COUNTABLE ALPHABET NON-AUTNOMOUS SELF-AFFINE SETS

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Abstract. We extend Falconer's formula from [1] by identifying the Hausdorff dimension of the limit sets of almost all contracting affine iterated function systems to the case of an infinite alphabet, non-autonomous choice of iterating matrices, and time dependent random choice of translations.

1. Introduction

In the seminal paper [1], given k contracting matrices $A_1, A_2, \ldots, A_k$, Ken Falconer has provided a close formula which gives the Hausdorff dimension of the limit sets of the iterated function system

$$\mathcal{S}_a = \{\mathbb{R}^q \ni x \mapsto A_i x + a_i\}_{i=1}^k$$

for Lebesgue almost every vector $a = (a_i)_{i=1}^k \in \mathbb{R}^{qk}$. In our article we extend Falconer's result in several directions.

- We allow k to be infinite; instead of Lebesgue measure we then consider appropriately defined product measure with infinitely many factors.
- Being in the iterating process we allow all the matrices A_i to depend on the time, i.e. making a new composition at a step n, we take the contracting matrices from an entirely new collection $A_1^{(n)}, \ldots, A_k^{(n)}$.
- We choose the vectors $(a_i)_{i=1}^k$ randomly according to some random process.

Roughly speaking we have either a finite or countable infinite alphabet E, the system $\mathcal{S}_a$ consists now of maps

$$\phi_e^{(n,a)}(x) = A_e^{(n)} x + a_e, \quad e \in E,$$

we have also a measurable transformation $\theta : X \to X$ preserving some Borel probability measure X, and smooth transformations $S_x : G^E \to G^E$ ($G \subset \mathbb{R}^q$), $x \in X$, with some additional technical properties. Each point $x \in X$ generates a non-autonomous iterative scheme

$$\phi_{\omega_1}^{(1,a)} \circ \phi_{\omega_2}^{(2,S_x(a))} \circ \ldots \circ \phi_{\omega_n}^{(n,S_x^n(a))} \circ \ldots, \quad \omega \in E^{\mathbb{N}},$$

where

$$S_x^n := S_x \circ S_{\theta(x)} \circ \ldots \circ S_{\theta^{n-1}(x)}.$$

This determines (see (2.2) for a rigorous definition) the limit set $J_{(x,a)}$, and our main result identifies the Hausdorff dimension of $J_{(x,a)}$ for m–a.e. $x \in X$ and

"Lebesgue"–a.e. $a \in G^E$. We do this by introducing the Falconer dimension $\mathrm{FD}(\mathcal{S})$, which depends only on matrices $A_e^{(n)}$, $e \in E$, $n \in \mathbb{N}$, and is independent of the maps $S_x : G^E \to G^E$. We prove the following main result

Theorem 1.1. *If $\mathcal{S}$ is an affine scheme on $\mathbb{R}^q$, then*

$$\mathrm{HD}\big(J_{(x,a)}\big) = \min\{q, \mathrm{FD}(\mathcal{S})\}.$$

for m-a.e. $x \in X$ and λ_G^E-a.e. $a \in G^E$.

which is Theorem 5.3 from the last section of our paper. We would like to add that another extension of Falconer's result, incorporating a different randomizing procedure, was treated in [3].

2. Affine Schemes

Fix E, a countable set, either finite or infinite; it will be called an alphabet in the sequel. Fix an integer $q \geq 1$ and two real numbers $\kappa, \xi \in (0,1)$. For every $n \geq 1$ and every $e \in E$ let $A_e^{(n)} : \mathbb{R}^q \to \mathbb{R}^q$ be an invertible linear map with

$$\big\|A_e^{(n)}\big\| \leq \kappa \quad \text{and} \quad \Big\|\big(A_e^{(n)}\big)^{-1}\Big\| \leq \xi^{-1}. \tag{2.1}$$

Let $G \subset \mathbb{R}^q$ be a bounded Borel subset of $\mathbb{R}^q$ with positive Lebesgue measure. Let λ_G be the normalized (so that $\lambda_G(G) = 1$) q-dimensional Lebesgue measure on G and let λ_G^E be the corresponding infinite product measure on G^E. This measure is uniquely determined by the requirement that

$$\lambda_G^E\left(\prod_{a\in\Gamma} F_a \times G^{E\setminus F}\right) = \prod_{a\in\Gamma} \lambda_G(F_a)$$

for every finite subset Γ of E and all Borel sets $F_a \subset G$, $a \in \Gamma$. Denote by R_G the largest radius $r > 0$ such that $G \subset B(0,r)$. For every $n \geq 1$, every $e \in E$, and every $a \in G^E$ consider the maps $\phi_e^{(n,a)} : \mathbb{R}^q \to \mathbb{R}^q$ given by respective formulas

$$\phi_e^{(n,a)}(x) = A_e^{(n)}x + a_e.$$

Since all the maps $A_e^{(n)}$ are uniform linear contractions and since the set G is bounded, there exists B, a sufficiently large closed ball in $\mathbb{R}^q$ centered at the origin such that

$$\phi_e^{(n,a)}(B) \subset B$$

for all $n \geq 1$, all $e \in E$, and all $a \in G^E$. Let $l_E^\infty(\mathbb{R}^q)$ be the Banach space of all bounded functions from E to $\mathbb{R}^q$, endowed with the supremum norm, i.e.

$$||a||_\infty = \sup\{||a_e|| : e \in E\}.$$

Of course, G^E is a subset of $l_E^\infty(\mathbb{R}^q)$. Let $(X, \mathcal{F}, m)$ be a probability space and let $\theta : X \to X$ be an invertible measurable map preserving the measure m. For every

$x \in X$ let $S_x : G^E \to G^E$ be a map for which there exists a bounded convex open set $\hat{G} \subset \mathbb{R}^q$ with the following properties.

