f)} \epsilon r |(f^s)'(z)|\right) \cap \text{Crit}(f^s) = \emptyset. \tag{4.13}$$

Proof. Since $z \notin \text{Crit}(f)$ and in view of Proposition 4.17, there exists a minimal number $s = s(\lambda, \epsilon, r, z)$ for which at least one of the following two conditions is satisfied

$$|f^s(z) - c| \leq \epsilon r |(f^s)'(z)| \tag{4.14}$$

for some $c \in \text{Crit}(J(f))$ or

$$u(\lambda, r, z) \text{ is well-defined and } s(\lambda, \epsilon, r, z) = u(\lambda, r, z) \tag{4.15}$$

Since $|(f^s)'(z)| \neq 0$, the parts (4.11) and (4.12) are proved.

In order to prove (4.13) notice first that no matter which of the two numbers s is, in view of Proposition 4.17 we always have

$$\epsilon r |(f^s)'(z)| \leq \epsilon \lambda^{-1} \theta \tau \tag{4.16}$$

Let us now argue that for every $0 \leq j \leq s$

$$\text{diam}\left(\text{Comp}(f^{s-j}(z), f^s(z), f^j, \epsilon r |(f^s)'(z)|)\right) \leq \beta \quad (4.17)$$

Indeed, if $s = u$, it follows immediately from Proposition 4.17 and (4.3) since $\epsilon \leq \lambda$. Otherwise $|f^s(z) - c| \leq \epsilon r |(f^s)'(z)| \leq \epsilon \lambda^{-1} \theta \tau < \theta$ and therefore, by (2.14), $f^s(z) \notin B(\Omega(f), \theta)$. Thus (4.17) follows from (2.16).

Now by (4.17) and (Lemma 2.8), there exists $0 \leq p \leq \#(\text{Crit}(f))$, an increasing sequence of integers $1 \leq k_1 < k_2 < \dots < k_p \leq s$ and mutually distinct critical points c_1, c_2, \dots, c_p of f such that

$$\{c_l\} = \text{Comp}(f^{s-k_l}(z), f^s(z), f^{k_l}, \epsilon r |(f^s)'(z)|) \cap \text{Crit}(f). \quad (4.18)$$

for every $l = 1, 2, \dots, p$ and if $j \notin \{k_1, k_2, \dots, k_p\}$, then

$$\text{Comp}(f^{s-j}(z), f^s(z), f^j, \epsilon r |(f^s)'(z)|) \cap \text{Crit}(f) = \emptyset. \quad (4.19)$$

Setting $k_0 = 0$ we shall show by induction that for every $0 \leq l \leq p$

$$\text{Comp}(f^{s-k_l}(z), f^s(z), f^{k_l}, (KA^2)^{-1} 2^{-l} \epsilon r |(f^s)'(z)|) \cap \text{Crit}(f^{k_l}) = \emptyset. \quad (4.20)$$

Indeed, for $l = 0$ there is nothing to prove. So, suppose that (4.20) is true for some $0 \leq l \leq p - 1$. Then by (4.19)

$$\text{Comp}(f^{s-(k_{l+1}-1)}(z), f^s(z), f^{k_{l+1}-1}, (KA^2)^{-1} 2^{-l} \epsilon r |(f^s)'(z)|) \cap \text{Crit}(f^{k_{l+1}-1}) = \emptyset.$$

So, if

$$c_{l+1} \in \text{Comp}(f^{s-k_{l+1}}(z), f^s(z), f^{k_{l+1}}, (KA^2)^{-1} 2^{-(l+1)} \epsilon r |(f^s)'(z)|)$$

then by Lemma 1.4 applied for holomorphic maps $H = f$, $Q = f^{k_{l+1}-1}$ and the radius $R = (KA^2)^{-1} 2^{-(l+1)} \epsilon r |(f^s)'(z)| < \gamma$ we get

$$\begin{aligned} |f^{s-k_{l+1}}(z) - c_{l+1}| &\leq KA^2 |(f^{k_{l+1}})'(f^{s-k_{l+1}}(z))|^{-1} (KA^2)^{-1} 2^{-(l+1)} \epsilon r |(f^s)'(z)| \\ &= 2^{-(l+1)} \epsilon r |(f^{s-k_{l+1}}(z))'| \\ &\leq \epsilon r |(f^{s-k_{l+1}}(z))'| \end{aligned}$$

which contradicts the definition of s and proves (4.20) for $l + 1$. In particular it follows from (4.20) that

$$\text{Comp}(z, f^s(z), f^s, (KA^2)^{-1} 2^{-\#(\text{Crit}(f))} \epsilon r |(f^s)'(z)|) \cap \text{Crit}(f^s) = \emptyset$$

The proof is finished. \blacksquare

4.3. Hausdorff and Conformal Measure.

Let m be a Borel probability measure on \mathcal{C} and let m_e be its Euclidean version, i.e. $\frac{dm_e}{dm}(z) = (1 + |z|^2)^t$. We will need in this and the next section the following.

Lemma 4.19. *If $z \in J(f)$, $r_n \searrow 0$ and $M = \lim_{n \rightarrow \infty} r_n^{-t} m_e(B(z, r_n))$, then*

$$\limsup_{n \rightarrow \infty} \frac{m(B_s(z, (2(1 + |z|^2))^{-1} r_n))}{((2(1 + |z|^2))^{-1} r_n)^t} \leq 2^t M$$

and

$$\liminf_{n \rightarrow \infty} \frac{m(B_s(z, 2(1 + |z|^2)^{-1} r_n))}{(2(1 + |z|^2)^{-1} r_n)^t} \geq 2^{-t} M$$

Proof. Since for every $r > 0$ sufficiently small

$$B(z, 2^{-1}(1 + |z|^2)r) \subset B_s(z, r) \subset B(z, 2(1 + |z|^2)r)$$

and since

$$\lim_{r \searrow 0} \frac{m_e(B(z, r))}{m(B(z, r))} = (1 + |z|^2)^t,$$

we get

$$\limsup_{n \rightarrow \infty} \frac{m(B_s(z, (2(1 + |z|^2))^{-1} r_n))}{((2(1 + |z|^2))^{-1} r_n)^t} \leq \lim_{n \rightarrow \infty} \frac{m(B(z, r_n))}{2^{-t}(1 + |z|^2)^{-t} r_n^t} = 2^t M$$

and

$$\liminf_{n \rightarrow \infty} \frac{m(B_s(z, 2(1 + |z|^2)^{-1} r_n))}{(2(1 + |z|^2)^{-1} r_n)^t} \geq \lim_{n \rightarrow \infty} \frac{m(B(z, r_n))}{2^t(1 + |z|^2)^{-t} r_n^t} = 2^{-t} M.$$

We are done. ■

Our first goal is to show that the h -conformal measure m proven to exist in Lemma 3.7 is atomless and that $H^h(J(f)) = 0$. We will consider almost t -conformal measures ν with $t \geq 1$. The notion of upper estimability introduced in Definition 4.5 is considered with respect to the Euclidean almost t -conformal measure ν_e . Recall that $l = l(f) \geq 1$ is the integer claimed in Lemma 2.20 and put

$$\begin{aligned} R_l(f) &= \inf\{R(f^j, c) : c \in \text{Crit}(f) \text{ and } 1 \leq j \leq l(f)\} \\ &= \min\{R(f^j, c) : c \in \text{Crit}(f) \cap \mathcal{R} \text{ and } 1 \leq j \leq l(f)\} < \infty \end{aligned}$$

and

$$\begin{aligned} A_l(f) &= \sup\{A(f^j, c) : c \in \text{Crit}(f) \text{ and } 1 \leq j \leq l(f)\} \\ &= \max\{A(f^j, c) : c \in \text{Crit}(f) \cap \mathcal{R} \text{ and } 1 \leq j \leq l(f)\} \end{aligned}$$

where the numbers $R(f^j, c)$ and $A(f^j, c)$ are defined just above Definition 1.1. Since the number of equivalence classes of the relation \sim is finite, looking at Lemma 2.20 and Lemma 4.15, the following lemma follows immediately from Lemma 4.13.

Lemma 4.20. *If $R_{i,1}^{(u)} > 0$ is a positive constant and $t \mapsto C_{t,i,1}^{(u)} \in (0, \infty)$, $t \in [1, \infty)$, is a continuous function such that all points $z \in \text{PC}(f)_i$ are $(r, C_{t,i,1}^{(u)})$ -t.u.e. with respect to any Euclidean almost t -conformal measure ν_e (with $t \geq 1$) for all $0 < r \leq R_{i,1}^{(u)}$, then there exists a continuous function $t \mapsto \tilde{C}_{t,i,1}^{(u)} > 0$, $t \in [1, \infty)$, such that all critical points $c \in Cr_{i+1}(f)$ are $(r, \tilde{C}_{t,i,1}^{(u)})$ -t.u.e. with respect to any Euclidean almost t -conformal measure ν_e for all $0 < r \leq A_l^{-1} R_{i,1}^{(u)}$.*

We shall now prove the following.

Lemma 4.21. *If $R_{i,2}^{(u)} > 0$ is a positive constant and $t \mapsto C_{t,i,2}^{(u)} \in (0, \infty)$, $t \in [1, \infty)$, is a continuous function such that all critical points $c \in S_i(f)$ are $(r, C_{t,i,2}^{(u)})$ -t.u.e. with respect to any Euclidean almost t -conformal measure ν_e (with $t \geq 1$) for all $0 < r \leq R_{i,2}^{(u)}$, then there exist a continuous function $t \mapsto \tilde{C}_{t,i,2}^{(u)} > 0$, $s \in [1, \infty)$, and $\tilde{R}_{i,2}^{(u)} > 0$ such that all points $z \in \text{PC}(f)_i$ are $(r, \tilde{C}_{t,i,2}^{(u)})$ -t.u.e. with respect to any Euclidean almost t -conformal measure ν_e (with $t \geq 1$) for all $0 < r \leq \tilde{R}_{i,2}^{(u)}$.*

Proof. We shall show that one can take

$$\tilde{R}_{i,2}^{(u)} = \min\{\tau\theta\|f'\|_{\text{PC}(f)}^{-1}\lambda^{-1}, R_{i,2}^{(u)}, 1\} \quad \text{and} \quad \tilde{C}_{t,i,2}^{(u)} = \max\{K^2 2^t C_{t,i,2}^{(u)}, K^{2t} B_s\}.$$

Indeed, denote $\#(\text{Crit}(J(f)))$ by $\#$. Put $\epsilon = 2K(KA^2)2^\#$ and then choose $\lambda > 0$ so large that

$$\epsilon < \lambda \min\{1, \tau^{-1}, \theta^{-1}\tau^{-1} \min\{\gamma, \rho, R_{i,2}^{(u)}/2\}\}. \quad (4.21)$$

Consider $0 < r \leq \tilde{R}_{i,2}^{(u)}$ and $z \in \text{PC}(f)_i$. If $z \in \text{Crit}(J(f))$, then $z \in S_i(f)$ and we are done. Thus, we may assume that $z \notin \text{Crit}(J(f))$. Let $s = s(\lambda, \epsilon, r, z)$. By the definition of ϵ ,

$$2Kr|(f^s)'(z)| = (KA^2)^{-1}2^{-\#}\epsilon r|(f^s)'(z)| \quad (4.22)$$

Suppose first that $u(\lambda, r, z)$ is well defined and $s = u(\lambda, r, z)$. Then by Proposition 4.17(4.4) or Proposition 4.17(4.6), applied with $\eta = 2K$, we see that the point $f^s(z)$ is $(2Kr|(f^s)'(z)|, B_t)$ -t.u.e.. Using (4.22), it follows from Proposition 4.18(4.13) and Lemma 4.10 that the point z is $(r, K^{2h}B_h)$ -h.u.e..

If either u is not defined or $s < u(\lambda, r, z)$, then in view of Proposition 4.18(4.13), there exists a critical point $c \in \text{Crit}(J(f))$ such that $|f^s(z) - c| \leq \epsilon r|(f^s)'(z)|$. Since $s \leq u$, by Proposition 4.17 and (4.21) we get

$$2Kr|(f^s)'(z)| \leq \epsilon r|(f^s)'(z)| < \epsilon\tau\theta\lambda^{-1} \min\{\rho, R_{i,2}^{(u)}/2\} \quad (4.23)$$

Since $z \in \text{PC}(f)_i$, it implies that $c \in S_i(f)$. Therefore using (4.23), the assumptions of Lemma 4.21, and (4.22) and then applying Proposition 4.18(4.13) and Lemma 4.10, we conclude that z is $(r, K^2 2^t C_{t,i,2}^{(u)})$ -t.u.e.. The proof is complete. ■

Lemma 4.22. *If $b \in f^{-1}(\infty)$, if ν is a Euclidean almost t -conformal measure with $t > \frac{2q_b}{q_b+1}$ such that $\nu(b) = 0$, and if m is the h -conformal measure proven to exist in Lemma 3.7, then*

$$\nu(B_b(R)) \preceq R^{2 - \frac{q_b+1}{q_b}t}$$

and

$$m_e(B(b, r)) \succeq r^{(q_b+1)h - 2q_b}$$

for all $0 < r \preceq 1$.

