Parabolic Cantor Sets

Mariusz Urbański[1]

Department of Mathematics, University of North Texas,
Denton, TX 76203-5116, USA
e-mail: urbanski@unt.edu

Abstract: The notion of a parabolic Cantor set is introduced allowing in the definition of hyperbolic Cantor sets some fixed points to have derivatives of modulus one. Such difference in the assumptions widely reflects in geometric properties of the Cantor set which are studied in detail. It turns out that if the Hausdorff dimension of this set is denoted by h, then its h-dimensional Hausdorff measure vanishes but the h-dimensional packing measure is positive and finite. This measure can be also dynamically characterized as the only h-conformal measure defined in a natural way appropriate in this context. It is relatively easy to see that any two parabolic Cantor sets formed with the help of the same alphabet are canonically topologically conjugate and we then discuss the rigidity problem of what are the possibly weakest sufficient conditions for this topological conjugacy to be "smoother". It turns out that if the conjugating homeomorphism preserves moduli of derivatives of periodic points, then the dimensions of both sets are equal and the homeomorphism is shown to be absolutely continuous with respect to the corresponding h-dimensional packing measures. This property in turn implies the conjugating homeomorphism to be Lipschitz continuous. Additionally the existence of the scaling function is shown and a version of rigidity theorem, expressed in terms of scaling functions, is proven. We also study the real analytic Cantor sets for which the stronger rigidity can be shown that the absolute continuity of the conjugating homeomorphism alone implies its real analyticity.

[1]Supported by NSF Grant DMS 9303888

§**1. Introduction.** The ultimate goal of this paper is to classify parabolic Cantor sets up to bi-Lipschitz and real analytic conjugacy. This is done in the last three sections of the paper. The first 6 sections forming the preparation for the classification part occupy a considerable part of the paper. In these sections we establish basic dynamical and geometric properties of a single parabolic Cantor set. The theory of parabolic Cantor sets takes roots from the theory of parabolic rational maps and expanding cookie-cutter Cantor sets. The former one shared larger part in this paper as a model and prototype for exploring properties of a single map. In particular it equipped us with the powerful method of conformal measures which turn out to be very convenient tools when hyperbolicity fails. One of our aims was to demonstrate in a relatively uncomplicated setting (extremely simple phase space - just the interval) how this machinery works. it has turned out to be very fruitful in various areas dealing with iterates of conformal maps. We mean here Kleinian groups, particularly the pioneering work of Patterson (see [Pa1], [Pa2]) who introduced the concept of conformal measures, and developing his approach work of Sullivan (see [Su3] and [Su5] for example). Sullivan has also brought the concept of conformal measures to the setting of rational functions (see [Su4], comp. [DU5] for example). This has resulted in a bunch of papers on the subject and subsequently, along with the "jump" construction (see Section 7), contributed to the recent development of the theory of conformal infinite iterated function systems (see [Ba] and [MU] for example).

On the other hand the theory of expanding Cantor sets (see for example [Be], [LS], [Pr2], [Pr3], [PT], [Su1], and [Su2]) where also a more complete collection of literature can be found) mainly provided us with the framework to investigate conjugacy classes for parabolic Cantor sets.

The part of the presentation of those properties of a single map which actually do not appeal to the one-dimensional and totally ordered structures of the interval is to high degree comparable with the presentation given in the papers [ADU], [DU1] - [DU4], [U1], and [U2] for rational functions. In this respect the technical difference between parabolic Cantor sets and parabolic rational functions is that these latter ones are not required to be analytic - we actually attempt to work here with as little amount of smoothness as possible. One of the primary tools as well here as in the setting of parabolic rational maps and expanding Cantor sets is the bounded distortion of derivatives along long inverse branches of iterates. It is the classical fact today that the distortion is bounded for expanding (hyperbolic) systems. In case of parabolic rational maps we have the Koebe distortion theorem at hands, and finally this is a technical problem which focuses our attention in the second section of this paper.

From the theory of expanding Cantor sets we mostly borrowed and adopted to our setting the concept of scaling function and the rigidity problem. I contrast to what is going on in the case of expanding Cantor sets, geometry of parabolic Cantor sets fails to be bounded. Nevertheless it continues to be determined, up to the level of bi-Lipschitz conjugacy, by the scaling function. The geometry is also determined (again up to bi-Lipschitz conjugacy) by the packing measure class and the Hausdorff dimension of the Cantor set. This much less evident than in the case of expanding sets. The point is that for expanding sets there is an extremely simple relation between conformal (equivalently packing) measure of a ball and the power of its radius, power taken with the exponent being the Hausdorff dimension of the Cantor set under consideration. Namely, these two quantities are almost proportional - their ratio stays bounded away from zero and infinity. For parabolic Cantor sets the relation between radii of balls and their conformal measures is more complex. Proving Lipschitz conjugacy becomes technically more involved.

Of special attention is Section 9, where dealing with real analytic systems, employing the methods of complex analytic functions and, indirectly the concept of nonlinearity (see [Su1] and [Pr3]), we prove a stronger version of rigidity that the absolute continuity (with respect to packing measures) of the conjugating homeomorphism alone implies its real analyticity.

§**2. Preliminaries.** Let S^1 denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and let l be the normalized Lebesgue measure on S^1, $l(S^1) = 1$. Let I be a finite set consisting of at least two elements and let $\{\Delta_j : j \in I\}$ be a finite collection of closed nondegenerate and not overlapping subarcs (their intersections contain at most one point) of S^1. Finally let $f : \bigcup_{j\in I} \Delta_j \to S^1$ be a C^1 continuous map, open onto its image with the following properties:

(2.1) If $i, j \in I$ and $\Delta_i \cap \Delta_j \neq \emptyset$, then $f|_{\Delta_i\cup\Delta_j}$ is injective.

(2.2) For every $j \in I$ the restriction $f|_{\Delta_j}$ is $C^{1+\theta}$ differentiable, that is the derivative function $f'|_{\Delta_j}$ is Hölder continuous with an exponent $\theta > 0$ which means that

$$|f'(y) - f'(x)| \le Q|y - x|^\theta$$

for some constant $Q > 0$ and all $x, y \in \Delta_j$.

(2.3) $|f'(x)| \ge 1$ for all $x \in \bigcup_{j\in I} \Delta_j$ but $|f'(x)| = 1$ may hold only if $f(x) = x$.

(2.4) If $f(\omega) = \omega$ and $|f'(\omega)| = 1$, then the derivative f' is monotone on each sufficiently small one-sided neighborhood of ω.

(2.5) There exists $L \ge 2$ such that if $f(\omega) = \omega$ and $|f'(\omega)| = 1$, then there exists $0 < \beta = \beta(\omega) < \theta/(1-\theta) (= \infty \text{ if } \theta = 1)$ such that

$$\frac{2}{L} \le \liminf_{x\to\omega} \frac{||f'(x)| - 1|}{|x-\omega|^\beta} \le \limsup_{x\to\omega} \frac{||f'(x)| - 1|}{|x-\omega|^\beta} \le \frac{L}{2}$$

(2.6) For every $i \in I$ there exists $I(i) \subset I$ such that $f(\Delta_i) \cap \bigcup_{j\in I} \Delta_j = \bigcup_{k\in I(k)} \Delta_k$.

The reader should notice that in the case when the intervals I_j are mutually disjoint, then without loosing generality the circle S^1 can be replaced by a compact subinterval of $\mathbb{R}$. In this case also the openness of $f : \bigcup_{j\in I} \Delta_j \to S^1$ and (2.1) follow automatically from other assumptions.

Coming back to the general case, property (2.3) describes a kind of hyperbolicity and requirement (2.6) establishes the Markov property which always gives rise to a nice symbolic representation of f. In the sequel we will need f to satisfy one condition more. In order to express it and in order to express various properties of objects introduced above let us prepare a suitable language. To begin with let $A : I \times I \to \{0,1\}$ be the matrix (called incidence matrix) defined by the requirement that $A_{ij} = 1$ if and only if $f(\Delta_i) \supset \Delta_j$. The last condition we need is that the matrix A is primitive which means that

(2.7) There exists $q \geq 1$ such that all entries of A^q are positive.

Let next $\Sigma_A^\infty \subset I^\infty$ be the space of all one-sided infinite sequences $\tau = \tau_0\tau_1\tau_2\ldots$ acceptable by A, that is such that $A_{\tau_j\tau_{j+1}} = 1$ for all $j = 0, 1, 2, \ldots$ and let Σ_A^* be the set of all finite sequences acceptable by A. We put $\Sigma_A = \Sigma_A^* \cup \Sigma_A^\infty$ and for every integer $n \geq 0$ we let Σ_A^n be the subset of Σ_A^* consisting of all words of length $n+1$. Going on with notation, given $\tau \in \Sigma_A$ and $n \geq 0$ we define $\tau|_n = \tau_0\tau_1\ldots\tau_n$ to consist of the first $n+1$ initial letters of τ; if $n+1$ exceeds the length of τ, then $\tau|_n$ is just τ. Notice that Σ_A^∞ is compact and by primitiveness of A it is nonempty. Notice also that Σ_A^∞ is forward invariant under the left-sided shift map (cutting out the first coordinate) which will be denoted by σ. For all words $\tau \in \Sigma_A^n$, $n \geq 0$ define

$$\Delta(\tau) = \Delta_{\tau_0} \cap f^{-1}(\Delta_{\tau_1}) \cap \ldots \cap f^{-n}(\Delta_{\tau_n})$$

Observe that $\Delta(\tau)$ is a nonempty closed subinterval of S^1. Fix $\tau \in \Sigma_A^\infty$ and consider the descending sequence $\{\Delta(\tau|_n) : n \geq 0\}$ of compact nonempty subintervals of S^1. Then the intersection $\bigcap_{n\geq 0} \Delta(\tau|_n)$ is a closed nonempty subinterval of S^1. We shall prove the following.

Lemma 2.1. For every $\tau \in \Sigma_A^\infty$ the set $\Delta(\tau) = \bigcap_{n\geq 0} \Delta(\tau|_n)$ is a singleton. Even more, the diameters of $\Delta(\tau|_n)$ tend to zero uniformly with respect to n.

Proof. Let $\Sigma_A^+ = \{\tau \in \Sigma_A^\infty : l(\Delta(\tau)) > 0\}$ and suppose that $\Sigma_A^+ \neq \emptyset$. Since for any two distinct elements $\tau, \tau \in \Sigma_A^\infty$ the intersection $\Delta(\tau) \cap \Delta(\tau)$ is either an empty set or a point, the family Σ_A^+ contains an element of largest length. So, the remark that if $\tau \in \Sigma_A^\infty$, then also $\sigma(\tau) \in \Sigma_A^\infty$ and $l(\Delta(\sigma(\tau))) = l(f(\Delta(\tau))) > l(\Delta(\tau))$, gives a contradiction and finishes the proof of the first part of the lemma.
In order to prove the second part suppose to the contrary that $\exists\, \varepsilon > 0\ \forall n \geq 0\ \exists\, \tau^{(n)} \in \Sigma_A^\infty\ \exists k_n \geq n$ such that $l(\tau|_{k_n}^{(n)}) \geq \varepsilon$. By compactness of Σ_A^∞ we can find an accumulation point $\tau \in \Sigma_A^\infty$ of the sequence $\{\tau^{(n)} : n \geq 1\}$. But keeping in mind that the sequence of lengths $\{l(\Delta(\tau|_n)) : n \geq 1\}$ is decreasing this yields $l(\Delta(\tau|_n)) \geq \varepsilon$ for all $n \geq 1$ and consequently $l(\Delta(\tau)) \geq \varepsilon$. This however contradicts the first part of the lemma and completes the proof. ♣

In view of Lemma 2.1 we can define a continuous map $\pi : \Sigma_A^\infty \to S^1$ putting $\pi(\tau) = \Delta(\tau)$. The range of this map, the set $J = J(f) = \pi(\Sigma_A^\infty)$ is called the dynamical Cantor set (DCS) associated to the dynamical system $(f, I; \Delta_j, j \in I)$. Although J may happen to be an interval, nevertheless we still choose the name Cantor set since we consider an interval as a degenerate Cantor set, and since, which is perhaps a more important reason, J is an interval in, in some sense, exceptional cases only (see Theorem 2.4 below). Let us now formulate the following obvious lemma.

Lemma 2.2. We have

(a) $J = \bigcap_{n\ge 0} \bigcup_{\tau\in\Sigma_A^n} \Delta(\tau)$

(b) J can be characterized as the set of those points of S^1 whose all positive iterates under f are defined (and therefore contained in $\bigcup_{j\in I}\Delta_j$).

(c) $f^{-1}(J) = J = f(J)$.

(d) $f\circ\pi = \pi\circ\sigma$

Proof. Properties (a) and (b) are obvious. The relations $f(J) \subset J = f^{-1}(J)$ follow immediately from (b), and the inclusion $f(J) \supset J$ follows from (b) and primitiveness of the matrix A. The property (d) follows from the definition of J. ♣

For every $\tau \in \Sigma_A$ define $J(\tau) = J \cap \Delta(\tau)$. For every $x \in S^1$ and $r > 0$ define $B_{S^1}(x,r)$ and $B_J(x,r)$ to be the balls centered at x with radius r respectively in the space S^1 and J. Additionally let $B(x,r)$ be the convex hull in S^1 containing $B_J(x,r)$. Note that if r is sufficiently small (independently of r), then $B(x,r) \subset \bigcup_{j\in I}\Delta_j$. The next lemma provides most basic "formal" properties of the sets $J(\tau)$. Its proof is of set-theoretic flavor and is left for the reader.

Lemma 2.3. We have

(a) $\forall_{(n\ge 0)}\ \ J = \bigcup_{\tau\in\Sigma_A^n} J(\tau)$.

(b) $\forall_{(\tau\in\Sigma_A^*\setminus\Sigma_A^0)}\ \ f(J(\tau)) = J(\sigma(\tau))$ and $f(\Delta(\tau)) = \Delta(\sigma(\tau))$.

(c) $\forall_{(i\in\Sigma_A^0)}\ \ f(J(i)) = \bigcup\{J(j) : A_{ij} = 1\}$.

(d) $\forall_{(n\ge 0)}\ \forall_{(\tau\in\Sigma_A^*)}\ \ J(\tau) = \bigcup_{j\in I} J(\tau j)$.

(e) If U is a nonempty open subset of J, then $f^n(U) = J$ for some integer $n \ge 0$.

(f) $\forall_{(x\in J)}\ \forall_{(n\ge 0)}\ \ \bigcup_{k>n} f^{-1}(\{x\}) \setminus \bigcup_{k\le n} f^{-1}(\{x\})$ is dense in J.

Perhaps only a few words about the proof of (e) would be in order. Indeed, by primitiveness of of A there exists $k \ge 0$ such that $f^{n+k}(J(\tau)) = J$ for every $\tau \in \Sigma_A^*$ and $n = |\tau| - 1$. So, the remark that each nonempty open subset of J contains a cylinder $J(\tau)$ for some $\tau \in \Sigma_A^*$ completes the argument.

Lemma 2.4. The set J is either a topological Cantor set (perfect, totally disconnected) or an interval. In particular, if $\bigcup_{j\in I}\Delta_j$ is not an interval, then J is a topological Cantor set.

Proof. First we shall show that J contains at least two distinct points. Indeed, suppose to the contrary that J is a singleton, say z. Then, as by the definition of J, all sets $C(i)$, $i \in I$, are nonempty, we deduce that I consists of exactly two elements, say i_1 and i_2, and $I_{i_1} \cap I_{i_2} = \{z\}$. Thus $f(z) = z$ and, as it follows from primitiveness of A that $f(\Delta_{i_j}) \supset \Delta_{i_1} \cup \Delta_{i_2}$ for $j = 1$ or $j = 2$, we deduce from (2.1) that $f(\Delta_{i_j}) = S^1$. Therefore applying (2.1) again we conclude that f is a homeomorphism of Δ_{i_j} onto S^1 which is a contradiction and finishes the proof that J contains at least two distinct points. Now, since by primitiveness of A, for every $\tau \in \Sigma_A^*$ there is an integer $n \ge 0$ such that $f^n(J(\tau)) = J$, each cylinder $J(\tau)$ contains at least two distinct points. Hence, applying Lemma 2.1 and Lemma 2.3(a) finishes the proof of perfectness of J.

In order to complete the proof of the first part it suffices now to show that if J is not totally disconnected, then it is an interval. Indeed, suppose that U is a nondegenerate interval contained in J. Then U has a nonempty interior in J and by Lemma 2.3(e), $J = f^q(U)$ for some $q \geq 0$. So, J as a continuous image of a connected set is also connected. The second part follows from the first one and the observation that by primitiveness of A, the set J intersects each interval Δ_j, $j \in I$. ♣

Let

$$\Omega = \Omega(f) = (f) = \{\omega \in J : f(\omega) = \omega \text{ and } |f'(\omega)| = 1\}$$

Each point $\omega \in \Omega$ is called a fixed parabolic point or shorter a parabolic point. For every $q \geq 1$ consider now the system $(f^q, I^q; \Delta(\tau), \tau \in I^q)$. We shall prove the following.

Lemma 2.5. The set I^q consists of at least two elements, $\{\Delta(\tau), \tau \in I^q\}$ is a finite collection of not-overlapping closed intervals, and $f^q : \bigcup_{\tau \in I^q} \Delta(\tau) \to S^1$ is continuous. Moreover,

(a) The system $(f^q, I^q; \Delta(\tau), \tau \in I^q)$ satisfies the the conditions (2.1) - (2.7).
(b) $J(f^q) = J(f)$.
(c) $\Omega(f^q) = \Omega(f)$.
(d) If $\tau \in I^2$ and $\omega \in \Omega(f) \cap \Delta(\tau)$, then $f^2|_{\Delta(\tau)}$ is orientation preserving.

Proof. The first part of this lemma is obvious. Let us now deal with the item (a). Condition (2.1) is satisfied since the composition of injective maps is injective and condition (2.2) holds since the composition of $C^{1+\theta}$ differentiable maps is $C^{1+\theta}$ differentiable. To prove (2.3) notice that by the chain rule $|(f^q)'(x)| \geq 1$ for all $x \in \bigcup_{\tau \in I^q} \Delta(\tau)$ and suppose that $|(f^q)'(x)| = 1$. Then by the chain rule and (2.3) (for f) we have $|f'(f^i(x))| = 1$ for all $i = 0, 1, \ldots, q-1$, and therefore, using the second part of (2.3) (for f), we conclude that $f(x) = x$. Hence we have proved (2.3) for f^q and simultaneously condition (c) of our lemma. In order to prove (2.4) and (2.5) consider first two functions g and h defined on a neighborhood of a point ω, both keeping it fixed and and satisfying (2.3), (2.4), and (2.5). Note that then both g and h have an inflection point at ω, $(g \circ h)'(y) - (g \circ h)'(x) = \big(g'(h(y) - g'(h(x))\big)h'(y) + g'(h(x))\big(h'(y) - h'(x)\big)$ and both summands have the same sign. So, $g \circ h$ has again monotone derivative on either side of ω. Hence (2.4) for f^q follows by induction. Also

$$\begin{aligned}
|(g \circ h)'(x) - 1| &= |(g \circ h)'(x) - g'(\omega)h'(\omega)| \\
&= |(g'(h(x)) - g'(\omega))h'(x) + g'(\omega)(h'(x) - h'(\omega))| \\
&= |h'(x)||g'(h(x)) - g'(\omega)| + |g'(\omega)||h'(x) - h'(\omega)| \\
&= |h'(x)||g'(h(x)) - 1| + |g'(\omega)||h'(x) - 1|
\end{aligned}$$

where the third equality sign has been written since both numbers $(g'(h(x)) - g'(\omega))h'(x)$ and $g'(\omega)(h'(x) - h'(\omega))$ are easily seen to have the same signs. Notice that by (2.5) for

h, it follows from Mean Value Theorem that $\lim_{x\to\omega}\frac{|h(x)-\omega|}{|x-\omega|}=1$ and as

$$\frac{|g'(h(x))-1|}{|x-\omega|^{\beta}}=\frac{|g'(h(x))-1|}{|h(x)-\omega|^{\beta}}\cdot\frac{|h(x)-\omega|^{\beta}}{|x-\omega|^{\beta}}$$

we conclude that (2.5) is satisfied for $g\circ h$. Now, condition for f^q also follows by induction.

Moving with the proof, condition (2.6) follows from Lemma 2.3(b) by induction and (2.7) is satisfied since products of primitive matrices are again primitive. Property (d) can be easily derived directly from definition. ♣

Besides of the formal value of Lemma 2.6 its practical advantage is that passing to the second iterate of f one keeps the same Cantor set, the same set of parabolic points, and f^2 "preserves" one-sided neighborhoods of parabolic points. Therefore from now on we will assume that already for f itself condition (d) of Lemma 2.5 is satisfied.

Lemma 2.6. For every $n\geq 1$ the set $\mathrm{Per}_n(f)=\{x\in J: f^n(x)=x\}$ is finite.

Proof. First note that in view of (2.3) and the left-hand side of (2.5) every point $\omega\in\Omega$ has an open neighborhood in $\bigcup_{j\in I}\Delta_j$ on which $|f'(x)|>1$ except for ω itself. Therefore Ω is countable. Suppose now that $\mathrm{Per}_n(f)$ is infinite for some $n\geq 1$ and let y be an accumulation point of $\mathrm{Per}_n(f)$. Since $\mathrm{Per}_n(f)$ is closed, $y\in\mathrm{Per}_n(f)$. Note that for every $z\in\mathrm{Per}_n(f)$ sufficiently close to y, the restriction $f^n|_{[z,y]}$ is well defined and injective. Pick one such $z\neq y$. Then $|z-y|=|f^n(z)-f^n(y)|=\int_y^z|(f^n)'(x)|\,dx$. But since Ω is countable, so is the set $\{x:|(f^n)'(x)|\leq 1\}$, hence the last integral is greater than $|z-y|$. This contradiction finishes the proof. ♣

Using our assumptions (2.1) - (2.7) and Lemma 2.6 we conclude that the number

$$\delta_1=\frac{1}{2}\min\begin{cases}\min\{l(I_i): i\in I\}\\ \min\{\mathrm{dist}(I_i,I_j): i,j\in I,\ I_i\cap I_j=\emptyset\}\\ \min\{|x-y|: x,y\in\mathrm{Per}_2(f), x\neq y\}\end{cases}$$

is positive.

Lemma 2.7. If $0<\delta\leq\delta_1$ and $x\in B(\mathrm{Per}_1(f),\delta)\setminus\mathrm{Per}_1(f)$, then there exists $n\geq 1$ such that $f^n(x)\notin B(\mathrm{Per}_1(f),\delta)$.

Proof. Suppose to the contrary that there exists $x\in B(\mathrm{Per}_1(f),\delta)\setminus\mathrm{Per}_1(f)$ such that $f^n(x)\in B(\mathrm{Per}_1(f),\delta)$ for all $n\geq 1$. Then there exists $z\in\mathrm{Per}_1(f)$ such that $f^n(x)\in B(z,\delta)$ for all $n\geq 1$. In view of (2.3) and (2.5) the sequence $f^{2n}(x)$ converges, say to y, and $y\neq z$. But then $f^2(y)=y$ which contradicts the choice of δ and finishes the proof. ♣

Recall that a continuous map $S:X\to X$ of a compact metric space X is expansive if there exists a positive η such that for all $x,y\in X$, $x\neq y$ there exists $n\geq 0$ such that $\mathrm{dist}(S^n(x),S^n(y))\geq\eta$.

Theorem 2.8. The map $f : J \to J$ is open and expansive, and any positive number $\eta \le \delta_1$ is an expansive constant for f.

Proof. In view of Lemma 2.2(c) we have $f(W \cap J) = f(W) \cap J$ for all subsets W of $\bigcup_{j\in I} \Delta_j$, and therefore the openness of $f : J \to J$ follows from the openness of $f : \bigcup_{j\in I} \Delta_j \to f\big(\bigcup_{j\in I} \Delta_j\big)$. Let us now prove expressiveness of $f : J \to J$. Take $0 < \delta \le \delta_1$ and suppose to the contrary that there are two distinct points x and y in J such that $|f^n(y) - f^n(x)| < \delta$ for all $n \ge 0$. Let $x = \pi(\tau)$ and $y = \pi(\rho)$, $\tau, \rho \in \Sigma_A^\infty$. Since $x \ne y$, there exists $q \ge 0$ such that $\tau_q \ne \rho_q$. Since $f^q(x) \in \Delta_{\tau_q}$ and $f^q(y) \in \Delta_{\rho_q}$, we get contradiction if Δ_{τ_q} and Δ_{ρ_q} are disjoint. So, $\Delta_{\tau_q} \cap \Delta_{\rho_q} \ne \emptyset$, and let z be the only point of this intersection. By the definition of δ_1, all the iterates $f^n|_{[f^q(x), f^q(y)]}$ are injective and $f^n(z)$ lies between $f^{qn}(x)$ and $f^{qn}(y)$. By (2.6) the point z, as well as all other points of the intervals Δ_{τ_q} and Δ_{ρ_q}, is eventually periodic, say $f^p(f^s(z)) = f^s(z)$. But then it follows from Lemma 2.7 that $f^k(x), f^k(y) \notin B(f^j(z), \delta)$ for some $s \le j \le s+p-1$, $k \ge q+s$, and simultaneously $f^j(z)$ lies between $f^k(x)$ and $f^k(y)$. This contradiction finishes the proof. ♣

As an immediate consequence of this theorem, Lemma 2.2 of [DU2] and [Ru, p. 128], (see also [PU]), we get the following.

Corollary 2.9. (Closing Lemma) For every $\varepsilon > 0$ there exists $\tilde{\varepsilon} > 0$ such that if $n \ge 0$ is an integer, $x \in J$, and $|f^n(x) - x| < \tilde{\varepsilon}$, then there exists a point $y \in J$ such that

$$f^n(y) = y \quad \text{and} \quad |f^j(y) - f^j(x)| < \varepsilon$$

for all $j = 0, 1, \dots, n-1$. ♣

The following last part of this section is devoted to prove the distortion properties of iterates of f. First observe that for every $\omega \in \Omega$ there is a continuous inverse branch $f_\omega^{-1} : B(\omega, \delta_1) \to S^1$ of f such that $f_\omega^{-1}(\omega) = \omega$. By (2.3)

$$f_\omega^{-1}(B(\omega, \delta_1)) \subset B(\omega, \delta_1)$$

and therefore all iterates $f_\omega^{-n}(B(\omega, \delta_1)) \subset B(\omega, \delta_1)$, $n \ge 1$, are well defined. Moreover, by property (d) of Lemma 2.5 the map f_ω^{-1} preserves one-sided neighborhoods of ω.
Now, the same argument as in the proof of Lemma 2.6 shows that every connected component of $\bigcup_{j\in I} \Delta_j$ may contain at most one fixed point. So, since the sequence $f_\omega^{-n}(x)$ is decreasing toward ω, we obtain

$$\lim_{n\to\infty} f_\omega^{-n}(x) = \omega \tag{2.8}$$

for all $\omega \in \Omega$ and all $x \in B(\omega, \delta_1)$. We shall prove the following.