(p1) $G \subset \hat{G}$ and $\operatorname{dist}(G, \mathbb{R}^q \setminus \hat{G}) > 0$; then $G^E \subset \operatorname{Int}_{l_E(\mathbb{R}^q)}\big(\hat{G}^E\big)$.
(p2) There exists a continuous map $\hat{S}_x : \hat{G}^E \to \hat{G}^E$ such that
(p3) $\hat{S}_x$ is differentiable throughout $\operatorname{Int}_{l_E(\mathbb{R}^q)}\big(\hat{G}^E\big)$.
(p4)
$$||DS_x||_\infty := \sup\{||D_a\hat{S}_x|| : a \in G^E\} < \infty$$
and
$$\beta := \operatorname{ess\,sup}\{||DS_x||_\infty : x \in X\} < \infty$$
is so small that
$$\kappa\beta < 1/3.$$
(p5) For m-a.e. $x \in X$ there exists a Borel probability measure μ_x on G^E equivalent (with bounded Radon-Nikodym derivatives) to λ_G^E such that
$$\mu_{\theta(x)} = \mu_x \circ S_x^{-1}.$$

Note that if the space X is a singleton, then we are talking about one mapping $S : G^E \to G^E$ (and its extension $\tilde{S} : \tilde{G}^E \to \tilde{G}^E$ preserving a Borel probability measure μ on G^E equivalent (with bounded Radon-Nikodym derivatives) to λ_G^E. This of course comprises the case of S being the identity map on G^E. This case is referred to as translation deterministic. $S = \operatorname{Id}_{G^E}$ was a part of Falconer's set up in [1]. He was also assuming that the alphabet E is finite and the linear contractions $A_e^{(n)} : \mathbb{R}^q \to \mathbb{R}^q$ are independent of n. We do not assume any of these. Now, the collection of maps
$$\big\{\phi_e^{(n,a)} : \mathbb{R}^q \to \mathbb{R}^q : n \geq 1,\, a \in G^E,\, e \in E\big\}$$
along with the map $\theta : X \to X$ and described above maps $S_x : G^E \to G^E$, $x \in X$, are referred to as an affine scheme $\mathcal{S}$. We classify affine schemes as follows.

(1) Autonomous if the affine contractions $A_e^{(n)} : \mathbb{R}^q \to \mathbb{R}^q$ are independent of n.
(1') Finitely autonomous if $\mathcal{S}$ is autonomous and the alphabet E is finite.
(2) Non-autonomous if $\mathcal{S}$ is not autonomous.
(3) Of dynamically deterministic type if the maps $S_x : G^E \to G^E$, $x \in X$, are independent of $x \in X$. Then the action $\theta : X \to X$ is irrelevant, and we may assume without loss of generality that X is a singleton.
(4) Deterministic if $\mathcal{S}$ is of dynamically deterministic type and $S : G^E \to G^E$ is the identity map on G^E.
(5) Of dynamically random type if $\mathcal{S}$ is not of dynamically deterministic type, meaning that $S_x : G^E \to G^E$ do depend on $x \in X$.

(6) A Falconer scheme if $\mathcal{S}$ is finitely autonomous and of dynamically deterministic type.

From now on $\mathcal{S}$ is an arbitrary affine scheme. As in the introduction, for every integer $k \geq 1$ and every $x \in X$ let

$$S_x^k := S_x \circ S_{\theta(x)} \circ \ldots \circ S_{\theta^{k-1}(x)}.$$

Given $n \geq 1$, $\omega \in E^n$, and $a \in G^E$, we define the maps

$$A_\omega := A_{\omega_1}^{(1)} \circ A_{\omega_2}^{(2)} \circ \ldots \circ A_{\omega_n}^{(n)} : R^q \to \mathbb{R}^q$$

and

$$\phi_\omega^{(x,a)} := \phi_{\omega_1}^{(1,a)} \circ \phi_{\omega_2}^{(2,S_x(a))} \circ \ldots \circ \phi_{\omega_n}^{(n,S_xn-1(a))} : B \to B.$$

Note that A_ω is the linear part of the affine map $\phi_\omega^{(x,a)}$. For every infinite word $\omega \in E^{\mathbb{N}}$ and every integer $n \geq 1$ we put

$$\omega|_n := \omega_1\omega_2 \ldots \omega_n.$$

Then $\big(\phi_{\omega|_n}^{(x,a)}(B)\big)_{n=1}^\infty$ is a descending sequence of non-empty compact subsets of B and

$$\operatorname{diam}\big(\phi_{\omega|_n}^{(x,a)}(B)\big) \leq \operatorname{diam}(B)\kappa^n.$$

So, the intersection

$$\bigcap_{n=1}^\infty \phi_{\omega|_n}^{(x,a)}(B)$$

is a singleton, and we denote its only element by $\pi_{(x,a)}(\omega)$. So, for every $x \in X$ and every $a \in G^E$ we have defined the projection map

$$\pi_{(x,a)} : E^{\mathbb{N}} \to B.$$

Slightly more generally, given any integer $k \geq 1$, we consider the maps

$$\phi_\omega^{(x,a;k)} := \phi_{\omega_1}^{(k,a)} \circ \phi_{\omega_2}^{(k+1,S_x(a))} \circ \ldots \circ \phi_{\omega_n}^{(k+n-1,S_x^{n-1}(a))} : B \to B.$$

and the corresponding projections

$$\pi_{(x,a)}^k : E^{\mathbb{N}} \to B.$$

In particular,

$$\pi_{(x,a)}^1 = \pi_{(x,a)}.$$

The set

$$J_{(x,a)} := \pi_{(x,a)}(E^{\mathbb{N}}) \subset B \subset \mathbb{R}^q \tag{2.2}$$

is called the limit set (or the attractor) of the affine scheme $\mathcal{S}$ at the point (x,a). Our goal is to determine the Hausdorff dimensions of these limit sets. Indeed, we will show that these dimensions are equal for m-almost all $x \in X$ and λ_G^E-almost all $a \in G^E$, and the resulting common value is directly expressible in terms of the sequence alone $\big(\{A_e^{(n)} : e \in E\}\big)_{n=1}^{\infty}$.