Proof. It follows from Lemma 4.15 that $m_e(\{z \in \mathcal{C} : R \leq |z| < 2R\}) \asymp R^2$ and $\nu(\{z \in \mathcal{C} : R \leq |z| < 2R\}) \preceq R^2$ for all $R > 0$ large enough. It therefore follows from (2.2) that

$$m_e\left((B_b(R) \setminus \overline{B_b(2R)})\right) \asymp R^2 R^{-\frac{q_b+1}{q_b}h}. \quad (4.24)$$

and

$$\nu\left((B_b(R) \setminus \overline{B_b(2R)})\right) \preceq R^2 R^{-\frac{q_b+1}{q_b}t}. \quad (4.25)$$

Fix now $r > 0$ so small that $R = (r/L)^{-q_b}$ is large enough for the formula (4.24) and (4.25) to hold. Using (2.4) and (4.25) we therefore get

$$\begin{aligned} \nu(B_b(R)) &= \nu\left(\bigcup_{j \geq 0} (B_b(2^j R) \setminus \overline{B_b(2^{j+1} R)})\right) = \sum_{j=0}^{\infty} \nu(B_b(2^j R) \setminus \overline{B_b(2^{j+1} R)}) \\ &\preceq \sum_{j=0}^{\infty} (2^j R)^2 (2^j R)^{-\frac{q_b+1}{q_b}t} = R^{2 - \frac{q_b+1}{q_b}t} \sum_{j=0}^{\infty} 2^j \left(2^{-\frac{q_b+1}{q_b}t}\right) \\ &= L^{q_b \left(2 - \frac{q_b+1}{q_b}t\right)} r^{(q_b+1)t - 2q_b} \sum_{j=0}^{\infty} 2^j \left(2^{-\frac{q_b+1}{q_b}t}\right) \asymp r^{(q_b+1)t - 2q_b}, \end{aligned}$$

where the last comparability sign was written since $\frac{q_b+1}{q_b}t > 2$. We are done with the first part of our lemma. Replace now in the above formula ν by m_e and t by h , which is greater than $\frac{2q_b}{q_b+1}$ due to Theorem 2.1. Since in this case the “ \preceq ” sign can be, due to (4.24), replaced by the comparability sign “ \asymp ”, since the first equality sign becomes “ \succeq ” (we do not rule out the possibility that $m_e(b) > 0$ yet), and since $m_e(B(b, r)) \geq \nu(B_b(R))$, we are also done in this case. ■

We shall prove now the following.

Lemma 4.23. *The h -conformal measure m for $f : J(f) \rightarrow J(f) \cup \{\infty\}$ proven to exist in Lemma 3.7 is atomless.*

Proof. Using the induction on $i = 0, 1, \dots, p$, it follows immediately from Lemma 4.21 (which is an inductive step and for $i = 0$ the first step of induction as $S_0(f) = \emptyset$), Lemma 4.20, and Lemma 2.19 that there exists a continuous function $t \mapsto C_t \in (0, \infty)$, $t \in [1, \infty)$, such that if ν is an arbitrary almost t -conformal measure on $J(f)$, then

$$\nu_e(B(x, r)) \leq C_t r^t \quad (4.26)$$

for all $x \in \text{PC}(f)$ and all $r \leq r_0$ for some $r_0 > 0$ sufficiently small. Consider now the almost t_n -conformal measures $m_n = m_{B_s(Y, 1/n)}$ (n is assumed to be so large that $B_s(Y, 1/n) \subset V$), where $t_n = S(B_s(Y, 1/n))$. Letting $n \rightarrow \infty$ and recalling that m is a weak limit of measures m_n , formula (4.26) gives

$$m_e(B(x, r)) \leq C_h r^h \quad (4.27)$$

for all $x \in \text{PC}(f)$ and all $r \leq r_0$. It now follows from Lemma 4.19 that

$$\limsup_{r \searrow 0} \frac{m(B(x, r))}{r^h} \leq 2^h C_h.$$

for all $x \in \text{PC}(f)$. In particular $m(\text{Crit}(f)) = 0$ and consequently

$$m \left(\bigcup_{n \geq 0} f^{-n}(\text{Crit}(f)) \right) = 0. \quad (4.28)$$

Fix now $b \in f^{-1}(\infty)$. Fix $t \in \left(\frac{2q_b}{q_b+1}, h\right)$. Consider all integers $n \geq 1$ so large that $t_n \geq t$. Since $m_n(f^{-1}(\infty)) \leq m_n(f^{-1}(B_s(Y, 1/n))) = 0$, it then follows from Lemma 4.22 that

$$m_n(B_b(R)) \leq R^{2 - \frac{q_b+1}{q_b} t_n} \leq R^{2 - \frac{q_b+1}{q_b} t}.$$

Hence $m_e(b) = 0$. Since m and m_e are equivalent on \mathcal{C} , this gives $m(b) = 0$. Since $\bigcup_{n \geq 0} f^{-n}(b) \cap \text{Crit}(f) = \emptyset$, this implies that $m\left(\bigcup_{n \geq 0} f^{-n}(b)\right) = 0$. Invoking now (4.28) and Lemma 3.7 finishes the proof. ■

Theorem 4.24. *There exists a unique atomless t -conformal measure m for $f : J(f) \rightarrow J(f) \cup \{\infty\}$. Then $t = h$, m is ergodic conservative and all other conformal measures are purely atomic, supported on $\text{Sing}^-(f)$ with exponents larger than h . Consequently $m(\text{Tr}(f)) = 1$.*

Proof. In view of Lemma 4.23 there exists an atomless h -conformal measure m for $f : J(f) \rightarrow J(f) \cup \{\infty\}$. Suppose that ν is an arbitrary t -conformal measure for f and some $t \geq 0$. By Lemma 3.1, $t \geq h$. Fix $z \in J(f) \setminus (I_\infty(f) \cup \text{Sing}^-(f))$. Then in view of Proposition 2.21 there exist a point $y(z) \in J(f)$ and an increasing sequence $\{n_k\}_{k=1}^\infty$ such that $y(z) = \lim_{k \rightarrow \infty} f^{n_k}(z)$. Define for every $l \geq 1$

$$Z_l = \{z \in J(f) \setminus (I_\infty(f) \cup \text{Sing}^-(f)) : |y(z)| \leq l \text{ and } \eta(z) \geq 1/l\},$$

fix $l \geq 1$ and $z \in Z_l$. Considering for k large enough the sets $f_z^{-n_k}(B(y, \frac{1}{4l}))$ and $f_z^{-n_k}(B(y, \frac{1}{4Kl}))$, where $f_z^{-n_k}$ is the holomorphic inverse branch of f^{n_k} defined on $B(y, \frac{1}{2l})$ and sending $f^{n_k}(z)$

to z , using conformality of the measure ν along with Koebe's distortion theorem, we easily deduce that

$$B(\nu, l)^{-1} c |(f^{n_k})^*(z)|^{-h} \leq \nu(B_s(z, c |(f^{n_k})^*(z)|^{-1})) \leq B(\nu, l) c |(f^{n_k})^*(z)|^{-h} \quad (4.29)$$

for all $k \geq 1$ large enough, where $K \geq 1$ is the constant appearing in the Koebe's distortion theorem and ascribed to the scale $1/2$ and $c > 0$ is some constant comparable with 1. Fix now E , an arbitrary bounded Borel set contained in Z_l . Since m is regular, for every $x \in E$ there exists a radius $r(x) > 0$ of the form from (4.29) such that

$$m\left(\bigcup_{x \in E} B_s(x, r(x)) \setminus E\right) < \epsilon. \quad (4.30)$$

Now by the Besicovič theorem (see [G]) we can choose a countable subcover $\{B_s(x_i, r(x_i))\}_{i=1}^{\infty}$, $r(x_i) \leq \epsilon$, from the cover $\{B_s(x, r(x))\}_{x \in E}$ of E , of multiplicity bounded by some constant $C \geq 1$, independent of the cover. Therefore by (4.29) and (4.30), we obtain

$$\begin{aligned} \nu(E) &\leq \sum_{i=1}^{\infty} \nu(B_s(x_i, r(x_i))) \leq B(\nu, l) \sum_{i=1}^{\infty} r(x_i)^t \\ &\leq B(\nu, l) B(m, l) \sum_{i=1}^{\infty} r(x_i)^{t-h} m(B_s(x_i, r(x_i))) \\ &\leq B(\nu, l) B(m, l) C \epsilon^{t-h} m\left(\bigcup_{i=1}^{\infty} B_s(x_i, r(x_i))\right) \\ &\leq C B(\nu, l) B(m, l) \epsilon^{t-h} (\epsilon + m(E)). \end{aligned} \quad (4.31)$$

In the case when $t > h$, letting $\epsilon \searrow 0$ we obtain $\nu(Z_l) = 0$. Since $J(f) \setminus (I_{\infty}(f) \cup \text{Sing}^{-}(f)) = \bigcup_{l=1}^{\infty} Z_l$, we therefore get $\nu(J(f) \setminus (I_{\infty}(f) \cup \text{Sing}^{-}(f))) = 0$ which by Lemma 3.2 implies that $\nu(\text{Sing}^{-}(f)) = 1$ and the last part of our theorem is proved. Suppose now that $t = h$. Since, in view of Lemma 3.2, $\nu(I_{\infty}(f) \setminus I_-(f)) = m(I_{\infty}(f)) = 0$, using (4.31) and letting $l \nearrow \infty$, we conclude that $\nu|_{J(f) \setminus \text{Sing}^{-}(f)} \ll m|_{J(f) \setminus \text{Sing}^{-}(f)}$. Exchanging the roles of m and ν we infer that the measures $\nu|_{J(f) \setminus \text{Sing}^{-}(f)}$ and $m|_{J(f) \setminus \text{Sing}^{-}(f)}$ are equivalent. Suppose that $\nu(\text{Sing}^{-}(f)) > 0$. Then there exists $y \in \text{Crit}(J(f)) \cup \Omega(f) \cup f^{-1}(\infty)$ such that $m(y) > 0$. But then

$$\sum_{\xi \in y^-} |(f^{n(\xi)})^*(\xi)|^{-h} < \infty,$$

where $y^- = \bigcup_{n \geq 0} f^{-n}(y)$ and for every $\xi \in y^-$, $n(\xi)$ is the least integer $n \geq 0$ such that $f^n(\xi) = y$. Hence,

$$\nu_y = \frac{\sum_{\xi \in y^-} |(f^{n(\xi)})^*(\xi)|^{-h} \delta_{\xi}}{\sum_{\xi \in y^-} |(f^{n(\xi)})^*(\xi)|^{-h}}$$

is an h -conformal measure supported on $y^- \subset \text{Sing}^{-}(f)$. This contradicts the proven fact that the measures $\nu_y|_{J(f) \setminus \text{Sing}^{-}(f)}$ and $m|_{J(f) \setminus \text{Sing}^{-}(f)}$ are equivalent and $m(J(f) \setminus \text{Sing}^{-}(f)) = 1$. Thus ν and m are equivalent.

Let us now prove that any h -conformal measure ν is ergodic. Indeed, suppose to the contrary that $f^{-1}(G) = G$ for some Borel set $G \subset J(f)$ with $0 < m(G) < 1$. But then the two conditional measures ν_G and $\nu_{J(f) \setminus G}$

$$\nu_G(B) = \frac{\nu(B \cap G)}{\nu(G)}, \quad \nu_{J(f) \setminus G}(B) = \frac{\nu(B \cap J(f) \setminus G)}{\nu(J(f) \setminus G)}$$

would be h -conformal and mutually singular; a contradiction.

If now ν is again an arbitrary h -conformal measures, then by a simple computation based on the definition of conformal measures we see that the Radon-Nikodym derivative $\phi = d\nu/dm$ is constant on grand orbits of f . Therefore by ergodicity of m we conclude that ϕ is constant m -almost everywhere. As both m and ν are probability measures, it implies that $\phi = 1$ a.e., hence $\nu = m$.