Lemma 2.10. For all $\omega \in \Omega$ and all $x \in B(\omega, \delta_1 ||f||^{-1}) \setminus \{\omega\}$ we have

$$\frac{|x-\omega|}{|f(x)-x|} \le \sum_{n=1}^{\infty} |(f_\omega^{-n})'(x)| \le \frac{f_\omega^{-1}(x)-\omega|}{|x-f_\omega^{-1}(x)|}$$

Proof. For every $x \in B(\omega, \delta_1 ||f||^{-1}) \setminus \{\omega\}$ and $k \ge 1$ let $F_k(x) = \sum_{n=1}^{k} |(f_\omega^{-n})'(x)|$. Then

$$\int_{f_\omega^{-1}(x)}^{x} F_k(t)\, dt = \sum_{n=1}^{k} |f_\omega^{-n}(x) - f_\omega^{-(n+1)}(x)| = |f_\omega^{-1}(x) - f_\omega^{-(k+1)}(x)|.$$

By (2.5) all the functions $|(f_\omega^{-n})'|$, $n \ge 1$, are decreasing on either side of ω in $B(\omega, \delta_1)$. In particular $F_k(f_\omega^{-1}(y)) \ge F_k(t) \ge F_k(x)$ for all $f_\omega^{-1}(x) \le t \le x$. Thus $|x - f_\omega^{-1}(x)|F_k(x) \le |f_\omega^{-1}(x) - f_\omega^{-(k+1)}(x)| \le |x - f_\omega^{-1}(x)|F_k(f_\omega^{-1}(x))$. Therefore, letting $k \to \infty$, using (2.8), and noting that $B(\omega, \delta_1 ||f||^{-1}) \subset f_\omega^{-1}(B(\omega, \delta_1))$, the required inequalities follow. ♣

Observe that in the proof of Lemma 2.10 we have not used the first part of (2.5) describing the infinitesimal behavior of f around parabolic points, and this is the main reason we decided to formulate and prove it. Incorporating formula (2.5) in its full strength we can prove more, that the series $\sum_{n=1}^{\infty} |(f_\omega^{-n})'(x)|^\theta$ converges. This will be done in several consecutive lemmas. We begin with a generalization of a result of Thaler (see [Th]) which actually goes back to the 19 th century. We provide a different more "dynamical" proof which is here due to L. Olsen.

Lemma 2.11. Let $p, a, A > 0$ be three positive numbers and let $\phi : (0, A) \to I\!R$ be a real-valued function. If $\phi(x) = x - ax^{p+1} + o(x^{p+1})$ as $x \to 0^+$, then there exists $\eta \in (0, A)$ such that

$$\lim_{n\to\infty} \frac{\phi^n(x)}{n^{-1/p}} = (pa)^{-1/p}$$

for all $x \in (0, \eta]$ and the convergence is uniform on compact subsets of $(0, \eta]$.

Proof. For every $a > 0$ define an auxiliary function $\phi_a : (0, \infty] \to (0, \infty]$ putting

$$\phi_a(x) = \frac{x}{(1 + apx^p)^{1/p}}$$

It is easy to check that with $a, b > 0$ the following conditions are satisfied.

(a) $0 < \phi_a(x) < x$ for all $x > 0$.
(b) $\phi_a \circ \phi_b = \phi_{a+b}$.
(c) ϕ_a is increasing.
(d) $\phi_a(x) = x - ax^{p+1} + O(x^{2p+1})$ as $x \to 0^+$.
(e) $\lim_{n\to\infty} n^{1/p} \phi_a^n(x) = (pa)^{-1/p}$ and the convergence is uniform on compact subsets of $(0, \infty)$.

Perhaps only property (e) requires a proof. Here it is. Using (b) we get

$$(pan)^{1/p}\phi_a^n(x) = (pan)^{1/p}\phi_{na}(x) = \frac{x}{\left(x^p + \frac{1}{pan}\right)^{1/p}} \to 1$$

uniformly on compact subsets of $(0, \infty)$.

Now note that if g and ψ are two real-valued functions defined on the same interval of $I\!R$, at least one of them, g or ψ is increasing, and if $g \le \psi$, then $g^n \le \psi^n$ for all $n \ge 1$. In order to prove Lemma 2.11 fix $0 < \varepsilon < a$. By (d) and the assumption on ϕ there exists $\eta \in (0, A)$ such that $\phi_{(a+\varepsilon)} < \phi(x) < \phi_{(a-\varepsilon)}(x)$ for all $x \in (0, \eta]$ and using (c) we therefore get $\phi^n_{(a+\varepsilon)} < \phi^n(x) < \phi^n_{(a-\varepsilon)}(x)$ for all $x \in (0, \eta]$ and all $n \ge 1$. Thus, using (e), we can write

$$\begin{aligned}(p(a+\varepsilon))^{-1/p} &= \lim_{n\to\infty} n^{1/p}\phi^n_{a+\varepsilon}(x) \le \liminf_{n\to\infty} n^{1/p}\phi^n(x) \le \limsup_{n\to\infty} n^{1/p}\phi^n(x) \\ &\le \lim_{n\to\infty} n^{1/p}\phi^n_{a-\varepsilon}(x) = (p(a-\varepsilon))^{-1/p}\end{aligned}$$

So, letting $\varepsilon \to 0$ finishes the proof. ♣

As an immediate consequence of Lemma 2.11 we get the following.

Corollary 2.12. Let $p, a, A > 0$ be three positive numbers and let $\phi : (0, A) \to I\!R$ be a real-valued function. If $\phi(x) \le x - ax^{p+1}$ for $x \in (0, A)$, then there exists $\eta \in (0, A)$ such that

(a) $$\limsup_{n\to\infty} \frac{\phi^n(x)}{n^{-1/p}} \le (pa)^{-1/p}$$

for all $x \in (0, \eta]$ and the convergence is uniform on compact subsets of $(0, \eta]$. If instead $\phi(x) \le x - ax^{p+1}$ for $x \in (0, A)$, then

(b) $$\liminf_{n\to\infty} \frac{\phi^n(x)}{n^{-1/p}} \ge (pa)^{-1/p}$$

Lemma 2.13. There exist constants $0 < \delta \le \delta_1$ and $L_1 \ge 2$ such that if $\omega \in \Omega$ and $z \in B(\omega, \delta)$, then

$$(2L_1)^{-1} \le \liminf_{n\to\infty} \frac{|f_\omega^{-n}(z) - \omega|}{n^{-1/\beta}} \le \limsup_{n\to\infty} \frac{|f_\omega^{-n}(z) - \omega|}{n^{-1/\beta}} \le L_1/2$$

and the convergence is uniform on compact subsets of $B(\Omega, \delta) \setminus \Omega$.

Proof. First notice that by the Mean Value Theorem and since f is C^1 differentiable we have $\lim_{x\to\omega} \frac{|f(x)-\omega|}{|x-\omega|} = 1$. Using this and boundedness of $|f'(x)|$ from below and from above we deduce that (2.5) remains true, perhaps with a bigger value of L, if we replace f by f_ω^{-1}. Then for every x sufficiently close to ω we have

$$
\begin{aligned}
|f_\omega^{-1}(x) - \omega| &= \int_\omega^x |(f_\omega^{-1})'(t)|\,dt \le \int_\omega^x (1 - L^{-1}|t-\omega|^\beta)\,dt \\
&\le |x-\omega| - L^{-1}(\beta+1)^{-1}|x-\omega|^{\beta+1} \qquad (2.a)
\end{aligned}
$$

and similarly

$$
\begin{aligned}
|f_\omega^{-1}(x) - \omega| &\ge \int_\omega^x (1 - L|t-\omega|^\beta)\,dt \\
&\ge |x-\omega| - L(\beta+1)^{-1}|x-\omega|^{\beta+1}. \qquad (2.b)
\end{aligned}
$$

So, employing Lemma 2.11 completes the proof. ♣

We have enumerated inequalities (2.a) and (2.b) since these will be frequently used in the sequel. As an immediate consequence of Lemma 2.13 we get the following.

Corollary 2.14. $\forall_{(\omega\in\Omega)}\forall_{(0<R\le\delta)}\ \exists_{(L_1(R)\ge1)}\ \forall_{(z\in B(\omega,\delta)\setminus B(\omega,R))}\ \forall_{(n\ge1)}$

$$L_1(R)^{-1} \le \frac{|f_\omega^{-n}(z) - \omega|}{n^{-1/\beta}} \le \frac{|f_\omega^{-n}(z) - \omega|}{n^{-1/\beta}} \le L_1(R)$$

Relying on this fact we shall prove the following.

Lemma 2.15. $\forall_{(\omega\in\Omega)}\forall_{(0<R\le\delta)}\ \exists_{(L_2(R)\ge2)}\ \forall_{(z\in B(\omega,\delta)\setminus B(\omega,R))}\ \forall_{(n\ge1)}$

$$L_2(R)^{-1} \le \frac{|(f_\omega^{-n})'(z)|}{n^{-\frac{\beta+1}{\beta}}} \le L_2(R).$$

Proof. Since all the functions $|(f_\omega^{-n})'(z)|$, $n\ge1$, are monotone nearby ω, we have

$$(2.9)\quad |(f_\omega^{-n})'(z)||f_\omega^{-1}(z) - z| \le |f_\omega^{-n}(z) - f_\omega^{-(n+1)}(z)| \le |(f_\omega^{-(n+1)})'(f_\omega^{-1}(z))||f_\omega^{-1}(z) - z|.$$

It follows from (2.a) and (2.b) that $L^{-1}(\beta+1)^{-1}|f_\omega^{-n}(z)-\omega|^{\beta+1} \le |f_\omega^{-n}(z)-f_\omega^{-(n+1)}(z)| \le L(\beta+1)^{-1}|f_\omega^{-n}(z)-\omega|^{\beta+1}$ for all $x\in B(\omega,\delta)$. Hence combining Corollary 2.14 and (2.9) we get

$$|(f_\omega^{-n})'(z)| \le \frac{L(\beta+1)^{-1}L_1(R)^{\beta+1}}{|f_\omega^{-1}(z) - z|} n^{-\frac{\beta+1}{\beta}}$$

and

$$|(f_\omega^{-(n+1)})'(z)| \ge \frac{L^{-1}(\beta+1)^{-1}L_1(R)^{-(\beta+1)}}{|f_\omega^{-1}(z) - z|} n^{-\frac{\beta+1}{\beta}}$$

The proof is completed. ♣

Since $\beta < \theta/(1-\theta)$, it follows from Lemma 2.15 that for every $\omega \in \Omega$ and every $x \in B(\omega, \delta)$

$$\sum_{n=1}^{\infty} |(f_\omega^{-n})'(x)|^\theta < \infty \tag{2.10}$$

and the convergence is uniform on compact subsets of $B(\Omega, \delta) \setminus \Omega$.

Now observe that for every $x \in S^1$ and every $n \geq 1$, if $f^n(x)$ is well-defined, then there exists a continuous inverse branch $f_x^{-n} : B(f^n(x), \delta_2) \to S^1$ of f^n sending $f^n(x)$ to x, where $\delta_2 = \min\{l(f(\Delta_i)) : i \in I\}$. We shall prove the following.

Lemma 2.16. $\forall_{t>0}\ \forall_{0<s<1}\ \exists_{K_1(t,s)>0}\ \exists_{M(t,s)>0}$ such that if $x \in S^1$, $n \geq 0$, $f^n(x)$ is well-defined, and $\mathrm{dist}(f^n(x), \Omega) \geq t$, then for all points $y, z \in B(f^n(x), \min\{\delta, st\})$

$$K_1(t,s)^{-1} \leq \frac{|(f_x^{-n})'(y)|}{|(f_x^{-n})'(z)|} \leq K_1(t,s)$$

and

$$\sum_{j=1}^{n-1} |(f^j)'(x)|^\theta \leq M(t,s).$$

Moreover for every $t > 0$ we have $\lim_{s\to 0} K_1(t,s) = 1$.

Proof. Set $r = \min\{\delta, (1-s)t\}$, $\lambda = \lambda(t,s) = \inf\{|f'(z)| : z \notin B(\Omega, r||f'||^{-1})\}$ and let $K = K(t,s) > 0$ be the supremum of the series appearing in (2.10) taken over the set $B(\Omega, r) \setminus B(\Omega, r||f'||^{-1})$. Fix $y \in B(f^n(x), \min\{\delta, st\})$, for every $0 \leq j \leq n$ put $y_j = f^j(f_x^{-n}(y))$ and let $p(j)$ be the number of integers $0 \leq i \leq n-1-j$ such that $f^i(y) \notin B(\Omega, r||f'||^{-1})$. Define also increasing sequences $0 \leq k_j < l_j \leq n$ determined by the requirements that

(a) $\{y_{k_j}, y_{k_j+1}, \ldots, y_{l_j}\} \subset B(\Omega, r)$

and

(b) If $i \notin \bigcup_j \{k_j, k_j+1, \ldots, l_j\}$, then $y_i \notin B(\Omega, r)$.

Since $y = y_n \notin B(\Omega, r)$, we conclude that for all j the point $y_{l_j} \in B(\Omega, r) \setminus B(\Omega, r||f'||^{-1})$. Thus $\sum_{i=k_j}^{l_j} |(f^{n-i})'(y_i)|^{-\theta} \leq (K+1)|(f^{n-l_j})'(y_{l_j})|^{-\theta} \leq (K+1)\lambda^{-\theta p(l_j)}$ and then

$$\begin{aligned}\sum_{i=0}^{n-1} |(f^{n-i})'(y_i)|^{-\theta} &\leq \sum_j (K+1)\lambda^{-\theta p(l_j)} + \sum_{i\in G} \lambda^{-\theta p(i)} \leq (K+1)\sum_{i=0}^{n-1} \lambda^{-\theta i} \\ &= (K+1)\frac{\lambda^\theta}{\lambda^\theta - 1},\end{aligned} \tag{2.11}$$

where the second inequality sign we could write since all the numbers $p(l_j)$ and $p(i)$, $i \in G$, are mutually distinct. So, the last part of the lemma is proven. As a matter of fact in what follows we will need a slightly stronger version of this estimate where we let the point y vary in $B(f^n(x), \min\{\delta, st\})$ with i. Let now z be another point in $B(f^n(x), \min\{\delta, st\})$. Then using (2.2) and the mean value theorem we see that for every j there exists $w^{(j)} \in [z, y]$ such that

$$\begin{aligned}
|\log|f'(z_j)| - \log|f'(y_j)|| &\le ||f'(z_j)| - |f'(y_j)|| \le Q|z_j - y_j|^\theta \\
&= Q|(f^{n-j})'(w_j^{(j)})|^{-\theta}|z - y|^\theta \\
&\le Q(2st)^\theta|(f^{n-j})'(w_j^{(j)})|^{-\theta}
\end{aligned}$$

Hence applying (2.11), in fact its stronger version discussed above, we get

$$\begin{aligned}
\big|\log|(f_x^{-n})'(y)| - \log|(f_x^{-n})'(z)|\big| &\le \sum_{j=0}^{n-1} |\log|f'(z_j)| - \log|f'(y_j)|| \\
&\le (2st)^\theta Q \sum_{j=0}^{n-1} |(f^{n-j})'(w_j^{(j)})|^{-\theta} \\
&\le (2st)^\theta Q(K+1)\frac{\lambda^\theta}{\lambda^\theta - 1}
\end{aligned}$$

So, the first part of the proof is finished setting

$$K_1(t, s) = \exp\left((2st)^\theta Q(K+1)\frac{\lambda^\theta}{\lambda^\theta - 1}\right).$$

In order to see that $\lim_{s\to 0} K_1(t, s) = 1$ it suffices to notice that

$$\lim_{s\to 0} \lambda(t, s) = \inf\{|f'(z)| : z \notin B(\Omega, \min\{\delta, t\})\} > 1$$

and $\lim_{s\to 0} K(t, s)$ is finite as the supremum of the series appearing in (2.10) over the set $B(\Omega, \min\{\delta, t\}) \setminus B(\Omega, \min\{\delta, t\}/||f'||)$. The proof is finished. ♣

Observe that given $\omega \in \Omega$ and $0 < t < \delta$, partitioning separately both connected components of $B(\omega, \delta) \setminus B(\omega, t)$ into finitely many segments of length $\le t/2$, and increasing $K_1(t, t/2)$ if necessary, we derive from Lemma 2.16 the following.

Corollary 2.17. For every $0 < t < \delta$ there exists $K_1(t) > 0$ such that if $x \in S^1$, $n \ge 0$, $f^n(x)$ is well-defined and belongs to $B(\omega, \delta) \setminus B(\omega, t)$, then

$$K_1(t)^{-1} \le \frac{|(f_x^{-n})'(y)|}{|(f_x^{-n})'(z)|} \le K_1(t)$$

for all points y, z lying in the same connected component of $B(\omega, \delta) \setminus B(\omega, t)$ as $f^n(x)$.

Lemma 2.18. For every $0 < s < 1$ there exists $K_2(s) > 1$ such that if $x \in S^1$, $n \geq 0$, and $f^n(x)$ is well-defined, then

$$K_2(s)^{-1} \leq \frac{|(f_x^{-n})'(y)|}{|(f_x^{-n})'(z)|} \leq K_2(s)$$

for all points $y, z \in B(f^n(x), \min\{s\mathrm{dist}(f^n(x), \Omega), \delta/4\})$.

Before starting the proof let us give a few words of comment on this lemma. First of all this is a substantial improvement of Lemma 2.16 since now the distortion constant $K_2(s)$ is independent of the distance from $f^n(x)$ to Ω; it depends only on the ratio of the radius of the ball around $f^n(x)$ and $\mathrm{dist}(f^n(x), \Omega)$. Note also that the lemma is vacuous if $f^n(x) \in \Omega$.

Proof of Lemma 2.18. If $\mathrm{dist}(f^n(x), \Omega) \geq \delta/2$, then

$$s\mathrm{dist}(f^n(x), \Omega) = \frac{s}{\delta/2}\mathrm{dist}(f^n(x), \Omega)\frac{\delta}{2} \leq \frac{s}{\delta/2}\mathrm{diam}(S^1)\frac{\delta}{2} = \frac{2s}{\delta}\frac{\delta}{2}$$

and therefore it follows from Lemma 2.16 that any constant $K_2(s) \leq K_1(\delta/2, 2s/\delta)$ works in this case. So, we can suppose that $\mathrm{dist}(f^n(x), \Omega) < \delta/2$ and let $\omega \in \Omega$ be the only point such that $|f^n(x) - \omega| < \delta/2$. Denote the ball $B\big(f^n(x), \min\{s\mathrm{dist}(f^n(x), \Omega), \delta/4\}\big)$ by $B(f^n(x))$. Since $B(f^n(x)) \subset B(f^n(x), s|f^n(x) - \omega|) \subset B(\omega, \delta)$, for every $y \in B(f^n(x))$ there exists a unique integer $k = k(y)$ such that $f^k(y) \in B(\omega, \delta)\backslash B(\omega, \delta/||f'||)$. Suppose now additionally that $f_x^{-n} = f_\omega^{-n}$. Then for every $y \in B(f^n(x))$ we have $f_x^{-n}(y) = f_\omega^{-(n+k)}(f^k(y))$, thus by Lemma 2.15 $L_2^{-1}(n+k)^{-\frac{\beta+1}{\beta}} \leq |(f_x^{-n})'(y)| \leq L_2(n+k)^{-\frac{\beta+1}{\beta}}$, where $L_2 = L_2(\delta/||f'||)$ is the constant produced in Lemma 2.15. Since $(1-s)|f^n(x) - \omega| \leq |y - \omega| \leq (1+s)|f^n(x) - \omega|$, it follows from Corollary 2.14 that $(1-s)|f^n(x) - \omega| \leq L_1 k^{-1/\beta}$ and $(1+s)|f^n(x) - \omega| \geq L_1^{-1} k^{-1/\beta}$, where $L_1 = L_1(\delta/||f'||)$. Thus

$$\frac{\max\{k(y) : y \in B(f^n(x))\}}{\min\{k(y) : y \in B(f^n(x))\}} \leq \left(L_1^2 \frac{1+s}{1-s}\right)^\beta$$

Denote the number in the right-hand side of this inequality by $a(s)^\beta \geq 1$. We then have for all $y, z \in B(f^n(x))$

$$\frac{|(f_x^{-n})'(y)|}{|(f_x^{-n})'(z)|} \leq \frac{L_2(n+k(y))^{-\frac{\beta+1}{\beta}}}{L_2^{-1}(n+k(z))^{-\frac{\beta+1}{\beta}}} = L_2^2\left(\frac{n+k(y)}{n+k(z)}\right)^{-\frac{\beta+1}{\beta}} \leq L_2^2 a(s)^{\beta+1}$$

and therefore we are done in this case. In the general case let $0 \leq j \leq n$ be the least integer such that $f^i(x) \in B(\Omega, \delta/2)$ for all $j \leq i \leq n$. Then $f^i(x) = f_\omega^{-(n-i)}(f^n(x))$ and $f_x^{-n} = f_x^{-(i-1)} \circ g \circ f_\omega^{-(n-i)}$ where g is the inverse branch of f sending $f^i(x)$ to $f^{i-1}(x)$. and $f_x^{-(i-1)}$ is the inverse branch of f^{i-1} sending $f^{i-1}(x)$ to x. Now, we have just proved that $f_\omega^{-(n-i)}$ has the distortion bounded by a number depending only on s, a uniform boundedness of distortion of g is obvious, and since the point $f^{(i-1)}(x)$ is far away from Ω (at least at the distance $\geq \delta/2$), a uniform bound of the distortion of $f_x^{-(i-1)}$ follows from the first part of the proof. We are done. ♣

As an immediate consequence of Lemma 2.18 we get the following.

Corollary 2.19. For every sufficiently small $0 < \gamma < 1$, for every $x \in S^1$, and $n \geq 0$, such that if $f^n(x)$ is well-defined, then

$$K_2^{-1}(\gamma) \leq \frac{|(f_x^{-n})'(y)|}{|(f_x^{-n})'(z)|} \leq K_2(\gamma)$$

for all points $y, z \in B(f^n(x), \gamma \mathrm{dist}(f^n(x), \Omega))$.

Our last result in this section is in some sense a partial improvement of Lemma 2.18 toward attempting to have $\lim_{s\to 0} K_2(s) = 1$.

Lemma 2.20. For every integer $q \geq 1$ there exists an increasing function $Q_q : (0, \delta) \to [1, \infty]$ such that $\lim_{t\to 0} Q_q(t) = 1$ and

$$Q_q^{-1}(t) \leq \frac{|(f_x^{-n})'(y)|}{|(f_x^{-n})'(z)|} \leq Q_q(t)$$

for all points $y, z \in \Delta$, where $\Delta \subset B(\Omega, t)$ is an arbitrary subarc of S^1 such that $\#\big(\Delta \cap \{f_\omega^{-j}(\partial B(\omega, \delta)) : j \geq 0\}\big) \leq q$ and x is any point in S^1 such that $f^n(x)$ is well defined and $f^n(x) \in B(\Delta, t)$.

Proof. Observe that without loosing generality one can assume $q = 1$. Take $w \in \partial B(\omega, \delta)$ such that $\Delta \subset [\omega, w]$. Suppose first that $x = \omega$ is a parabolic point. Take any $v \in B(\omega, t)$. In view of (2.4) we have $|(f_\omega^{-n})'(v)| \leq |(f_\omega^{-n})'(f_\omega^{-1}(v))|$ for all $n \geq 1$. On the other hand

$$|(f_\omega^{-n})'(f_\omega^{-1}(v))| = |(f_\omega^{-n})'(v)| \cdot \frac{|f'(f_\omega^{-1}(v))|}{|f'(f_\omega^{(-n+1)}(v))|} \leq |(f_\omega^{-n})'(v)| \cdot |f'(f_\omega^{-1}(v))|$$

Hence

$$1 \leq \frac{|(f_\omega^{-n})'(f_\omega^{-1}(v))|}{|(f_\omega^{-n})'(v)|} \leq |f'(f_\omega^{-1}(v))|$$

for all $n \geq 1$. Since, by continuity of f', we have $\lim_{v\to\omega} |f'(f_\omega^{-1}(v))| = |f'(\omega)| = 1$, it follows from (2.12) and (2.4) (monotonicity of f') the existence of a function $K_1(t)$ claimed in the lemma as long as only the inverse branches of the form f_ω^{-n}, $\omega \in \Omega$, are involved. In the general case using what has been proved above, one repeats the argument described in the last part of the proof of Lemma 2.18. ♣

Frequently in the sequel, if there will be no specific requirements of how small $\gamma > 0$ is to be we will drop the dependence of $K_2(\gamma)$ on γ writing K_2 for $K_2(\gamma)$. We end up this section fixing the following notation

$$R(\omega) = B(\omega, \delta) \setminus B(\omega, \delta/||f'||).$$

§3. **Pressure and dimensions.** This section is somewhat sketchy, of rather general character and consists of two parts first of which is devoted to describe and discuss some general facts from geometric measure theory, while the second one provides quick introduction to the thermodynamic formalism and establishes some its basic applications in geometric measure theory. This part mostly overlaps with respect to the contents as well as with respect to the methods used with the paper [DU1].

To begin with given a subset A of a compact metric space (X, d), a countable family $\{B(x_i, r_i)\}_{i=1}^{\infty}$ of open balls centered at points of A is said to be a packing of A if and only if for any pair $i \neq j$

$$d(x_i, x_j) \geq r_i + r_j.$$

Given a nondecreasing function $g : (0, \varepsilon) \to (0, \infty)$ for some $\varepsilon > 0$, the g-dimensional outer Hausdorff measure $\mathrm{H}_g(A)$ of the set A is defined as

$$\mathrm{H}_g(A) = \sup_{\varepsilon > 0} \inf \bigl\{ \sum_{i=1}^{\infty} g\bigl(\mathrm{diam}(A_i)\bigr) \bigr\},$$

where infimum is taken over all countable covers $\{A_i : i \geq 1\}$ of A by arbitrary sets whose diameters do not exceed ε. If g is of the form x^t instead of writing H_{x^t} we write H_t and speak about t-dimensional outer Hausdorff measure. In this case one will get comparable numbers (in the sense that ratios are bounded away from zero and infinity) if instead of covering A by arbitrary sets one considers only open balls centered at points of A.