3. The Singular Value Function

let $A : \mathbb{R}^q \to \mathbb{R}^q$ be an invertible linear contraction and let

$$1 > \alpha_1 \geq \alpha_1 \geq \ldots \geq \alpha_q > 0$$

be the square roots of (necessarily positive) eigenvalues of the self-adjoint map $A^*A : \mathbb{R}^q \to \mathbb{R}^q$. Geometrically, the numbers $\alpha_1, \ldots \alpha_q$ are the lengths of the (mutually perpendicular) principle semi-axes of $A(\overline{B}(0,1))$, where $\overline{B}(0,1)$ is the closed ball in R^q centered at 0 and of radius 1. These numbers are called singular values of the map $A : \mathbb{R}^q \to \mathbb{R}^q$. Following Falconer ([1]) we define

$$\alpha^t(A) := \alpha_1\alpha_2 \ldots \alpha_{k-1}\alpha_k^{t-(k-1)}$$

if $0 \leq t \leq q$, where k is the least integer greater than or equal to s, i.e. $k-1 < t \leq k$, and

$$\alpha^t(A) := (\alpha_1\alpha_2 \ldots \alpha_{k-1}\alpha_q)^{t/q}$$

if $t > q$. Denote by $L_*(\mathbb{R}^q)$ the set of all invertible linear contractions from $\mathbb{R}^q$ onto itself. Note that $L_*(\mathbb{R}^q)$ is closed under the compositions of maps. We quote from [1] the following two lemmas.

Lemma 3.1. *For each $t \geq 0$ the function $\alpha_t : L_*(\mathbb{R}^q) \to L_*(\mathbb{R}^q)$ is submultiplicative, meaning that*

$$\alpha^t(AC) \leq \alpha^t(A)\alpha^t(C)$$

for all $A, C \in L_*(\mathbb{R}^q)$.

and

Lemma 3.2. *Given a non-integral real number $0 < t < q$ and a real number $R > 0$ there exists a constant $c < +\infty$ (depending on all of then q, t, and R) such that*

$$\int_{\overline{B}(0,1)} \frac{d\,\lambda_q(x)}{||Ax||^t} \leq \frac{c}{\alpha^t(A)}$$

for all $A \in L_*(\mathbb{R}^q)$, where λ_q denotes q-dimensional Lebesgue measure on $\mathbb{R}^q$.

4. Falconer Dimension

Let $\mathcal{S}$ be an affine scheme. Fix $t \geq 0$. Define the metric $\rho_F^{(t)}$ on $E^{\mathbb{N}}$ as follows.

$$\rho_F^{(t)}(\omega, \tau) := \begin{cases} \alpha^t\big(A_{\omega\wedge\tau}\big) & \text{if } \omega \neq \tau \\ 0 & \text{if } \omega = \tau. \end{cases}$$

To check that $\rho_F^{(t)}$ is a metric indeed only triangle inequality requires an argument. For this take also $\gamma \in E^{\mathbb{N}}$. Then $|\omega\wedge\tau| \geq \min\{|\omega\wedge\gamma|, |\tau\wedge\gamma|\}$. Assume without loss of generality that $\omega\wedge\tau \geq |\omega\wedge\gamma|$. Then $\omega\wedge\tau = (\omega\wedge\gamma)\theta$ with some $\theta \in E^*$, say $\theta \in E^k$. Denote $n := |\omega\wedge\tau|$. We then have,

$$\begin{aligned}\rho_F^{(t)}(\omega, \tau) = \alpha^t\big(A_{\omega\wedge\tau}\big) = \alpha^t\big(A_{(\omega\wedge\gamma)\theta}\big) &\leq \alpha^t\big(A_{\omega\wedge\gamma}\big)\alpha^t\big(A_{\theta_1}^{(n+1)}A_{\theta_2}^{(n+1)}\dots A_{\theta_k}^{(n+1)}\big) \\ &\leq \alpha^t\big(A_{\omega\wedge\gamma}\big) = \rho_F^{(t)}(\omega,\gamma) \\ &\leq \max\{\rho_F^{(t)}(\omega,\gamma), \rho_F^{(t)}(\gamma,\tau)\}.\end{aligned}$$

So, $\rho_F^{(t)}$ is a metric indeed, in fact we have proved the following.

Proposition 4.1. *For every $t \geq 0$, $\rho_F^{(t)}$ is an ultra-metric on $E^{\mathbb{N}}$.*

Let H_F^t be the 1-dimensional Hausdorff measure on $E^{\mathbb{N}}$ generated by the metric $\rho_F^{(t)}$. Of course if $s < t$ and $\mathrm{H}_F^s(E^{\mathbb{N}}) < +\infty$, then $\mathrm{H}_F^t(E^{\mathbb{N}}) = 0$. Therefore,

$$\inf\{t \geq 0 : \mathrm{H}_F^t(E^{\mathbb{N}}) = 0\} = \sup\{t \geq 0 : \mathrm{H}_F^t(E^{\mathbb{N}}) = +\infty\}.$$

Call this common number the Falconer dimension of the scheme $\mathcal{S}$ and denote it by $\mathrm{FD}(\mathcal{S})$. Note that it in fact depends only on the sequence $\big(\{A_e^{(n)} : e \in E\}\big)_{n=1}^\infty$ and is entirely independent of the vectors a_e, $e \in E$, or the maps $S_x : G^E \to G^E$.