Let us show now that m is conservative. We shall prove first that every forward invariant ($f(E) \subset E$) subset E of $J(f)$ is either of measure 0 or 1. Indeed, suppose to the contrary that $0 < m(E) < 1$. Since $m(I_\infty(f) \cup \text{Sing}^-(f)) = 0$, it suffices to show that

$$m(E \setminus (I_\infty(f) \cup \text{Sing}^-(f))) = 0.$$

Denote by Z the set of all points $z \in E \setminus (I_\infty(f) \cup \text{Sing}^-(f))$ such that

$$\lim_{r \rightarrow 0} \frac{m(B(z, r) \cap (E \setminus (I_\infty(f) \cup \text{Sing}^-(f))))}{m(B(z, r))} = 1. \quad (4.32)$$

In view of the Lebesgue density theorem (see for example Theorem 2.9.11 in [Fe]), $m(Z) = m(E)$. Since $m(E) > 0$ we find at least one point $z \in Z$. Since $z \in J(f) \setminus (I_\infty(f) \cup \text{Sing}^-(f))$, let $x \in J(f)$, $\eta(z) > 0$, and an increasing sequence $\{n_k\}_{k=1}^\infty$ be given by Proposition 2.21.

$$\delta = \eta(z)/8.$$

Suppose that $m(B(x, \delta) \setminus E) = 0$. By conformality of m , $m(f(Y)) = 0$ for all Borel sets Y such that $m(Y) = 0$. Hence,

$$\begin{aligned} 0 &= m(f^n(B(x, \delta) \setminus E)) \geq m(f^n(B(x, \delta)) \setminus f^n(E)) \\ &\geq m(f^n(B(x, \delta)) \setminus E) \geq m(f^n(B(x, \delta))) - m(E) \end{aligned} \quad (4.33)$$

for all $n \geq 0$. Since $J(f) = \overline{\bigcup_{n \geq 1} f^{-n}(\infty)}$, for some $p \geq 2$, the image $f^{p-1}(B(x, \delta))$ contains an open neighbourhood of ∞ . thus, it contains at least one (in fact infinitely many) copy of the fundamental parallelogram \mathcal{R} and consequently $f^p(B(x, \delta)) = \overline{\mathcal{U}}$. In particular $m(f^p(B(x, \delta))) = 1$. Then (4.33) implies that $0 \geq 1 - m(E)$ which is a contradiction. Consequently $m(B(x, \delta) \setminus E) > 0$. Hence for every $j \geq 1$ large enough, $m(B(f^{n_j}(z), 2\delta) \setminus E) \geq m(B(x, \delta) \setminus E) > 0$. Therefore, as $f^{-1}(J(f) \setminus E) \subset J(f) \setminus E$, the standard application of Koebe's Distortion Theorem shows that

$$\limsup_{r \rightarrow 0} \frac{m(B(z, r) \setminus E)}{m(B(z, r))} > 0$$

which contradicts (4.32). Thus either $m(E) = 0$ or $m(E) = 1$.

Now conservativity is straightforward. One needs to prove that for every Borel set $B \subset J(f)$ with $m(B) > 0$ one has $m(G) = 0$, where

$$G = \{x \in J(f) : \sum_{n \geq 0} \chi_B(f^n(x)) < +\infty\}.$$

Indeed, suppose that $m(G) > 0$ and for all $n \geq 0$ let

$$G_n = \{x \in J(f) : \sum_{k \geq n} \chi_B(f^k(x)) = 0\} = \{x \in J(f) : f^k(x) \notin B \text{ for all } k \geq n\}.$$

Since $G = \bigcup_{n \geq 0} G_n$, there exists $k \geq 0$ such that $m(G_k) > 0$. Since all the sets G_n are forward invariant we conclude that $m(G_k) = 1$. But on the other hand all the sets $f^{-n}(B)$, $n \geq k$, are of positive measure and are disjoint from G_k . This contradiction finishes the proof of conservativity of m . Consequently $m(\text{Tr}(f)) = 1$. Since, by Lemma 3.1, $H^h \prec m$, we thus see that $H^h(J(f) \setminus \text{Tr}(f)) = 0$. We are done. ■

The proof of part (a) of Theorem 4.1. Let m be the unique h -conformal atomless measure proven to exist in Theorem 4.24. Consider an arbitrary point $z \in \text{Tr}(f)$. Fix a pole $b \in f^{-1}(\infty)$. Since $b \notin \overline{O_+(\text{Crit}(f))}$, there exists $\gamma > 0$ such that

$$B(b, \gamma) \cap O_+(\text{Crit}(f)) = \emptyset. \quad (4.34)$$

Since $z \in \text{Tr}(f)$, there exists an infinite increasing sequence $\{n_j\}_{j=0}^\infty$ such that

$$\lim_{j \rightarrow \infty} f^{n_j}(z) = b \text{ and } |f^{n_j}(z) - b| < \gamma/4 \quad (4.35)$$

for every $j \geq 1$. It follows from this and (4.34) that for every $j \geq 1$ there exists a holomorphic inverse branch $f_z^{-n_j} : B(f^{n_j}(z), 3\gamma/4) \rightarrow \mathcal{C}$ of f^{n_j} sending $f^{n_j}(z)$ to z . Using now Koebe's Distortion Theorem (Euclidean version) and Lemma 4.22, we conclude that

$$\begin{aligned} m_e(z, B(K|(f^{n_j})'(z)|^{-1}2|f^{n_j}(z) - b|)) &\geq m_e(B(f^{n_j}(z), 2|f^{n_j}(z) - b|)) |(f^{n_j})'(z)|^{-h} \\ &\geq m_e(B(b, |f^{n_j}(z) - b|)) |(f^{n_j})'(z)|^{-h} \\ &\geq |f^{n_j}(z) - b|^{(q_b+1)h-2q_b} |(f^{n_j})'(z)|^{-h} \\ &= \left(K|(f^{n_j})'(z)|^{-1}|f^{n_j}(z) - b|\right)^h K^{-h} |f^{n_j}(z) - b|^{q_b(h-2)}. \end{aligned}$$

Since $h < 2$, using (4.35), this implies that $\overline{\lim}_{r \rightarrow 0} r^{-h} m_e(B(z, r)) = \infty$. Hence $H^h(\text{Tr}(f)) = 0$ in view of Theorem 4.3. Since by Theorem 4.24 $m_e(J(f) \setminus \text{Tr}(f)) = 0$, it follows from Lemma 3.1 that $H^h(J(f) \setminus \text{Tr}(f)) = 0$. In conclusion $H^h(J(f)) = 0$ and the proof is complete. ■

Proposition 4.25. *The conformal measure m is absolutely continuous with respect to the packing measure Π^t and moreover, the Radon-Nikodym derivative $dm/d\Pi^t$ is uniformly bounded away from infinity. In particular $\Pi^t(J(f)) > 0$.*

Proof. Since $J(f) \cap \omega(\text{Crit}(f) \setminus \text{Crit}(J(f))) = \Omega(f)$, we conclude from Lemma 2.9 that there exists $y \in J(f)$ at a positive distance, say 8η , from $O_+(\text{Crit}(f))$. Fix $z \in \text{Tr}(f)$. Then there exists an infinite sequence $n_j \geq 1$ of increasing integers such that $f^{n_j}(z) \in B(y, \eta)$. Therefore $B(f^{n_j}(z), 4\eta) \cap O_+(\text{Crit}(f)) = \emptyset$ and consequently

$$\text{Comp}(z, f^{n_j}(z), f^{n_j}, \eta/2) \cap \text{Crit}(f^{n_j}) = \emptyset$$

Hence, it follows from Lemma 1.2 and Lemma 4.10 that

$$\liminf_{r \rightarrow 0} \frac{m_e(B(z, r))}{r^h} \leq B$$

for some constant $B \in (0, \infty)$ and all $z \in \text{Tr}(f)$. Applying Lemma 4.19 we therefore get that

$$\liminf_{r \rightarrow 0} \frac{m(B_s(z, r))}{r^h} \leq 2^h B.$$

Hence, by Theorem 4.4(1), the measure $m|_{\text{Tr}(f)}$ is absolutely continuous with respect to $\Pi^h|_{\text{Tr}(f)}$. Since, by Theorem 4.24, $m(J(f) \setminus \text{Tr}(f)) = 0$, we are done. ■

Lemma 4.26. *If $\Omega(f) \neq \emptyset$, then $\Pi^h(J(f)) = +\infty$.*

Proof. Fix $\omega \in \Omega$. Since $\bigcup_{n \geq 0} f^{-n}(\omega)$ is dense in $J(f)$ and, by Lemma 2.9, $\omega(\text{Crit}(f))$ is non-where dense in $J(f)$, there exist an integer $s > 0$, a real number $\eta > 0$, and a point $y \in f^{-s}(\omega) \setminus B(\bigcup_{n \geq 0} f^n(\text{Crit}(f)), \eta)$. Since by Theorem 2.1, $h > 1$, it follows from Lemma 2.5 and Lemma 4.13 (y may happen to be a critical point of f^s !) that

$$\liminf_{r \rightarrow 0} \frac{m_e(B(y, r))}{r^h} = 0. \quad (4.36)$$

Consider now a transitive point $z \in J(f)$, i.e. $z \in \text{Tr}(f)$. Then there exists an infinite increasing sequence $n_j = n_j(z) \geq 1$ of positive integers such that

$$\lim_{j \rightarrow \infty} |f^{n_j}(z) - y| = 0 \quad \text{and} \quad r_j = |f^{n_j}(z) - y| < \eta/7$$

for every $j = 1, 2, \dots$. By the choice of y , for all $j \geq 1$ there exist holomorphic inverse branches $f_z^{-n_j} : B(f^{n_j}(z), 6r_j) \rightarrow \overline{\mathcal{U}}$ sending $f^{n_j}(z)$ to z . So, applying Lemma 1.2 and Lemma 4.10 with $R = 3r_j$, we conclude from (4.36) that

$$\liminf_{r \rightarrow 0} \frac{m_e(B(z, r))}{r^h} = 0.$$

Applying Lemma 4.19, we conclude that the same formulas remain true with m_e replaced by m and $B(z, r)$ by $B_s(z, r)$. Therefore, it follows from Theorem 4.24 ($m(\text{Tr}(f)) = 1$) and Theorem 4.4(1) that $\Pi^h(J(f)) = +\infty$. We are done. ■

From now on let m denote the unique atomless h -conformal measure m proven to exist in Theorem 4.24.

Since the number of equivalence classes of the relation \sim is finite, looking at Lemma 2.20 and Lemma 4.15, the following lemma follows immediately from Lemma 4.14

Lemma 4.27. *If $C_{i,1}^l > 0$, $0 < R_{i,1}^l \leq R_l(f)/3$, and $0 < \sigma \leq 1$ are three real numbers such that all points $z \in \text{PC}(f)_i$ are $(r, \sigma, C_{i,1}^l)$ -h.s.l.e. with respect to the measure m_e , then there exists $\tilde{C}_{i,1}^l > 0$ such that all critical points $c \in Cr_{i+1}(f)$ are $(r, \tilde{\sigma}, \tilde{C}_{i,1}^l)$ -h.s.l.e. with respect to the measure m_e for all $0 < r \leq A_l^{-1}R_{i,1}^l$.*

Let us prove the following.

Lemma 4.28. *Suppose that $\Omega(f) = \emptyset$. Assume that $C_{i,2}^{(l)} > 0$, $R_{i,2}^{(l)} > 0$ and $0 < \sigma \leq 1$ are three real numbers such that all critical points $c \in S_i(f)$ are $(r, \sigma, C_{i,2}^{(l)})$ -h.s.l.e. with respect to the measure m_e for all $0 < r \leq R_{i,2}^{(l)}$. Then there exist $\tilde{C}_{i,2}^{(l)} > 0$, $\tilde{R}_{i,2}^{(l)} > 0$ and such that all points $z \in \text{PC}(f)_i$ are $(r, 8K^3A^22^{\#(\text{Crit}(f))}\sigma, \tilde{C}_{i,2}^{(l)})$ -h.s.l.e. with respect to the measure m_e for all $0 < r \leq \tilde{R}_{i,2}^{(l)}$.*

Proof. We shall show that this time one can take

$$\tilde{R}_{i,2}^{(l)} = \min\{\tau\theta\|f'\|_F^{-1}\lambda^{-1}, R_{i,2}^{(l)}, 1\} \quad \text{and} \quad \tilde{C}_{i,2}^{(l)} = (8(KA^2)2^{\#})^h C_{i,2}^{(l)},$$

where $\|f'\| = \|f'\|_{\text{PC}(f)_i}$. Indeed, denote again $\#(\text{Crit}(f))$ by $\#$. Take $\epsilon = 4K(KA^2)2^{\#}$ and then choose $\lambda > 0$ so large that

$$\epsilon < \lambda \min\{1, \tau^{-1}, \theta^{-1}\tau^{-1} \min\{\gamma, \rho, R_{i,2}^{(l)}/2\}\}. \quad (4.37)$$

Consider $0 < r \leq \tilde{R}_{i,2}^{(l)}$ and $z \in \text{PC}(f)_i$. If $z \in \text{Crit}(J(f))$, then $z \in S_i(f)$ and we are done. Thus, we may assume that $z \notin \text{Crit}(J(f))$. Let $s = s(\lambda, \epsilon, r, z)$. By the definition of ϵ

$$4Kr|(f^s)'(z)| = (KA^2)^{-1}2^{-\#}\epsilon r|(f^s)'(z)|. \quad (4.38)$$

Suppose first that $u(\lambda, r, z)$ is well defined and $s = u(\lambda, r, z)$. Then by Proposition 4.17(4.5) or Proposition 4.17(4.6), applied with $\eta = K$, we see that the point

$$f^s(z) \text{ is } (Kr\|(f^s)'(z)\|, \sigma/K^2, W_h(\sigma/K^2)) - \text{h.s.l.e.}$$