The g–dimensional outer packing measure $\Pi_g(A)$ of the set A is defined as

$$\Pi_g(A) = \inf_{\cup A_i = A} \bigl\{ \sum_i \Pi_g^*(A_i) \bigr\}$$

(A_i are arbitrary subsets of A), where Π_g^*, the g-packing premeasure is given by:

$$\Pi_g^*(A) = \inf_{\varepsilon > 0} \sup \bigl\{ \sum_{i=1}^{\infty} g(r_i) \bigr\}.$$

Here the supremum is taken over all packings $\{B(x_i, r_i)\}_{i=1}^{\infty}$ of the set A by open balls centered at points of A with radii which do not exceed ε. Similarly as in the case of Hausdorff measures if g is of the form x^t instead of writing Π_{x^t} we write Π_t and speak about t-dimensional outer packing measure. These two outer measures H_g and Π_g define countable additive measures on Borel σ-algebra of X. For additional properties of packing measures and a comprehensive discussion of this and related notions the reader is referred to the paper [TT] and [Ma] and [PU] books.

The definitions of the Hausdorff dimension $\mathrm{HD}(A)$ of A and packing dimension $\mathrm{PD}(A)$ are the following

$$\mathrm{HD}(A) = \inf\{t : \mathrm{H}_t(A) = 0\} = \sup\{t : \mathrm{H}_t(A) = \infty\}$$

and

$$\mathrm{PD}(A) = \inf\{t : \Pi_t(A) = 0\} = \sup\{t : \Pi_t(A) = \infty\}.$$

Let now ν be a Borel probability measure on X. Define the function $\rho = \rho_t(\nu) : X \times (0, \infty) \to (0, \infty)$ by

$$\rho(x, r) = \frac{\nu(B(x, r))}{r^t}.$$

The following two theorems (see [DU3], [Fa], [Ma], [PU], and [TT] for example) are for our aims the key facts from geometric measure theory. Their proofs are an easy consequence of Besicovič covering theorem (see [Gu]).

Theorem 3.1. Assume that X is a compact subspace of an d-dimensional euclidean space. Then there exists a constant $b(d)$ depending only on d with the following properties: If A is a Borel subset of X and $C > 0$ is a positive constant such that

(1) for all (but countably many) $x \in A$

$$\limsup_{r \to 0} \rho(x, r) \geq C^{-1},$$

then for every Borel subset $E \subset A$ we have $\mathrm{H}_t(E) \leq b(d)C\nu(E)$ and, in particular, $\mathrm{H}_t(A) < \infty$.

or

(2) for all $x \in A$

$$\limsup_{r \to 0} \rho(x, r) \leq C^{-1},$$

then for every Borel subset $E \subset A$ we have $\mathrm{H}_t(E) \geq Cb(d)^{-1}\nu(E)$.

Theorem 3.2. Assume that X is a compact subspace of an d-dimensional euclidean space. Then there exists a constant $b(d)$ depending only on d with the following properties: If A is a Borel subset of X and $C > 0$ is a positive constant such that

(1) for all $x \in A$

$$\liminf_{r \to 0} \rho(x, r) \leq C^{-1},$$

then for every Borel subset $E \subset A$ we have $\Pi_t(E) \geq Cb(d)^{-1}\nu(E)$,

or

(2) for all $x \in A$

$$\liminf_{r \to 0} \rho(x, r) \geq C^{-1},$$

then for every Borel subset $E \subset A$ we have $\Pi_t(E) \leq b(d)C\nu(E)$ and, consequently, $\Pi_t(A) < \infty$.

(1') If ν is non-atomic then (1) holds under the weaker assumption that the hypothesis of part (1) is satisfied on the complement of a countable set.

Let us now pass to the dynamics and thermodynamic formalism. Let $S : X \to X$ be a continuous map of a compact metric space X and let $\phi : X \to I\!\!R$ be a continuous function. Given an $\varepsilon > 0$ and an integer $n \geq 1$ we say that a set $F \subset X$ is (n, ε)-separated if and only if for all $x, y \in F$, $x \neq y$ there exists $0 \leq k \leq n-1$ such that $\mathrm{dist}(S^k(x), S^k(y)) > \varepsilon$. Let

$$E_n = \inf_F \sum_{x \in F} \exp\big(\sum_{j=0}^{n-1} \phi \circ S^j(x)\big),$$

where the infimum is taken over all maximal (in the sense of inclusion) (n, ε)-separated sets. The topological pressure $\mathrm{P}(S, \phi)$ of the map S and the function (potential) ϕ is defined as the following limit.

$$\mathrm{P}(S, \phi) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log E_n$$

In the case when the function f is identically equal to 0 the quantity E_n is the maximal cardinality of an (n, ε)-separated set. The pressure $\mathrm{P}(S, 0)$ is then rather called the topological entropy of S and is denoted by $\mathrm{h_{top}}(S)$.

A Borel measure μ is said to be S-invariant if and only if $\mu \circ S^{-1} = \mu$. The measure μ is said to be ergodic if and only if all invariant sets A, that is satisfying equality $\mu(A) = \mu(S^{-1}(A))$, are either of measure 0 or their complements are of measure 0. If μ is a Borel probability measure invariant under S then (see [BK]) the following limit exists for μ a.e. $x \in X$

$$\mathrm{h}_\mu(x) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon)), \tag{3.1}$$

where $B_n(x, \varepsilon) = \{y \in X : \mathrm{dist}(S^j(y), S^j(x)) < \varepsilon \text{ for all } j = 0, 1, \ldots, n-1\}$. The integral $\int \mathrm{h}_\mu(x)\, d\mu(x)$ is called the metric entropy of S with respect to the measure μ and is denoted by $\mathrm{h}_\mu(S)$. If μ is ergodic almost all numbers $\mathrm{h}_\mu(x)$ are equal to $\mathrm{h}_\mu(S)$. Usually in the literature a different approach is used to define metric entropy, which is based on the concept of partition. Formula (3.1) is then a deep theorem, called Brin-Katok formula whose proof uses heavily Breiman-Shannon-McMillan theorem. By $M(S)$, $M_e(S)$, $M^+(S)$ and $M_e^+(S)$ we denote respectively the set of all Borel probability measures invariant under S, its subset of ergodic measures, measures of positive entropy, and ergodic measures of positive entropy. The following formula

$$\mathrm{P}(S, \phi) = \sup_{M(S)} \left\{ \mathrm{h}_\mu(S) + \int \phi \, d\mu \right\} = \sup_{M_e(S)} \left\{ \mathrm{h}_\mu(S) + \int \phi \, d\mu \right\} \tag{3.2}$$

called the variational principle for topological pressure, or just variational principle, establishes basic relationship between the notions of pressure and entropy, and has been proven in [Wa].

Coming back to our continuous map $f : J \to J$ we recall first that the Lyapunov exponent $\chi_\mu(f)$ of f with respect to a measure $\mu \in M_e(f)$ is defined as

$$\chi_\mu(f) = \int \log |f'| \, d\mu.$$

We shall prove the following.

Proposition 3.3. If $\mu \in M_e(f)$, then $\chi_\mu(f) \geq 0$. Additionally $\chi_\mu(f) = 0 \ \Leftrightarrow \mu(\Omega) = 1 \ \Leftrightarrow \mu(\Omega) > 0 \ \Leftrightarrow \mu(\mu(\{\omega\}) = 1$ for some $\omega \in \Omega$.

Proof. That $\chi_\mu(f) \geq 0$ we see immediately from (2.3). The equivalence of the three last properties follows from ergodicity of μ, and $\mu(\Omega) = 1$ obviously implies that $\chi_\mu(f) = 0$. If $\mu(\Omega) = 0$, then there is a compact set $K \subset J$ of positive μ measure disjoint from Ω (and hence of positive distance from Ω apart) and therefore $|f'|_K > \lambda$ for some $\lambda > 1$. Since by the Birkhoff ergodic theorem every typical point of μ visits K with positive frequency, keeping in mind (2.3) we deduce that $\chi_\mu(f) > 0$. ♣

Let us now define the pressure function $\mathrm{P}(t)$, $t \in [0, \infty)$ putting

$$\mathrm{P}(t) = \mathrm{P}(f|_J, -t \log |f'|).$$

Some basic elementary properties of this function are collected in the following proposition.

Proposition 3.4. The function $t \mapsto \mathrm{P}(t)$ is continuous, non-increasing, and non-negative if $\Omega \neq \emptyset$.

Proof. The continuity follows immediately from general facts about topological pressure (see [Wa]). In order to prove that $\mathrm{P}(t)$ is decreasing, consider $0 \leq t_1 \leq t_2$. We see from Proposition 3.3 that for $\mu \in M(f)$, $h_\mu(f) - t_1\chi_\mu \geq h_\mu(f) - t_2\chi_\mu$. Hence, applying the variational principle it follows that $\mathrm{P}(t)$ is non-increasing. If $\Omega \neq \emptyset$, we can consider an f-invariant probability measure ν concentrated on a forward orbit of some point $\omega \in \Omega$. Obviously $h_\nu(f) = \chi_\nu = 0$. Hence, again by the variational principle, $\mathrm{P}(t) \geq h_\nu(f) - t\chi_\nu = 0$ for every $t \in [0, \infty)$. This completes the proof of the proposition. ♣

Recall that

$$\mathrm{HD}(\mu) = \inf\{\mathrm{HD}(Y) : \mu(Y) = 1\}.$$

By definition, $\mathrm{HD}(\mu) \leq \mathrm{HD}(J) \leq 2$ and hence $\sup\{\mathrm{HD}(\mu) : \mu \in M_e^+(f)\} \leq 1$. This supremum is in the literature denoted by $\mathrm{DD}(J)$ and called dynamical dimension of J (see [DU5], comp. [PU]). Let us recall also the famous formula for the Hausdorff dimension of an ergodic measure of positive entropy invariant under a conformal map whose origins go back to Billingsley's work and probably even earlier. Up to our knowledge, in the context of real one-dimensional dynamics, this formula has been proved by F. Hofbauer and P. Raith in [HR] under possibly weakest assumptions, much weaker than required here. It reads that if $\mu \in M_e^+(f)$, then $\chi_\mu(f) > 0$ and

$$\mathrm{HD}(\mu) = \frac{\mathrm{h}_\mu(f)}{\chi_\mu(f)} \tag{3.3}$$

We shall prove the following.

Lemma 3.5. We have
(a) $\mathrm{P}(t) > 0$ for every $t \in [0, \mathrm{DD}(J))$.
(b) If $\Omega = \emptyset$, then $\mathrm{P}(t) < 0$ for every $t \in (\mathrm{DD}(J), \infty)$. If $\Omega \neq \emptyset$, then $\mathrm{P}(t) = 0$ for every $t \in [\mathrm{DD}(J), \infty)$.
(c) $\mathrm{P}|_{[0,\mathrm{DD}(J)]}$ is injective.

Proof. For the sake of this proof let us denote the dynamical dimension $\mathrm{DD}(J)$ by s. If $t < s$ then by (3.3) there exists $\mu \in M_+(f)$ such that $t < h_\mu(f)/\chi_\mu$. Hence $\mathrm{P}(t) \geq \mathrm{h}_\mu(f) - t\chi_\mu > 0$ and (a) is proved. In order to prove (ii) consider any $t \geq 0$ and suppose that $\mathrm{P}(t) > 0$. Then by (3.2), the variational principle, there exists $\mu \in M(f)$ such that $\mathrm{h}_\mu(f) - t\chi_\mu > 0$. So, in view of Proposition 3.3, $\mathrm{h}_\mu(f) > 0$, and by (3.3), $s \geq \mathrm{HD}(\mu) = \mathrm{h}_\mu(f)/\chi_\mu > t$. So, $\mathrm{P}(t) \leq 0$ for $t \geq s$ and (b) follows from Proposition 3.4. We will show (c). Assume that $\mathrm{P}(t_1) = \mathrm{P}(t_2)$ for some $0 \leq t_1 < t_2 \leq s$. As $f|_J$ is expansive, there exist $\mu_1, \mu_2 \in M(f)$ (see e.g. [Wa]) such that $\mathrm{h}_{\mu_1}(f) - t_1\chi_{\mu_1} = \mathrm{P}(t_1) = \mathrm{P}(t_2) = \mathrm{h}_{\mu_2}(f) - t_2\chi_{\mu_2}$. If $\chi_{\mu_2} > 0$ then $t_1\chi_{\mu_2} < t_2\chi_{\mu_2}$. This implies that $\mathrm{P}(t_1) \geq \mathrm{h}_{\mu_2}(f) - t_1\chi_{\mu_2} > \mathrm{h}_{\mu_2}(f) - t_2\chi_{\mu_2} = \mathrm{P}(t_2)$ – a contradiction. Therefore $\chi_{\mu_2} = 0$ and by (3.3), $\mathrm{h}_{\mu_2}(f) = 0$. Thus $\mathrm{P}(t_1) = \mathrm{P}(t_2) = 0$ which contradicts part (a). ♣

It follows from this lemma that in the case when $\Omega \neq \emptyset$, the graph of the pressure function $\mathrm{P}(t)$ looks like on the Figure 1; it has a phase transition at the point $s = \mathrm{DD}(J)$. An intriguing problem arises of what kind this phase transition is. Is for example $\mathrm{P}(t)$ differentiable at s or not? We will come back to this point at the end of Section 7; at this moment we we shall prove the following.

Theorem 3.6. The function $\mathrm{P}(t)$ is differentiable at $t = \mathrm{DD}(J)$ if and only if there is no equilibrium state of positive entropy for the potential $-\mathrm{DD}(J)\log|f'|$.

Proof. For the sake of this proof set $s = \mathrm{DD}(J)$. Of course we only need to consider the left-hand side neighborhood of s. On the right-hand side $\mathrm{P}(t)$ is perfectly analytic and Lemma 3.5(b) shows that if $\mathrm{P}'(s)$ exists, then it must be equal to 0. So, suppose that there is μ, an equilibrium state for $-s\log|f'|$ with $\mathrm{h}_\mu > 0$. Then by (3.3), $\chi_\mu > 0$ and by (3.2), the variational principle, for every $t > 0$, we have $\mathrm{P}(t) - \mathrm{P}(s) \geq \mathrm{h}_\mu - t\chi_\mu - (\mathrm{h}_\mu - s\chi_\mu) = -(t-s)\chi_\mu$. Hence

$$\limsup_{t \nearrow s} \frac{\mathrm{P}(t) - \mathrm{P}(s)}{t - s} \leq -\chi_\mu < 0$$

and $\mathrm{P}'(s)$ does not exist.

If, on the other hand $\mathrm{P}'(s)$ does not exist, then there exist a sequence $t_n \nearrow s$ and a number $\sigma > 0$ such that

$$\mathrm{P}(t_n) - \mathrm{P}(s) \geq \sigma(s - t_n). \tag{3.4}$$

Without loosing generality we may assume that the sequence μ_n of equilibrium states for $-t_n\log|f'|$ converges in the weak topology of measures to an f-invariant measure μ.

Since, by Theorem 2.8, the map $f : J \to J$ is expansive, it follows from [Wa] that the entropy function $\nu \to \mathrm{h}_\nu(f)$ is upper semi-continuous. This and the continuity of $\mathrm{P}(t)$ imply that μ is an equilibrium state for the potential $-s\log|f'|$. We shall now show that $\mathrm{h}_\mu > 0$ completing the proof. Indeed, it follows from (3.4) that for all $n \ge 1$ we have $\mathrm{h}_{\mu_n} - t_n\chi_{\mu_n} \ge \mathrm{P}(s) + \sigma(s - t_n) \ge \mathrm{h}_{\mu_n} - s\chi_{\mu_n}$. Hence $\chi_{\mu_n} \ge \sigma$ for all $n \ge 1$. By Lemma 3.5 we have $\mathrm{h}_{\mu_n} - t_n\chi_{\mu_n} = \mathrm{P}(t_n) \ge 0$, whence $\liminf_{n\to\infty} \mathrm{h}_{\mu_n} \ge s\sigma$. Thus, applying the upper semi-continuity of the entropy function again, we get $\mathrm{h}_\mu \ge s\sigma > 0$ which completes the proof. ♣

§4. **Conformal measures and dimensions.** This section constitutes a natural extension of the previous one enriching its results by employing the method of conformal measures along the lines worked out in [DU1], [DU5], and [U1] (see also [PU]). We begin this section with the definition of conformal measures. Let $t \ge 0$ be a real number. A Borel probability measure m on the Cantor set J is called t-conformal for f if and only if

$$m(f(A)) = \int_A |f'|^t \, dm \tag{4.1}$$

for every special set $A \subset J$, that is a Borel subset of J such that $f|_A$ is injective.

Notice that if m is t-conformal, then

$$m(f(A)) \le \int_A |f'|^t \, dm \tag{4.2}$$

for every Borel set $A \subset J$. Observe also that for a measure m to be conformal it is enough to check (4.1) for Borel subsets of elements of partition $\{\Delta_j : j \in I\}$. From (2.2) and primitiveness of the incidence matrix A we immediately get the following.

Lemma 4.1. Any conformal measure for f is positive on nonempty open subsets of J.

Lemma 4.2. Let $x \in J \setminus \bigcup_{n=0}^\infty f^{-n}(\Omega)$. Then there exist an increasing sequence $\{n_j = n_j(x) : j \ge 1\}$ of positive integers, a sequence $\{r_j(x)\}_{j=1}^\infty$ of positive reals decreasing to 0, and an element $y \in \omega(x) \setminus B(\Omega, \delta)$ with the following properties:

(a) $y = \lim_{j\to\infty} f^{n_j}(x)$.

(b) $f^{n_j}(x) \notin B(\Omega, \delta)$.

(c) If m is a t-conformal measure for f, $t \ge 0$, then there exists a constant $B(m) \ge 1$ such that

$$B(m)^{-1} \le \frac{m(B(x, r_j(x)))}{r_j(x)^t} \le B(m).$$

for all $j \ge 1$.

Proof. In view of Lemma 4.1

$$M = \inf\{m(B(z, K_2^{-2}\gamma\delta)) : z \in J\} > 0.$$

It follows from Theorem 2.8 that if $x \in J \setminus \bigcup_{n=0}^{\infty} f^{-n}(\Omega)$, then there exists a sequence $\{n_j = n_j(x) : j \geq 1\}$ such that $f^{n_j(x)} \notin B(\Omega, \delta)$. Let $f_x^{-n_j} : B(f^{n_j}(x), \gamma\delta) \to S^1$ be the continuous inverse branch of f^{n_j} sending $f^{n_j}(x)$ to x. Then it follows from Corollary 2.19, that $f_x^{-n_j}\big(B(f^{n_j}(x), \gamma\delta)\big) \supset B\big(x, r_j\big)$ and $f^{n_j}(B(x, r_j)) \supset B(f^{n_j}(x), K_2^{-2}\gamma\delta)$, where

$$r_j = r_j(x) = K_2^{-1}|(f_x^{-n_j})'(f^{n_j}(x))|\gamma\delta = K_2^{-1}\gamma\delta|(f^{n_j})'(x)|^{-1}.$$

Using conformality of m and Corollary 2.19 we can estimate

$$\begin{aligned} 1 \geq m(f^{n_j}(B(x, r_j))) &= \int_{B(x,r_j)} |(f^{n_j})'|^t \, dm \geq K_2^{-t}|(f^{n_j})'(x)|^t m(B(x, r_j)) \\ &= (\gamma\delta)^t K_2^{-2t} r_j^{-t} m(B(x, r_j)) \end{aligned}$$

and

$$\begin{aligned} M \leq m(f^{n_j}(B(x, r_j))) &= \int_{B(x,r_j)} |(f^{n_j})'|^t \, dm \leq K_2^t|(f^{n_j})'(x)|^t m(B(x, r_j)) \\ &= (\gamma\delta)^t r_j^{-t} m(B(x, r_j)). \end{aligned}$$

Therefore $M(\gamma\delta)^{-t} \leq \frac{m(B(x,r_j))}{r_j^t} \leq K_2^{2t}(\gamma\delta)^{-t}$. Also, using (2.3) we can easily deduce that $\lim_{j\to\infty} |(f_\nu^{-n_j})'(f^{n_j}(x))| = 0$ and consequently, $r_j(x) = K_2^{-1}|(f^{n_j})'(x)|^{-1}\gamma\delta \to 0$. Since J is compact, passing to a subsequence of j, property (a) will be also satisfied. ♣

Let us now give a proof of the following well-known fact from the geometric measure theory.

Lemma 4.3. Let μ and ν be Borel probability measures on Y, a bounded subset of a Euclidean space. Suppose that there are a constant$M > 0$ and for every point $x \in Y$ a decreasing to zero sequence $\{r_j(x) : j \geq 0\}$ of positive radii such that for all $j \geq 1$ and all $x \in Y$

$$\mu(B(x, r_j(x)) \leq M\nu(B(x, r_j(x)).$$

Then the measure μ is absolutely continuous with respect to ν and the Radon-Nikodym derivative $d\mu/d\nu$ is uniformly bounded away from infinity .

Proof. Consider a Borel set $E \subset Y$ and fix $\varepsilon > 0$. Since $\lim_{j\to\infty} r_j(x) = 0$ and since ν is regular, for every $x \in E$ there exists a radius $r(x)$ being of the form $r_j(x)$ such that $\nu(\bigcup_{x\in E} B(x, r(x)) \setminus E) < \varepsilon$. Now by the Besicovič theorem (see [Gu]) we can choose a countable subcover $\{B(x_i, r(x_i))\}_{i=1}^{\infty}$ from the cover $\{B(x, r(x))\}_{x\in E}$ of E, of multiplicity bounded by some constant $C \geq 1$, independent of the cover. Therefore we obtain

$$\begin{aligned} \mu(E) &\leq \sum_{i=1}^{\infty} \mu(B(x_i, r(x_i))) \leq M \sum_{i=1}^{\infty} \nu(B(x_i, r(x_i))) \\ &\leq MC\nu\Big(\bigcup_{i=1}^{\infty} B(x_i, r(x_i))\Big) \\ &\leq MC(\varepsilon + \nu(E)). \end{aligned}$$

Letting $\varepsilon \searrow 0$ we obtain $\nu(E) \le MC\nu(E)$. So μ is absolutely continuous with respect to ν with Radon–Nikodym derivative bounded by MC. ♣

Let $X = J \setminus \bigcup_{n=0}^{\infty} f^{-n}(\Omega)$. As a direct consequence of the two previous lemmas we get the following.

Lemma 4.4. Any two t-conformal measures for $f : J \to J$ restricted to the set X are equivalent. Moreover their Radon-Nikodym derivative $\phi : X \to I\!\!R$ is bounded away from zero and infinity, and satisfies $\phi(f(x)) = \phi(x)$ for almost every $x \in X$.

Proof. Indeed, in view of Lemma 4.2 and Lemma 4.3 only the equation $\phi(f(x)) = \phi(x)$ requires a proof, which is obtained by direct computation. ♣

Lemma 4.5. If H_t is the t–dimensional Hausdorff measure on J and m is a t-conformal measure for $T : J \to J$ then H_t is absolutely continuous with respect to m such that the Radon–Nikodym derivative is bounded from above. Consequently $t \ge \mathrm{HD}(J)$ and there is no t-conformal measure for $t < \mathrm{HD}(J)$.

Proof. Let $F \subset J$ be any Borel set. Put $E = X \cap F = F \setminus \bigcup_{n=0}^{\infty} T^{-n}(\Omega)$. Since the set $\bigcup_{n=0}^{\infty} f^{-n}(\Omega)$ is at most countable, $\mathrm{H}_t(E) = \mathrm{H}_t(F)$. Fix $\eta, \varepsilon > 0$. Since m is regular, similarly to the argument used in the proof of Lemma 4.4, we can find a countable cover $\{B(x_i, r(x_i))\}_{i=1}^{\infty}$ of E of multiplicity bounded by $M \ge 1$ such that $x_i \in E$, the radius $0 < r(x_i) < \eta$ is of the form $r_j(x_i)$ for every $i = 1, 2, \ldots$ (defined in Lemma 4.2) and such that $m\big(\bigcup_{i=1}^{\infty} B(x_i, r(x_i)) \setminus E\big) < \varepsilon$. Hence, applying Lemma 4.2 to the measure m, we obtain

$$
\begin{aligned}
\sum_{i=1}^{\infty} r(x_i)^t &\le B(m) \sum_{i=1}^{\infty} m(B(x_i, r(x_i))) \le B(m) M m\Big(\bigcup_{i=1}^{\infty} B(x_i, r(x_i))\Big) \\
&\le MB(m)(\varepsilon + m(E))
\end{aligned}
$$

Letting $\varepsilon \searrow 0$ and then $\eta \searrow 0$ we get $\mathrm{H}_t(F) = \mathrm{H}_t(E) \le CB(m)m(E) \le CB(m)m(F)$. ♣

Let $e(J)$ be the infimum of all exponents $t \ge 0$ such that a t-conformal measure exists and let $\delta(J)$ be the first zero of the pressure function $\mathrm{P}(t)$. The main result of this section is the following.

Theorem 4.6. We have $\mathrm{DD}(J) = \delta(J) = e(J) = \mathrm{HD}(J)$ and an h-conformal measure exists, where h denotes the common value of these three numbers.

Proof. That $\delta(J) = \mathrm{DD}(J) \le \mathrm{HD}(J) \le e(J)$ we see from Lemma 3.5 and Lemma 4.5. So, in order to complete the proof it suffices to find a $\delta(J)$-conformal measure on J. But since by Theorem 2.8 the mapping $f : J \to J$ is open and expansive, and since $\mathrm{P}(\delta(J)) = 0$, the existence of such measure follows from Theorem 3.12 of [DU6]. ♣

It seems interesting to ask about other conformal measures for $t > h$. If $\Omega = \emptyset$, then no such measures exist. In the opposite case the do exist (since $\mathrm{P}(f) = 0$) but are concentrated on the backward orbit of $\Omega(f)$. We will never make use of this remark and the reader interested in proofs is suggested to look at the paper [DU1].

§5. **Local behavior around parabolic points.** In this section we examine the local behavior of conformal measures around parabolic points. For every $\omega \in \Omega$ let

$$\alpha(\omega) = h + \beta(\omega)(h-1).$$

We begin with proving the following.