We now define an auxiliary dimension $\mathrm{FD}_*(\mathcal{S})$. For every $l \geq 1$ and every set $\Gamma \subset E^{\mathbb{N}}$ define

$$\mathrm{F}_l^t(\Gamma) := \inf\left\{\sum_{\omega\in\mathcal{A}_l} \alpha^t(A_\omega)\right\},$$

where the infimum is taken over the family $\mathcal{A}_l$ of all countable covers of Γ by cylinders $[\omega]$ of length $\geq l$. The sequence $\big(\mathrm{F}_l^t(\Gamma)\big)_{l=1}^\infty$ is monotone increasing, and therefore the following limit

$$\mathrm{F}^t(\Gamma) = \lim_{l\to\infty} F_l^t(\Gamma)$$

exists and is equal to

$$\sup\{\mathrm{F}_l^t(\Gamma) : l \geq 1\}.$$

Note that if $s < t$ and $\mathrm{F}^s(E^{\mathbb{N}}) < +\infty$, then $\mathrm{F}^t(E^{\mathbb{N}}) = 0$. Therefore,

$$\inf\{t \geq 0 : \mathrm{F}^t(E^{\mathbb{N}}) = 0\} = \sup\{t \geq 0 : \mathrm{F}^t(E^{\mathbb{N}}) = +\infty\}$$

Denote this common number by $\mathrm{FD}_*(\mathcal{S})$. Note that as in the case of $\mathrm{FD}(\mathcal{S})$ it in fact depends only on the sequence $\big(\{A_e^{(n)} : e \in E\}\big)_{n=1}^{\infty}$ and is entirely independent of the vectors a_e, $e \in E$, or the maps $S_x : G^E \to G^E$. We shall prove the following.

Proposition 4.2. *If $\mathcal{S}$ an affine scheme and*

$$\varliminf_{e\to\infty} \left\|A_e^{(n)}\right\| > 0$$

for all $n \geq 1$, then

$$\mathrm{FD}_*(\mathcal{S}) = \mathrm{FD}(\mathcal{S}).$$

Proof. Obviously,

$$\mathrm{FD}(E^{\mathbb{N}}) \leq \mathrm{FD}_*(E^{\mathbb{N}}).$$

In order to prove the opposite inequality fix $\delta > 0$ and consider $\mathcal{A}$, an arbitrary cover of $E^{\mathbb{N}}$ by sets of diameters (with respect to $\rho_F^{(t)}$) $\leq \delta$. For every $\Gamma \in \mathcal{A}$ let $\omega_\gamma \in E_A^*$ be a longest word such that

$$\Gamma \subset [\omega_\Gamma].$$

Then of course

$$\mathrm{diam}_{\rho_F^{(t)}}(\Gamma) \leq \mathrm{diam}_{\rho_F^{(t)}}\big([\omega_\Gamma]\big) \tag{4.1}$$

but, more importantly for us at the moment, there exist two elements $\beta, \gamma \in \Gamma$ such that $\beta|_{|[\omega_\Gamma]|+1} \neq \gamma|_{|[\omega_\Gamma]|+1}$. As also $\beta|_{|[\omega_\Gamma]|} = \gamma|_{|[\omega_\Gamma]|}$, we thus get

$$\mathrm{diam}_{\rho_F^{(t)}}(\Gamma) \geq \rho_F^{(t)}(\beta,\gamma) = \alpha^t\big(A_{\beta\wedge\gamma}\big) = \alpha^t\big(A_{\omega_\Gamma}\big) = \mathrm{diam}_{\rho_F^{(t)}}\big([\omega_\Gamma]\big).$$

Along with (4.1) this yields

$$\mathrm{diam}_{\rho_F^{(t)}}\big([\omega_\Gamma]\big) = \mathrm{diam}_{\rho_F^{(t)}}(\Gamma) \tag{4.2}$$

Hence $\{[\omega_\Gamma]\}_{\Gamma\in\mathcal{A}}$ is also a cover of $E^{\mathbb{N}}$ by sets with diameter (with respect to the metric $\rho_F^{(t)}$) $\leq \delta$. Therefore, we are done since, by our hypothesis, $sup\{|\omega_\Gamma| : \gamma \in \mathcal{A}\}$ converges to zero if $\delta \to 0$. $\square$

As an immediate consequence of this proposition we get the following.

Corollary 4.3. *If $\mathcal{S}$ is a finitely autonomous affine scheme, then*

$$\mathrm{FD}_*(\mathcal{S}) = \mathrm{FD}(\mathcal{S}).$$

We also define

$$\underline{\mathrm{P}}_{\mathcal{S}}(t) := \varliminf_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} \alpha^t(A_\omega),$$

and call $\underline{\mathrm{P}}_{\mathcal{S}}(t)$ the lower topological pressure of the affine scheme $\mathcal{S}$ at the parameter t. Let

$$\theta_{\mathcal{S}}^{-} := \inf\{t \geq 0 : \underline{\mathrm{P}}_{\mathcal{S}}(t) < +\infty\}$$

and

$$\theta_{\mathcal{S}}^{+} := \inf\{t \geq 0 : \underline{\mathrm{P}}_{\mathcal{S}}(t) = -\infty\}.$$

Since for $0 \leq s < t$, we have $\alpha^t(A_\omega) \leq \alpha_1^{t-s}(A_\omega)\alpha^s(A_\omega) \leq \kappa^{t-s}\alpha^s(A_\omega)$, we immediately get the following.

Proposition 4.4. *If $\mathcal{S}$ is an affine scheme, then*

(a) *the function $[0,+\infty) \ni t \mapsto \underline{\mathrm{P}}_{\mathcal{S}}(t) \in [-\infty,+\infty]$ is monotone decreasing,*
(b) *the function $(\theta_{\mathcal{S}}^{-}, \theta_{\mathcal{S}}^{+}) \ni t \mapsto \underline{\mathrm{P}}_{\mathcal{S}}(t)$ is strictly decreasing.*

Proposition 4.5. *If $\mathcal{S}$ is a finitely autonomous affine scheme, then the following numbers are equal.*

(a) $\mathrm{FD}(\mathcal{S})$,
(b) $\mathrm{FD}_*(\mathcal{S})$
(c) $\inf\{t \geq 0 : \underline{\mathrm{P}}_{\mathcal{S}}(t) \leq 0\}$,
(d) $\inf\{t \geq 0 : \sum_{\omega \in E^*} \alpha^t(A_\omega) < +\infty\} = \sup\{t \geq 0 : \sum_{\omega \in E^*} \alpha^t(A_\omega) = +\infty\}$.