Using (4.38) it follows from Proposition 4.18(4.13) and Lemma 4.11 that the point z is $(r, \sigma, W_h(\sigma/K^2))$ -h.s.l.e.. If either u is not defined or $s \leq u(\lambda, r, z)$, then in view of Proposition 4.18(4.12), there exists a critical point $c \in \text{Crit}(f)$ such that $|f^s(z) - c| \leq \epsilon r|(f^s)'(z)|$. Since $s \leq u$, by Proposition 4.17 and (4.37) we get

$$4Kr|(f^s)'(z)| \leq \epsilon r|(f^s)'(z)| < \epsilon\tau\theta\lambda^{-1} \min\{\rho, R_{i,2}^{(l)}/2\}. \quad (4.39)$$

Since $z \in \text{PC}(f)_i$, it implies that $c \in S_i(f)$. Therefore, by the assumptions of Lemma 4.28 and by (4.39) we conclude that c is $(2\epsilon r|(f^s)'(z)|, \sigma, C_{i,2}^{(l)})$ -h.s.l.e.. Consequently, in view of

Lemma 4.7, the point $f^s(z)$ is $(\epsilon r |(f^s)'(z)|, 2\sigma, 2^h C_{i,2}^{(l)})$ -h.s.l.e.. So, by Lemma 4.8 this point is

$$(Kr |(f^s)'(z)|, 2\sigma\epsilon/K, (2\epsilon K^{-1})^h C_{i,2}^{(l)}) - h.s.l.e.$$

Using now formula (4.38) and Proposition 4.18(4.13) it follows from Lemma 4.11 that the point z is $(r, 2K\epsilon\sigma, (2\epsilon K^{-1})^h C_{i,2}^{(l)}) - h.s.l.e..$ If $z \in \text{Crit}(J(f))$, then by the definition of $\text{PC}(f)_i$ we see that $z \in S_i(f)$ and we are done in view of the assumption of the lemma and in view of the definitions of $\tilde{R}_{i,2}^{(l)}$ and $\tilde{C}_{i,2}^{(l)}$. The proof is completed. ■

Lemma 4.29. *If $\Omega(f) = \emptyset$, then $\Pi_e^h(F) < \infty$ for every bounded Borel set $F \subset \mathcal{C}$.*

Proof. Let

$$q_{\min} = \min\{q_b : b \in f^{-1}(\infty)\}.$$

Take $\kappa \in (0, 1)$ so small that if $z \in \mathcal{C}$, then $f|_{B(z,d)}$ is 1-to-1 for every $d \leq \kappa \text{dist}(z, \text{Crit}(f) \cup f^{-1}(\infty))$. Using induction on $i = 0, 1, \dots, p$, it follows immediately from Lemma 4.28 (which is an inductive step and for $i = 0$ the first step of induction as $S_0(f) = \emptyset$), Lemma 4.27, and Lemma 2.19 that each point $z \in \text{PC}(f)$ is $(r, \sigma, G) - h.s.l.e.$ for some $\sigma \in (0, 1)$, $G > 0$, $R > 0$ and all $r \in (0, R)$. Without loss of generality we may assume $R \in (0, 1)$ to be so small that

$$\xi^{-1}|z - b|^{-q_b} \leq |f(w)| \leq \xi|z - b|^{-q_b} \quad (4.40)$$

and

$$\sup\{|w| : w \in \text{PC}(f)\} \leq \xi^{-1}R^{-q_{\min}} - 8R \quad (4.41)$$

for all $b \in f^{-1}(\infty)$, all $z \in B(b, R)$ and some $\xi \geq 1$. Fix a point $z \in F \setminus \text{Sing}^-(f)$ and $r \in (0, R)$. In view of Corollary 2.22 there exists the least $n \geq 1$ such that either

$$\text{dist}(f^n(z), \text{PC}(f)) \leq 8(K\kappa)^{-1}r|(f^n)'(z)| \quad \text{or} \quad r|(f^n)'(z)| \geq \frac{1}{8}\kappa R.$$

There are the following three possibilities.

1⁰

$$K^{-1}r|(f^n)'(z)| < \frac{1}{8}\kappa R.$$

This in particular implies that

$$\text{dist}(f^n(z), \text{PC}(f)) \leq 8(K\kappa)^{-1}r|(f^n)'(z)|.$$

2⁰

$$K^{-1}r|(f^n)'(z)| \geq \frac{1}{8}\kappa R \quad \text{and} \quad \text{dist}(f^n(z), \text{PC}(f)) > 8(K\kappa)^{-1}r|(f^n)'(z)|.$$

3⁰

$$K^{-1}r|(f^n)'(z)| \geq \frac{1}{8}\kappa R \quad \text{and} \quad \text{dist}(f^n(z), \text{PC}(f)) \leq 8(K\kappa)^{-1}r|(f^n)'(z)|.$$

Let us consider the case 1^0 . Since $8(K\kappa)^{-1}r|(f^{n-1})'(z)| < \text{dist}(f^{n-1}(z), \text{PC}(f))$, we get

$$8K^{-1}r|(f^{n-1})'(z)| < \kappa \text{dist}(f^{n-1}(z), \text{Crit}(f)). \quad (4.42)$$

Suppose now that

$$8K^{-1}r|(f^{n-1})'(z)| \geq \kappa \text{dist}(f^{n-1}(z), f^{-1}(\infty)).$$

This implies that there exists $b \in f^{-1}(\infty)$ such that $|f^{n-1}(z) - b| < R$. Hence, using (4.40), we get

$$|f^n(z)| \geq \xi^{-1}|f^{n-1}(z) - b|^{-q_b} \geq \xi^{-1}R^{-q_b}.$$

On the other hand, using (4.41), we obtain

$$\begin{aligned} |f^n(z)| &\leq \sup\{|w| : w \in \text{PC}(f)\} + \text{dist}(f^n(z), \text{PC}(f)) \\ &\leq \sup\{|w| : w \in \text{PC}(f)\} + 8(K\kappa)^{-1}r|(f^n)'(z)| \\ &\leq \sup\{|w| : w \in \text{PC}(f)\} + 8R \leq \xi^{-1}R^{-q_b}. \end{aligned}$$

This contradiction shows that

$$8K^{-1}r|(f^{n-1})'(z)| < \kappa \text{dist}(f^{n-1}(z), f^{-1}(\infty)).$$

Along with (4.42) and the definition of κ , this implies that the map f restricted to the ball $B(f^{n-1}(z), 8K^{-1}r|(f^{n-1})'(z)|)$, is univalent. It therefore follows from Koebe's $\frac{1}{4}$ -theorem that

$$f\left(B(f^{n-1}(z), 8K^{-1}r|(f^{n-1})'(z)|)\right) \supset B(f^n(z), 2K^{-1}r|(f^n)'(z)|). \quad (4.43)$$

Thus, there exists a unique holomorphic inverse branch $f_*^{-1} : B(f^n(z), 2K^{-1}r|(f^n)'(z)|) \rightarrow B(f^{n-1}(z), 8K^{-1}r|(f^{n-1})'(z)|)$ of f sending $f^n(z)$ to $f^{n-1}(z)$. Since

$$B(f^{n-1}(z), 8K^{-1}r|(f^{n-1})'(z)|) \cap \text{PC}(f) = \emptyset$$

there exists a unique holomorphic inverse branch

$$f_z^{-(n-1)} : B(f^{n-1}(z), 8K^{-1}r|(f^{n-1})'(z)|) \rightarrow \mathcal{C}$$

of f^{n-1} sending $f^{n-1}(z)$ to z . Therefore, the composition

$$f_z^{-n} = f_z^{-(n-1)} \circ f_*^{-1} : B(f^n(z), 2K^{-1}r|(f^n)'(z)|) \rightarrow \mathcal{C}$$

is a well-defined holomorphic inverse branch of f^n sending $f^n(z)$ to z . As $\text{dist}(f^n(z), \text{PC}(f)) < R$, since $K^{-1}r|(f^n)'(z)| < R$ and since each point $z \in \text{PC}(f)$ is (r, σ, G) -*hs.l.e.*, we obtain that

$$m_e\left(B(f^n(z), K^{-1}r|(f^n)'(z)|)\right) \geq G(K^{-1}r|(f^n)'(z)|)^h.$$

Using now Koebe's distortion theorem, we conclude that

$$\begin{aligned} m_e(B(z, r)) &\geq m_e\left(f_z^{-n}\left(B(f^n(z), K^{-1}r|(f^n)'(z)|)\right)\right) \\ &\geq K^{-h}|(f^n)'(z)|^{-h} m_e\left(B(f^n(z), K^{-1}r|(f^n)'(z)|)\right) \\ &\geq K^{-h}|(f^n)'(z)|^{-h} G^h K^{-h} r^h |(f^n)'(z)|^h = (GK^{-2}r)^h. \end{aligned} \quad (4.44)$$

Let us now deal with the case 2⁰. In this case the holomorphic inverse branch $f_z^{-n} : B(f^n(z), 2K^{-1}r|(f^n)'(z)|) \rightarrow \mathcal{C}$ of f^n sending $f^n(z)$ to z is well-define. Using Koebe's distortion theorem and Lemma 4.15, we get

$$\begin{aligned}
 m_e(B(z, r)) &\geq m_e\left(f_z^{-n}\left(B(f^n(z), K^{-1}r|(f^n)'(z)|)\right)\right) \\
 &\geq K^{-h} |(f^n)'(z)|^{-h} m_e(B(f^n(z), K^{-1}r|(f^n)'(z)|)) \\
 &\geq K^{-h} |(f^n)'(z)|^{-h} C_h \left(\frac{1}{8}R\kappa\right) (K^{-1}r|(f^n)'(z)|)^h \\
 &= C_h \left(\frac{1}{8}R\kappa\right) K^{-2h} r^h
 \end{aligned} \tag{4.45}$$

Case 3³. Suppose first that

$$|f^{n-1}(z) - b| \leq \frac{1}{2}K^{-1}r|(f^{n-1})'(z)|$$

for some pole $b \in f^{-1}(\infty)$. Then

$$B(f^{n-1}(z), K^{-1}r|(f^{n-1})'(z)|) \supset B(b, \frac{1}{2}K^{-1}r|(f^{n-1})'(z)|). \tag{4.46}$$

Since $\frac{1}{2}K^{-1}r|(f^{n-1})'(z)| \leq \frac{1}{16}\kappa R$, it follows from Lemma 4.22 that

$$m_e\left(B(b, \frac{1}{2}K^{-1}r|(f^{n-1})'(z)|)\right) \geq C \left(\frac{1}{2}K^{-1}r|(f^{n-1})'(z)|\right)^{(q_b+1)h-2q_b} \tag{4.47}$$

for some universal constant $C > 0$. Since $2K^{-1}r|(f^{n-1})'(z)| < 8K^{-1}\kappa^{-1}r|(f^{n-1})'(z)| \leq \text{dist}(f^{n-1}(z), \text{PC}(f))$, we see that there exists a unique holomorphic inverse branch $f_z^{-(n-1)} : B(f^{n-1}(z), 2K^{-1}r|(f^{n-1})'(z)|) \rightarrow \mathcal{C}$ of f^{n-1} sending $f^{n-1}(z)$ to z . Therefore, applying (4.46), (4.47) and Koebe's distortion theorem, we obtain

$$\begin{aligned}
 m_e(B(z, r)) &\geq m_e\left(f_z^{-(n-1)}\left(B(f^{n-1}(z), K^{-1}r|(f^{n-1})'(z)|)\right)\right) \\
 &\geq K^{-h} |(f^{n-1})'(z)|^{-h} m_e\left(B(f^{n-1}(z), K^{-1}r|(f^{n-1})'(z)|)\right) \\
 &\geq K^{-h} |(f^{n-1})'(z)|^{-h} m_e\left(B(b, \frac{1}{2}K^{-1}r|(f^{n-1})'(z)|)\right) \\
 &\geq K^{-h} C |(f^{n-1})'(z)|^{-h} \left(\frac{1}{2}K^{-1}r|(f^{n-1})'(z)|\right)^{(q_b+1)h-2q_b} \\
 &\geq CK^{-h} \left(K^{-1}r|(f^{n-1})'(z)|\right)^{q_b(h-2)} r^h \\
 &\geq CK^{-h} \left(\frac{1}{8}\kappa R\right)^{q_{\max}(h-2)} r^h.
 \end{aligned} \tag{4.48}$$