Lemma 5.1. If m is an h-conformal measure for $f : J \to J$, then $\exists_{(C_1 \geq 1)} \forall_{(\omega \in \Omega)} \forall_{(0 < r \leq 1)}$

$$C_1^{-1} \leq \frac{m(B(\omega, r) \setminus \{\omega\})}{r^{\alpha(\omega)}} \leq C_1.$$

Proof. Fix a constant $L_1 = L_1(\delta/||f'||)$, where the function L_1 is described in Corollary 2.14 and for every $n \geq 1$ define $R_n = \{z \in B(\omega, \delta) : L_1^{-1} n^{-1/\beta} \leq |f_\omega^{-n}(z) - \omega| \leq L_1 n^{-1/\beta}\}$. By definition of $R(\omega)$ and since δ is an expansive constant for $f : J \to J$, we conclude that for every $z \in B(\omega, \delta) \cap J \setminus \{\omega\}$ there exists $l \geq 0$ such that $f^l(z) \in R(\omega)$. Therefore, the set $J \cap R(\omega)$ is nonempty and since J is perfect, it has nonempty interior in J. Hence at least one of the connected components of $R(\omega)$, denote it by $R_0(\omega)$, has positive measure m. By Corollary 2.14 there is $n_0 \geq 1$ such that $f_\omega^{-n}(z) \in R_n$ for every $n \geq n_0$ and every $z \in R(\omega)$. In other words this means that $R_n \supset f_\omega^{-n}(R(\omega))$ for $n \geq n_0$. Thus

$$B\left(\omega, \frac{L_1}{n^{1/\beta}}\right) \supset \bigcup_{k=n}^{\infty} R_k \supset \bigcup_{k=n}^{\infty} f_\omega^{-k}(R(\omega)) \supset \bigcup_{k=n}^{\infty} f_\omega^{-n}(R_0(\omega)).$$

On the other hand, for any $z \in B(\omega, \delta) \setminus \{\omega\}$ let $l(z) \geq 0$ be the smallest integer such that $f^l(z) \in R(\omega)$. Take $n_1 \geq n_0$ so large that if $z \in B(\omega, L_1 n_1^{-1/\beta})$, then $l(z) \geq n_0$. Consider now any $z \in B(\omega, L_1 n^{-1/\beta}) \setminus \{\omega\}$ with $n \geq n_1$. Since $l(z) \geq n_0$ and $f^{l(z)}(z) \in R(\omega)$ we conclude that $z = (f^{l(z)}(z))_{l(z)} \in R_{l(z)}$. Therefore $L_1^{-1} l(z)^{-1/\beta} \leq L_1 n^{-1/\beta}$ and consequently $l(z) \geq L_1^{-2\beta} n$. Hence

$$B\left(\omega, \frac{L_1}{n^{1/\beta}}\right) \subset sbt\{\omega\} \cup \bigcup_{l \geq L_1^{-2\beta} n} f_\omega^{-l}(R(\omega)).$$

In view of Lemma 2.15 and the conformality of the measure m we get

$$\begin{aligned} m\big(B(\omega, L_1 n^{-1/\beta})\big) &\geq \sum_{k=n}^{\infty} m(f_\omega^{-k}(R_0(\omega)) \geq \sum_{k=n}^{\infty} L_2^{-h} (k^{-\frac{\beta+1}{\beta}})^h m(R_0(\omega)) \\ &\geq m(R_0(\omega)) L_2^{-h} \sum_{k=n}^{2n} (k^{-\frac{\beta+1}{\beta}})^h \geq m(R_0(\omega)) L_2^{-h} n((2n)^{-\frac{\beta+1}{\beta}h}) \\ &= 2^{-\frac{\beta+1}{\beta}h} m(R_0(\omega)) K^{-h} L_2^{-h} (n^{-1/\beta})^{\alpha(\omega)}, \end{aligned}$$

where $L_2 = L_2(\delta/\|f'\|)$, and (using continuity of m in addition)

$$m\big(B(\omega, L_1 n^{-1/\beta})\big) \le m\big(m\big(\{\omega\} \cup \bigcup_{l \ge L_1^{-2\beta} n} f_\omega^{-l}(R(\omega))\big) \le L_2^h \sum_{l \ge L_1^{-2\beta} n} l^{-\frac{\beta+1}{\beta}h}$$
$$\le L_1'(L_1 n^{-1/\beta})^{\alpha(\omega)}$$

where $L_1' > 0$ denotes some constant. The proof is finished observing that the limit of $\frac{(n+1)^{-1/\beta}}{n^{-1/\beta}}$ is 1. ♣

Now we shall prove a result which can be viewed as an improvement of Lemma 5.1.

Lemma 5.2. $\forall_{(\zeta>0)}\ \exists_{(C_2=C_2(\zeta)\ge 1)}\ \forall_{(\omega\in\Omega)}\ \forall_{(z\in J)}$

$$C_2^{-1}|z-\omega|^{\alpha(\omega)} \le m(B(z, \zeta|z-\omega|)) \le C_2|z-\omega|^{\alpha(\omega)}$$

Proof. Let us first prove the right-hand side of this lemma. Consider $z \in B(\omega, \delta)$ such that $(1+\zeta)|z-\omega| \le 1$. Then in view of Lemma 5.1, we have

$$m(B(z, \zeta|z-\omega|)) \le m(B(\omega, (1+\zeta)|z-\omega|)) \le C(1+\zeta)^{\alpha(\omega)}|z-\omega|^{h+p(\omega)(h-1)}$$
$$\le C_1(1+\zeta)^h|z-\omega|^h.$$

If $|z-\omega| > (1+\zeta)^{-1}$ or $z \notin B(\omega, \delta)$, it is enough to apply the obvious estimate $m(B(z, \zeta|z-\omega|)) \le 1$.

In order to prove the left-hand side inequality suppose first that $z \in B(\omega, \delta)$ for some $\omega \in \Omega$ and even more that $|z-\omega| \le (\zeta L^{-1}/2)^{1/\beta}$ where $\beta = \beta(\omega)$. Let $k \ge 0$ be the largest integer such that $f_\omega^{-k}(z) \in B(z, \zeta|z-\omega|)$. By a simple integration argument contained for example in the proof of Lemma 2.13, it follows from (2.5) that

$$L^{-1}(\beta+1)^{-1}|x-\omega|^{\beta+1} \le |f_\omega^{-1}(x) - x| \le L(\beta+1)^{-1}|x-\omega|^{\beta+1} \tag{5.1}$$

for every $x \in B(\omega, \delta)$. Therefore

$$\zeta|z-\omega| \le \sum_{j=0}^{k} |f_\omega^{-(j+1)}(z) - f_\omega^{-j}(z)| \le (k+1)L(\beta+1)^{-1}|z-\omega|^{\beta+1},$$

whence $k+1 \ge L^{-1}(\beta+1)\zeta|z-\omega|^{-\beta}$. Thus by (5.1) we get $k \ge l+1$ where $l = L^{-1}\beta\zeta|z-\omega|^{-\beta}$. Letting now $n \ge 0$ be the least integer with $f^n(z) \in R(\omega)$ and setting $y = f^n(z)$, it follows from Lemma 2.15 that

$$m(B(z, \zeta|z-\omega|)) \ge \sum_{j=0}^{k} m([f_\omega^{-(j+1)}(z), f_\omega^{-j}(z)]) \ge \sum_{j=0}^{l} m([f_\omega^{-(j+1)}(z), f_\omega^{-j}(z)])$$
$$= \sum_{q=n}^{n+l} m([f_\omega^{-(q+1)}(y), f_\omega^{-q}(y)]) \ge \sum_{q=n}^{n+l} L_2(\delta/||f'||^2)^{-h} q^{-\frac{\beta+1}{\beta}h}$$
$$\ge L_2(\delta/||f'||^2)^{-h} l(n+l)^{-\frac{\beta+1}{\beta}h} \tag{5.2}$$

Now it follows from Corollary 2.14 that with $L_1 = L_1(\delta/||f'||)$, we have $n^{-1/\beta} \geq L_1^{-1}|z-\omega|$, whence $n \leq L_1^\beta |z-\omega|^{-\beta}$. Thus, combining this, (5.2), and since $k \geq l+1$ we get

$$\begin{aligned} m(B(z,\zeta|z-\omega|)) &\geq L_2(\delta/||f'||^2)^{-h}L^{-1}\beta\zeta|z-\omega|^{-\beta}(L_1^\beta + L^{-1}\beta\zeta)^{-\frac{\beta+1}{\beta}h}(|z-\omega|^{-\beta})^{-\frac{\beta+1}{\beta}h} \\ &\geq C|z-\omega|^{-\beta}|z-\omega|^{(\beta+1)h} = C|z-\omega|^{\alpha(\omega)} \end{aligned}$$

with a universal constant C depending on ζ. Thus the proof is finished since the case $|z-\omega| \geq (\zeta L^{-1}/2)^{1/\beta}$ for all $\omega \in \Omega$ is taken care by the observation that the infimum of all measures $m(B(z,\zeta(\zeta L^{-1}/2)^{1/\beta})$, $z \in J$, is positive which in turn follows from Lemma 4.1. ♣

We want to end up this section with the following two results which although of global character, are proved by employing a local argument. Moreover the second result is a starting point for our all next considerations.

Theorem 5.3. We have $h = \mathrm{HD}(J) > \max\{\beta(\omega)/(\beta(\omega)+1) : \omega \in \Omega\}$.

Proof. Fix fix $\omega \in \Omega$. Since δ is an expansive constant for f, the interior of at least one of the two connected components of $R(\delta)$, has a nonempty intersection with the set J. Call it by $R_0(\omega)$. Since by Theorem 4.6 there exists an h-conformal measure m for $f : J \to J$, it follows from Lemma 2.15 that

$$1 \geq \sum_{n=1}^\infty m(f_\omega^{-n}(R_0(\omega)) \geq L_2(\delta/\|f'\|)^{-h} m(R_0(\omega)) \sum_{n=1}^\infty n^{-\frac{\beta+1}{\beta}h}.$$

Since $m(R_0(\omega)) > 0$, this formula implies that the series $\sum_{n=1}^\infty n^{-\frac{\beta+1}{\beta}h}$ converges. Therefore, $h > \beta(\omega)/(\beta(\omega)+1)$. The proof is finished. ♣

Theorem 5.4. There exists a unique (up to equivalence of measures) h-conformal measure. Moreover this measure is continuous.

Proof. By Theorem 4.6 and Theorem 3.12 in [DU6] there is an h-conformal measure for f. By Lemma 4.4, this measure, if continuous, is unique up to equivalence of measures. From Lemma 2.15 and Theorem 5.3 we deduce that there exist constants $\sigma > 0$ and $C > 0$ such that for every fixed point $\omega \in \Omega$ and every point $\omega \in R(\omega)$ $\exists_{(C(\omega,z)\geq 1)}$ $\forall_{(t \geq h-\sigma)}$ $\forall_{(k\geq 1)}$

$$\sum_{n=k}^\infty |(f_\omega^{-n})'(z)|^t \leq C \sum_{n=k}^\infty \frac{1}{n^{1+\sigma}}.$$

Let us now construct a special sequence of neighborhoods of Ω. To this end fix $\omega \in \Omega$, $n \geq 1$, and consider the two connected components V_ω^1 and V_ω^2 of $S^1 \setminus f^{-n}(\{\omega\})$ that are adjacent to ω. Define then $W_n = \Omega \cup \bigcup_{\omega\in\Omega} V_\omega^1 \cup V_\omega^2$ which is an open neighborhood of Ω and let $K_n = \{z \in J : f^k(z) \notin W_n \text{ for every } k \geq 0\}$. The sets K_n are closed and forward invariant under f. Moreover the maps $f|_{K_n} : K_n \to K_n$ are open. Since additionally, by Theorem 2.8, these are expansive, it follows from Theorem 3.12 in [DU6] and Theorem 4.6 that for every $n \geq 1$ there exists a number $h_n \leq h$ and an h_n-conformal measure for $f|_{K_n}$. Notice that then $m_n(f(A)) \geq \int_A |f'|^{t_n}\, dm_n$ for every special set $A \subset J$ and $m_n(f(A)) = \int_A |f'|^{t_n}\, dm_n$ for every special set $A \subset J$ disjoint from $\overline{W_n}$. Let m be an arbitrary weak accumulation point of the sequence $\{m_n\}_{n=1}^\infty$ in the weak-$*$ topology on J. Fix $k \geq 1$. Since $f : J \to J$ is an open map it easily follows, see for ex. lemma 3.3 in [DU5], that $m(f(A)) = \int_A |f'|^u\, dm$ for every special set $A \subset J$ disjoint from $\overline{W_k}$ where $t_n \to u$. Therefore, since $\{W_n : n \geq 1\}$ is a descending sequence of sets such that $J \cap \bigcap_{n\geq 1} W_n = \Omega$, letting $k \to \infty$ we conclude that this formula spreads out to every special set $A \subset J$ disjoint from Ω. And since $|f'(\omega)| = 1$ for every $\omega \in \Omega$, it is true for every special set $A \subset J$. Consequently m is a u–conformal measure for $f : J \to J$. As $u \leq h$, it follows from Theorem 4.6 that $u = h$.

In order to conclude the proof it is sufficient to show that $m(\Omega) = 0$. Since δ is an expansive constant for f, we conclude that for every $\omega \neq x \in B(\omega, \delta) \cap J$ there exists the least integer $n(x) \geq 0$ such that $f^{n(x)} \in R(\omega)$. Thus, for every open neighborhood $V \subset B(\omega, \delta)$ of ω we have $V \cap J \subset sbt\{\omega\} \cup \bigcup_{n \geq n(V)} f_\omega^{-n}(R(\omega))$, where $n(V) = \min\{n(x) : \omega \neq x \in V \cap J\}$. Using the properties of $\{m_n\}$ and the definition of σ, we therefore conclude that for every $k \geq 1$ large enough

$$m_k(V) \leq Cm(R(\omega)) \sum_{n=n(V)}^{\infty} \frac{1}{n^{1+\sigma}}.$$

Thus letting $k \to \infty$ we get $m(V) \leq Cm(R(\omega)) \sum_{n=n(V)}^\infty \frac{1}{n^{1+\sigma}}$ which proves that $m(\omega) = 0$, since $n(V) \to \infty$ as V shrinks to ω. ♣

In Section 8 we shall show more, that there is only one such measure.

§6. **Geometric measures.** In this section following the ideas and exposition contained in [DU3], [DU4], and [U2] we deal with geometric properties of the set J. Recall that in Section 4 we have defined X to be $J \setminus \bigcup_{n=0}^\infty f^{-n}(\Omega)$.

Lemma 6.1. If $F \subset J$ is a closed nowhere dense (relative to J) forward invariant subset of J, then $m(F) = 0$.

Proof. Since m is nonatomic it suffices to show that $m(F \setminus X) = 0$. Denote by Z the set of all points $z \in F \setminus X$ such that

$$\lim_{r \to 0} \frac{m(B(z,r) \cap (F \setminus X))}{m(B(z,r))} = 1.$$

In view of the Lebesgue density theorem (see for example Theorem 2.10.11 in [Fa]), $m(Z) = m(Y)$. Suppose now that $m(Z) > 0$ and fix $x \in Y$. Let y and the sequence $\{n_j\}$ be the objects associated to x produced in Lemma 4.2. Since F is nowhere dense in J and since m is positive on nonempty open sets of J for every j large enough $m\big(B(f^{n_j}(x), \gamma\delta) \setminus F\big) \geq m(B(x, \gamma\delta/2) \setminus F) > 0$. Therefore, as $f^{-1}(J \setminus F) \subset J \setminus F$, the standard way of application of the bounded distortion property (Corollary 2.19 in our case) and conformality of m gives

$$\limsup_{r\to 0}\frac{m(B(z,r)\setminus F)}{m(B(z,r))}>0$$

which contradicts the definition of the set Z and finishes the proof. ♣

A point $z\in J$ is said to be transitive if $\omega(z)=J$. Consider a countable basis $\{V_n\}_{n=1}^\infty$ of topology on J. By Lemma 6.1 and Lemma 2.3(e) every set $K_n=\{z: f^k(z)\notin V_n\}$ is of m measure zero and therefore $m\big(\bigcup_{k\ge 0}\bigcup_{n\ge 0} f^{-k}(V_n)\big)=0$. Since the complement of this set consists of transitive points, we obtain the following.

Lemma 6.2. $m(\{z:\omega(z)=J\})=1$.

Lemma 6.3. For every $C_3>0$ there exists $C_4>0$ such that if $n\ge 0$, $f^n(z)\in B(\omega,\delta)$, $\omega\in\Omega$, and $f^{n-1}(z)\notin B(\omega,\delta)$ (in case $n\ge 1$), then for every $r>0$ satisfying $r|(f^n)'(z)|\le \gamma\delta K_2^{-1}$ and $r|(f^n)'(z)|\ge C_3|f^n(z)-\omega|$ we have

$$C_4^{-1}(r|(f^n)'(z)|)^{\beta(\omega)(h-1)}\le \frac{m(B(z,r))}{r^h}\le C_4(r|(f^n)'(z)|)^{\alpha(\omega)(h-1)}.$$

Proof. Since $f^{n-1}(z)\notin B(\omega,\delta)$, applying Corollary 2.19 to the continuous inverse branch $f_z^{-n}: B(f^n(z),\gamma\delta)\to S^1$ of f^n sending $f^n(z)$ to z. we obtain

$$\begin{aligned}(||f'||K_2)^{-h}|(f^n)'(z)|^{-h}m(B(f^n(z),K_2^{-1}r|(f^n)'(z)|))&\le\\ \le m(B(z,r))&\le\\ \le (||f'||K_2)^{h}|(f^n)'(z)|^{-h}m(B(f^n(z),K_2r|(f^n)'(z)|)).\end{aligned}\tag{6.1}$$

It follows from the last assumption of our lemma that the ball $B(f^n(z),K_2r|(f^n)'(z)|)$ is contained in the ball $B(\omega,(K_2+C_3^{-1})r|(f^n)'(z)|)$. Thus, in view of Lemma 5.1,

$$m(B(z,r))\le C_1(||f'||K_2)^h(K_2+C_3^{-1})^{\alpha(\omega)}|(f^n)'(z)|^{-h}(r|(f^n)'(z)|)^{\alpha(\omega)}.$$

Hence

$$\frac{m(B(z,r))}{r^h}\le C_1(||f'||K_2)^h(K_2+C_3^{-1})^{\beta(\omega)}(r|(f^n)'(z)|)^{\alpha(\omega)(h-1)}.\tag{6.2}$$

If $\frac12K_2^{-1}r|(f^n)'(z)|\ge|f^n(z)-\omega|$ then $B(\omega,\frac12K_2^{-1}r|(f^n)'(z)|)\subset B(f^n(z),K_2^{-1}r|(f^n)'(z)|)$ and by similar arguments as before we obtain

$$\frac{m(B(z,r))}{r^h}\ge C_1^{-1}(||f'||K_2)^{-h}(2K_2)^{-\alpha(\omega)}(r|(f^n)'(z)|)^{\beta(\omega)(h-1)}.\tag{6.3}$$

If $\frac12K_2^{-1}r|(f^n)'(z)|\le|f^n(z)-\omega|$ then using (6.1), assumption (b), and Lemma 5.2 with $\xi=C_3K_2^{-1}$ we get

$$\begin{aligned}m(B(z,r))&\ge (C_2(\xi))^{-1}(||f'||K_2)^{-h}|(f^n)'(z)|^{-h}|f^n(z)-\omega|^{\alpha(\omega)}\\ &\ge (C_2(\xi))^{-1}2^{-\alpha(\omega)}(||f'||K_2)^{-h}K_2^{-\alpha(\omega)}|(f^n)'(z)|^{-h}(r|(f^n)'(z)|)^{\alpha(\omega)}\end{aligned}$$

and therefore

$$\frac{m(B(z,r))}{r^h} \geq (C_2(\xi)2^{\alpha(\omega)}||f'||^h K_2^{h+\alpha(\omega)})^{-1}(r|(f^n)'(z)|)^{\beta(\omega)(h-1)}.$$

This, (6.3), and (6.2) prove the lemma. ♣

Now we shall construct (positive) integer valued functions $n(z,r)$, $k(z,r)$ and $u(z,r)$, ($z \in J$, $0 < r < 1$), simultaneously proving their properties listed in Theorem 6.4 below.

Theorem 6.4. There exists $Q \geq 1$ such that for every pair (z,r), $z \in J$, $0 < r < 1$, there exists a number $\beta(z,r) \in \{\beta(\omega) : \omega \in \Omega\} \cup \{0\}$ such that

$$Q^{-1}(r|(f^u)'(z)|)^{\beta(z,r)(h-1)} \leq \frac{m(B(z,r))}{r^h} \leq Q(r|(f^u)'(z)|)^{\beta(z,r)(h-1)}.$$

Moreover $\gamma\delta(K_2\|f'\|)^{-1}|f^u(z)-\omega| \leq r|(f^u)'(z)| \leq \gamma\delta K_2^{-1}$ and there is a continuous inverse branch $f_z^{-u} : B(f^u(z), r|(f^u)'(z)|) \to S^1$ sending $f^u(z)$ to z.

Proof. Suppose first that $\sup_{n\geq 0}\{r|(f^n)'(z)|\} > \gamma\delta(K_2\|f'\|)^{-1}$ and let $n = n(z,r) \geq 0$ be a minimal integer such that $r|(f^n)'(z)| > \gamma\delta(K_2\|f'\|)^{-1}$. Then also $r|(f^n)'(z)| \leq \gamma\delta K_2^{-1}$. We say that the pair (z,r) belongs to the family $\Re$ if $f^n(z) \notin B(\Omega,\delta)$. Since the conformal measure m is positive on nonempty open sets, $\inf\{m(B(x,\gamma\delta K_2^{-2}\|f'\|^{-1}) : x \in J\} > 0$. Therefore, using Corollary 2.19 we conclude the existence of a constant $C_5 > 0$ independent of $(z,r) \in \Re$ and such that

$$C_5^{-1} \leq \frac{m(B(z,r))}{r^h} \leq C_5. \tag{6.4}$$

So, in this case our theorem is proved setting $u(z,r) = n(z,r)$.
Let $\omega \in \Omega$. We say that $(z,r) \in \Re(\omega)$ if $f^n(z) \in B(\omega,\delta)$. Let $0 \leq k = k(z,r) \leq n$ be the least integer such that $f^j(z) \in B(\omega,\delta)$ for every $j = k, k+1, \ldots, n$. Consider all the numbers $r_i = |f^i(z) - \omega||(f^i)'(z)|^{-1}$ where $i = k, k+1, \ldots, n$. By the definition of $n(z,r)$ we have $r_n = |f^n(z) - \omega||(f^n)'(z)|^{-1} \leq K_2||f'||(\gamma\delta)^{-1}r$ and therefore there exists a minimal $k \leq u = u(z,r) \leq n$ such that $r_u \leq K_2\|f'\|(\gamma\delta)^{-1}r$. Then

$$\gamma\delta(K_2\|f'\|)^{-1}|f^u(z) - \omega| \leq r|(f^u)'(z)| \leq \gamma\delta K_2^{-1}. \tag{6.5}$$

Thus, if $u = k$, then it follows from Lemma 6.3 with $C_3 = \gamma\delta(K_2\|f'\|)^{-1}$ that there exists a constant $C_6 > 0$ such that

$$C_6^{-1}(r|(f^u)'(z)|)^{\beta(\omega)(h-1)} \leq \frac{m(B(z,r))}{r^h} \leq C_6(r|(f^u)'(z)|)^{\beta(\omega)(h-1)}. \tag{6.6}$$

So, we are done in this case. If $u > k$ then $r_{u-1} > K_2\|f'\|(\gamma\delta)^{-1}r$ and therefore, using (2.3) and (2.4), we get

$$r_u = \frac{|f^u(z) - \omega|}{|f^{u-1}(z) - \omega|}|f'(f^{u-1}(z))|^{-1}r_{u-1} \geq \|f\|^{-1}r_{u-1} \geq K_2(\gamma\delta)^{-1}r.$$

Thus

$$r|(f^u)'(z)| \le \gamma\delta K_2^{-1}|f^u(z)-\omega|. \tag{6.7}$$

Let $f_z^{-u}: B(f^u(z), \gamma|f^u(z)-\omega|) \to S^1$ be the continuous inverse branch of f^u which sends $f^u(z)$ to z. Applying Lemma 5.2, it follows from formulas (6.7), (6.5), and Corollary 2.19 that formula (6.6) continues to hold in case $u > k$, with a possibly bigger constant than C_6.

It remains to deal with the case when $\sup_{n\ge 0}\{r|(f^n)'(z)|\} \le \gamma\delta(K_2\|f'\|)^{-1}$. Then by (2.3), $z \in J \setminus \bigcup_{j=1}^\infty f^{-j}(\Omega)$. Let $u = u(z,r) \ge 0$ be the minimal integer such that $T^u(z) \in \Omega$ and let $f_z^{-u} : B(f^u(z), K_2 r|(f^u)'(z)|) \to S^1$ be a continuous inverse branch sending $T^u(z)$ to z. Applying Corollary 2.19 we therefore obtain

$$\begin{aligned} K_2^{-h}|(f^u)'(z)|^{-h} m(B(f^u(z), K_2^{-1} r|(f^u)'(z)|)) \le& \\ \le m(B(z,r)) \le& \\ \le K_2^h|(f^u)'(z)|^{-h} m(B(f^n(z), K_2 r|(f^u)'(z)|)).& \end{aligned}$$

and employing Lemma 5.1 finishes the proof. ♣

Lemma 6.5. There exists $\xi > 0$ sufficiently small such that if $x \in J \setminus X$, q is a positive integer, $f^q(x) \in B(\omega, \xi)$, $\omega \in \Omega$, and $f^{q-1}(x) \notin B(\Omega, \delta)$, then

$$u\big(x, \gamma\delta(K||f'||)^{-1}|f^q(x)-\omega||(f^q)'(x)|^{-1}\big) = q.$$

Proof. We need to determine how small $\xi > 0$ should be and our requirements are that $\xi \le \delta/||f'||$ and $\xi \le (L_2L_1||f'||)^{-1/\beta}$, where $L_2 = L_2(\delta/||f'||)$ and $L_1 = L_1(\delta/||f'||)$ are constants taken from Lemma 2.15 and Corollary 2.14 respectively. Set

$$r = \gamma\delta(K||f'||)^{-1}|f^q(x)-\omega||(f^q)'(x)|^{-1}.$$

Then $q \le n(x,r)$. Let $l \ge 1$ be the minimal integer such that $f^{q+l}(x) \in R(\omega)$. Then by Corollary 2.14, $|f^q(x)-\omega| \ge L_2^{-1}l^{-1/\beta}$. Hence, by Lemma 2.15 we get

$$|(f^l)'(f^q(x))| \ge L_2^{-1}l^{\frac{\beta+1}{\beta}} \ge (L_2L_1)^{-1}|f^q(x)-\omega|^{-(\beta+1)}$$

Thus

$$r|(f^{q+l})'(x)| \ge \gamma\delta(K||f'||)^{-1}(L_2L_1)^{-1}|f^q(x)-\omega|^{-\beta} > \gamma\delta K^{-1}$$

So, $n(z,r) < q+l$ and therefore $k(z,r) = q$. Finally from this, the definition of r, and $u(z,r)$ we conclude that $u(z,r) = q$. ♣

Theorem 6.6. We have $0 < \Pi_h(J) < \infty$ and $\mathrm{H}_h(J) < \infty$. Additionally $\mathrm{H}_h(J) = 0$ if and only if $h < 1$. Moreover Π_h is equivalent to m with Radon-Nikodym derivative bounded away from zero and infinity.