Proof. Because of Corollary 4.3 it suffices to prove that the numbers in (b), (c), and (d) are all equal. Indeed, if $s < t$ and $\sum_{\omega \in E^*} \alpha^s(A_\omega) < +\infty$, then $\inf\{t \geq 0 : \sum_{\omega \in E^*} \alpha^t(A_\omega) < +\infty\}$. Therefore, the equality in (d) is proved. The equality of numbers in (c) and (d) is a direct consequence of Proposition 4.4(b). Now, if $\Gamma :=: \sum_{\omega \in E^*} \alpha^t(A_\omega) < +\infty$, then for every $l \geq 1$, $\sum_{\omega \in E^l} \alpha^t(A_\omega) \leq \Gamma$, and therefore $F_l^t(E^{\mathbb{N}}) \leq \Gamma$. consequently, $F^t(E^{\mathbb{N}}) \leq \Gamma < +\infty$, and so (b)$\leq$(d). The implication (c)$\leq$(b) requires the system $\mathcal{S}$ to be finitely autonomous and is established in [1]. $\square$

The proof of the following lemma is an adaptation of the proof of Lemma 3 in [3].

Lemma 4.6. *If $\mathcal{S}$ is an affine scheme and $F^t(E^{\mathbb{N}}) = +\infty$, then there exist a finite Borel measure ν on $E^{\mathbb{N}}$ and a constant $C > 0$ such that*

$$\nu([\omega]) \leq C\alpha^t(A_\omega)$$

for all $\omega \in E^*$.

Proof. Because of Proposition 4.1 it follows from Theorem 57(c) in [4] that there exists a compact set $\Gamma \subset E^{\mathbb{N}}$ such that

$$0 < \mathrm{H}_F^t(\Gamma) < +\infty.$$

Since $\mathrm{diam}_{\rho_F^{(t)}}([\omega]) = \alpha^t(A_\omega)$, the proof is thus completed by invoking Theorem 8.17 in [Ma]. □

5. Main Theorem; The Proof

The proof of our main theorem, Theorem 1.1 will consist of several lemmas. We start with the following.

Lemma 5.1. *let $\mathcal{S}$ be an affine scheme acting on $\mathbb{R}^q$. Let $0 < t < q$ be a non-integral number. Then there exists a constant $C \in (0, +\infty)$ such that*

$$\int_X \int_{G^E} \frac{d\lambda_G^E(a)\, dm(x)}{||\pi_{(x,a)}(\omega) - \pi_{(x,a)}(\tau)||^t} \leq \frac{C}{\alpha^t\big(A_{\omega\wedge\tau}\big)}$$

for all $\omega, \tau \in E^{\mathbb{N}}$ with $\omega \neq \tau$.

Proof. Let

$$\rho := \omega \wedge \tau$$

and let $k := |\omega \wedge \tau| < +\infty$. Let $\omega' := \sigma^k(\omega)$ and $\tau' := \sigma^k(\tau)$. Then

$$\begin{aligned}
I(\omega,\tau) &:= \int_X \int_{G^E} \frac{d\mu_x(a)\, dm(x)}{||\pi_{(x,a)}(\omega) - \pi_{(x,a)}(\tau)||^t} \\
&= \int_X \int_{G^E} \frac{d\mu_x(a)\, dm(x)}{\Big\|\phi_\rho^{(a)}\big(\pi^{(k+1)}_{\theta^k(x),S_x^k(a)}(\omega')\big) - \phi_\rho^{(a)}\big(\pi^{(k+1)}_{\theta^k(x),S_x^k(a)}(\tau')\big)\Big\|^t} \\
&= \int_X \int_{G^E} \frac{d\mu_{\theta^{-k}(x)}(a)\, dm(x)}{\Big\|\phi_\rho^{(S_x^{-k}(a))}\big(\pi^{(k+1)}_{(x,a)}(\omega')\big) - \phi_\rho^{(S_x^{-k}(a))}\big(\pi^{(k+1)}_{(x,a)}(\tau')\big)\Big\|^t} \qquad (5.1)\\
&= \int_X \int_{G^E} \frac{d\mu_{\theta^{-k}(x)}(a)\, dm(x)}{\Big\|A_\rho\big(\big(\pi^{(k+1)}_{(x,a)}(\omega')\big) - \pi^{(k+1)}_{(x,a)}(\tau')\big)\Big\|^t} \\
&\asymp \int_X \int_{G^E} \frac{d\mu_x(a)\, dm(x)}{\Big\|A_\rho\big(\big(\pi^{(k+1)}_{(x,a)}(\omega')\big) - \pi^{(k+1)}_{(x,a)}(\tau')\big)\Big\|^t}.
\end{aligned}$$

Now,

$$\pi^{(k+1)}_{(x,a)}(\omega')\big) - \pi^{(k+1)}_{(x,a)}(\tau') = a_{\omega_1'} - a_{\tau_1'} + F(a),$$

where $F : G^E \to \mathbb{R}^q$ is given by the formula:

$$F(a) := \sum_{j=1}^{\infty} A_{\omega'|_{j^{(k+1)}}}\big((S_x^j(a))_{\omega'_{j+1}}\big) - \sum_{j=1}^{\infty} A_{\tau'|_{j^{(k+1)}}}\big((S_x^j(a))_{\tau'_{j+1}}\big). \qquad (5.2)$$

Now consider the product measure

$$\ell_{\omega_1'} := \lambda_q \otimes \prod_{e\in E\setminus\{\omega_1'\}} = \lambda_q \otimes \lambda_G^{E\setminus\{\omega_1'\}}$$

on $\mathbb{R}^q$, where, we recall, λ_q is the q-dimensional Lebesgue measure on $\mathbb{R}^q$. Let $H : \hat{G}^E \to \mathbb{R}^q \times G^{E\setminus\{\omega_1'\}}$ be given by the following formula:

$$H(a)_j := \begin{cases} a_{\omega_1'} - a_{\tau_1'} + F(a) & \text{if } j = \omega_1' \\ a_j & \text{if } j \neq \omega_1'. \end{cases} \tag{5.3}$$

We shall prove the following.

Claim 1: The map $H : \hat{G}^E \to \mathbb{R}^q \times G^{E\setminus\{\omega_1'\}}$ is injective.