So, suppose finally that

$$|f^{n-1}(z) - b| > \frac{1}{2}K^{-1}r|(f^{n-1})'(z)|.$$

for all poles $b \in f^{-1}(\infty)$. Since also

$$\text{dist}(f^{n-1}(z), \text{PC}(f)) > 4K^{-1}\kappa^{-1}r|(f^{n-1})'(z)|, \quad (4.49)$$

we conclude that the map $f : \mathcal{C} \rightarrow \overline{\mathcal{C}}$, restricted to the ball $B(f^{n-1}(z), \frac{1}{2}\kappa K^{-1}r|(f^{n-1})'(z)|)$, is univalent. It therefore follows from Koebe's $\frac{1}{4}$ -theorem that

$$f\left(B(f^{n-1}(z), \frac{1}{2}\kappa K^{-1}r|(f^{n-1})'(z)|)\right) \supset B\left(\frac{1}{8}\kappa K^{-1}r|(f^n)'(z)|\right).$$

Hence, there exists a unique holomorphic inverse branch $f_*^{-1} : B(f^n(z), \frac{1}{8}\kappa K^{-1}r|(f^n)'(z)|) \rightarrow B(f^{n-1}(z), \frac{1}{2}\kappa K^{-1}r|(f^{n-1})'(z)|)$ of f sending $f^n(z)$ to $f^{n-1}(z)$. In view of (4.49) there exists a unique holomorphic inverse branch $f_z^{-(n-1)} : B(f^{n-1}(z), \frac{1}{2}\kappa K^{-1}r|(f^{n-1})'(z)|) \rightarrow \mathcal{C}$ of f^{n-1} sending $f^{n-1}(z)$ to z . Hence, the composition

$$f_z^{-n} = f_z^{-(n-1)} \circ f_*^{-1} : B(f^n(z), \frac{1}{8}\kappa K^{-1}r|(f^n)'(z)|) \rightarrow \mathcal{C}$$

is a well-defined holomorphic inverse branch of f^n sending $f^n(z)$ to z . Since $\frac{1}{16}\kappa K^{-1}r|(f^n)'(z)| > 2^{-7}\kappa^2 R$, applying Koebe's distortion theorem and Lemma 4.15, we get

$$\begin{aligned} m_e(B(z, r)) &\geq m_e\left(f_z^{-(n-1)}\left(B(f^n(z), \frac{1}{16}\kappa K^{-1}r|(f^n)'(z)|)\right)\right) \\ &\geq K^{-h}|(f^n)'(z)|^{-h} m_e\left(B(f^n(z), (16K)^{-1}\kappa r|(f^n)'(z)|)\right) \\ &\geq K^{-h}C_h(2^{-7}\kappa^2 R)|(f^n)'(z)|^{-h}\left((16K)^{-1}\kappa r|(f^n)'(z)|\right)^h \\ &= (16)^{-1}K^{-2h}\kappa C_h(2^{-7}\kappa^2 R)r^h. \end{aligned}$$

Combining this inequality along with (4.44) (4.45) and (4.48), we conclude that $\Pi_e^h(F) < \infty$. We are done. ■

Our last lemma in this section is this.

Lemma 4.30. *If $\Omega(f) = \emptyset$, then the spherical packing measure $\Pi^h(J(f))$ is finite.*

Proof. Since the packing measure Π^h is Λ -invariant, it follows from Lemma 4.29 and Proposition 4.25 that $\Pi_e^h(J(f) \cap (B(0, 2R) \setminus \overline{B(0, R)})) \preceq R^2$ for all $R \geq 1$. Since in addition $\frac{d\Pi_e^h}{d\Pi_e^h}(z) = (1 + |z|^2)^{-h}$ and since $h > 1$, we get

$$\begin{aligned} \Pi^h(J(f) \cap (\mathcal{C} \setminus \overline{B(0, 1)})) &= \sum_{n=0}^{\infty} \Pi^h(J(f) \cap (B(0, 2^{n+1}) \setminus \overline{B(0, 2^n)})) \\ &\asymp \sum_{n=0}^{\infty} 2^{-2hn} \Pi_e^h(J(f) \cap (B(0, 2^{n+1}) \setminus \overline{B(0, 2^n)})) \\ &\asymp \sum_{n=0}^{\infty} 2^{-2hn} 2^{2n} = \sum_{n=0}^{\infty} 2^{(2-2h)n} < \infty. \end{aligned}$$

We are done. ■

The proof of Theorem 4.1 is therefore complete. ■

5. INVARIANT MEASURES

In this section we deal with σ -finite invariant measures equivalent to the conformal measure m . We prove their existence, ergodicity, conservativity and we detect the points around which these measures are finite or infinite. This allows us to provide sufficient conditions for their finiteness.

5.1. σ -finite invariant measures equivalent to the conformal measure m . In order to prove Theorem 5.2 below we apply a general sufficient condition for the existence of σ -finite absolutely continuous invariant measure proven in [15]. In order to formulate this condition suppose that X is a σ -compact metric space, ν is a Borel probability measure on X , positive on open sets, and that a measurable map $f : X \rightarrow X$ is given with respect to which measure ν is quasi-invariant, i.e. $\nu \circ f^{-1} \ll \nu$. Moreover we assume the existence of a countable partition $\alpha = \{A_n : n \geq 0\}$ of subsets of X which are all σ -compact and of positive measure ν . We also assume that $\nu(X \setminus \bigcup_{n \geq 0} A_n) = 0$, and if additionally for all $m, n \geq 1$ there exists $k \geq 0$ such that

$$\nu(f^{-k}(A_m) \cap A_n) > 0,$$

then the partition α is called irreducible. Martens' result comprising Proposition 2.6 and Theorem 2.9 of [15] reads the following.

Theorem 5.1. *Suppose that $\alpha = \{A_n : n \geq 0\}$ is an irreducible partition for $T : X \rightarrow X$. Suppose that T is conservative and ergodic with respect to the measure ν . If for every $n \geq 1$ there exists $K_n \geq 1$ such that for all $k \geq 0$ and all Borel subsets A of A_n*

$$K_n^{-1} \frac{\nu(A)}{\nu(A_n)} \leq \frac{\nu(f^{-k}(A))}{\nu(f^{-k}(A_n))} \leq K_n \frac{\nu(A)}{\nu(A_n)},$$

then T has a σ -finite T -invariant measure μ absolutely continuous with respect to ν . In addition, μ is equivalent with ν , conservative and ergodic, and unique up to a multiplicative constant. Moreover, for every Borel set $A \subset X$

$$\mu(A) = \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n \nu(f^{-k}(A))}{\sum_{k=0}^n \nu(f^{-k}(A_0))}.$$

The first result of this section is the following.

Theorem 5.2. *There exists a σ -finite f -invariant measure μ absolutely continuous with respect to h -conformal measure m . In addition, μ is equivalent with m and ergodic.*

Proof. Let $\xi \in \mathcal{C}$ be a periodic point of f with some period $p \geq 3$. We put

$$P_3(f) = \overline{O_+(f(\text{Crit}(f)))} \cup \{\xi, f(\xi), \dots, f^{p-1}(\xi)\}.$$

Since $\overline{O_+(f(\text{Crit}(f)))}$ is a forward-invariant nowhere-dense subset of $J(f)$ and since the h -conformal measure m is positive on nonempty open subsets of $J(f)$, it follows from ergodicity and conservativity of m (see Theorem 4.24) that $m(\overline{O_+(f(\text{Crit}(f)))}) = 0$. Since m has no atoms (see Theorem 4.24) we therefore obtain that $m(P_3(f)) = 0$. We shall now construct the partition α of the set $J(f) \setminus P_3(f)$. We shall check next that it satisfies the assumptions of Theorem 5.1. We first define the family of balls

$$\left\{ B \left(z, \frac{1}{2} \text{dist}(z, P_3(f)) \right) \right\}_{z \in \mathcal{C} \setminus P_3(f)}.$$

This family obviously covers $\mathcal{C} \setminus P_3(f)$. Since $\mathcal{C} \setminus P_3(f)$ is an open set, it is a Lindelöf space, and therefore we can choose a countable subcover of $\mathcal{C} \setminus P_3(f)$, which we denote by

$$\left\{ B \left(z_i, \frac{1}{2} \text{dist}(z_i, P_3(f)) \right) \right\}_{i=1}^{\infty}.$$

We inductively define a partition $\mathcal{A} = \{A_i\}_{i=0}^{\infty}$ of $\mathcal{C} \setminus P_3(f)$ as follows. Let

$$A_0 = \left\{ B \left(z_0, \frac{1}{2} \text{dist}(z_0, P_3(f)) \right) \right\}.$$

Assume that we have defined the set A_1, \dots, A_n such that

$$A_j \subset \left\{ B \left(z_j, \frac{1}{2} \text{dist}(z_j, P_3(f)) \right) \right\}$$

and

$$\text{Int} A_j \neq \emptyset.$$

Then A_{n+1} we define as

$$A_{n+1} = \left\{ B \left(z_{n+1}, \frac{1}{2} \text{dist}(z_{n+1}, P_3(f)) \right) \right\} \setminus \bigcup_{j=1}^n A_j.$$

The set A_{n+1} is disjoint with the sets A_1, \dots, A_n and

$$A_{n+1} \subset B \left(z_{n+1}, \frac{1}{2} \text{dist}(z_{n+1}, P_3(f)) \right) \setminus \bigcup_{j=1}^n B \left(z_j, \frac{1}{2} \text{dist}(z_j, P_3(f)) \right).$$

Thus either $A_{n+1} = \emptyset$ or $\text{Int} A_{n+1} \neq \emptyset$ and we remove all the empty sets.

We shall now check that the partition is irreducible. And indeed, it follows from the construction of the sets $\{A_i\}_{i=0}^{\infty}$ and continuity of the measure m that it suffices to demonstrate that if $z \in \mathcal{C}$, $r > 0$ and $K \subset \mathcal{C}$ is a compact set, then there exists $n \geq 1$ such that

$$f^n \left(B(z, r) \setminus \bigcup_{k \geq 0} f^{-k}(\infty) \right) \supset K \setminus \bigcup_{k \geq 0} f^{-k}(\infty).$$

Since the set of repelling periodic points is dense in the Julia ([2], comp. [5]), there thus exists a periodic point $x \in B(z, r)$, say of period $q \geq 1$. Since x is repelling there exists $s > 0$ so small that $B(x, s) \subset B(z, r)$ and $f^q(B(x, s)) \supset B(x, s)$. Since $\bigcup_{j \geq 1} f^{qj}(B(x, s)) \supset \mathcal{C}$, since K is a compact subset of \mathcal{C} and since $\{f^{qj}(B(x, s))\}_{j=1}^{\infty}$ is an increasing family of open sets, there thus exists $k \geq 1$ such that $f^{qk}(B(x, s)) \supset K$.

Let us check now the distortion assumption of Theorem 5.1. And indeed, in view of Koebe's distortion theorem there exists a constant $K \geq 1$ such that if $f_*^{-n} : B(z_i, \text{dist}(z_i, P_3(f))) \rightarrow \mathcal{C}$ is a holomorphic branch of f^{-n} , then for every $k \geq 0$ and all $x, y \in A_k \subset B(z_i, \frac{1}{2} \text{dist}(z_i, P_3(f)))$ we have

$$\frac{|(f_*^{-n})'(y)|}{|(f_*^{-n})'(x)|} \leq K. \quad (5.1)$$

We therefore obtain for all Borel sets $A, B \subset A_k$ with $m(B) > 0$ and all $n \geq 0$ that

$$\frac{m(f_*^{-n}(A))}{m(f_*^{-n}(B))} = \frac{\int_A |(f_*^{-n})'|^h dm}{\int_B |(f_*^{-n})'|^h dm} \leq \frac{\sup_{A_k} \{|(f_*^{-n})'|^h\} m(A)}{\inf_{A_k} \{|(f_*^{-n})'|^h\} m(B)} \leq K^h \frac{m(A)}{m(B)}.$$

and similarly

$$\frac{m(f_*^{-n}(A))}{m(f_*^{-n}(B))} \geq K^{-h} \frac{m(A)}{m(B)}.$$

Since by Theorem 4.24 the measure is conservative ergodic, all the assumptions of Theorem 5.1 have been checked and we are done. ■

The following lemma easily follows from Theorem 5.1.

Lemma 5.3. *For every $n \geq 0$ we have $0 < \mu(A_n) < \infty$.*

We say that the f -invariant measure μ produced in Theorem 5.2 is of finite condensation at $x \in J(f)$ if and only if there exists an open neighborhood V of x such that $\mu(V) < \infty$. Otherwise μ is said to be of infinite condensation at x . We respectively say that x is a point of finite or infinite condensation of μ . We end this subsection with the following obvious results.

Lemma 5.4. *If x is a point of infinite condensation of μ , then each point of the closure $\overline{\{f^n(x) : n \geq 0\}}$ is also of infinite condensation of μ .*

Lemma 5.5. *The set of points of infinite condensation of measure μ is contained in the union $O_+(\text{Crit}(f)) \cup \Omega \cup \{\infty\}$.*

Proof. If $z \notin \overline{O_+(\text{Crit}(f)) \cup \Omega \cup \{\infty\}}$, then by local finiteness of the family $\{A_n : n \geq 0\}$ there exist an open neighborhood V of z and an integer $k \geq 0$ such that $m(V \setminus \bigcup_{j=0}^k A_j) = 0$.