Proof. The inequalities $\mathrm{H}_h(J) < \infty$, $0 < \Pi_h(J)$, and a uniform boundedness of $dm/d\Pi_h$ follow from Lemma 4.5. Let $\alpha = \max\{\alpha(\omega) : \omega \in \Omega\}$. Since $h \leq 1$, it follows from Theorem 6.4 that $\liminf_{r\to 0} m(B(z,r)/r^h \geq Q^{-1}(\gamma\delta K_2^{-1})^{\alpha(h-1)}$ for all $z \in J$. Therefore in view of Theorem 3.2(2), $d\Pi_h/dm \leq b(1)Q(\gamma\delta K_2^{-1})^{\alpha(1-h)}$ and $\Pi_h(J) < \infty$. Now it is left to show that $\mathrm{H}_h(J) = 0$ if $h < 1$. Let $J_0 = \{z \in J : \omega(z) \cap \Omega = \emptyset\}$. It follows from Lemma 6.2 and Lemma 4.5 that $\mathrm{H}_h(J_0) = 0$, whence we only need to show that $\mathrm{H}_h(X \setminus J_0) = 0$, but this follows immediately from Lemma 6.5, Theorem 6.4, and Theorem 3.1(1). The proof is finished. ♣

The next result can be considered as a completion of Theorem 6.6.

Theorem 6.7. If J is disconnected, then $h = \mathrm{HD}(J) < 1$. In particular the Lebesgue measure of J is equal to 0.

Proof. First we shall show that $l(J) = 0$. Indeed, if $l(J) > 0$, then in view of the Lebesgue density theorem $l\big(B(x,r)\cap J\big)/2r \to 1$ for l-a.e. $x \in J$. Fix one such point x which additionally does not belong to the countable set $\bigcup_{n=0}^\infty f^{-n}(\Omega)$. Let $\{r_j(x)\}_{j=1}^\infty$ be the sequence of radii produced in Lemma 4.2 and let $n_j = n_j(x)$, $j \geq 1$, be the sequence of positive integers produced there. Recall that $r_j = K_2^{-1}\gamma\delta|(f^{n_j})'(x)|^{-1}$. Since, by Lemma 2.4 J is a compact nowhere dense subset of S^1, the exists an arc $\Delta \subset B(f^{n_j}(x),\delta) \setminus J$ for every j sufficiently large. But then by Corollary 2.19, $l\big(f_x^{-n_j}(\Delta)\big) \geq K_2^{-1}|(f^{n_j})'(x)|^{-1}l(\Delta) \geq \gamma^{-1}\delta^{-1}r_j$ and $B(x,r_j) \cap J \subset f_x^{-n_j}\big(B(f^{n_j}(x),\gamma\delta) \cap J\big)$. Therefore we get

$$l(B(x,r_j) \cap J) \leq 2r_j - l(f_x^{-n_j}(\Delta)) \leq 2r_j\left(1 - \frac{1}{2}\frac{l(\Delta)}{\delta}\right)$$

which contradicts our choice of x and shows that $l(J) = 0$. Hence $\mathrm{H}_1(J) = 0$ and the proof is completed applying the middle part of Theorem 6.6. ♣

Remark 6.8. We would like to end up this section with the remark that making use of the concept of the jump transformation (see the next section) one could prove, essentially as in [DU4], that the box counting dimension of J exists and coincides with $\mathrm{HD}(J)$.

§**7. Schweiger's formalism and jump transformation.** This section has rather abstract character and is self-contained. We closely follow here Section 3 of [DU2] which in turn is based on Schweiger approach given in [Sc]. A much more complete treatment of the subject is presented in [ADU].

So, let $(B,\mathcal{F},\lambda)$ be a probability space and let $T : B \to B$ be a measurable and nonsingular map. We assume that the transformation T admits a countable measurable partition $\Re = \{B(k) : k \in I\}$ such that for every $k \in I$

$$T(B(k)) = \bigcup_{i\in I'(k)} B(i) \qquad \text{for some } I'(k) \subset I.$$

Any partition with this property is also called a Markov partition for T. We assume that the family

$$\{T(B(k)) : k \in I\} \text{ is finite.} \tag{7.1}$$

The transition matrix $A = (A_{ij})_{i,j\in I}$ associated to the Markov partition $\Re$ is defined by

$$A_{ij} = \begin{cases} 1 & \text{if } T(B(i)) \supset B(j) \\ 0 & \text{if } T(B(i)) \cap B(j) = \emptyset \end{cases} \qquad i, j \in I.$$

A sequence $\tau = \tau_0, \tau_1, \ldots, \tau_n$, $n \geq 1$, is said to be A-admissible if $A_{\tau_i \tau_{i+1}} = 1$ for every $i = 0, 1, \ldots, n-1$. The matrix A is assumed to be irreducible, i.e. for all $i, j \in I$ there exists an A-admissible sequence that begins with i and ends with j.

We also assume that for every $k \in I$ there exists a measurable and nonsingular map $T_k^{-1} : T(B(k)) \to B(k)$ which is the inverse to $T|_{B(k)}$. In particular, $T : B(k) \to T(B(k))$ is injective. For any A-admissible sequence $\tau = \tau_0, \tau_1, \ldots, \tau_n$ define

$$\begin{aligned} &T_\tau^{-n} : T(B(\tau_{n-1})) \to B(\tau_0) \\ &T_\tau^{-n} = T_{\tau|_{n-1}}^{-(n-1)} \circ T_{\tau_n}^{-1}, \\ &B(\tau) = T_{\tau|_n}^{-n}(B(\tau_n)) = \bigcap_{j=0}^{n} T^{-j}(B(\tau_j)). \end{aligned} \tag{7.2}$$

Let $\mathcal{L}^{(n)}$ denote the family of all cylinders $B(\tau)$ of length n. The family $\mathcal{L} = \bigcup_{n=1}^\infty \mathcal{L}^{(n)}$ is supposed to generate the σ-algebra $\mathcal{F}$. We put

$$\phi_\tau(x) = \frac{d\lambda T_\tau^{-n}}{d\lambda}(x)$$

for the Jacobian (with respect to λ) of the mapping T_τ^{-n} at the point $x \in T(B(\tau_{n-1}))$. Fix a constant $C \geq 1$. A cylinder $B(\tau)$ is called an R-cylinder if it satisfies "Rényi's condition"

$$\text{ess sup}\{\phi_\tau(x) : x \in T(B(\tau_{n-1}))\} \leq C \text{ ess inf}\{\phi_\tau(x) : x \in T(B(\tau_{n-1}))\}.$$

The set of all R-cylinders with constant C is denoted by $\mathcal{G}(C, T)$. We assume that there exists a constant $C \geq 1$ and a class $\Re(C, T) \subset \mathcal{G}(C, T)$ such that:

$$\text{If } B(\tau) \in \Re(C, T) \quad \text{then} \quad B(\rho\tau) \in \Re(C, T)$$

for any A-admissible sequence ρ such that $A_{\rho_{|\rho}\tau_1} = 1$. Note that for any $B(\tau) \in \Re(C, T)$ with $\lambda(B(\tau)) > 0$ and for any admissible sequence $\rho\tau_1$, one also obtains $\lambda(B(\rho\tau)) = \lambda\big(T_\rho^{-|\rho|}(B(\tau))\big) > 0$. For $n \geq 1$ let

$$D_n = \{B(\tau) \in \mathcal{L}^{(n)} : B(\tau|_s) \in \mathcal{L} \setminus \Re(C, T) \text{ for all } 0 \leq s \leq n\}.$$

Our last assumption here is that

$$\lim_{n\to\infty} \sum_{D_n} \lambda(B(\tau)) = 0.$$

The proofs of the following two results are elementary and go back to [Sc].

Lemma 7.1. Let E be a measurable set. Then, for any $B(\tau) \in \mathcal{L}^{(n)} \cap \Re(C,T)$, we have

$$\lambda(T^{-n}(E) \cap B(\tau)) \geq C^{-1} \lambda_{T(B(\tau_{n-1}))}(E) \cdot \lambda(B(\tau)),$$

where $\lambda_{T(B(\tau_{n-1}))}$ denotes the conditional measure of λ on $T(B(\tau_{n-1}))$.

Lemma 7.2. Any cylinder is (mod λ) a disjoint union of elements of $\Re(C,T)$. Consequently, the family $\Re(C,T)$ generates the σ-algebra $\mathcal{F}$ mod λ.

In order to prove ergodicity of T with respect to λ (see Theorem 1 of [Sc]), additional arguments are required, involving the primitiveness of the matrix A. We therefore give a full proof.

Theorem 7.3. The transformation T is ergodic with respect to the measure λ.

Proof. Suppose that $T^{-1}(E) = E$ and $\lambda(E) > 0$. Then it follows from Lemma 7.2 that there exists $l \geq 1$ and $\tau \in \Re(C,T)$ of length l such that

$$\lambda(E \cap B(\tau)) > 0. \tag{7.3}$$

Since T is nonsingular, we also have $\lambda(T^l(E \cap B(k_0, k_1, \dots, k_l)) > 0$. Since $T^l(E) \subset E$ and by (7.2) it follows that

$$\lambda(E \cap T(B(\tau_{l-1}))) \geq \lambda(T^l(E) \cap T^l(B(\tau))) \geq \lambda(T^l(E \cap B(\tau))) > 0. \tag{7.4}$$

Since the matrix A is irreducible, for every $k \in I$ there exists $B(k\rho) \in \mathcal{L}^{(s+1)}$ such that $A_{\rho_s \tau_1} = 1$, where $s = |\rho|$. Therefore $B(k\rho\tau) \in \Re(C,T)$ and by (7.3), (7.4), and Lemma 7.1

$$\begin{aligned}\lambda(E \cap B(k)) &\geq \lambda(E \cap B(k\rho\tau)) = \lambda(T^{-(s+1+l)}(E) \cap B(k\rho\tau)) \\ &\geq C^{-1} \lambda_{T(B(\tau_{l-1}))}(E) \cdot \lambda(B(k\rho\tau)) > 0.\end{aligned}$$

Consequently, for any $j \in I$ we have $\lambda(E \cap T(B(j)) > 0$, and using (7.1), we see that $\alpha = \min\{\lambda_{T(B(q))}(E) : q \in I\} > 0$. Hence by Lemma 7.1 again, one obtains for any $Z \in \mathcal{L}^{(n)} \cap \Re(C,T)$ that $\lambda(E \cap Z) = \lambda(T^{-n}(E) \cap Z) \geq C^{-1}\alpha\lambda(Z)$. Therefore, using Lemma 7.2, the indicator function of the set E is λ-a.e. positive, which means that $\lambda(E) = 1$. The proof is finished. ♣

Let $I(x) = \{k \in I : x \in T(B(k))\}$. Let us recall the following two elementary facts.

Lemma 7.4. The transformation T admits a σ–finite T–invariant measure equivalent to λ if and only if there exists a measurable function ψ such that

$$\psi(x) = \sum_{k \in I(x)} \psi(T_k^{-1}(x))\phi_k(x) \qquad \text{a.e.}$$

Lemma 7.5. If there is a constant $D > 0$ such that $\mathcal{G}(D, T) = \mathcal{L}$, then T admits a finite T–invariant measure equivalent to λ such that the Radon–Nikodym derivative is uniformly bounded away from zero and infinity.

The right hand side of the formula in Lemma 7.4 may be regarded as the value of the Perron–Frobenius operator associated to the measure λ and applied to the function f. Also notice that in the proof of Lemma 7.5, formula (7.1) is also used.

Let us now, for every $n \geq 1$, introduce the class

$$B_n = \{B(\tau) \in \Re(C, T) : B(\tau|_{n-1}) \in D_{n-1}\}.$$

Define the jump transformation $T^* : B \to B$ by

$$T^*(x) = T^n(x) \qquad \text{if} \quad x \in B(\tau) \quad \text{and} \quad B(\tau) \in B_n.$$

It follows that T^* is almost everywhere defined. Moreover, it is nonsingular and ergodic with respect to λ. Since for every $B(\tau) \in B_n$ we have $T^*(B(\tau)) = T(B(\tau_n))$, we conclude that the family $\bigcup_{j=1}^{\infty} B_n$ is a Markov partition for T^* (usually infinite, even if $\Re$ was finite). The corresponding transition matrix is irreducible and (7.1) is also satisfied. Moreover:

Proposition 7.6. $\mathcal{G}(C, T^*) = \mathcal{L}^*$ and there exists a unique, ergodic, T^*-invariant probability measure μ^* equivalent to λ. Moreover, the Radon–Nikodym derivative $\psi^* = d\mu^*/d\lambda$ satisfies $D^{-1} \leq \psi^* \leq D$ for some constant $D > 0$.

The first statement of this proposition is contained in Lemma 5 of [Sc]. The existence of μ^* follows then from Lemma 7.5. Uniqueness and ergodicity of μ^* follow from standard arguments and from ergodicity of T^* with respect to λ. The main result of Schweiger's theory is the following.

Theorem 7.7. The transformation T admits a unique (up to a multiplicative constant), σ-finite, invariant measure μ equivalent to λ with Radon–Nikodym derivative $\frac{d\mu}{d\lambda}$ given by the formula

$$\frac{d\mu}{d\lambda}(x) = \psi^*(x) + \sum_{n=1}^{\infty} \sum_{D_n(x)} \psi^*(T_\tau^{-n}(x))\phi_\tau(x),$$

where $D_n(x) = \{B(\tau) \in D_n : x \in T(B(\tau_n))\}$.

The existence of μ is shown as in the proof of Theorem 2 in [Sc] – up to some minor changes. Uniqueness of μ (not included there) follows easily from ergodicity of T with respect to

λ. Finally, following [Sc], we state the following necessary and sufficient condition for the finiteness of the measure μ.

Proposition 7.8. The measure μ is finite (or equivalently the Radon-Nikodym derivative $\frac{d\mu}{d\lambda}$ is integrable) if and only if

$$\sum_{n=1}^{\infty} \sum_{D_n} \lambda(B(\tau)) < \infty.$$

Let us also mention the following technical result.

Lemma 7.9. Let ϕ_T, resp. ϕ_{T^*}, denote the Jacobian of T, resp. the Jacobian of T^*, (with respect to λ). Then

$$\log \phi_T \in L_1(\mu) \text{ if and only if } \log \phi_{T^*} \in L_1(\mu^*)$$

and in this case $\int_B \log \phi_T \, d\mu = \int_B \log \phi_{T^*} \, d\mu^*$.

Remark 7.10. The irreducibility of the transition matrix has only been used to prove ergodicity and uniqueness of invariant measures. All other results of this section are true without this assumption.

Remark 7.11. In particular, in the context of a dynamical system $(f, I; \Delta_j, j \in I)$ taking as $\Re(C, f)$ the family of all the cylinders $\tau = \tau_0, \tau_1, \ldots, \tau_n \in \Sigma_A^*$ such that

$$\bigcup_j f(\Delta_j) \cap \Omega = \emptyset,$$

where the union is taken over all the indexes j with $A_{j\tau_n} = 1$. all the results obtained in this section apply to the h-conformal measure m and the map $f : J \to J$. As one of the consequences of this remark observe that combining Lemma 4.5, Theorem 5.4, and Theorem 7.3 we get the following.

Theorem 7.12. There exists a unique h-conformal measure m for the map $f : J \to J$. Moreover this measure is continuous.

Now, as an immediate consequence of Theorem 7.7 we get the following.

Theorem 7.13. The map $f : J \to J$ admits a unique (up to a multiplicative constant) f-invariant σ-finite measure μ equivalent (or equivalently absolutely finite) to the conformal measure m.

Lemma 7.14. If F is a Borel subset of J and $\overline{F} \cap \Omega = \emptyset$, then $\mu(F) < +\infty$.

Proof. First notice that defining the jump transformation we can also use the cylinder sets $\{\Delta_\tau : \tau \in \Sigma_A^q\}$ with a fixed integer $q \geq 1$. This is possible since by Lemma 2.3 this family forms a Markov partition for f. Although in that way we will be getting mutually different jump transformations, these will give raise to the same measure μ up to multiplicative constants. By the definition of $\Re(C,T)$ it follows from Lemma 2.1 and Theorem 7.7 that $d\mu/dm(x) = \psi^*(x)$ out of some neighborhood of Ω shrinking to Ω if $q \to \infty$. Hence, invoking Proposition 7.6 finishes the proof. ♣

In the context of dynamical Cantor sets Proposition 7.8 leads to the following much more effective criterion for the finiteness of the invariant measure μ.

Theorem 7.15. The f-invariant σ-finite measure μ, equivalent to the conformal measure m, is finite if and only if

$$h > 2\max\left\{\frac{\beta(\omega)}{\beta(\omega)+1} : \omega \in \Omega\right\}$$

Proof. It follows from Lemma 2.15 and the choice of the family $\Re(C,f)$ that

$$\sum_{n=1}^{\infty}\sum_{D_n} m(B(\tau)) \asymp \sum_{n=1}^{\infty}\sum_{\omega\in\Omega}\sum_{k\geq n} n^{-\frac{\beta(\omega)+1}{\beta(\omega)}h}$$

$$= \sum_{\omega\in\Omega}\sum_{n=1}^{\infty} n\cdot n^{-\frac{\beta(\omega)+1}{\beta(\omega)}h} = \sum_{\omega\in\Omega}\sum_{n=1}^{\infty} n^{1-\frac{\beta(\omega)+1}{\beta(\omega)}h}$$

and this series converges if and only if $1 - \frac{\beta(\omega)+1}{\beta(\omega)}h < -1$ for all $\omega \in \Omega$. Applying Proposition 7.8 finishes the proof. ♣

Combining this theorem and Theorem 3.6 we get the following.

Corollary 7.16. The following three conditions are equivalent.

(a) $\mathrm{P}'(t)$ does not exist.

(b) There exists a probability f-invariant measure ν absolutely continuous with respect th the conformal measure m.

(c) There exists an equilibrium state of positive entropy associated to the potential $-h\log|f'|$.

Proof. The equivalence of conditions (a) and (b) follows from Theorem 3.6 and Theorem 4.6.
In order to see that (c) implies (b) notice first that, in view of Theorem 7.13 measure ν is equivalent to m and therefore $\mathrm{HD}(\nu) = h$. Since ν is non-atomic and ergodic, using the Birkhoff ergodic theorem we conclude that $\chi_\nu > 0$. Hence by (3.3), $\mathrm{h}_\nu(f) > 0$ and $\mathrm{h}_\nu(f) - h\chi_n u = 0$. Since by Lemma 3.5, $\mathrm{P}(t) \geq 0$for all $t \geq 0$, we have shown that ν is an equilibrium state for $-h\log|f'|$.
The implication $(b) \Rightarrow (c)$ can be proven proceeding as in [Le]. ♣

Corollary 7.17. If $\beta(\omega) = 1$ for all $\omega \in \Omega$, in particular if $f \in C^2$, then the measure μ is infinite.

Corollary 7.18. If the family $t \mapsto f_t$ is a local perturbation of f around points of Ω such that $\lim_{t \to t_0} \beta_t(\omega) = 0$ for some t_0, then for every t sufficiently close to t_0 the corresponding invariant measure μ_t is finite.

Proof. For the proof it suffices to notice that the local perturbations around Ω keep the Hausdorff dimension of J_{f_t} away from 0. ♣

The sections 8,9, and 10, the last three sections of this paper, are devoted to study the rigidity problem for parabolic Cantor sets. To be more precise we explore the problem of what are necessary and sufficient conditions for two parabolic Cantor sets which are topologically conjugate to be conjugate in a smoother manner like bi-Lipschitz continuous or real analytic. In Section 8 we resolve this problem (see Theorem 8.1) in terms of spectra of moduli of multipliers of periodic points as well as in terms of measure classes of of packing measures and Hausdorff dimensions.
In Section 9 dealing with real analytic systems we prove (see Theorem 9.9) a much stronger rigidity result that absolute continuity with respect to packing measures (the equality of Hausdorff dimensions is not required!) implies that the conjugating homeomorphism is real analytic.
In the last section, Section 10, we undertake the most geometrical approach defining and proving the existence of the scaling function. We then express a partial solution of the rigidity problem in terms of the these functions.
Our approach to the rigidity problem of parabolic Cantor sets is motivated by the results and ideas used in the setting of hyperbolic systems. See for example [Su1], [Su2], [Pr2], [Pr3], [PT], [LS], and [Be] where also a more complete collection of literature can be found.

§**8. Rigidity of dynamical Cantor sets.** In this section we deal with two dynamical systems $(f, I; \Delta_{f,j}, j \in I)$ and $(g, I; \Delta_{g,j}, j \in I)$ assuming that these are set-theoretically equivalent, that is that $\Delta_{f,i} \cap \Delta_{f,j} \neq \emptyset$ if and only if $\Delta_{g,i} \cap \Delta_{g,j} \neq \emptyset$. Then the map $\phi : J_f \to J_g$ given by the formula

$$\phi(\pi_f(\tau)) = \pi_g(\tau)$$

is well defined (that is for all $x \in J_f$ it does not depend on the choice of $\tau \in \pi_f^{-1}(x)$) and moreover it can be easily checked that ϕ is a homeomorphism. Our main aim in this section is to prove the following rigidity theorem.

Theorem 8.1. The following three conditions are equivalent.

(a) If $z \in \mathrm{Per}_n(f)$, then $|(g^n)'(\phi(z))| = |(f^n)'(z)|$.

(b) The dimensions $h_f = \mathrm{HD}(J_f)$ and $h_g = \mathrm{HD}(J_g)$ are equal and the homeomorphism ϕ transports the measure class of the packing measure Π_{h_f} on J_f onto the measure class of the packing measure Π_{h_g} on J_g.

(c) Both homeomorphisms ϕ and ϕ^{-1} are Lipschitz continuous.

We shall also provide the proof of the following theorem which sheds some light on what is going on in the general case.

Theorem 8.2. The conjugacy $\phi : J_f \to J_g$ is Hölder continuous if and only if either both Cantor sets J_f and J_g are hyperbolic or both are parabolic.

Since the proofs of Theorem 8.2 and the implication $(b) \Rightarrow (c)$ have a considerable overlap, we partially proceed with them simultaneously. In fact we begin with two general technical lemmas, then we prove the implication $(c) \Rightarrow (a)$ of Theorem 8.1 and we begin the proof that $(b) \Rightarrow (c)$ including there the proof of Theorem 8.2. We end the section with the implication $(a) \Rightarrow (b)$.

The definition we intend to give now and the lemma following it involve only one single dynamical system $(f, I; \Delta_j, j \in I)$ and therefore formulating these and proving Lemma 8.4 we skip the subscript "f" when dealing with the objects associated with this dynamical system.

Definition 8.3. Suppose that a positive number $\zeta \le \delta$ is given. If $\omega \in \Omega$ we set $R_\zeta(\omega) = B(\omega, \zeta) \setminus B(\omega, \zeta/||f'||)$. If x and y (not necessarily different) are in the closure of the same connected component of $B(\omega, \delta) \setminus \{\omega\}$, then we let $z \in \{x, y\}$ be the point lying farther from ω. By $0 \le q = q(x, y) \le \infty$ we denote the largest integer such that $f_\omega^{-q}(z) \in [x, y]$ and by $p = p(\zeta, x, y) \ge 0$ we denote the least integer such that $f^p(z) \in R_\zeta(\omega)$.

Lemma 8.4. $\forall_{(0<\zeta\le\delta)}\ \forall_{(0<\xi\le\zeta)}\ \forall_{(\omega\in\Omega)}\ \forall_{(x\ne y\in S^1)}\ \exists_{(C(\zeta,\xi))}$ such that the following holds: If x and y belong to the closure of the same connected component of $B(\omega, \delta) \setminus \{\omega\}$ and $|f^p(y) - f^p(x)| \ge \xi$, then

$$C(\zeta,\xi)^{-1} \sum_{j=0}^{q} (p+j)^{-\frac{\beta(\omega)+1}{\beta(\omega)}} \le |y - x| \le C(\zeta,\xi) \sum_{j=0}^{q} (p+j)^{-\frac{\beta(\omega)+1}{\beta(\omega)}},$$

where we assume $0^{-1} = 1$.