Proof. Suppose that $H(a') = H(a)$. Then immediately $a_e' = a_e$ for all $e \in E\setminus\{\omega_1'\}$. Since $\tau_1' \neq \omega_1'$, this entails $a_{\tau_1'}' = a_{\tau_1'}$. So,

$$F(a') - F(a) = a_{\omega_1'} - a_{\omega_1'}'.$$

It then follows from (5.2), (p4), linearity of the maps $A_{\omega'|j^{(k+1)}}$ and $A_{\tau'|j^{(k+1)}}$, and Q-quasi-convexity of $\hat{G}$, that

$$\begin{aligned}
&||a' - a||_\infty = \\
&= ||a_{\omega_1'}' - a_{\omega_1'}|| = ||F(a') - F(a)|| \\
&= \Big\| \sum_{j=1}^\infty A_{\omega'|j^{(k+1)}}\big((S_x^j(a'))_{\omega_{j+1}'}\big) - (S_x^j(a))_{\omega_{j+1}'}\big) - \\
&\qquad - \sum_{j=1}^\infty A_{\tau'|j^{(k+1)}}\big((S_x^j(a'))_{\tau_{j+1}'}\big) - (S_x^j(a))_{\tau_{j+1}'}\big)\Big\| \\
&\le \sum_{j=1}^\infty \kappa^j ||(S_x^j(a'))_{\omega_{j+1}'}) - S_x^j(a))_{\omega_{j+1}'}|| + \sum_{j=1}^\infty \kappa^j ||(S_x^j(a'))_{\tau_{j+1}'}) - S_x^j(a))_{\tau_{j+1}'}|| \\
&\le 2\sum_{j=1}^\infty \kappa^j ||S_x^j(a') - S_x^j(a)||_\infty \\
&\le 2\sum_{j=1}^\infty \kappa^j \beta ||a' - a||_\infty \\
&= 2Q\kappa\beta(1-\kappa\beta)^{-1}||a' - a||_\infty \\
&< ||a' - a||_\infty,
\end{aligned} \tag{5.4}$$

where the last equality followed from the assumption (see (p4)) that $\kappa\beta < 1/3$. This contradiction finishes the proof of Claim 1. □

In the same vein let us prove now the existence and estimate the norm of of the partial derivative $D_{\omega_1'}F(a)$ at every point $a \in G^E$. Indeed, it again follows from

(5.2), (p4), and linearity of both $A_{\omega'|j^{(k+1)}}$ and $A_{\tau'|j^{(k+1)}}$, that

(5.5)

$$\begin{aligned}
||D_{\omega_1'}F(a)|| &= \\
&= \Big\| \sum_{j=1}^{\infty} A_{\omega'|j^{(k+1)}} \circ D_{\omega_1'}(p_{\omega_{j+1}'} \circ S_x^j)(a) - \sum_{j=1}^{\infty} A_{\omega'|j^{(k+1)}} \circ D_{\omega_1'}(p_{\tau_{j+1}'} \circ S_x^j)(a) \Big\| \\
&\le 2\sum_{j=1}^{\infty} \kappa^j ||DS_x^j||_\infty \le 2\sum_{j=1}^{\infty}(\kappa\beta)^j \\
&= \frac{2\kappa\beta}{1-\kappa\beta},
\end{aligned}$$

i. e. $D_{\omega_1'}F(a)$ exists and (5.5) holds. So, because of the special form (5.3), we now conclude that the map $H : \hat{G}^E \to \mathbb{R}^q \times G^{E\setminus\{\omega_1'\}}$ is non-singular with respect to the measure $\ell_{\omega_1'}$, and its Jacobian is given by the formula

$$J_H^*(a) = |\det\big(\mathrm{Id}_{\mathbb{R}^q} + D_{\omega_1'}F(a)\big)| \ge (1 - ||D_{\omega_1'}F(a)||)^q \ge \left(1 - \frac{2\kappa\beta}{1-\kappa\beta}\right)^q .$$

So, if we consider the measure $\ell_{\omega_1'}$ on $H(G^E)$ but the measure λ_G^E on G^E, then $J_{H^{-1}}(a)$, the corresponding Jacobian of the map $H^{-1} : H(G^E) \to G^E$, is

$$J_{H^{-1}}(b) = \frac{1}{\lambda_q(G)} J_{H^{-1}}^*(b) \le \gamma := \left(\lambda_q(G)\left(1 - \frac{2\kappa\beta}{1-\kappa\beta}\right)^q\right)^{-1}$$

for all $b \in H(G^E)$. Therefore, we can single out the inner integral in (5.1) to get

$$\begin{aligned}
I_x(\omega,\tau) &:= \int_{H^{-1}(H(G^E))} \frac{d\lambda_G^E(a)}{\Big\| A_\rho\big(\big(\pi_{(x,a)}^{(k+1)}(\omega')\big) - \pi_{(x,a)}^{(k+1)}(\tau')\big)\Big\|^t} \\
&= \int_{H(G^E)} \frac{J_{H^{-1}}(b)}{||A_\rho((b)_{\omega_1'})||^t}\, d\ell_{\omega_1'}(b) \\
&\le \gamma \int_{H(G^E)} \frac{d\ell_{\omega_1'}(b)}{||A_\rho((b)_{\omega_1'})||^t} \\
&= \gamma \int_{p_*(H(G^E))} \int_{p_{\omega_1'}(p_*^{-1}(b))} \frac{d\lambda_q(y)}{||A_\rho(y)||^t}\, d\lambda_G^{E\setminus\{\omega_1'\}}(b),
\end{aligned}$$

where $p_* : (\mathbb{R}^q)^E \to (\mathbb{R}^q)^{E\setminus\{\omega_1'\}}$ is the canonical projection onto $(\mathbb{R}^q)^{E\setminus\{\omega_1'\}}$, i. e. $p_*((b_e)_{e\in E} = ((b_e)_{e\in E\setminus\{\omega_1'\}}$, and, we recall, $p_{\omega_1'} : (\mathbb{R}^q)^E \to \mathbb{R}^q$ is the canonical projection onto ω_1'th coordinate. Now, if $a \in G^E$, then

$$\begin{aligned}
||(H(a))_{\omega_1'}|| &= ||a_{\omega_1'} - a_{\tau_1'} + F(a)|| \le ||a_{\omega_1'}|| + ||a_{\tau_1'}|| + ||F(a)|| \\
&\le 2R_G + R_G\kappa(1-\kappa)^{-1} \\
&= (2 + \kappa(1-\kappa)^{-1})R_G,
\end{aligned}$$

where the estimate $||F(a)|| \le R_G\kappa(1-\kappa)^{-1}$ is a simplification of the calculation from (5.4). Therefore, for every $b \in p_*(H(G^E))$, we have that $p_{\omega_1'}(p_*^{-1}(b)) \subset$