Hence, in view of Lemma 5.3 and Theorem 5.2 ($\mu \prec m$) we get $\mu(A) \leq \sum_{j=0}^k \mu(A_j) < \infty$. The proof is finished. ■

5.2. ∞ is a Point of Finite Condensation of μ .

The goal of this subsection is to prove that ∞ is a point of a finite condensation of the measure μ . We start with the following.

Lemma 5.6. *For every $R > 1$ large enough there exists a constant $C_1(R) > 0$ such that $m(B_b(R)) \leq C_1(R) \text{diam}_s^h(B_b(R))$.*

Proof. For every $k \geq 0$ let $A_{k,R} = \{z \in \mathcal{C} : 2^k R \leq |z| < 2^{k+1} R\}$. As in the proof of Lemma 3.2 let

$$B_R^+ = \{z \in B_R \setminus \{\infty\} : \text{Im}z > 0\}$$

$$B_R^1 = \{z \in B_R \setminus \{\infty\} : \text{Im}z < 1\} \quad \text{and} \quad B_R^+ = \{z \in B_R \setminus \{\infty\} : \text{Im}z > 0\}.$$

We also put $A_{k,R}^+ = A_{k,R} \cap B_R^+$ and $A_{k,R}^- = A_{k,R} \cap B_R^-$. Using formula (2.3) we can write for all $b \in f^{-1}(\infty)$, all $j \in \{1, \dots, q_b\}$ and all $k \geq 0$ that

$$m(f_{b,B_R^+,j}^{-1}(A_{k,R}^+)) = \int_{A_{k,R}^+} |(f_{b,B_R^+,j}^{-1})^*|^h dm \asymp (1 + |b|^2)^{-h} (2^k R)^{\frac{q_b-1}{q_b}h} m(A_{k,R}^+)$$

and similarly

$$m(f_{b,B_R^+,j}^{-1}(A_{k,R}^-)) \asymp (1 + |b|^2)^{-h} (2^k R)^{\frac{q_b-1}{q_b}h} m(A_{k,R}^-)$$

Thus

$$m(f_{b,R,j}^{-1}(A_{k,R})) = m(f_{b,R,j}^{-1}(A_{k,R}^+)) + m(f_{b,R,j}^{-1}(A_{k,R}^-)) \asymp (1 + |b|^2)^{-h} (2^k R)^{\frac{q_b-1}{q_b}h} m(A_{k,R}).$$

Summing now over all $j \in \{1, \dots, q_b\}$, we get

$$m(A_{k,R,b}) \asymp (1 + |b|^2)^{-h} (2^k R)^{\frac{q_b-1}{q_b}h} m(A_{k,R}) \tag{5.2}$$

where $A_{k,R,b} = B_b(R) \cap f^{-1}(A_{k,R})$. Therefore, putting $S = \sum_{w \in \Lambda} (1 + |w|^2)^{-h} < \infty$ (since by Theorem 2.1 $h > 1$), we obtain

$$\begin{aligned} m(f^{-1}(A_{k,R})) &= \sum_{b \in f^{-1}(\infty)} m(A_{k,R,b}) \\ &= \sum_{b \in \mathcal{R} \cap f^{-1}(\infty)} \sum_{w \in \Lambda} m(A_{k,R,b+w}) \\ &\asymp \sum_{b \in \mathcal{R} \cap f^{-1}(\infty)} \sum_{w \in \Lambda} (1 + |b+w|^2)^{-h} (2^k R)^{\frac{q_b-1}{q_b}h} m(A_{k,R}) \\ &\asymp m(A_{k,R}) \sum_{b \in \mathcal{R} \cap f^{-1}(\infty)} (2^k R)^{\frac{q_b-1}{q_b}h} \sum_{w \in \Lambda} (1 + |b+w|^2)^{-h} \\ &\asymp m(A_{k,R}) S (2^k R)^{\frac{q-1}{q}h} \end{aligned}$$

Hence $m(A_{k,R}) \asymp (2^k R)^{\frac{1-q}{q}h} S^{-1} m(f^{-1}(A_{k,R}))$ where $q = \max\{q_b : b \in \mathcal{R} \cap f^{-1}(\infty)\}$. Combining this and (5.2), we get for every $b \in f^{-1}(\infty)$ that

$$\begin{aligned} m(A_{k,R,b}) &\asymp (1 + |b|^2)^{-h} (2^k R)^{(1-\frac{1}{q})h} (2^k R)^{(\frac{1}{q}-1)h} S^{-1} m(f^{-1}(A_{k,R})) \\ &\leq (1 + |b|^2)^{-h} S^{-1} m(f^{-1}(A_{k,R})) \asymp (1 + |b|^2)^{-h} m(f^{-1}(A_{k,R})) \end{aligned}$$

Summing now over all $k \geq 0$ we get $m(B_b(R)) \preceq (1 + |b|^2)^{-h} m(f^{-1}(B_R)) \leq (1 + |b|^2)^{-h}$. Combining in turn this with (2.4) we get

$$m(B_b(R)) \preceq L^h R^{\frac{h}{q}} \text{diam}_s^h(B_b(R)) \quad (5.3)$$

The proof is complete. ■

Lemma 5.7. *Fix $R > 2$ sufficiently large. Re-numerating the elements of the partition $\{A_j\}_{j=0}^\infty$, we may assume that $A_0 \subset B_R$ and $\text{diam}_s(A_0) = 1$. For every $b \in f^{-1}(\infty)$ and every $n \geq 0$ let $A^{(n)} = f^{-n}(A_0) \cap B_n$, where B_n is a connected component of $f^{-n}(B_R)$. Then there exists a constant $C_2 > 0$ such that $m(B_n) \leq C_2(R)m(A^{(n)})$.*

Proof. It follows from the construction of the partition $\{A_n\}_{n \geq 0}$ that

$$m(A^{(n)}) \asymp \text{diam}_s^h(A^{(n)}) \quad (5.4)$$

Since $\text{dist}(0, A_0) \geq R > 2$ and since $\text{diam}(A_0) = 1$ using (2.3), and (2.4), we get for every pole $b \in f^{-1}(\infty)$ that

$$\frac{\text{diam}_s(A_{0,b})}{\text{diam}_s(B_b(R))} \asymp \frac{(1 + |b|^2)^{-1} \text{dist}(0, A_0)^{\frac{q_b-1}{q_b}} \text{diam}_s(A_0)}{(1 + |b|^2)^{-1} R^{\frac{-1}{q_b}}} \geq R^{\frac{q_b-1}{q_b}} R^{\frac{1}{q_b}} = 1, \quad (5.5)$$

where $A_{0,b} = f^{-1}(A_0) \cap B_b(R)$. Since $\omega(\text{Crit}(f))$ is a compact subset of the complex plane \mathcal{C} , $\text{dist}(\omega(\text{Crit}(f)), f^{-1}(\infty)) > 0$. Therefore there exists $r > 0$ such that for all $R > 1$ large enough $B_b(R) \subset B(b, r)$ and $B(b, 2r) \cap O_+(\text{Crit}(f)) = \emptyset$. Since $B_n = f_*^{-(n-1)}(B_b(R))$ for an appropriate holomorphic inverse branch $f_*^{-(n-1)} : B(b, 2r) \rightarrow \mathcal{C}$ of $f^{(n-1)}$, it follows from Koebe's distortion theorem and (5.5)

$$\frac{\text{diam}_s(A^{(n)})}{\text{diam}_s(B_n)} = \frac{\text{diam}_s(f_*^{-(n-1)}(A_{0,b}))}{\text{diam}_s(f_*^{-(n-1)}(B_b(R)))} \asymp \frac{\text{diam}_s(A_{0,b})}{\text{diam}_s(B_b(R))} \succeq 1$$

and that

$$\frac{\text{diam}_s^h(B_n)}{m(B_n)} = \frac{\text{diam}_s^h(f_*^{-(n-1)}(B_b(R)))}{m(f_*^{-(n-1)}(B_b(R)))} \asymp \frac{\text{diam}_s^h(B_b(R))}{m(B_b(R))}.$$

Combining the last two formulas and (5.4) we get

$$m(A^{(n)}) \succeq \text{diam}_s^h(B_n) \asymp \left(\frac{\text{diam}_s^h(B_b(R))}{m(B_b(R))} \right) m(B_n) \succeq m(B_n)$$

The proof is complete. ■

We are ready now to prove the main result of this section.

Theorem 5.8. ∞ is the point of finite condensation of the measure μ .

Proof. Take $R > 0$ so large as required in Lemma 5.7. It follows from this lemma that $m(f^{-k}(B_R)) \leq C_2(R)m(f^{-1}(A_0))$ for every $k \geq 0$. Thus, applying Theorem 5.1, we get

$$\mu(B_R) = \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n m(f^{-1}(B_R))}{\sum_{k=0}^n m(f^{-1}(A_0))} \leq C_2(R) < \infty.$$

We are done. ■

5.3. All Points of Finite and Infinite Condensation. We say that $z \in J(f) \setminus \Omega$ is geometrically good if

$$m(B_n) \preceq \text{diam}^h(B_n) \tag{5.6}$$

for every set B of sufficiently small diameter containing x , every $n \geq 0$ and every connected component B_n of $f^{-n}(B)$. The direction of the inequality above makes that checking geometrical goodness one can assume the sets B to be balls centered at x . The most general sufficient condition for finite condensation is the following.

Lemma 5.9. *If $z \in J(f) \setminus \Omega$ is geometrically good, then z is a point of finite condensation of measure μ .*

Proof. Since $z \notin \Omega$, taking $\theta > 0$ sufficiently small, $z \notin B(\Omega, \theta)$. Set $B = B(z, \gamma)$. Since $m(B) > 0$ and $m(\bigcup_{n \geq 0} A_n) = 1$, there exists $i \geq 0$ such that $m(B \cap A_i) > 0$. Since $B \cap A_i \cap J(f)$ has a non-empty interior relative to $J(f)$, there exists an open ball $F \subset B \cap A_i$ having nonempty intersection with $J(f)$. Of course $m(F) > 0$. For every $n \geq 0$ let B_n be a connected component of $f^{-n}(B)$ and let $F_n \subset B_n$ be some connected component of $f^{-n}(F)$ contained in B_n . Using Koebe's Distortion Theorem, I (Euclidean version) and the fact that the point z is geometrically good, we get

$$m(F_n) \asymp \text{diam}^h(F_n) = \left(\frac{\text{diam}(F_n)}{\text{diam}(B_n)} \right)^h \text{diam}^h(B_n) \succeq m(B_n) \left(\frac{\text{diam}(F_n)}{\text{diam}(B_n)} \right)^h$$

Applying now Lemma 2.12 to the connected sets F and B we obtain

$$m(F_n) \succeq m(B_n) \left(\frac{\text{diam}(F)}{\text{diam}(B)} \right)^h.$$

Thus

$$\sum_{k=0}^n m(f^{-k}(B)) \preceq \sum_{k=0}^n m(f^{-k}(F)) \preceq \sum_{k=0}^n m(f^{-k}(A_i)).$$

Hence, using Lemma 5.3, we get $\mu(B) \preceq \mu(A_i) < \infty$ and therefore z is a point of finite condensation of μ . ■

In order to make use of this lemma we need to provide sufficient conditions for points to be geometrically good. This is done below.

Lemma 5.10. *If μ is h -upper estimable at every point $z \in J(f)$ with the same estimability constant, then every point $z \in J(f)$ is geometrically good.*

Proof. The proof of this lemma follows by a straightforward inductive argument incorporating Koebe's Distortion Theorem, Lemma 4.13, finiteness of the equivalence classes of the relation \sim on the set of critical points of f , Lemma 2.8, and equivalently (2.16). ■

Theorem 5.11. *The set of points of infinite condensation of μ is contained in the set of parabolic points $\Omega(f)$.*

Proof. The proof of Lemma 4.23 shows that each point $z \in J(f)$ is upper estimable with respect to the Euclidean h -conformal measure m_e and so, also with respect to the measure m . Therefore the proof of Theorem 5.11 is completed by applying Lemma 5.10 and Lemma 5.9. ■

Corollary 5.12. *If $\Omega = \emptyset$, then there exists an f -invariant probability measure μ equivalent to m .*

§6. Invariant measure - Parabolic points. From what we have shown in the previous section, it is clear that in order to localize the points of infinite condensation of μ we have to look at the parabolic points. Proceeding in exactly the same way as in Section 6 of [22], we can prove the following.