Proof. Without loosing generality we may assume that $z = y$, where z is described in Definition 8.3. Suppose first that $q \ge 1$. Then by the definitions of q and p we have

$$\bigcup_{j=0}^{q-1} f_\omega^{-(p+j)}\big([f_\omega^{-1}(f^p(y)), f^p(y)]\big) \subset [x, y]$$

and

$$\bigcup_{j=0}^{q} f_\omega^{-(p+j)}\big([f_\omega^{-1}(f^p(y)), f^p(y)]\big) \supset [x, y]$$

Since $f^p(y) \in R_\zeta(\omega)$ it follows from (2.b) that $|f_\omega^{-1}(f^p(y)) - \omega| \ge r(\zeta)$, where $r(\zeta) = \zeta/||f'|| - L(\beta(\omega)+1)^{-1}\zeta^{\beta(\omega)+1}$. Hence $[f_\omega^1(f^p(y)), f^p(y)] \subset B(\omega, \delta) \setminus B(\omega, r(\zeta))$ and applying Lemma 2.15 we get

$$r(\zeta) L_2^{-1}(r(\zeta)) \sum_{j=0}^{q-1} (p+j)^{-\frac{\beta(\omega)+1}{\beta}} \le |f_\omega^{-1}(f^p(y)) - f^p(y)| L_2^{-1}(r(\zeta)) \sum_{j=0}^{q-1} (p+j)^{-\frac{\beta(\omega)+1}{\beta}} \le |x - y|$$

and

$$|x - y| \le |f_\omega^{-1}(f^p(y)) - f^p(y)| L_2(r(\zeta)) \sum_{j=0}^{q} (p+j)^{-\frac{\beta(\omega)+1}{\beta}} \le \delta L_2(r(\zeta)) \sum_{j=0}^{q} (p+j)^{-\frac{\beta(\omega)+1}{\beta}}$$

Since $(p+q)^{-\frac{\beta(\omega)+1}{\beta}} \le (p+q-1)^{-\frac{\beta(\omega)+1}{\beta}}$, combining the last two displays we get

$$\frac{1}{2}L_2(r(\zeta)^{-1}\sum_{j=0}^{q}(p+j)^{-\frac{\beta(\omega)+1}{\beta}} \le |x-y| \le \delta L_2(r(\zeta))\sum_{j=0}^{q}(p+j)^{-\frac{\beta(\omega)+1}{\beta}}$$

and we are done in the case $q \ge 1$.
If $q = 0$, then we have $[x,y] \subset f_\omega^{-p}([f_\omega^{-1}(y),y])$ and similarly as above we get $|x-y| \le \delta L_2(r(\zeta))p^{-\frac{\beta(\omega)+1}{\beta}}$. On the other hand in this case $[x,y] = f_\omega^{-p}([f_\omega^{-1}(y),y])$. Since $[f^p(x), f^p(y)] \subset [f_\omega^{-1}(f^p(y)), f^p(y)]$, similarly as before we get $[f^p(x), f^p(y)] \subset B(\omega,\delta) \setminus B(\omega, r(\zeta))$. Therefore, in view of Lemma 2.15

$$|x-y| \ge |f^p(x) - f^p(y)|L_2(r(\zeta))^{-1}p^{-\frac{\beta(\omega)+1}{\beta}} \ge \xi L_2(r(\zeta))^{-1}p^{-\frac{\beta(\omega)+1}{\beta}}$$

The proof is finished. ♣

Lemma 8.5. If J_f and J_g are two dynamical Cantor sets and $\phi : J_f \to J_g$ is the canonical topological conjugacy between them, then $\phi(\Omega_f) = \Omega_g$ if and only if $\exists_\kappa \ge 1\ \forall x \in J_f\ \forall n \ge 1$

$$\kappa^{-1}\log|(f^n)'(x)| \le \log|(g^n)'(\phi(x))| \le \kappa \log|(f^n)'(x)|.$$

Proof. A straightforward computation shows that if the second part of our equivalence is satisfied, then $\phi(\Omega_f) = \Omega_g$. In order to prove the converse implication it is of course sufficient to show only one of these two inequalities, say the second one. Take $0 < \zeta_1 < \delta_f$ so small that $\phi(B(\Omega_f,\zeta_1)) \subset B(\Omega_g,\delta_g)$. Then there exist two universal constants $0 < W_f \le 1$ and $W_g \ge 1$ such that for all $\omega \in \Omega_f$ and all $x \in B(\omega,\zeta_1) \setminus \Omega_f$

$$|f'(x)| \ge W_f p(x,\zeta_1)^{\frac{\beta_f(\omega)+1}{\beta_f(\omega)}} \text{ and } |g'(x)| \le W_g p(x,\zeta_1)^{\frac{\beta_g(\omega)+1}{\beta_g(\omega)}}$$

Let now $0 < \zeta_2 \le \zeta_1$ be so small that for every $\omega \in \Omega_f$ and for every $x \in B(\omega,\zeta_2)$ the number $p(x,\zeta_1)$ is so large that

$$\frac{\log W_f}{\log p(x,\zeta_1)} + 1 \ge \frac{1}{2}$$

Let $\kappa_1 = \inf\{|f'(x)| : x \in J_f \setminus B(\Omega_f,\zeta_1)\} > 1$. Using also the fact that $f(\Omega_f) = \Omega_g$, we can therefore conclude that

$$\log|g'(\phi(x)| \le \max\left\{\frac{\log||g'||}{\log\kappa_1}, \sup\left\{\frac{\beta_f(\omega)+1}{\beta_f(\omega)}\frac{\beta_g(\omega)+1}{\beta_g(\omega)}\right\}2(1+\log W_g)\right\}\log|f'(x)|$$

Now the straightforward application of the chain rule completes the proof. ♣

Proof that $(c) \Rightarrow (a)$**.** Indeed, suppose that there is a periodic point z of period n such that $|(g^n)'(\phi(z))| \ne |(f^n)'(z)|$. Then without loosing generality we can suppose that

$|(g^n)'(\phi(z))| < |(f^n)'(z)|$. Fix two numbers $\lambda_1, \lambda_2 > 1$ such that $|(g^n)'(\phi(z))| < \lambda_2 < \lambda_1 < |(f^n)'(z)|$ and take $0 < \varepsilon_f < \delta_f$ and $0 < \varepsilon_g < \delta_g$ so small that $|(f^n)'(x)| \geq \lambda_1$ for all $x \in B(z, \varepsilon_f)$, $|(g^n)'(y)| \leq \lambda_2$ for all $y \in B(\phi(z), \varepsilon_g)$, and $\phi(J_f \cap B(z, \varepsilon_f)) \subset J_g \cap B(\phi(z), \varepsilon_g)$. Fix $x \in J_f \cap B(z, \varepsilon_f) \setminus \{z\}$. Then for all $k \geq 1$ we have $|f_z^{-nk}(x) - z| \leq \lambda_1^{-k}|x - z|$ and $|g_{\phi(z)}^{-nk}(\phi(x)) - \phi(z)| \geq \lambda_2^{-k}|\phi(x) - \phi(z)|$. Therefore

$$\lim_{k\to\infty} \frac{|g_{\phi(z)}^{-nk}(\phi(x)) - \phi(z)|}{|f_z^{-nk}(x) - z|} \geq \lim_{k\to\infty} \left(\frac{\lambda_1}{\lambda_2}\right)^k \frac{|\phi(x) - \phi(z)|}{|x - z|} = \infty$$

and since $g_{\phi(z)}^{-nk}(\phi(x)) = \phi(f_z^{-nk}(x))$, this shows that ϕ is not Lipschitz continuous. ♣

Proof that $(b) \Rightarrow (c)$**.** Since the two measures m_g and $m_f \circ \phi^{-1}$ are equivalent, the measures μ_g^* and $\mu_f^* \circ \phi^{-1}$ are also equivalent, whence, in view of Proposition 7.6 these are equal as equivalent ergodic probability g^*-invariant measures. Therefore, it follows from the last part of this proposition that there exists $M \geq 1$ such that

$$M^{-1} \leq \frac{m_g(\phi(A))}{m_f(A)} \leq M \tag{8.1}$$

for all Borel subsets A of J_f. In order to continue the proof we need the following.

Lemma 8.6. If (b) is satisfied and $\omega \in \Omega_f$, then $\phi(\omega) \in \Omega_g$ and $\beta(\phi(\omega)) = \beta(\omega)$.

Proof. Take $\varepsilon_f, \varepsilon_g > 0$ so small that $\phi(J_f \cap B(\omega, \varepsilon_f)) \subset B(\phi(\omega), \varepsilon_g)$. Suppose now that $|g'(\phi(\omega))| > 1$ and fix $1 < \lambda < |g'(\phi(\omega))|$. Take $0 < \varepsilon \leq \varepsilon_g$ so small that $|g'(z)| \geq \lambda$ for all $z \in B(\phi(\omega), \varepsilon)$. Fix $y \in J_g \cap B(\phi(\omega), \varepsilon)$. By conformality of m_g we have for all $n \geq 0$

$$m_g([g_{\phi(\omega)}^{-(n+1)}(y), g_{\phi(\omega)}^{-n}(y)]) \leq \lambda^{-n} m_g([g_{\phi(\omega)}^{-1}(y), y]) \leq \lambda^{-n}$$

On the other hand, in view of Lemma 2.15, for all $n \geq 0$ we get

$$m_f([f_\omega^{-(n+1)}(\phi^{-1}(y)), f_\omega^{-n}(\phi^{-1}(y))]) \geq L_{2,f}(R) n^{-\frac{\beta(\omega)+1}{\beta(\omega)} h_f} m_f([f_\omega^{-1}(\phi^{-1}(y)), \phi^{-1}(y)]),$$

where $R = |\omega - \phi^{-1}(y)|$. Therefore

$$\begin{aligned}\frac{m_g([g_{\phi(\omega)}^{-(n+1)}(y), g_{\phi(\omega)}^{-n}(y)])}{m_f([f_\omega^{-(n+1)}(\phi^{-1}(y)), f_\omega^{-n}(\phi^{-1}(y))])} &\leq \\ &\leq \big(L_{2,f}(R) m_f([f_\omega^{-1}(\phi^{-1}(y)), \phi^{-1}(y)])\big)^{-1} \lambda^{-n} n^{\frac{\beta(\omega)+1}{\beta(\omega)} h_f}\end{aligned}$$

Since $\lim_{n\to\infty} \lambda^{-n} n^{\frac{\beta(\omega)+1}{\beta(\omega)} h_f} = 0$ and $m_f([f_\omega^{-1}(\phi^{-1}(y)), \phi^{-1}(y)]) > 0$ we arrive at a contradiction with (8.1) and the proof of the first part of Lemma 8.6 is finished.

In order to prove the second part of the lemma we apply Lemma 2.15 again, this time to the both maps f and g obtaining as a result the existence of a constant $M>0$ such that for all $n\ge 1$

$$M^{-1}\le n^{-\frac{\beta(\phi(\omega))+1}{\beta(\phi(\omega))}h_g+\frac{\beta(\omega)+1}{\beta(\omega)h_f}}\le M$$

Thus $h_g(\beta(\phi(\omega))+1)/\beta(\phi(\omega))=h_f(\beta(\omega)+1)/\beta(\omega)$. Since the dimensions h_g and h_f are equal, we get $\beta(\phi(\omega))=\beta(\omega)$ which finishes the proof of Lemma 8.6. ♣

Now, let us continue the proof of the implication $(b)\Rightarrow(c)$ including the proof of Theorem 8.2. Fix $0<\eta<\delta_f/4$ so small that if $x,y\in J_f$ with $|x-y|\le\eta$, then $|\phi(x)-\phi(y)|<\delta_g/4$. Let $\tau>0$ be so small that $|x-y|\le\tau$ implies $|\phi^{-1}(x)-\phi^{-1}(y)|<\eta/||f'||$ and let $\eta_1>0$ be so small that $|x-y|\le\eta_1$ implies that $|\phi(x)-\phi(y)|<\tau/2$. Finally let $\tau_1>0$ be so small that if $|x-y|\le\tau$, then $|\phi^{-1}(x)-\phi^{-1}(y)|<\eta_1/||f'||$.
Consider now an arbitrary pair of points $x\ne y\in J_f$ with $|x-y|<\eta_1/||f'||$. Since by Lemma 2.4 J_f has no isolated points, in order to prove the Lipschitz continuity of ϕ we may assume that $m_f([x,y])>0$. Then also $m_g([x,y])>0$. Let $n=n(x,y)\ge 1$ be the least integer such that $|f^n(y)-f^n(x)|\ge\eta_1/||f'||$. Then $|f^n(y)-f^n(x)|\le\eta_1$. We will consider several cases.
Case 1. $\{f^n(y),f^n(x)\}\cap\big(J_f\setminus B(\Omega_f,\eta/||f'||)\big)\ne\emptyset$. Without loosing generality we may assume that $f^n(x)\in J_f\setminus B(\Omega_f,\eta/||f'||)$ whence in view of Lemma 8.6 and the choice of τ we have $g^n(x)\in J_g\setminus B(\Omega_g,\tau)$. Thus, applying Lemma 2.18 we get

$$K_{f,2}^{-1}(1/2)|(f^n)'(x)|\le\frac{|f^n(y)-f^n(x)|}{|y-x|}\le K_{f,2}(1/2)|(f^n)'(x)|,$$

$$K_{g,2}^{-1}(1/2)|(g^n)'(\phi(x))|\le\frac{|g^n(\phi(y))-g^n(\phi(x))|}{|\phi(y)-\phi(x)|}\le K_{g,2}(1/2)|(g^n)'(\phi(x))|,$$

Using these two formulas and applying also Lemma 8.5 we now get

$$|\phi(y)-\phi(x)|\le\frac{1}{2}K_g(1/2)K_{f,2}(1/2)(||f'||\eta_1^{-1})^{1/\kappa}|y-x|^{1/\kappa}$$

which ends the proof of Hölder continuity in this case.
To continue the proof of the implication $(b)\Rightarrow(c)$ notice that we get two similar inequalities for conformal measures

$$K_{f,2}^{-h_f}(1/2)|(f^n)'(x)|^{h_f}\le\frac{m_f([f^n(x),f^n(y)])}{m_f([x,y])}\le K_{f,2}^{h_f}(1/2)|(f^n)'(x)|^{h_f},$$

and

$$K_{g,2}^{-h_g}(1/2)|(g^n)'(\phi(x))|^{h_g}\le\frac{m_g([g^n(\phi(x)),g^n(\phi(y))])}{m_g([\phi(x),\phi(y)])}\le K_{g,2}^{h_g}(1/2)|(g^n)'(\phi(x))|^{h_g}.$$

It follows now from the above inequalities for measures, from (8.1) and since $h_f=h_g$ that

$$(K_{g,2}^hM^2K_{f,2}^h(1/2))^{-1}\frac{|(f^n)'(x)|^h}{|(g^n)'(\phi(x))|^h}\le K_{g,2}^hM^2K_{f,2}^h(1/2).$$

Hence applying the inequalities involving distances we get

$$\frac{|\phi(y)-\phi(x)|}{|y-x|} \geq K_{g,2}^{-1}K_{f,2}^{-1}(1/2)\frac{|g^n(\phi(y))-g^n(\phi(x))|}{|f^n(y)-f^n(x)|} \geq \big(K_{g,2}K_{f,2}(1/2)\big)^{-2}M^{-2/h}\frac{\tau_1}{\eta_1}$$

and

$$\begin{aligned}\frac{|\phi(y)-\phi(x)|}{|y-x|} &\leq K_{g,2}(1/2)K_{f,2}(1/2)\frac{|g^n(\phi(y))-g^n(\phi(x))|}{|f^n(y)-f^n(x)|}\frac{|(f^n)'(x)|}{|(g^n)'(\phi(x))|}\\ &\leq \big(K_{g,2}K_{f,2}(1/2)\big)^{2}M^{2/h}\frac{\tau||f'||}{2\eta_1}.\end{aligned}$$

So, we are done in this case.

Case 2. $\{f^n(y), f^n(x)\} \subset B(\Omega_f, \eta/||f'||)$. Since $|\phi(y)-\phi(x)| \leq \eta_1 \leq \eta/2 \leq \delta_f/2$ there is $\omega \in \Omega_f$ such that $f^n(x), f^n(y) \in B(\omega, \eta/||f'||)$. Let us consider

Case 2.1. The two points $f^n(y)$ and $f^n(x)$ are in the same connected component of $B(\omega, \eta/||f'||) \setminus \{\omega\}$. Let $0 \leq k = k(x,y) \leq n$ be the least integer such that $[f^j(x), f^j(y)] \subset B(\omega, \eta/||f'||)$ for all $k \leq j \leq n$. Finally let $q = q(f^k(x), f^k(y))$ and $p = p(\eta, f^k(x), f^k(y))$. Since $p \geq n-k$, we get $|f^{p+k}(y)-f^{p+k}(x)| \geq |f^n(y)-f^n(x)| \geq \eta_1/||f'||$. Since $\eta_1/||f'|| \leq \eta$, it follows from Lemma 2.14 that with the constant $C_f = C(\eta, \eta_1/||f'||) > 0$ and $\beta = \beta(\omega)$ we have

$$C_f^{-1}\sum_{j=0}^{q}(p+j)^{-\frac{\beta+1}{\beta}} \leq |f^k(y)-f^k(x)| \leq C_f\sum_{j=0}^{q}(p+j)^{-\frac{\beta+1}{\beta}}. \tag{8.2}$$

Now, since ϕ is a topological conjugacy between f and g, we have $q(g^k(\phi(x)), g^k(\phi(y))) = q(f^k(x), f^k(y))$. Let S be closure of the connected component of $B(\omega, \eta/||f'||) \setminus \{\omega\}$ that has non-empty intersection with $\{f^k(x), f^k(y)\}$ and let $\kappa = \kappa(\omega) > 0$ be the diameter of $\phi(S \cap J_f)$. Note that then $p(\kappa||g'||, g^k(\phi(x)), g^k(\phi(y))) = p(\eta, f^k(x), f^k(y))$, and as $|g^k(\phi(x)) - g^k(\phi(y))| \geq \tau_1$, using Lemma 8.6 and applying Lemma 8.4 for the map g, we have

$$C_{g,\omega}^{-1}\sum_{j=0}^{q}(p+j)^{-\frac{\beta+1}{\beta}} \leq |g^k(\phi(y))-g^k(\phi(x))| \leq C_{g,\omega}\sum_{j=0}^{q}(p+j)^{-\frac{\beta+1}{\beta}},$$

where $C_{g,\omega} = C(\kappa(\omega)||g'||, \min\{\tau_1, \kappa(\omega)||g'||\})$ is the constant produced in Lemma 8.4 associated with the map g. Combining this formula and (8.2) we get

$$(C_fC_g)^{-1} \leq \frac{|g^k(\phi(y))-g^k(\phi(x))|}{|f^k(y)-f^k(x)|} \leq C_fC_g, \tag{8.3}$$

where $C_g = \max\{C_{g,\omega} : \omega \in \Omega_f\}$. Observe now that by the definition of n and k we have $|f^{k-1}(y) - f^{k-1}(x)| \leq \eta_1/||f'||$ and $\mathrm{dist}\big(\Omega_f, \{f^{k-1}(y), f^{k-1}(x)\}\big) \geq \delta_f/||f'||$. Hence $|g^{k-1}(\phi(y)) - g^{k-1}(\phi(x))| \leq \tau/2$ and $\mathrm{dist}\big(\Omega_g, \{g^{k-1}(\phi(y)), g^{k-1}(\phi(x))\}\big) \geq \tau$. So, representing inverse branches f_x^{-k} and $g_{\phi(x)}^{-k}$ respectively as the compositions $f_x^{-(k-1)} \circ f_{f^{k-1}(x)}^{-1}$ and $g_{\phi(x)}^{-(k-1)} \circ g_{g^{k-1}(\phi(x))}^{-1}$, it follows from Lemma 2.16 that

$$(K_{f,1}(1/2)||f'||)^{-1}|(f^k)'(x)| \leq \frac{|f^k(y)-f^k(x)|}{|y-x|} \leq K_{f,1}(1/2)||f'|||(f^k)'(x)|$$

and

$$(K_{g,1}(1/2)||g'||)^{-1}|(g^k)'(\phi(x))| \leq \frac{|g^k(\phi(y)) - g^k(\phi(x))|}{|\phi(y) - \phi(x)|} \leq K_{g,1}(1/2)||g'|||(g^k)'(\phi(x))|,$$

So similarly as in the Case 1, applying Lemma 8.5 and using (8.3), we get

$$|\phi(y) - \phi(x)| \leq C|y - x|^{1/\kappa},$$

where C is a universal constant which finishes the proof of Hölder continuity in this case. Similarly for conformal measures

$$(K_{f,1}(1/2)||f'||)^{-h}|(f^k)'(x)|^h \leq \frac{m_f([f^k(y), f^k(x)])}{m_f([y,x])} \leq (K_{f,1}(1/2)||f'||)^h|(f^k)'(x)|^h$$

and

$$\begin{aligned}(K_{g,1}(1/2)||g'||)^{-h}|(g^k)'(\phi(x))|^h &\leq \frac{m_g([g^k(\phi(y)), g^k(\phi(x))])}{m_g([\phi(y), \phi(x)])} \\ &\leq (K_{g,1}(1/2)||g'||)^h|(g^k)'(\phi(x))|^h.\end{aligned}$$

From the last two inequalities (involving measures) and from (8.1) we derive

$$\begin{aligned}&\big((K_{f,1}(1/2)||f'||)^h M^2 (K_{g,1}(1/2)||g'||)^h\big)^{-1} \\ &\qquad \leq \frac{|(f^k)'(x)|^h}{|(g^k)'(\phi(x))|^h} \\ &\qquad \leq (K_{f,1}(1/2)||f'||)^h M^2 (K_{g,1}(1/2)||g'||)^h.\end{aligned}$$

Hence, applying the estimates for distances and (8.3), we get

$$\begin{aligned}\frac{|\phi(y) - \phi(x)|}{|y - x|} &\leq K_{g,1}(1/2)||g'||K_{f,1}(1/2)||f'||\frac{|g^k(\phi(y)) - g^k(\phi(x))|}{|f^k(y) - f^k(x)|}\frac{|(f^k)'(x)|^h}{|(g^k)'(\phi(x))|^h} \\ &\leq \big(K_{g,1}(1/2)||g'||K_{f,1}(1/2)||f'||\big)^2 M^{2/h} C_f C_g\end{aligned}$$

and

$$\begin{aligned}\frac{|\phi(y) - \phi(x)|}{|y - x|} &\geq \big(K_{g,1}(1/2)||g'||K_{f,1}(1/2)||f'||\big)^{-1}\frac{|g^k(\phi(y)) - g^k(\phi(x))|}{|f^k(y) - f^k(x)|}\frac{|(f^k)'(x)|^h}{|(g^k)'(\phi(x))|^h} \\ &\geq \big(K_{g,1}(1/2)||g'||K_{f,1}(1/2)||f'||\big)^{-2} M^{-2/h} (C_f C_g)^{-1}\end{aligned}$$

Therefore the proof is also finished in this case.

Case 2.2. The two points $f^n(y)$ and $f^n(x)$ are in different connected components of $B(\omega, \eta/||f'||) \setminus \{\omega\}$. Then also $f^k(y)$ and $f^k(x)$ are in different connected components of $B(\omega, \eta/||f'||) \setminus \{\omega\}$. Since the map $f^k|_{[x,y]}$ (even more, the map $f^n|_{[x,y]}$) is well defined there exists a (unique) point $v \in (x, y)$ such that $f^k(v) = w$, in particular $v \in J_f$. Now, note that since $n(x, v), n(y, v) \geq n(x, y)$, both pairs (x, v) and (y, v) fall in the Case 2.1 (although it would not hurt us, the Case 1 is forbidden for the pairs (x, v) and (y, v) since, by the choice of η_1 and η_2 the nth iterates of both points must be then out of Ω_f and therefore the numbers $|\phi(x) - \phi(y)|$ and $|x - v|$ are comparable as well as the distances $|\phi(y) - \phi(v)|$ and $|y - v|$ are. Combining these together finishes the proof of Theorem 8.3 and the implication $(b) \Rightarrow (c)$. ♣

In order to prove the implication $(a) \Rightarrow (b)$ let us introduce the following notation. For every $x \in J_f$ let

$$\eta(x) = \log |g'(\phi(x))| - \log |f'(x)|$$

and if $x \in J_f$ is a transitive point of the map $f : J_f \to J_f$, which means that the closure $\overline{\{f^n(x) : n \geq 0\}}$ of the forward trajectory of x is equal to J_f, then for every $n \geq 0$ set

$$u(f^n(x)) = \sum_{j=0}^{n-1} \eta(f^j(x)) \tag{8.4}$$

We shall first prove the following technical result.

Lemma 8.7. If x is a transitive point of f, then for every $0 < t < \delta/2$ the function u restricted to the set $(J_f \setminus B(\Omega_f, t)) \cap \{f^n(x) : n \geq 0\}$ is uniformly continuous.

Proof. Fix $0 < \varepsilon < 1/2$ and let $0 < \zeta < \varepsilon t$ be a number less than the number produced in Corollary 2.9 associated with εt. Consider two points $f^m(x), f^n(x) \in J_f \setminus B(\Omega_f, t)$ with $|f^n(x) - f^m(x)| < \zeta$. Without loosing generality we may assume that $m \leq n$. Then in view of Corollary 2.9 there exists a point $y \in J_f$ such that $f^{n-m}(f^m(y)) = f^m(y)$ and $|f^{m+j}(x) - f^{m+j}(y)| < \varepsilon t$ for all $j = 0, 1, \ldots, n - m$. Since by the assumption $\sum_{j=m}^{n-1} \eta(f^j(y)) = 0$, we therefore get

$$\begin{aligned}
u(f^n(x)) - u(f^m(x)) &= \sum_{j=m}^{n-1} \eta(f^j(x)) = \sum_{j=m}^{n-1} \big(\eta(f^j(x)) - \eta(f^j(y))\big) \\
&= \sum_{j=m}^{n-1} \Big(\big(\log |g'(\phi(g^j(x)))| - \log |g'(\phi(g^j(y)))|\big) - \big(\log |f'(f^j(x))| - \log |f'(f^j(y))|\big)\Big) \\
&= \log \left| \frac{(g^{n-m})'(\phi(g^m(x)))}{(g^{n-m})'(\phi(g^m(y)))} \right| - \log \left| \frac{(f^{n-m})'(f^m(x))}{(f^{n-m})'(f^m(y))} \right|
\end{aligned}$$

Thus, in order to show that $|u(f^n(x)) - u(f^m(x))|$ is small if $\zeta > 0$ is small it suffices to prove that both numbers $|\log |(g^{n-m})'(\phi(g^m(x)))/(g^{n-m})'(\phi(g^m(y)))|$ and the the number $|\log |(f^{n-m})'(f^m(x))/(f^{n-m})'(f^m(y))|$ are small. Since ϕ is a homeomorphism it is enough to establish this property for the latter number. And indeed, Since $\varepsilon t < \delta/4 < \delta$, it follows from the properties of y that $f^m(y) = f_{f^m(x)}^{-(n-m)}(f^n(y))$, where $f_{f^m(x)}^{-(n-m)}$, the continuous inverse branch of f^{n-m} sending $f^n(x)$ to $f^m(x)$ is defined on $B(f^n(x), \delta)$. Therefore, since $|f^n(x) - f^m(x)| < \zeta \le \delta/4$, since $|f^n(x) - f^m(x)| < \varepsilon t$, and since $\mathrm{dist}(f^n(x), \Omega_f) \ge t$, it follows from Lemma 2.16 that

$$\left|\log\left|\frac{(f^{n-m})'(f^m(x))}{(f^{n-m})'(f^m(y))}\right|\right| \le |\log K_1(t, \varepsilon)|$$

and $\lim_{\varepsilon\to 0} |\log K_1(t, \varepsilon)| = 0$. The proof is finished. ♣

Proceeding with the proof of the implication $(a) \Rightarrow (b)$ we shall show the following.