$B(0,(2+\kappa(1-\kappa)^{-1})R_G)$. So, by virtue of Lemma 3.2, there exists a constant $C>0$ such that

$$
\begin{aligned}
I_x(\omega,\tau) &\le \gamma \int_{p_*(H(G^E))}\int_{\overline{B}(0,(2+\kappa(1-\kappa)^{-1})R_G)} \frac{d\lambda_q(y)}{||A_\rho(y)||^t}\, d\lambda_G^{E\setminus\{\omega_1'\}}(b) \\
&\le \frac{C}{\alpha^t(A_\rho)}\int_{p_*(H(G^E))} d\lambda_G^{E\setminus\{\omega_1'\}} \\
&= \frac{C}{\alpha^t(A_\rho)}\lambda_G^{E\setminus\{\omega_1'\}}\big(p_*(H(G^E))\big) \\
&\le \frac{C}{\alpha^t(A_\rho)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_X\int_{G^E}\frac{d\lambda_G^E(a)\,dm(x)}{||\pi_{(x,a)}(\omega)-\pi_{(x,a)}(\tau)||^t} &\asymp \int_X\int_{G^E}\frac{d\mu_x(a)\,dm(x)}{||\pi_{(x,a)}(\omega)-\pi_{(x,a)}(\tau)||^t} \\
&\asymp \int_X I_x(\omega,\tau)\,dm(x) \\
&\le \frac{C}{\alpha^t\big(A_{\omega\wedge\tau}\big)}\int_X dm(x) \\
&= \int_X I_x(\omega,\tau)\,dm(x).
\end{aligned}
$$

The proof of our lemma is complete. ∎

the proof of the following proposition goes, with almost no changes, as the proof of Proposition 5.1 in [1]

Proposition 5.2. *If $\mathcal{S}$ is an affine scheme and $\mathrm{H}_F^t(E^{\mathbb{N}})<+\infty$, then $\mathrm{H}^t(J_{(x,a)})<+\infty$ for all $x\in X$ and all $a\in G^E$.*

Proof. Begin in the same way as in the proof of Proposition 4.2. Fix $\delta>0$ and consider $\mathcal{A}$, an arbitrary cover of $E^{\mathbb{N}}$ by sets of diameters (with respect to the metric $\rho_F^{(t)}$) $\le\delta$. For every $\Gamma\in\mathcal{A}$ let $\omega_\gamma\in E_A^*$ be a longest word such that

$$\Gamma\subset[\omega_\Gamma].$$

Then of course

$$\operatorname{diam}_{\rho_F^{(t)}}(\Gamma)\le \operatorname{diam}_{\rho_F^{(t)}}\big([\omega_\Gamma]\big) \tag{5.6}$$

but, more importantly for us at the moment, there exist two elements $\beta,\gamma\in\Gamma$ such that $\beta|_{||[\omega_\Gamma]|+1}\ne\gamma|_{||[\omega_\Gamma]|+1}$. As also $\beta|_{||[\omega_\Gamma]|}=\gamma|_{||[\omega_\Gamma]|}$, we thus get

$$\operatorname{diam}_{\rho_F^{(t)}}(\Gamma)\ge \rho_F^{(t)}(\beta,\gamma)=\alpha^t\big(A_{\beta\wedge\gamma}\big)=\alpha^t\big(A_{\omega_\Gamma}\big)=\operatorname{diam}_{\rho_F^{(t)}}\big([\omega_\Gamma]\big).$$

Along with (5.6) this yields

$$\operatorname{diam}_{\rho_F^{(t)}}\big([\omega_\Gamma]\big) = \operatorname{diam}_{\rho_F^{(t)}}(\Gamma) \tag{5.7}$$

Hence $\{[\omega_\Gamma]\}_{\Gamma\in\mathcal{A}}$ is also a cover of $E^{\mathbb{N}}$ by sets with diameter (with respect to the metric $\rho_F^{(t)}$) $\leq \delta$. Therefore, for all $x \in X$ and all $a \in G^E$ we have that

$$J_{(x,a)} \subset \bigcup_{\Gamma\in\mathcal{A}} \phi_{\omega_\Gamma}^{(x,a)}(B).$$

But each set $\phi_{\omega_\Gamma}^{(x,a)}(B)$ is contained in a rectangular box with sides of length

$$2\mathrm{diam}(B)\alpha_1(A_\omega),\ 2\mathrm{diam}(B)\alpha_2(A_\omega), \dots, 2\mathrm{diam}(B)\alpha_q(A_\omega).$$