Proposition 5.13. *If $\omega \in \Omega \setminus \overline{O_+(\text{Crit}(f))}$, then μ is of infinite condensation at ω if and only if $h \leq \frac{2p(\omega)}{p(\omega)+1}$.*

Corollary 5.14. *If*

$$\max\{q_b : b \in \mathcal{R} \cap f^{-1}(\infty)\} \geq \sup \left\{ \frac{2p(\omega)}{p(\omega)+1} : \omega \in \Omega \right\},$$

then the invariant measure μ is finite.

Proposition 5.15. *If $\omega \in \Omega$ and $h \leq \frac{2p(\omega)}{p(\omega)+1}$, then μ has infinite condensation at ω .*

Theorem 5.16. *If $c \in J(f)$ is a critical point of f of order q , $\omega = f(c) \in \Omega$, and $h \leq \frac{2qp(\omega)}{p(\omega)+1}$, then ω is of infinite condensation of measure μ .*

6. APPENDIX

The goal of this appendix is to provide a proof of Theorem 2.6. We first prove a version of Przytycki's lemma from [19] for the sake of completeness and then we prove a version of Mane's theorem from [13]. We decided to provide a full proof of this theorem since its original Mane's proof contains some minor misprints and it would be very difficult to indicate in which places and in which way one needs to modify it.

Lemma 6.1. *For every integer $K \geq 0$ and every $0 < \lambda < 1$ the following holds. For every $\epsilon > 0$ and every $\kappa > 0$ there exists $\delta_0 = \delta_0(K, \epsilon, \lambda, \kappa) > 0$ such that for every disk $B(x, \delta)$ with $\delta \leq \delta_0$ and every $x \in \mathcal{C}$ in the distance at least κ apart from the set of parabolic points and attracting points, for every $n \geq 0$ and every connected component $W = \text{Comp}f^{-n}(B(x, \delta))$ such that $f|_W^n$ has at most K critical points counted with multiplicities, for every component $W' = \text{Comp}(f^{-n}(B'))$ in W , for the disc $B' = B(x, \lambda\delta)$ we have*

$$\text{diam}W' \leq \epsilon$$

$\text{diam}W' \rightarrow 0$ for $n \rightarrow \infty$ uniformly (i.e. independently of the choices of B and W').

Proof. Suppose on the contrary that there exist a sequence $\{x_n\}_{n=1}^{\infty}$ of points in the distance at least κ apart from the set of parabolic points and attracting points, a sequence $\delta_n \searrow 0$, a sequence of components $W_n = \text{Comp}f^{-k_n}(B_n)$ with $k_n \rightarrow \infty$, as $n \rightarrow \infty$ such that the number of critical points of each map f^{k_n} on W_n is bounded by K and W'_n , the sequence associate to W_n as in the statement of the lemma, such that $\lim_{n \rightarrow \infty} \text{diam}(W'_n) = 0$. Then for each n there exists $L = L(n) : 0 \leq L \leq K$ such that there is no critical value of $f|_{W_n}^{k_n}$ in

$$P(n) := B(x_n, \delta_n(\lambda + (1 - \lambda)\frac{L+1}{K+1})) \setminus B(x_n, \delta_n(\lambda + (1 - \lambda)\frac{L}{K+1})).$$

Without losing generality we may assume that all the components W'_n intersect the fundamental region \mathcal{R} . Put

$$W_n^{(1)} := \text{Comp}f^{-k_n}B(x_n, \delta_n(\lambda + (1 - \lambda)\frac{L(n)}{K+1}))$$

$$W_n^{(2)} := \text{Comp}f^{-k_n}B(x_n, \delta_n(\lambda + (1 - \lambda)\frac{L(n)+1}{K+1}))$$

the components containing W'_n ,

$$P_n := W_n^{(2)} \setminus W_n^{(1)}$$

and for every $0 \leq m \leq k_n$, $i = 1, 2$,

$$W_{n,m}^{(i)} = f^{k_n-m}(W_n^{(i)}), \quad P_{n,m} := f^{k_n-m}(P_n) = W_{n,m}^{(2)} \setminus W_{n,m}^{(1)}.$$

Let, for each n , the number $m = m(n) \leq k_n$ be the least integer such that

$$\text{diam}W_{n,m}^{(1)} \geq \min(\eta, \inf_{c_1, c_2 \in \text{Crit}(f), c_1 \neq c_2} \text{dist}(c_1, c_2)),$$

So for every $0 \leq t < m(n)$ the set $P_{n,t}$ is a topological annulus. That is so because at each step back by f^{-1} from $P_{n,t-1}$ to $P_{n,t}$ there is at most one branch point for f^{-1} from $W_{n,t-1}^{(i)}$ to $W_{n,t}^{(i)}$, $i = 1, 2$. Now, all the annuli $P_{n,m(n)-1}$'s have moduli bounded below by $2^{-K}(1-\lambda)\frac{1}{K+1}$. Since in addition all the components W'_n intersect the fundamental region \mathfrak{R} , it follows from Montel's Theorem that there exists a topological (maybe not geometric) annulus P contained in all $P_{n,m(n)-1}$'s for a subsequence n_s , which bounds a topological disk D . So $D \subset W_{n_s, m(n_s)-1}^{(2)}$. Hence $f^{m(n_s)-1}(D) \subset B(x, \delta_n)$. Passing to yet another subsequence we may assume that the sequence x_n converges to a point $y \in \mathcal{C}$ at the distance at least κ apart from the set of parabolic points and attracting points. Since $\delta_n \rightarrow 0$, we have also $m(n) \rightarrow \infty$. Thus D cannot intersect the Julia set $J(f)$. If the were contained in a preimage of a Siegel disk or a Herman ring, the limit of diameters of iterate $f^{m(n_s)-1}(D)$ would be positive. Thus D is either contained in the basin of attraction to an attracting periodic orbit or a parabolic periodic orbit. In either case the limit of the sets $f^{m(n_s)-1}(D)$ would be contained in either an an attracting periodic orbit or a parabolic periodic orbit. Since this limit would coincide with y , we get a contradiction. The proof is complete. ■

Remark 6.2. *Obviously this lemma remains true (with the proof required only minor modifications) if instead of the disk $B(x, \delta)$ one takes the square centered at x and with edges of length δ . This is the version we will need in the next theorem.*

Theorem 6.3. *Let $f : \mathcal{C} \rightarrow \overline{\mathcal{C}}$ be an elliptic function. If a point $x \in J(f) \setminus \Omega(f)$ is not contained in the ω -limit set of a recurrent critical point, then for all $\epsilon > 0$ there exists a neighbourhood U of x such that:*

- (a) *For all $n \geq 0$, every connected component of $f^{-n}(U)$ has diameter $\leq \epsilon$;*
- (b) *There exists $N > 0$ such that for all $n \geq 0$ and every connected component V of $f^{-n}(U)$, the degree of $f|_V^n$ is $\leq N$;*

Proof. The core of the theorem is (a), from which the property (b) will easily follow. Given an open set $U \subset \overline{\mathcal{C}}$ denote $c(U, n)$ the set of connected components of $f^{-n}(U)$. Observe that $V \in c(U, n)$ implies $f^j(V) \in c(U, n-j)$ for all $0 \leq j \leq n$. If $V \in c(U, n)$ define $\Delta(V, n) = \#\{x \in V; (f^n)'(x) = 0\}$ counted with algebraic multiplicity. A square is the set S of the form $S = \{z \in \mathcal{C} : |\Re(z-p)| < \delta, |\Im(z-p)| < \delta\}$. The point p is the center and δ

its radius. Given a square S with center p and radius δ , then, given $k > 0$, denote by S^k the square with center p and radius $k\delta$. If S is a square with radius δ , denote by $\mathcal{L}(S)$ the family of squares contained in $S^{\frac{3}{2}} - S$ and having radius $\delta/4$. Denote by $\mathcal{L}^*(S)$ the family of squares $S_0^{\frac{3}{2}} - S$ and having radius $\delta/4$. Denote by $\mathcal{L}^*(S)$ the family of squares $S_0^{\frac{3}{2}}$ with $S_0 \in \mathcal{L}(S)$. Suppose that x is not a parabolic point or is contained in the ω -limit set of recurrent critical point. Then there exists $\delta_0 > 0$ such that

- (1) There is no critical point c of f such that there exists $0 \leq n_1 \leq n_2$ satisfying

$$|f^{n_1}(c) - c| < \delta_0$$

and

$$|f^{n_2}(c) - c| < \delta_0$$

- (2) $|x - p| > 10\delta_0$ for every parabolic or attracting periodic point p .

Given $\epsilon > 0$ take $\epsilon_1 > 0$ satisfying

- (3) $0 < \epsilon_1 < \min\{\epsilon/10, \delta_0/10\}$
(4) if U is an open connected set with $\text{diam}(U) \leq 2\epsilon_1$ then $\text{diam}(W) \leq \delta_0$ for all $W \in c(U, 1)$

Let N_0 be the number of equivalence classes of the relation \sim between critical points of f . Take $N_1 > 2$ such that

- (5) If S is a square and $V \in c(S, n)$ satisfies $\Delta(V, n) \leq N_0 + 1$ then the number of connected components of $f^{-n}(S^{\frac{2}{3}})$ contained in V is $\leq N_1$.

Finally, take δ given by

- (6) $\delta = \min\{\delta_0/10, \epsilon_1/10, \delta(2N_0, \frac{\epsilon_1}{20N_1}, \frac{2}{3}, \delta_0)\}$ where $\delta(2N_0, \frac{\epsilon_1}{20N_1}, \frac{2}{3}, \delta_0)$ is given by Lemma 6.1.

Let S_0 be the square of center x and radius δ . Suppose that Theorem 6.3 fails for $U = S_0$. Then there exists $n > 0$ and $V \in c(S_0, n)$ with $\text{diam}V \geq \epsilon \geq 10\epsilon_1$. On the other hand, by (1), $\text{diam}S_0 = 2\sqrt{2}\delta < 3\delta < \epsilon_1$. Hence there exists an integer $n_0 \geq 0$ such that there exists $V_0 \in c(S_0^{\frac{3}{2}}, n_0)$ satisfying

- (7) $\text{diam}(f^{-(n_0-i)}(S_0) \cap f^i(V_0)) \leq \epsilon_1$ for all $1 \leq i \leq n_0$, and
(8) $\text{diam}(f^{-n_0}(S_0) \cap V_0) > \epsilon_1$

Since $\text{diam}S_0 < \epsilon_1$ it follows that $n_0 > 0$. Now, starting with S_0 we shall construct a sequence of squares S_0, S_1, S_2, \dots and strictly positive integers $n_0 \geq n_1 \geq n_2 \dots$ satisfying

- (9) $S_{j+1} \in \mathcal{L}^*(S_j)$
(10) there exists $V_j \in c(S_j^{\frac{3}{2}}, n_j)$ such that

$$\text{diam}(f^{(-n_j-i)}(S_j) \cap f^i(V_j)) \leq \epsilon_1$$

for all $i \leq i \leq n_j$ and

$$\text{diam}(f^{-n_j}(S_j) \cap V_j) > \epsilon_1.$$

From (7) and (8), it follows that S_0 satisfies (10). If we construct such a sequence of squares and integers, then Theorem 6.3 will be proved by contradiction because the condition $n_0 \geq n_1 \geq n_2 \dots \geq n_m \geq \dots > 0$ implies that $n_j = n_i$ for all $j \geq i$ for a certain i . But (a) implies that the radius of S_j is $(\frac{3}{8})^j \delta$; in particular $\text{diam}(S_j) \rightarrow 0$ when $j \rightarrow +\infty$. But by (10),

$$\epsilon_1 < \text{diam}(f^{-n_j}(S_j) \cap V_j) = \text{diam}(f^{-n_i}(S_j) \cap V_j),$$

$$V_j \in c(S_j^{\frac{3}{2}}, n_j) = c(S_j^{\frac{3}{2}}, n_i)$$

Taking $j \rightarrow +\infty$, and recalling that i is contained and $\lim_{j \rightarrow \infty} \text{diam} S_j = 0$, we conclude that the inequality above cannot hold.

The sequence $\{S_j\}$ and $\{n_j\}$ will be constructed by induction starting with S_0 . Suppose S_i and n_i constructed for $0 \leq i \leq j$. To find S_{j+1} and n_{j+1} we begin by observing that from (a) it follows that if $p \in S \in \mathcal{L}^*(S_j)$, then, by

$$|p - x| \leq \sum_{i=0}^j \text{diam}(S_i) = \sum_{i=0}^{j+1} \left(\frac{3}{8}\right)^i \text{diam}(S_0) = 2\sqrt{2} \sum_{i=0}^{j+1} \left(\frac{3}{8}\right)^i \delta \leq 4\sqrt{2}\delta.$$

Hence, if a point q satisfies $\text{dist}(q, S) \leq \delta_0$, we have

$$|q - x| \leq 4\sqrt{2}\delta + \delta_0 \leq 2\delta_0.$$

By (2), this means that

$$(11) \quad \text{dist}(q, S) > \delta_0 \text{ for all } S \in \mathcal{L}^*(S_j) \text{ and all parabolic or attracting periodic point } q.$$

For the induction step (i.e. the construction of S_{j+1} and n_{j+1}), we shall use the following lemma.