Lemma 8.8. The functions $\log|f'(x)|$ and $\log|g'(\phi(x))|$ are cohomologoous in the class of continuous functions on J_f, that is there exists a continuous function $u : J_f \to I\!\!R$ such that

$$\log|g'(\phi(z))| - \log|f'(z)| = u(f(z)) - u(z)$$

for all $z \in J_f$.

Proof. It follows from Lemma 2.3(e) that there exists a transitive point $x \in J_f$. We shall show that u defined by (8.4) on the forward trajectory of x extends continuously to J_f and satisfies the cohomological equation required in Lemma 8.8. First note that by (8.4)

$$\eta(z) = u(f(z)) - u(z) \tag{8.5}$$

for all $z \in \{f^n(x) : n \ge 1\}$ and in view of Lemma 8.7 u extends continuously to the set $J_f \setminus \Omega_f$. Therefore (8.5) holds for all $z \in J_f \setminus (\Omega_f \cup f^{-1}(\Omega))$. Using these two facts we shall now show that u extends continuously to J_f and that then (8.5) holds for all $z \in \Omega_f$. Indeed, let $\omega \in \Omega_f$. Take $x \in f^{-1}\{\omega\} \setminus \{\omega\}$ and define $u(\omega)$ by the formula

$$u(\omega) = \eta(x) + u(x).$$

We want to show first that u is continuous at ω and that $u(\omega)$ is independent of of the choice of $x \in f^{-1}\{\omega\} \setminus \{\omega\}$. So, let $y_n \to \omega$, $y_n \ne \omega$. Since by Theorem 2.8 the map $f : J_f \to J_f$ is open there exists a sequence $x_n \to x$ such that $f(x_n) = y_n$ and therefore

$$\lim_{n\to\infty} u(y_n) = \lim_{n\to\infty} (\eta(x_n) + u(x_n)) = \eta(x) + u(x) = u(\omega).$$

The continuity of u at ω is therefore proven. In order to prove the independence of $x \in f^{-1}\{\omega\} \setminus \{\omega\}$ actually the same argument is employed. Take $z \in f^{-1}\{\omega\} \setminus \{\omega\}$. Since J_f is perfect there is a sequence of points $z_n \in J_f \setminus \{z\}$, $n \ge 1$, tending to z. Since by Theorem 2.8 the map $f : J_f \to J_f$ is open, there exists a sequence $v_n \in J_f$ of points tending to x and such that $f(v_n) = f(z_n)$ for all $n \ge 1$. Hence

$$\begin{aligned}
u(\omega) = \eta(x) + u(x) &= \lim_{n\to\infty} (\eta(v_n) + u(v_n)) \\
&= \lim_{n\to\infty} u(f(v_n)) = \lim_{n\to\infty} u(f(z_n)) = \lim_{n\to\infty} (\eta(z_n) + u(z_n)) \\
&= \eta(z) + u(z).
\end{aligned}$$

We have therefore obtained that u extends continuously to J_f and that (8.5) holds for all $z \in \Omega_f \setminus \Omega_f$. But since the functions appearing in (8.5) are continuous and $\Omega_f \setminus \Omega_f$ is dense in J_f, we conclude that (8.5) continuous to be true for all $z \in J_f$. ♣

Proof of the implication $(a) \Rightarrow (b)$**.** The proof we present here is similar to the Proof of Lemma 4.2. In view of Lemma 8.8 we conclude the existence of a constant $Q \geq 1$ such that for all $z \in J_f$ and all $n \geq 1$ we have

$$Q^{-1} \leq \frac{|(g^n)'(\phi(z))|}{|(f^n)'(z)|} \leq Q, \tag{8.6}$$

We shall show that the measure $m_g \circ \phi$ is absolutely continuous with respect to the measure m_f. So, take $\eta > 0$ so small that if $|x - y| \leq \eta$, then $|\phi^{-1}(x) - \phi^{-1}(y)| < \delta$. Fix $\gamma_g > 0$ so small as required in Lemma 2.19 for the map g and then take $\gamma_f > 0$ so small as required in Lemma 2.19 for the map f and moreover so small that if $|x - y| \leq \gamma_f \delta$, then $|\phi(x) - \phi(y)| < \gamma_g \eta$. As in the proof of Lemma 4.2 it follows from Theorem 2.8 that for every $x \in J_f \setminus \bigcup_{n=0}^{\infty} f^{-n}(\Omega)$, there exists a sequence $\{n_j = n_j(x) : j \geq 1\}$ such that $f^{n_j(x)} \notin B(\Omega, \delta)$. Let $f_x^{-n_j} : B(f^{n_j}(x), \gamma\delta) \to S^1$ be the continuous inverse branch of f^{n_j} sending $f^{n_j}(x)$ to x. Then it follows from Corollary 2.19, that $f^{n_j}(B(x, r_j)) \supset B(f^{n_j}(x), K_2^{-2}(\gamma_f)\gamma_f \delta)$ and

$$m_f(B(x, r_j)) \geq K_2^{-h}(\gamma_f) P |(f^{n_j})'(x)|^{-h}, \tag{8.7}$$

where $P = \inf\{m(B(z, K_2^{-2}(\gamma_f)\gamma_f\delta)) : z \in J\} > 0$ and

$$r_j = r_j(x) = K_2^{-1}(\gamma_f)|(f_x^{-n_j})'(f^{n_j}(x))|\gamma\delta = K_2^{-1}(\gamma_f)\gamma_f\delta|(f^{n_j})'(x)|^{-1}.$$

Since also $B\big(x, r_j\big) \subset f_x^{-n_j}\big(B(f^{n_j}(x), \gamma_f\delta)\big)$, by the choice of γ_f we get

$$\phi\big(B\big(x, r_j\big)\big) \subset \phi\big(f_x^{-n_j}\big(B(f^{n_j}(x), \gamma_f\delta)\big)\big) \subset g_{\phi(x)}^{-n_j}\big(B(g^{n_j)}(\phi(x), \gamma_g\eta)\big)$$

Since by the property (a), $\phi(\Omega_f) = \Omega_g$ and since $\text{dist}\big(f_j^n(x), \Omega_f\big) \geq \delta_f$, it follows from the choice of η that $\text{dist}\big(g^{n_j}(\phi(x)), \Omega_g\big) > \eta$. Hence, applying Lemma 2.19 for g, using (8.6) and (8.7) we get

$$\begin{aligned}
m_g\big(\phi(B(x, r_j(x))\big) &\leq m_g\big(g_{\phi(x)}^{-n_j}(B(g^{n_j}(\phi(x)), \gamma_g\eta))\big) \\
&\leq K_{2,g}^h(\gamma_g) m_g\big(B(g^{n_j}(\phi(x)), \gamma_g\eta)\big)|(g^{n_j})'(\phi(x))|^{-h} \\
&\leq K_{2,g}^h(\gamma_g)|(g^{n_j})'(\phi(x))|^{-h} \leq K_{2,g}^h(\gamma_g)Q^h|(f^{n_j})'(x)|^{-h} \\
&\leq K_{2,f}^h(\gamma_f)K_{2,g}^h(\gamma_g)Q^h P^{-1} m_f\big(\phi(B(x, r_j(x))\big)
\end{aligned}$$

So, applying Lemma 4.3 finishes the proof. ♣

§**9. Real analytic systems.** In this section we consider parabolic Cantor sets generating by dynamical systems $(f, I; \Delta_j, j \in I)$ with f being real analytic on each set Δ_j. It turns out that then the rigidity theorem, Theorem 8.1, takes on a much stronger form, namely in the condition (b) the assumption of equality of Hausdorff dimensions can be dropped. In order to meet this aim we work first with complex analytic extensions of f to get analyticity of the Radon-Nikodym derivative $d\mu/dm$. This in turn, with the help of complex analytic methods, implies real analyticity of the Jacobian of the map $f : J \to J$ with respect to the measure μ. The last step indirectly employing the concept of nonlinearity of expanding dynamical Cantor sets due to Sullivan shows that the Jacobian is not everywhere locally constant which constitutes the last major ingredient of the proof of real analyticity of the conjugacy ϕ. We begin with the following.

Definition 9.1. A dynamical system $(f, I; \Delta_j, j \in I)$ is said to be real analytic if the map $f : \bigcup_{j \in I} \Delta_j \to S^1$ has a real analytic extension onto an open neighborhood of $f : \bigcup_{j \in I} \Delta_j$ in S^1.

The remark that enables us to take advantage of the theory of complex analytic functions is that for any real analytic dynamical system there exists an open in $\mathbb{C}$, the set of complex numbers, neighborhood H of $f : \bigcup_{j \in I} \Delta_j$ and an $\mathbb{C}$-analytic function on H whose restriction to $f : \bigcup_{j \in I} \Delta_j$ coincides with f. We call this function the (complex) analytic extension of f and we keep for it the same symbol f. Our exposition begins with citing the following improved version of the Koebe Distortion Theorem proven in [Pr1] (for the classical version and some discussion of the subject see [Po] for example).

Lemma 9.2. (The Koebe distortion Theorem) Given an open bounded subset G of the complex plane $\mathbb{C}$ there exists a constant $K > 1$ such that if $B(z, \delta) \subset G$ and $H : B(z, \delta) \to G$ is a holomorphic univalent map, then for every $0 < \lambda < 1$ and every $x \in B(z, \delta)$ we have

$$\frac{|H'(x)|}{|H'(z)|}, \ \frac{|H'(z)|}{|H'(x)|} < K(1 - \lambda)^{-1}$$

Switching to the setting of parabolic Cantor sets and using some ideas from [Pr1] we shall prove the following.

Lemma 9.3. Let $V \subset J$ be an open neighborhood of Ω. Then there exists an $r > 0$ such that for every $x \in J \setminus V$, every $n \geq 0$ and every $z \in J \cap f^{-n}(x)$ there is an inverse $\mathbb{C}$-analytic branch $f_z^{-n} : B_{\mathbb{C}}(x, 2r) \to \mathbb{C}$ of f^n sending x to z. Additionally the diameters of the sets $f_z^{-n}\big(B_{\mathbb{C}}(x, 2r)\big)$ converge to 0 uniformly with respect to variables n, $x \in J \setminus V$, and $z \in J \cap f_z$.

Proof. Since $f : H \to \mathbb{C}$ as analytic is open and since J is compact,

$$\eta = \mathrm{dist}\big(J, \partial(H \cap f(H))\big) > 0.$$

Hence, using compactness of J again we see that there exists $s > 0$ such that all the inverse branches of f are well defined on the balls $B(x, s)$, $x \in J$. Suppose now additionally that $x \notin V$ and consider an arbitrary infinite sequence $x_n \in J$, $n \geq 0$, such that $f(x_{n+1}) = x_n$ and $x_0 = x$. Set

$$b_n = \frac{1}{2} M(t, 1/2)^{-1} |(f_{x_{n+1}}^{-(n+1)})'(x)|,$$

where $t = \mathrm{dist}(\Omega, J \setminus V)$ and $M(t, 1/2)$ is taken from Lemma 2.16. In view of Lemma 2.16 $\sum_{n=0}^{\infty} \leq 1/2$ and therefore the product $\Pi_{n\geq 0}(1 - b_n)^{-1}$ converges. In fact it lies between 1 and e. Hence there exists $r > 0$ independent of x so small that

$$2r\Pi_{n\geq 0}(1 - b_n)^{-1} \leq \min\{s, \delta, t/2, s(2KM(t, 1/2))^{-1}\} \tag{9.1}$$

We shall show by induction that for every $n \geq 1$ there is an analytic inverse branch $f_{x_n}^{-n} : B\big(x, 2r\Pi_{k\geq n}(1 - b_k)^{-1}\big) \to \mathbb{C}$ sending x to x_n and

$$f_{x_n}^{-n}\big(B\big(x, 2r\Pi_{k\geq n}(1 - b_k)^{-1}\big)\big) \subset B(x_n, s)$$

Indeed, for $n = 0$, $f_{x_0}^{-0}$ is the identity map and our assertion follows from (9.1). So, fix some $n \geq 0$ and suppose that the assertion is true for this n. Then by the definition of s the inverse branch $f_{x_{n+1}}^{-(n+1)} : B\big(x, 2r\Pi_{k\geq n}(1 - b_k)^{-1}\big) \to \mathbb{C}$ is also well-defined and by Lemma 9.2 (the Koebe Distortion Theorem), the definition of b_n's and (9.1)

$$\begin{aligned}
f_{x_{n+1}}^{-(n+1)}&\big(B\big(x, 2r\Pi_{k\geq n+1}(1 - b_k)^{-1}\big)\big) \subset \\
&\subset B\big(x_{n+1}, 2r\Pi_{k\geq n+1}(1 - b_k)^{-1} K b_n^{-1} |(f_{x_n+1}^{-(n+1)})'(x)|\big) \\
&\subset B\big(x_{n+1}, 2r\Pi_{k\geq 0}(1 - b_k)^{-1} K 2M(t, 1/2)\big) \\
&\subset B\big(x_{n+1}, s\big)
\end{aligned}$$

Thus, the inductive reasoning is completed and as for every n, $\Pi_{k\geq n}(1 - b_k)^{-1} \geq 1$, the first part of the lemma is proven. The second part follows now immediately from Lemma 2.1 and Lemma 9.2 (the Koebe Distortion Theorem). ♣

As an immediate consequence of Lemma 9.3 and Lemma 9.2 (the Koebe distortion theorem) we get the following.

Corollary 9.4. $\forall_{\lambda>1}\ \exists_q\ \forall_{n\geq q}\ \forall_{z\in J\setminus V}$ if $f_\nu^{-n} : B(z, 2r) \to \overline{\mathbb{C}}$ is an inverse branch of f^n then $|(f_\nu^{-n})'(x)| < \lambda^{-1}$ for every $x \in B(z, r)$.

Our next goal is to show that the Radon-Nikodym derivative $d\mu/dm$ allows a real analytic extension, that is in fact even a complex analytic extension. In order to cope with this problem we need to go back to Section 8 to examine the way the σ-finite measure μ has been constructed. So, first we defined the jump transformation $f^* : J_f \setminus \Omega$ setting

$$f^*(x) = f^{n(x)+1}(x),$$

where $n(x) \geq 0$ is the least integer $n \geq 0$ such that $f^n(x) \notin \bigcup_{j \in I(\Omega)} \Delta_j$. In Proposition 7.6 we claimed that there exists a unique, ergodic, f^*-invariant probability measure μ^* equivalent to m and $\psi^* = d\mu^*/dm$ satisfies $D^{-1} \leq \psi^* \leq D$ for some constant $D > 0$. Now proceeding essentially as in the proof of Lemma 4.6 of [U1], we shall prove the following .

Lemma 9.5. If (f, J_f) is real analytic, then there exists a $\mathbb{C}$-analytic extension of $\psi^* = d\mu^*/dm$ onto an open neighborhood of $\bigcup_{j \in J} \Delta_j$.

Proof. Let

$$\mathcal{L} : L^1(m) \to L^1(m), \quad \mathcal{L}(\theta)(z) = \sum_{x \in (f^*)^{-1}(z)} \frac{1}{|(f^*)'(x)|^h} \, \theta(x)$$

be the Perron-Frobenius operator of the mapping f with respect to the measure m, that is $\mathcal{L}(\theta) = d\big((\theta m) \circ (f^*)^{-1}\big)/dm$. Therefore it follows from Proposition 7.6 that $\psi^* = d\mu^*/dm$ is the only positive fixed point of $\mathcal{L}$. An easy computation shows that for every $n \geq 1$

$$\mathcal{L}^n(\theta)(z) = \sum_{x \in ((f^*)^n)^{-1}(z)} \frac{\theta(x)}{|((f^*)^n)'(x)|^h} \tag{9.2}$$

For every $x \in S^1$ and $k \geq 1$ let $n(x,k) = n(x) + 1 + n(f^*(x)) + 1 + n((f^*)^2(x)) + 1 + \ldots + n((f^*)^{k-1}(x)) + 1$ (we make the convention $n(\omega) = \infty$) for $\omega \in \Omega$). Then $(f^*)^k(x) = f^{n(x,k)}(x)$. In view of Lemma 9.3 and the definition of the jump transformation there exists $0 < R \leq r$ such that for every $k \geq 1$, every $z \in J$, and every $x \in (f^*)^{-k}(z)$ there exists a unique holomorphic inverse branch $f_{\nu(x,k)}^{-n(x,k)} : B(z, 2R) \to \overline{\mathbb{C}}$ of $f^{n(x,k)}$ determined by the condition $f_{\nu(x,k)}^{-n(x,k)}(z) = x$. Since the map f mapping $\bigcup_{j \in I} \Delta_j$ onto its image is open, using (9.2), we can write

$$\mathcal{L}^k(\theta)(y) = \sum_{x \in (f^*)^{-k}(z)} |(f_{\nu(x,k)}^{-n(x,k)})'(y)|^h \theta(f_{\nu(x,k)}^{-n(x,k)}(y))$$

for every $k \geq 1$ and $y \in S^1 \cap B(z, 2R)$. Since $f_{\nu(x,k)}^{-n(x,k)}(S^1 \cap B(z, 2R)) \subset S^1$, we have

$$|(f_{\nu(x,k)}^{-n(x,k)})'(y)| = \frac{y \cdot (f_{\nu(x,k)}^{-n(x,k)})'(y)}{f_{\nu(x,k)}^{-n(x,k)}(y)} \quad \text{for all } \ y \in S^1 \cap B(z, 2R).$$

Thus

$$\mathcal{L}^k(1)(y) = \sum_{x \in (f^*)^{-k}(y)} \left(\frac{y \cdot f_{\nu(x,k)}^{-n(x,k)\prime}(y)}{f_{\nu(x,k)}^{-n(x,k)}(y)} \right)^h \quad \text{for every } \ y \in S^1 \cap B(z, 2R), \tag{9.3}$$

where raising to the h-th power we have chosen the unique analytic branch sending z to $|(f_{\nu(x,k)}^{-n(x,k)})'(z)|^h$ which is well-defined since the set $B(z, 2R)$ is simply connected. Let

$M(z) = 2(m(B(z,R)))^{-1} < +\infty$. Since $1 \ge m((f^*)^{-m}(B(z,R))) = \int_{B(z,R)} \mathcal{L}^k(1)dm$, there exists a point $y_k \in B(z,R)$ such that

$$\mathcal{L}^k(1)(y_k) \le M(z). \tag{9.4}$$

In view of Lemma 9.3 there exists a constant $N > 0$ such that

$$\left| \frac{y}{f_{\nu(x,k)}^{-n(x,k)}(y)} \right| \le N \quad \text{for every} \quad k \ge 1,\ y \in B(z,R) \quad \text{and} \quad x \in (f^*)^{-k}(z), \tag{9.5}$$

and in view of Lemma 9.2 (the Koebe Distortion Theorem)

$$|(f_{\nu(x,k)}^{-n(x,k)})'(y)| \le K|(f_{\nu(x,k)}^{-n(x,k)})'(y_k)|$$

for every $k \ge 1$, $y \in B(z,R)$, and $x \in (f^*)^{-k}(z)$. Therefore, using (9.3), (9.4), and (9.5), we get

$$\begin{aligned}
\sum_{x \in (f^*)^{-k}(z)} \left| \frac{y \cdot (f_{\nu(x,k)}^{-n(x,k)})'(y)}{f_{\nu(x,k)}^{-n(x,k)}(y)} \right|^h &\le (NK)^h \sum_{x \in (f^*)^{-k}(z)} |(f_{\nu(x,k)}^{-n(x,k)})'(y_k)|^h \\
&= (NK)^h \mathcal{L}^k(1)(y_k) \\
&\le (NK)^h M(z)
\end{aligned}$$

for every $k \ge 1$ and $y \in B(z,R)$. Hence the series appearing in (9.3) defines on $B(z,R)$ a holomorphic function for which we keep also the name $\mathcal{L}^k(1)$. It follows again from the last display that

$$\left| \frac{1}{k} \sum_{j=0}^{k-1} \mathcal{L}^j(1)(y) \right| \le (NK)^h M(z) \quad \text{for every} \quad k \ge 1 \quad \text{and} \quad y \in B(z,R)$$

Thus, by Vitali's theorem, the family $\{\frac{1}{k} \sum_{j=0}^{k-1} \mathcal{L}^j(1)\}_{k=1}^{\infty}$ of holomorphic functions on $B(z,R)$ is normal in the sense of Montel and therefore one can find an increasing to infinity subsequence $\{k_s\}_{s=1}^{\infty}$ such that $\frac{1}{k_s} \sum_{j=0}^{k_s-1} \mathcal{L}^j(1)$ converges on $B(z,R/2)$ uniformly to an analytic function, say $H : B(z,R/2) \to \mathbb{C}$. Hence $\psi^* = H$ almost everywhere on $S^1 \cap B(z,R/2)$. Thus the proof is finished since analyticity is a local property. ♣

Now, as an immediate consequence of Lemma 9.5 and Theorem 7.7, along with real analyticity of $1/|f'|^h$, and Lemma 2.15, we get the following.

Lemma 9.6. The Radon-Nikodym derivative $\psi = d\mu/dm$ has a real analytic extension to the set $\bigcup_{j \in J} \Delta_j \setminus \Omega$.

Let now ρ_μ denote the Jacobian of the map f with respect to the measure μ. Since $\rho_\mu(x) = |f'(x)|^h\psi(f(x))/\psi(x)$, we derive from Lemma 9.6 the following main technical result about real analyticity.

Lemma 9.7. The Jacobian ρ_μ has a real analytic extension to the set $\bigcup_{j\in J}\Delta_j \setminus \Omega$.

Our first consequence of Lemma 9.7 is the following.

Lemma 9.8. If (f, Δ_j, I) is a real-analytic parabolic system, then there is $i \in I$ such that the Jacobian ρ_μ of f with respect to the invariant measure μ is not locally constant at any point of Δ_i.

Proof. Suppose to the contrary that every interval Δ_j contains a point (not necessarily lying in J) around which the Jacobian ρ_μ is constant. Then it follows from Lemma 9.7 that ρ_μ is constant on each whole interval Δ_j, $j \in I$. Denote this common value by ρ_j. Since μ is invariant $\sum_{y\in f^{-1}(x)} \rho_i^{-1}(y) = 1$ for μ almost every $x \in J$, and since each point of J has at least two distinct preimages under f and since μ is positive on non-empty open sets, it follows that $\rho_\mu(y) > 1$ for all $y \in f^{-1}(x)$. Hence $\lambda = \min\{\rho_j : j \in I\} > 1$. Take now an arbitrary point $\omega \in \Omega$ and choose one point $z \in J \cap B(\omega, \delta) \setminus \{\omega\}$. In view of Lemma 7.14, $\mu\big([f_\omega^{-1}(z), z)\big) < \infty$. Thus

$$\mu([\omega, z)) = \sum_{n\ge 0} \mu\big(f_\omega^{-n}([f_\omega^{-1}(z), z))\big) \le \sum_{n\ge 0} \lambda^{-n}\mu\big([f_\omega^{-1}(z), z)\big) = \frac{1}{1-\lambda}\mu\big([f_\omega^{-1}(z), z)\big) < \infty$$

Choosing if necessary one point in $J \cap B(\omega, \delta) \setminus \{\omega\}$ locating on the other side of ω, we therefore conclude that ω has a neighborhood of finite μ measure. Since Ω is finite the same continues to be true for the whole set Ω. Combining this fact and Lemma 7.14 we deduce that $\mu(J) < \infty$. But this contradicts Corollary 7.17 and finishes the proof of the lemma. ♣

Let us now proof the main result of this section.

Theorem 9.9. Let (J_f, f) and (J_g, g) be two real-analytic parabolic systems and let $\phi : J_f \to J_g$ be the corresponding canonical topological conjugacy. If the homeomorphism ϕ transports the measure class of the packing measure Π_{h_f} on J_f onto the measure class of the packing measure Π_{h_g} on J_g, then ϕ and ϕ^{-1} extend to real analytic maps on open neighborhoods in S^1 respectively of J_f and J_g. In particular $\mathrm{HD}(J_f) = \mathrm{HD}(J_g)$.

Proof. Fix an f-invariant measure μ_f equivalent to the conformal measure m_f. Since ϕ transports measure class of m_f to the measure class of conformal measure m_g, the measure $\mu_g = \mu_f \circ \phi^{-1}$ is g-invariant and equivalent with m_g. Since ϕ is invertible it equivalently means that ρ_ϕ, the Jacobian of ϕ with respect to the measures μ_f and μ_g is equal to 1. The formula $g \circ \phi = \phi \circ f$ combined with the chain rule therefore give

$$\rho_g \circ \phi = \rho_f \quad \mu_f - \text{a.e.},$$

where ρ_g and ρ_f denote respectively the Jacobians of the maps g and f with respect to the measures μ_g and μ_f. Since the measure μ_f is positive on non-empty open subsets of J_f and since by Lemma 9.7, both sides of this equality are continuous on $J_f \setminus \big(\Omega_f \cup \phi^{-1}(\Omega_g)\big)$, we get

$$\rho_g \circ \phi(x) = \rho_f(x) \tag{9.5}$$

for all $x \in J_f \setminus \big(\Omega_f \cup \phi^{-1}(\Omega_g)\big)$. Now Lemma 9.8 applied to the real analytic system (g, J_g) produces an open arc $V \subset S^1$ such that $V \cap J)g \neq \emptyset$ and $\rho_g|_V$ is injective. Let $W = \phi^{-1}(V \cap J_g)$. Since W is a non-empty subset of J_f and since $\rho_g(V)$ is an open subset of $\mathbb{R}$, using (9.5), we deduce the existence of an open subset U of $S^1 \setminus \big(\Omega_f \cup \phi^{-1}(\Omega_g)\big)$ such that $\emptyset \neq U \cap J_f \subset W$, $\rho_f(U) \subset \rho_g(V)$ and

$$\phi(x) = (\rho_g|_V)^{-1} \circ \rho_f(x) \tag{9.6}$$

for all $x \in J_f \cap U$. In particular $\phi|_{J_f \cap U}$ has a real analytic extension on U. Take now an arbitrary point $z \in J_f$. In view of Lemma 2.3(f) there exist $y \in J_f \cap U$ and $n \geq 0$ such that $f^n(y) = z$. Take $r > 0$, depending on y and n, so small that there exists $f_y^{-n} : B(z, r) \to S^1$, a continuous inverse branch of f^n sending z to y. We may additionally require $r > 0$ to be so small that $f_y^{-n}(B(z, r)) \subset U$ and $g^n(\rho_g|_V)^{-1} \circ \rho_f f_y^{-n}(B(z, r))$ is well defined. From $\phi \circ f^n = g^n \circ \phi$ (on J_f) we deduce that $\phi = g^n \circ \phi \circ f_y^{-n}$ on $J_f \cap B(z, r)$. So, since f_y^{-n} on $B(z, r)$ is real analytic and since g^n is real analytic on any arc where it is well defined, using (9.6) we deduce that $g^n \circ (\rho_g|_V)^{-1} \circ \rho_f \circ f_y^{-n} : B(z, r) \to S^1$ gives a real analytic extension of $\phi|_{J_f \cap B(z,r)}$ to the ball $B(z, r)$. Thus we have proved that every point of J_f has an open connected neighborhood in S^1 to which ϕ can be extended in a real analytic fashion. Now, to conclude the proof, it suffices to remark that any two of such real analytic extensions, defined on respective intervals having non-empty intersections, coincide. ♣

§**10. The scaling function.** In this section we collect some basic properties of the scaling function associated with a cookie-cutter Cantor set construction, stressing differences between parabolic and hyperbolic case. Next we formulate a rigidity theorem in terms of scaling functions. Throughout the section we assume that the basic sets Δ_j, $j \in I$, are mutually disjoint which implies that $\Sigma_A^\infty = \Sigma^\infty$ is the full shift space over $d = \#I$ elements, $\pi : \Sigma^\infty \to J$ is a homeomorphism, and J is a topological Cantor set. Moreover we require that for all $j \in I$

$$f(\Delta_j) \supset \bigcup_{i \in I} \Delta_i \tag{10.1}$$

and the endpoints of the interval $f(\Delta_j)$ are contained in the union $\bigcup_{i \in I} \Delta_i$, hence are the same for all $j \in I$.

Recall that in Section 1 by $\Delta(\tau)$, $\tau \in \Sigma^n$, we have denoted the interval $\Delta_{\tau_0} \cap f^{-1}(\Delta_{\tau_1}) \cap \ldots \cap f^{-n}(\Delta_{\tau_n})$. Now we want to extend this definition letting τ be of the form $\rho\gamma$, where

$\rho \in \Sigma^*$ and γ ranges over the set $\mathcal{G}$ (consisting of $d-1$ elements) of gaps between the elements Δ_j, $j \in I$. We set

$$\Delta(\rho\gamma) = \Delta(\rho) \cap f^{-(|\rho|+1)}(\gamma)$$

and now we are in position to define the function $S : \Sigma^* \to [0,1]^{2d-1}$ putting for all $\tau \in \Sigma^*$ and $j \in I \cup \mathcal{G}$

$$S(\tau)(j) = S_j(\tau) = \frac{|\Delta(\tau_j)|}{\Delta(\tau)|}$$

Note that $\sum_j S(\tau)(j) = 1$. We will also consider functions S defined on the dual shift space $\tilde{\Sigma}^*$ consisting of all left-infinite words $\ldots\tau_n\tau_{n-1}\ldots\tau_0$, $\tau_i \in I$. Given $n \geq 0$ and $\tau \in \tilde{\Sigma}^*$ we define $S_n(\tau) = S(\tau_n\tau_{n-1}\ldots\tau_0)$. So, $S_n : \tilde{\Sigma}^* \to [0,1]^{2d-1}$. Our first aim is to prove the following.

Theorem 10.1. The sequence $\{S_n : \tilde{\Sigma}^* \to [0,1]^{2d-1} : n \geq 1\}$ converges uniformly. The limit function $S : \tilde{\Sigma}^* \to [0,1]^{2d-1}$, called the scaling function, is continuous.

Proof. Take $j \in I \cap \mathcal{G}$. Fix also integers $k, n \geq 0$. Take an auxiliary $x \in \Delta(\tau|_{n+k})$. In view of the Mean Value Theorem there exist $y \in \Delta(\tau_k j)$ and $z \in \Delta(\tau|_k)$ such that $|\Delta(\tau|_{n+k}j)| = |(f_x^{-n})'(y)| \cdot |\Delta(\tau_k j)|$ and $||\Delta(\tau|_{n+k})| = |(f_x^{-n})'(z)| \cdot |\Delta(\tau_k)|$. Therefore

$$\begin{aligned}
|S_{n+k}(\tau)(j) - S_k(\tau)(j)| &= \left| \frac{|\Delta(\tau|_{n+k}j)|}{|\Delta(\tau|_{n+k})|} - \frac{|\Delta(\tau_k j)|}{|\Delta(\tau_k)|} \right| \\
&= \left| \frac{|(f_x^{-n})'(y)| \cdot |\Delta(\tau_k j)|}{|(f_x^{-n})'(z)| \cdot |\Delta(\tau_k)|} - \frac{|\Delta(\tau_k j)|}{|\Delta(\tau_k)|} \right| \\
&= \frac{|\Delta(\tau_k j)|}{|\Delta(\tau_k)|} \left| \frac{|(f_x^{-n})'(y)|}{|(f_x^{-n})'(z)|} - 1 \right| \\
&\leq \left| \frac{|(f_x^{-n})'(y)|}{|(f_x^{-n})'(z)|} - 1 \right| \qquad (10.2)
\end{aligned}$$

With the help of (10.2) we shall prove that all the sequences $S_n(.)(j)$, $j \in I \cup G$, satisfy the uniform Cauchy condition. Indeed, fix again $j \in I \cup G$ and $\varepsilon > 0$. Take $\psi > 0$ so small that $\max\{Q_1(2\psi) - 1, 1 - Q_1(2\psi)^{-1}\} < \varepsilon$, where Q_1 is the function produced in Lemma 2.20. Now fix $A(\varepsilon) > 0$ so small that setting

$$K_1 = K_1(\delta/||f'||, L_1(\delta/||f'||)^{\beta+1})L_2(\delta/||f'||)\delta^{-1}||f'||\psi^{-(\beta+1)}A(\varepsilon),$$

where the function K_1 is produced in Lemma 2.16, it holds $\max\{K_1 - 1, 1 - K_1^{-1}\} < \varepsilon/2$. Finally, in view of Lemma 2.1 we can fix $k \geq 1$ so large that

$$\operatorname{diam}(\Delta(\tau|_k)) < A(\varepsilon) \qquad (10.3)$$

for all $\tau \in \tilde{\Sigma}$.

Take now an arbitrary $\tau \in \tilde{\Sigma}$ and suppose that

$$\operatorname{dist}(\Omega, \Delta(\tau|_k)) \geq \psi.$$

Let $t \geq 0$ be the least integer such that $\Delta(\tau|_k) = f_\omega^{-(k-t)}(\Delta(\tau|_t))$ for some $\omega \in \Omega$. Since ψ is positive, $\mathrm{dist}(\Omega, \Delta(\tau|_t)) \geq \delta/||f'||$. If $t = k$, then $\mathrm{diam}(\Delta(\tau|_t)) < A(\varepsilon)$. Otherwise, using Corollary 2.14 we conclude that $\mathrm{dist}(\Omega, \Delta(\tau|_k)) \leq L_1(\delta/||f'||)(k-t)^{-1/\beta}$. Hence $L_1(\delta/||f'||)(k-t)^{-1/\beta} \geq \psi$ and therefore $k - t \leq (L_1(\delta/||f'||)\psi^{-1})^\beta$. Thus by Lemma 2.15 we get

$$\begin{aligned}\mathrm{diam}(\Delta(\tau|_k)) &\geq L_2(\delta/||f'||)^{-1}(k-t)^{-\frac{\beta+1}{\beta}}\mathrm{diam}(\Delta(\tau|_t)) \\ &\geq L_2(\delta/||f'||)^{-1}L_1(\delta/||f'||)^{-(\beta+1)}\psi^{\beta+1}\mathrm{diam}(\Delta(\tau|_t))\end{aligned}$$

which implies that

$$\begin{aligned}\mathrm{diam}(\Delta(\tau|_t)) &\leq L_2(\delta/||f'||)L_1(\delta/||f'||)^{\beta+1}\psi^{-(\beta+1)}\mathrm{diam}(\Delta(\tau|_k)) \\ &\leq L_2(\delta/||f'||)L_1(\delta/||f'||)^{\beta+1}\psi^{-(\beta+1)}A(\varepsilon).\end{aligned}$$

Hence applying (10.2) and Lemma 2.16, it follows from the choice of k and ψ that for every $n \geq 0$ we have

$$\begin{aligned}|S_{n+k}(\tau)(j) - S_k(\tau)(j)| &\leq |S_{n+k}(\tau)(j) - S_t(\tau)(j)| + |S_t(\tau)(j) - S_k(\tau)(j)| \\ &\leq 2\max\{|K_1 - 1|, |1 - K_1^{-1}|\} < \varepsilon\end{aligned} \tag{10.4}$$

So, we can assume that

$$\mathrm{dist}(\Omega, \Delta(\tau|_k)) < \psi.$$

Then $\Delta(\tau|_k) \subset B(\Omega, 2\psi)$. Therefore if $\tau|_k$ does not consist only of indices corresponding to one parabolic point (so the assumptions of Lemma 2.20 are satisfied with $q = 1$), the it follows from (10.2), Lemma 2.20, and the choice of ψ that for every $n \geq 0$

$$|S_{n+k}(\tau)(j) - S_k(\tau)(j)| \leq \max\{Q_1(2\psi) - 1, 1 - Q_1^{-1}(2\psi)\} < \varepsilon.$$

Now, the only case left is when $\tau|_k$ consists of indices j_ω only for some $\omega \in \Omega$, where $j_\omega \in I$ is determined by the requirement that $\omega \in \Delta_{j_\omega}$. Since by the Mean Value Theorem $\lim_{x\to\omega} \frac{|f_\omega^{-1}(x)-\omega|}{|x-\omega|} = 1$ and in view of Corollary 2.14 and Lemma 2.15 we deduce that $\lim_{n\to\infty} S_n(j_\omega^n)(j)$ is equal to 1 if $j = j_\omega$ and 0 otherwise. Hence taking k sufficiently large, larger than required in (10.3) perhaps we see that $|S_{n+k}(\tau)(j) - S_k(\tau)(j)| < \varepsilon$ if $\tau|_{n+k} = j_\omega^{n+k}$. Otherwise look at the largest number q such that $\tau|_q = j_\omega^q$. Then $k \leq q < n + k$ and

$$\begin{aligned}|S_{n+k}(\tau)(j) - S_k(\tau)(j)| \leq\ &|S_{n+k}(\tau)(j) - S_{q+1}(\tau)(j)| + |S_{q+1}(\tau)(j) - S_q(\tau)(j)| + \\ &+ |S_q(\tau)(j) - S_k(\tau)(j)|\end{aligned}$$

As above $|S_q(\tau)(j) - S_k(\tau)(j)| < \varepsilon$. Moreover the first summand $|S_{n+k}(\tau)(j) - S_{q+1}(\tau)(j)|$ is estimated from above by ε similarly as the two summands in (10.4) ($q+1$ corresponds to t) and in view of (10.2) applied with $n = 1$ the second summand $|S_{q+1}(\tau)(j) - S_q(\tau)(j)|$ is less than ε if and only if $\mathrm{diam}(\Delta(\tau|_k))$, and consequently also $\mathrm{diam}(\Delta(\tau|_k))$ is sufficiently small. Then $|S_{n+k}(\tau)(j) - S_k(\tau)(j)| < 3\varepsilon$ which completes the proof of the uniform convergence of the sequence S_n. Since all the functions S_n are obviously continuous the limit function is also continuous and the proof is finished. ♣

Now we shall prove the fact, actually already proven in the course of the proof of Theorem 10.1 which describes some differences between parabolic and hyperbolic dynamical Cantor sets in the language of scaling functions.

Lemma 10.2. *$S(\tau)(j) = 0$ if and only if for all $n \geq 0$, $\Delta(\tau_n)$ is the (only) element containing some $\omega \in \Omega$ and Δ_j does not contain ω.*

Proof. Suppose first that for all $\omega \in \Omega$ not all the elements $\Delta(\tau_n)$, $n \geq 0$, contain ω. If $\Delta(\tau_0) \cap \Omega \neq \emptyset$, set $q = 0$. Otherwise there exists a least finite number $q \geq 1$ such that $\tau_q \neq \tau_0$. In any case $\text{dist}(\Omega, \Delta(\tau|_q)) \geq \delta/2$. In view of the Mean Value Theorem there exist $y \in \Delta(\tau|_q j) \subset \Delta(\tau|_q)$ and $z \in \Delta(\tau|_q)$ such that $|\Delta(\tau|_{q+n} j)| = |(f_t^{-n})'(y)| \cdot |\Delta(\tau|_q j)|$ and $|\Delta(\tau|_{q+n})| = |(f_t^{-n})'(z)| \cdot |\Delta(\tau|_q)|$, where f_t^{-n} denotes the inverse branch of f^n sending $\Delta(\tau|_q)$ to $|\Delta(\tau|_{q+n})$. Therefore

$$S_{q+n}(\tau)(j) = \frac{|\Delta(\tau|_{q+n} j)|}{|\Delta(\tau|_{q+n})|} = \frac{|(f_t^{-n})'(y)|}{|(f_t^{-n})'(z)|} S_q(\tau)(j)$$

and applying Corollary 2.17 we get $S_{q+n}(\tau)(j) \geq K_1(\delta/2)^{-1} S_q(\tau)(j)$. So, letting $n \to \infty$ (and employing Theorem 10.1 of course), we get $S(\tau)(j) \geq K_1(\delta//2)^{-1} S_q(\tau)(j) > 0$.

Now suppose that $\Delta(\tau|_n) = f_\omega^{-n}(\Delta_{\tau_0})$ for all $n \geq 0$ and some $\omega \in \Omega$. If j is taken such that $\omega \notin \Delta_j$, then in view of Lemma 2.15 and Corollary 2.14

$$S_n(\tau)_j = \frac{|\Delta(\tau|_n j)|}{|\Delta(\tau|_n)|} \leq \frac{l_2(\delta/2) n^{-\frac{\beta+1}{\beta}}}{l_1^{-1}(\delta/2) n^{-\frac{1}{\beta}}} = L_1(\delta/2) L_2(\delta/2) n^{-1}.$$

Hence $S(\tau)(j) = 0$. Since $\sum_j S(\tau)(j) = 1$, the proof is completed. ♣

Corollary 10.3. *If two dynamical Cantor sets J_f and generated respectively by dynamical systems $(f, I, \Delta_{f,j}, j \in I)$ and $(g, I, \Delta_{g,j}, j \in I)$ have the same scaling functions, then the topological conjugacy $\phi : J_f \to J_g$ sends the set of parabolic points of f onto set of parabolic points of g.*

Theorem 10.4. *If two dynamical Cantor sets J_f and J_g generated respectively by dynamical systems $(f, I, \Delta_{f,j}, j \in I)$ and $(g, I, \Delta_{g,j}, j \in I)$ have the same scaling functions, then the topological conjugacy $\phi : J_f \to J_g$ is Lipschitz continuous.*
Conversely, if the conjugacy $\phi : J_f \to J_g$ is a C^1 diffeomorphism, then the Cantor sets J_f and J_g have the same scaling functions.

Proof. Let us prove first the second part of this theorem.Indeed, keep the same notation ϕ for a C^1 extension of ϕ to an open neighborhood of J_f. Decreasing this neighborhood if necessary we can assume that ϕ', the derivative of ϕ nowhere vanishes. Therefore for every $n \geq 0$ sufficiently large and every $\tau \in \Sigma^n$, the map $\phi|_{\Delta_f(\tau)}$ is well defined and $\phi(\Delta_f(\tau)) = \Delta_g(\tau)$. Now, in view of the Mean Value Theorem, for every $\tau \in \tilde{\Sigma}$, every $j \in I$ and every sufficiently large $n \geq 0$, there are $y \in \Delta_f(\tau|_n j) \subset \Delta_f(\tau|_n)$ and $z \in \Delta_f(\tau|_n)$ such that $|\Delta_g(\tau|_n j)| = |\phi'(y)|c|\Delta_f(\tau|_n j)|$ and $|\Delta_g(\tau|_n)| = |\phi'(z)|c|\Delta_f(\tau|_n)|$. Thus

$$S_{g,n}(\tau)(j) = \frac{|\phi'(y)|}{|\phi'(z)|} S_{f,n}(\tau)(j).$$

Since $\lim_{n\to\infty} |\Delta(\tau|_n)| = 0$, it follows from positiveness and continuity of ϕ' that

$$S_g(\tau)(j) = \lim_{n\to\infty} S_{g,n}(\tau)(j) = \lim_{n\to\infty} S_{f,n}(\tau)(j) = S_f(\tau)(j)$$

finishing the proof of the second part of the theorem.

In order to prove the first part of this theorem we will show that condition (a) of Theorem 8.1 is satisfied, that is that the spectra of moduli of periodic points of f and g are the same. So, let z be an arbitrary periodic point of f, say of period $q \geq 1$. For $0 \leq j \leq q-1$ let $f^j(z) \in \Delta_f(\tau_j)$ and let $\tau = \tau_0\tau_1 \ldots \tau_{q-1}$. Our aim is to show that $|(g^q)'(z)| = |(f^q)'(z)|$. In view of Corollary 10.3 we may assume that neither z nor $\phi(z)$ are parabolic. Denoting by τ^n the concatenation of n words τ, we get

$$\begin{aligned}
&\frac{|\Delta_g(\tau^{n+1}\tau_0)|}{|\Delta_g(\tau^n\tau_0)|} : \frac{|\Delta_f(\tau^{n+1}\tau_0)|}{|\Delta_f(\tau^n\tau_0)|} = \\
&\qquad = \frac{|\Delta_g(\tau^n\tau_1\ldots\tau_{q-1}\tau_0)|}{|\Delta_g(\tau^n\tau_1\ldots\tau_{q-1})|} \cdot \frac{|\Delta_g(\tau^n\tau_1\ldots\tau_{q-1})|}{|\Delta_g(\tau^n\tau_1\ldots\tau_{q-2})|} \cdot \ldots \cdot \frac{|\Delta_g(\tau^{n+1}\tau_0\tau_1)|}{|\Delta_g(\tau^{n+1}\tau_0)|} : \\
&\qquad\qquad : \frac{|\Delta_f(\tau^n\tau_1\ldots\tau_{q-1}\tau_0)|}{|\Delta_f(\tau^n\tau_1\ldots\tau_{q-1})|} \cdot \frac{|\Delta_f(\tau^n\tau_1\ldots\tau_{q-1})|}{|\Delta_f(\tau^n\tau_1\ldots\tau_{q-2})|} \cdot \ldots \cdot \frac{|\Delta_f(\tau^{n+1}\tau_0\tau_1)|}{|\Delta_f(\tau^{n+1}\tau_0)|} \\
&\qquad = S_g(\tau^n\tau_1\ldots\tau_{q-1})(\tau_0) S_g(\tau^n\tau_1\ldots\tau_{q-2})(\tau_{q-1}) \ldots S_g(\tau^n\tau_1\ldots\tau_{q-2})(\tau_{q-1}) \cdot \\
&\qquad\qquad \cdot S_f^{-1}(\tau^n\tau_1\ldots\tau_{q-1})(\tau_0) S_f^{-1}(\tau^n\tau_1\ldots\tau_{q-2})(\tau_{q-1}) \ldots S_f^{-1}(\tau^n\tau_0)(\tau_1)
\end{aligned}$$

Thus, denoting by $\tau^\infty \in \tilde{\Sigma}$ the infinite concatenation of τ's we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{|\Delta_g(\tau^{n+1}\tau_0)|}{|\Delta_g(\tau^n\tau_0)|} : \frac{|\Delta_f(\tau^{n+1}\tau_0)|}{|\Delta_f(\tau^n\tau_0)|}\right) = \\
&\qquad = \frac{S_g(\tau^\infty\tau_1\ldots\tau_{q-1})(\tau_0)}{S_f(\tau^\infty\tau_1\ldots\tau_{q-1})(\tau_0)} \cdot \frac{S_g(\tau^\infty\tau_1\ldots\tau_{q-2})(\tau_{q-1})}{S_f(\tau^\infty\tau_1\ldots\tau_{q-2})(\tau_{q-1})} \cdot \ldots \cdot \frac{S_g(\tau^\infty\tau_0)(\tau_1)}{S_f(\tau^\infty\tau_0)(\tau_1)} \\
&\qquad = 1 \qquad\qquad (10.5)
\end{aligned}$$

On the other hand, since $\Delta_f(\tau^n\tau_0) = f^q(\Delta_f(\tau^{n+1}\tau_0)$ and $\Delta_g(\tau^n\tau_0) = g^q(\Delta_g(\tau^{n+1}\tau_0)$, by the Mean Value Theorem there are two points $x_n \in \Delta_f(\tau^{n+1}\tau_0)$ and $y_n \in \Delta_g(\tau^{n+1}\tau_0)$ such that $|\Delta_f(\tau^n\tau_0)| = |(f^q)'(x_n)||\Delta_f(\tau^{n+1}\tau_0)|$ and $|\Delta_g(\tau^n\tau_0)| = |(g^q)'(y_n)||\Delta_g(\tau^{n+1}\tau_0)|$. Combining these equalities and (10.5) we get

$$\frac{|(g^q)'(\phi(z))|}{|(f^q)'(z)|} = \lim_{n\to\infty} \frac{|(g^q)'(y_n)|}{|(f^q)'(x_n)|} = 1.$$

Applying now Theorem 8.1 completes the proof. ♣

References

[ADU] J. Aaronson, M. Denker, M. Urbański, Ergodic theory for Markov fibered systems and parabolic rational maps, Transactions A.M.S. 337 (1993), 495-548.

[Ba] K. Barański, Hausdorff dimension and measures on Julia sets of some meromorphic functions, Preprint 1994.

[Be] T. Bedford, Applications of dynamical systems theory to fractals - a study of cookie-cutter Cantor sets, Preprint 1989.

[BK] M. Brin, A. Katok, On local entropy, in Geometric Dynamics, Lect. Notes in Math. 1007, (1983), 30-38, Springer-Verlag.

[DU1] M. Denker, M. Urbański, Hausdorff and conformal measures on Julia sets with a rationally indifferent periodic point, J. London Math. Soc. 43 (1991), 107-118.

[DU2] M. Denker, M. Urbański, On absolutely continuous invariant measures for expansive rational maps with rationally indifferent periodic points, Forum Math. 3(1991), 561-579.

[DU3] M. Denker, M. Urbański, Geometric measures for parabolic rational maps, Ergod. Th. and Dynam. Sys. 12 (1992), 53-66.

[DU4] M. Denker, M. Urbański, The Capacity of parabolic Julia sets, Math. Zeitsch. 211 (1992), 73-86.

[DU5] M. Denker, M. Urbański, On Sullivan's conformal measures for rational maps of the Riemann sphere, Nonlinearity 4 (1991), 365-384.

[DU6] M. Denker, M. Urbański, On the existence of conformal measures, Trans. A.M.S. 328 (1991), 563-587.

[Fa] K. J. Falconer, The geometry of fractal sets, Cambridge University Press, Cambridge, 1985.

[Gu] M. Guzmán, Differentiation of integrals in $\mathbb{R}^n$. Lect. Notes in Math. 481, Springer Verlag, New York.

[HR] F. Hofbauer, P. Raith, The Hausdorff dimension of an ergodic invariant measure for piecewise monotonic maps of the interval, Canad. Math. Bull. 35 (1992) 84-98.

[Le] F. Ledrappier: Quelques propriétés ergodiques des applications rationelles. C.R. Acad. Paris Sér. I Math. 299 (1984), 37-40.

[LS] R. de la Llave, R. Philip Schafer, Rigidity properties of one dimensional expanding maps, Preprint 1994.

[Ma] P. Mattila, Geometry of sets and measures in euclidean spaces, to appear.

[MU] D. Mauldin, M. Urbański, Dimensions and measures in infinite iterated function systems, to appear Proc. London Math. Soc.

[Pa1] S. J. Patterson, The limit set of a Fuchsian group, Acta Math. 136, (1976), 241-273.

[Pa2] S. J. Patterson, Lectures on measures on limit sets of Kleinian groups, in "Analytical and geometric aspects of hyperbolic space, LMS Lect. Notes Series 111, Cambridge University Press, 1989.

[Po] Ch. Pommerenke, Boundary behaviour of conformal maps, Springer-Verlag, Berlin, Heidelberg, 1992.

[Pr1] F. Przytycki, Iterations of holomorphic Collet-Eckmann maps: conformal and invariant measures, Preprint 1995.

[Pr2] F. Przytycki, $C^{1+\varepsilon}$ Cantor repellors in the line, scaling functions, an application to Feigenbaum's universality, Preprint 1991.

[Pr3] F. Przytycki, Sullivan's classification of conformal expanding repellors, Preprint 1991.

[PT] F. Przytycki, F. Tangerman, Cantor sets in the line: Scaling functions of the shift map, Preprint 1992.

[PU] F. Przytycki, M. Urbański, Fractals in the complex plane - ergodic theory methods, to appear.

[Ru] D. Ruelle, Thermodynamic formalism, Addison Wesley, 1978.

[Sc] F. Schweiger, Number theoretical endomorphisms with σ–finite invariant measures. Isr. J. Math. 21, (1975), 308-318.

[Su1] D. Sullivan, Quasiconformal homeomorphisms in dynamics, topology, and geometry, Proc. International Congress of Mathematicians, A.M.S. (1986), 1216-1228.

[Su2] D. Sullivan, Differentiable structures on Fractal-like sets, Determined by Intrinsic Scaling functions on Dual Cantor sets, The Mathematical Heritage of Herman Weyl, A.S Proc. Symp. Pure Math. 48 (1988).

[Su3] D. Sullivan, The density at infinity of a discrete group of hyperbolic motions, Publ. IHES, 50 (1979), 171-121.

[Su4] D. Sullivan, Conformal dynamical systems. In: Geometric Dynamics. Lect. Notes in Math. 1007 (1983), 725 – 752, Springer Verlag.

[Su5] D. Sullivan, Entropy, Hausdorff measures old and new, and limit sets of geometrically Kleinian groups, Acta Math. 153 (1984), 259 - 277.

[Th] M. Thaler: Estimates of the invariant densities of endomorphisms with indifferent fixed points. Isr. J. Math. 37 (1980), 303-314.

[TT] S.J. Taylor, C. Tricot, Packing measure, and its evaluation for a Brownian path, Trans. A.M.S. 288 (1985), 679-699.

[U1] M. Urbański, On Hausdorff dimension of Julia set with a rationally indifferent periodic point, Studia Math. 97 (1991), 167-188.

[U2] M. Urbański, Rational functions with no recurrent critical points, to appear Ergod. Th. & Dynam. Sys. 14 (1994), 391-414.

[Wa] P. Walters, A variational principle for the pressure of continuous transformations, Amer. J. Math. 97 (1975), 937 - 971.