Lemma 6.4. *If $U \subset \mathcal{C}$ is an open set and $V \in c(U, n)$ satisfies*

$$\text{diam} f^i(V) \leq \delta_0, \quad 0 \leq i \leq n$$

then

$$\Delta(V, n) \leq N_0.$$

Proof. If $\Delta(V, n) \geq N_0 + 1$, there exists $N_0 + 1$ different points x_i , $1 \leq i \leq N_0 + 1$, in V_j such that $(f^{n_j})'(x_j) = 0$. This means that for each $1 \leq i \leq N_0 + 1$ there exist $1 \leq m_i < n$, such that $f^{m_i}(x_i)$ is a critical point. Recalling that N_0 is the number of the equivalence classes of the equivalence relation \sim , it follows that there exists two different points in the set $\{x_i; 1 \leq i \leq N_0 + 1\}$, that we shall denote by x_1, x_2 , and two critical points c_1 and c_2 in the same equivalence class of the equivalence relation \sim , such that $f^{m_1}(x_1) = c_2$ and $f^{m_2}(x_2) = c_2$. Assume without loss of generality that $0 \leq m_1 \leq m_2$. Then by the choice of δ_0 , $m_1 < m_2$ and

$$|f^{m_2-m_1}(c_1), c_1| = |f^{m_2-m_1}(c_2), c_1| = |f^{m_2}(x_1) - f^{m_2}(x_2)| \leq \text{diam} f^{m_2}(V_j) \leq \delta_0$$

and

$$|f^{n-m_1}(c_1), x| = |f^{n-m_1}(f^{m_1}(x_1)) - x| = |f^n(x_1) - x| \leq \delta_0$$

contradicting property (1) of δ_0 . ■

Now, to find S_{j+1} and n_{j+1} we first claim that there exists $S \in \mathcal{L}(S_j)$ that for some $0 < n \leq n_j$ has $V \in c(S, n)$ with $\text{diam} V \geq \frac{\epsilon_1}{10N_1}$. Suppose that the claim is false. Then, for all $1 \leq i \leq n_j$,

$$\begin{aligned} \text{diam}(f^i(V_j)) &\leq \text{diam}\left(f^{-(n_j-i)}(S_j) \cap f^i(V_j)\right) \\ &\quad + \sup\{\text{diam}(W); W \in c(S, n_j - i), S \in \mathcal{L}(S_j)\} \\ &\leq \epsilon_1 + \frac{\epsilon_1}{10N_1} \leq 2\epsilon_1 \end{aligned}$$

From this inequality applied to $i = 1$ and property (4), we have

$$\text{diam}(V_j) \leq \delta_0$$

Moreover, since $2\epsilon_1 \leq \delta_0$ (by (3)),

$$\text{diam}(f^i(V_j)) \leq \delta_0$$

for all $1 \leq i \leq n_j$, hence for all $0 \leq i \leq n_j$. By Lemma 6.4. This proves that $\Delta(V_j, n_j) \leq N_0$. Then, since $V_j \in c(S_j^{\frac{2}{3}}, n_j)$ it follows from (5), (11) and Lemma 6.4 that

$$W \in c(S_j, n_j), W \subset V_j \rightarrow \text{diam}(W) \leq \frac{\epsilon_1}{10N_1}$$

Moreover, by the way N_1 was chosen, we have

$$\#\{W \in c(S_j, n_j); W \subset V_j\} \leq N_1$$

and we are assuming that

$$S \in \mathcal{L}(S_j), U \in (S, n_j) \rightarrow \text{diam}(U) \leq \frac{\epsilon}{10N_1}.$$

Now observe that V_j is the union of sets $U \in c(S, n_j), U \subset V_j, S \in \mathcal{L}(S_j)$ and the sets $W \in c(S_j, n_j), W \subset V_j$. Moreover, for any two sets W', W'' in this family there exist $W' = W_0, W_1, \dots, W_k = W''$ in $c(S_j, n_j)$ and contained in V_j such that for all $0 \leq i < k$ there exists $S_i \in \mathcal{L}(S_j)$ and $U_i \in c(S_i, n_j)$ such that $\overline{U}_i \cap \overline{W}_i \neq \emptyset$ $\overline{U}_i \cap \overline{W}_{i+1} \neq \emptyset$. Then

$$\text{diam}(V_j) \leq N_1 \left(\frac{\epsilon_1}{10N_1} + \frac{\epsilon_1}{10N_1} \right) = \frac{\epsilon_1}{5}$$

contradicting the last inequality in condition (10). This completes the proof of the claim. Now we can take $S \in \mathcal{L}(S_j)$ such that $\text{diam}(V) \geq \frac{\epsilon}{10N_1}$ for some $V \in c(S, n), 0 \leq n \leq n_j$. Take $\tilde{V} \in c(S^{\frac{3}{2}}, n)$ containing V . Suppose that $\Delta(\tilde{V}, n) \leq N_0$. Then by Lemma 6.1 and condition (6)

$$\text{diam}(V) \leq \frac{\epsilon_1}{20N_1}$$

since $V \in c((S^{\frac{3}{2}})^{\frac{2}{3}}, n)$ and is contained in \tilde{V} . This contradicts the fact that

$$\text{diam}(V) \geq \frac{\epsilon_1}{10N_1}$$

and proves $\Delta(\tilde{V}, n) \geq N_0 + 1$. From Lemma 6.4, it follows that

$$\text{diam}(f^i(\tilde{V})) > \delta_0$$

for some $0 \leq i \leq n$. Now we define $S_{j+1} = S^{\frac{3}{2}}$. Then $f^i(\tilde{V}) \in c(S^{\frac{3}{2}}, n-i)$ and $\text{diam}(f^i(\tilde{V})) > \delta_0 \geq 10\epsilon_1$. Moreover $\text{diam}S_{j+1}^{\frac{3}{2}} \leq 2\delta < \epsilon_1$. Then there exists $0 \leq n_{j+1} \leq n-i \leq n_j-i$ and $V_{j+1} \in c(S_{j+1}^{\frac{3}{2}}, n_{j+1})$ such that

$$\text{diam}(f^{-n_{j+1}}(S_{j+1}) \cap V_{j+1}) > \epsilon_1$$

and

$$\text{diam}(f^{-n_{j+1}+i}(S_{j+1}) \cap f^i(V_{j+1})) \leq \epsilon_1.$$

Observe that $n_{j+1} > 0$ since $\text{diam}(S_{j+1}) < 2\delta < \epsilon_1$. This completes the construction of the sequence $\{S_j\}$ and $\{n_j\}$ and the proof of part (a) of Theorem 6.3. Property (b) of Theorem 6.3 follows from (a) and Lemma 6.4. ■

Our destination in this appendix is the following.

Proof of Theorem 2.6 If X is a compact subset of the complex plane \mathcal{C} the theorem immediately follows from Theorem 6.3 and compactness of X . So suppose that $X \subset J(f) \setminus \Omega(f)$ is a closed subset of \mathcal{C} . Let $\Delta = \text{dist}(\Omega(f), f^{-1}(\infty)) > 0$. In view of (2.2) and (2.4) there exists $R > 0$ so large that if $|f(z)| \geq R/2$, then for some $b \in f^{-1}(\infty), z \in B_b(R/2)$

$$|f'(z)| \geq 2 \text{ and } \text{diam}(B_b(R/2)) \leq \Delta/2. \quad (6.1)$$

Consider now the compact set $Y = X \cup (\mathcal{C} \setminus B(\Omega(f), \Delta/2)) \setminus B_R$ and the corresponding number $0 < \delta \leq \min\{\epsilon, R/2\}$ ascribed to Y and the number $\min\{\epsilon, R/2\}$. In order to complete the proof it suffices to show that if $x \in B_R$, then the diameter of each connected component $C_n(x)$ of $f^{-n}(B(x, \delta))$ does not exceed ϵ for every $\epsilon > 0$. And indeed, fix $w \in f^{-n}(x) \cap C_n(x)$ and let $1 \leq k \leq n$ be the least integer such that $f^{n-k}(w) \notin B_R$ provided it exists. Otherwise, set $k = n$. We shall show by mathematical induction that

$$\text{diam}(f^{n-j}(C_n(x))) \leq \delta \leq \min\{\epsilon, R/2\} \quad (6.2)$$

for every $0 \leq j \leq k$. For $j = 0$ this formula is true since $f^n(C_n(x)) = B(x, \delta)$. So, suppose that it is true for some $0 \leq j \leq k-1$. Since $f^{n-j}(w) \in B_R$ and since $\text{diam}(f^{n-j}(C_n(x))) \leq R/2$, we conclude that

$$f^{n-j}(C_n(x)) \subset B_{R/2}. \quad (6.3)$$

It therefore follows from the first part of formula (6.1) that

$$\text{diam}(f^{n-j+1j}(C_n(x))) \leq \frac{1}{2} \text{diam}(f^{n-j}(C_n(x))) \leq \delta.$$

This proves formula (6.2). It follows from (6.3) and the second part of formula (6.1) that $f^{n-k}(C_n(x)) \subset \mathcal{C} \setminus B(\Omega(f), \Delta/2)$. Since we also know that $f^{n-k}(w) \notin B_R$, we conclude that $f^{n-k}(w) \in Y$, we see that $\text{diam}(C_n(x)) \leq \min\{\epsilon, R/2\} \leq \epsilon$. We are done. ■

REFERENCES

- [1] J. Aaronson, M. Denker, M. Urbański, Ergodic theory for Markov fibered systems and parabolic rational maps, *Transactions of A.M.S.* 337 (1993), 495-548.
- [2] I. N. Baker, J. Kotus, Y. Lü, Iterates of meromorphic functions I, *Ergod. Th. and Dynam. Sys.* 11 (1991), 241-248.
- [3] I. N. Baker, J. Kotus, Y. Lü, Iterates of meromorphic functions IV, *Results Math.* 22 (1992), 651-656.
- [4] A.F. Beardon, *Iteration of Rational Maps*, Springer-Verlag, New York, 1991.
- [5] W. Bergweiler, Iteration of meromorphic functions. *Bull. Amer. Math. Soc.*, 29:2 (1993), 151-188.
- [6] M. Denker, M. Urbański, On the existence of conformal measures, *Trans. A.M.S.* 328 (1991), 563-587.
- [7] M. Denker, M. Urbański, On Sullivan's conformal measures for rational maps of the Riemann sphere, *Nonlinearity* 4 (1991), 365-384.
- [8] M. Denker, M. Urbański, Geometric measures for parabolic rational maps, *Ergod. Th. and Dynam. Sys.* 12 (1992), 53-66.
- [9] M. Denker, M. Urbański, The Capacity of parabolic Julia sets, *Math. Zeitsch.* 211, (1992), 73-86
- [10] M. Guzmán: Differentiation of integrals in \mathbb{R}^n . *Lect. Notes in Math.* 481, Springer Verlag.
- [11] J. Hawkins, L. Koss, Ergodic properties and Julia sets of Weierstrass elliptic functions, Preprint 2001.
- [12] J. Kotus, M. Urbański, Hausdorff dimension and Hausdorff measures of Elliptic Functions, to appear *Bull. London Math. Soc.*
- [13] R. Mané, On a theorem of Fatou, *Bol. Soc. Bras. Mat.*, 24 (1933), 1-11.
- [14] R. Mané, The Hausdorff dimension of invariant probabilities of rational maps, *Dynamical Systems, Valparaiso 1986*, *Lect. Notes in Math.* 1331, Springer-Verlag (1988), 86-117.
- [15] M. Martens, The existence of σ -finite invariant measures, Applications to real one-dimensional dynamics, Preprint.
- [16] R.L. Mauldin, M. Urbański, Dimensions and measures in infinite iterated function systems. *Proc. London Math. Soc.* 73:3 (1996), 105-154.
- [17] J. Milnor, *Dynamics in one Complex Variable, Introductory lectures*, Vieweg 2000.
- [18] F. Przytycki, Hausdorff dimension of harmonic measure on the boundary of an attractive basin for a holomorphic map, *Invent. Math.* 80 (1985), 169-171.
- [19] F. Przytycki, Iterations of holomorphic Collet-Eckmann maps: conformal and invariant measures. *Trans. AMS* 350.2 (1998), 717-742.
- [20] P.J. Rippon, G.M. Stallard, Iteration of a class of hyperbolic meromorphic functions, *Proc. Amer. Math. Soc.* 127 (1999), 3251-3258.
- [21] M. Urbański, Rational functions with no recurrent critical points, *Ergod. Th. and Dynam. Sys.* 14 (1994), 391-414.
- [22] M. Urbański, Geometry and ergodic theory of conformal non-recurrent dynamics, *Ergod. Th. and Dynam. Sys.* 17 (1997), 1449-1476.

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