If k is the least integer greater than or equal to t, then each such box can be divided into at most

$$\left(4\mathrm{diam}(B)\frac{\alpha_1(A_\omega)}{\alpha_k(A_\omega)}\right)\cdot\left(4\mathrm{diam}(B)\frac{\alpha_2(A_\omega)}{\alpha_k(A_\omega)}\right)\cdot\ldots\cdot\left(4\mathrm{diam}(B)\frac{\alpha_2(A_\omega)}{\alpha_k(A_\omega)}\right)\cdot \\ \cdot\left(4\mathrm{diam}(B)\right)^{q-k+1}$$

rectangular cubes with sides of length α_k, that is of diameter $\sqrt{q}\alpha_k$. Therefore, fixing $\eta > 0$, there exists, because of (2.1) and (5.7), $\delta_\eta > 0$ such that $\mathrm{diam}\big(\phi_{\omega_\gamma}(B)\big) \leq \eta$ for all $\Gamma \in \mathcal{A}$. Hence,

$$\begin{aligned}
&\mathrm{H}^t_\eta\big(J_{(x,a)}\big) \leq \\
&\leq \sum_{\Gamma\in\mathcal{A}}\left(4\mathrm{diam}(B)\frac{\alpha_1(A_\omega)}{\alpha_k(A_\omega)}\right)\cdot\left(4\mathrm{diam}(B)\frac{\alpha_2(A_\omega)}{\alpha_k(A_\omega)}\right)\cdot\ldots\cdot\left(4\mathrm{diam}(B)\frac{\alpha_2(A_\omega)}{\alpha_k(A_\omega)}\right)\cdot \\
&\qquad\cdot\left(4\mathrm{diam}(B)\right)^{q-k+1}(\sqrt{q}\alpha_k)^t \\
&\preceq \sum_{\Gamma\in\mathcal{A}}\alpha_1\big(A_{\omega_\Gamma}\big)\alpha_2\big(A_{\omega_\Gamma}\big)\ldots\alpha_{k-1}\big(A_{\omega_\Gamma}\big)\alpha_k^{t-(k-1)}\big(A_{\omega_\Gamma}\big) \\
&\preceq \sum_{\Gamma\in\mathcal{A}}\alpha^t\big(A_{\omega_\Gamma}\big).
\end{aligned}$$

Therefore,

$$\mathrm{H}^t\big(J_{(x,a)}\big) \preceq \mathrm{H}^t_F(E^{\mathbb{N}}).$$

So,

$$\mathrm{H}^t\big(J_{(x,a)}\big) = \lim_{\eta\to 0}\mathrm{H}^t_\eta\big(J_{(x,a)}\big) \preceq \lim_{\eta\to 0}\mathrm{H}^t_F(E^{\mathbb{N}}) < +\infty.$$

The proof is complete. □

Now we can prove the main theorem of our paper.

Theorem 5.3. *If $\mathcal{S}$ is an affine scheme on $\mathbb{R}^q$, then*

$$\mathrm{HD}\big(J_{(x,a)}\big) = \min\{q, \mathrm{FD}(\mathcal{S})\}.$$

for m-a.e. $x \in X$ and λ_G^E-a.e. $a \in G^E$.

Proof. Because of the previous proposition we only have to prove that

$$\mathrm{HD}\big(J_{(x,a)}\big) = \min\{q, \mathrm{FD}(\mathcal{S})\}.$$

for m-a.e. $x \in X$ and λ_G^E-a.e. $a \in G^E$. Indeed, fix a non-integral number $0 < s < \min\{q, \mathrm{FD}(\mathcal{S})\}$. Take then arbitrary $0 < s < t < \min\{q, \mathrm{FD}(\mathcal{S})\}$. So, $F^t(E^{\mathbb{N}}) = +\infty$, and, by Lemma 4.6, there is a finite Borel measure ν on $E^{\mathbb{N}}$ such that

$$\nu([\omega]) \le C\alpha^t\big(A_\omega\big) \tag{5.8}$$

for all $\omega \in E^*$. Applying Lemma 5.1, formula (5.8), and the observation that $\nu \otimes \nu$ does not charge the diagonal, we get

$$\begin{aligned}
I &:= \int_X \int_{E^{\mathbb{N}}} \int_{E^{\mathbb{N}}} \int_{G^E} \frac{dm(x)\, d\lambda_G^E(a)\, d\nu(\omega)\, d\nu(\tau)}{||\pi_{(x,a)}(\omega) - \pi_{(x,a)}(\tau)||^s} \le \int_{E^{\mathbb{N}}} \int_{E^{\mathbb{N}}} \frac{d\nu(\omega)\, d\nu(\tau)}{\alpha^s\big(A_{\omega\wedge\tau}\big)} \\
&\preceq \sum_{n=0}^{\infty} \sum_{|\gamma|=n} \iint_{\substack{\omega,\tau\in E^{\mathbb{N}} \\ \omega\wedge\tau=\gamma}} \alpha^s\big(A_\gamma\big)^{-1} d\nu(\omega)\, d\nu(\tau) \\
&= \sum_{n=0}^{\infty} \sum_{|\gamma|=n} \alpha^s\big(A_\gamma\big)^{-1} \nu\otimes\nu(A_\gamma) \\
&\le \sum_{n=0}^{\infty} \sum_{|\gamma|=n} \alpha^s\big(A_\gamma\big)^{-1} \nu^2([\gamma]) \\
&\le \sum_{n=0}^{\infty} \sum_{|\gamma|=n} \alpha^s\big(A_\gamma\big)^{-1} \alpha^t\big(A_\gamma\big)\nu([\gamma]).
\end{aligned} \tag{5.9}$$

Now, with k being the least integer greater than or equal to s and l being the least integer greater than or equal to t, we get

$$\begin{aligned}
\alpha^t\big(A_\gamma\big)\alpha^s\big(A_\gamma\big)^{-1} &= \alpha_1(A_\gamma)\alpha_2(A_\gamma)\alpha_{k-1}(A_\gamma)\ldots\alpha_k(A_\gamma)\alpha_{k+1}(A_\gamma)\ldots\alpha_{l-1}(A_\gamma)\alpha_l(A_\gamma)^{t-l+1} \\
&\qquad \alpha_1(A_\gamma)^{-1}\ldots\alpha_{k-1}^{-1}(A_\gamma)\alpha_k(A_\gamma)^{-s+k-1} \\
&= \alpha_k(A_\gamma)^{k-s}\alpha_{k+1}(A_\gamma)\ldots\alpha_{l-1}(A_\gamma)\alpha_l(A_\gamma)^{t-l+1}.
\end{aligned}$$

Since $t - l + 1 \ge 0$ and since $k - s > 0$, we further get

$$\alpha^t\big(A_\gamma\big)\alpha^s\big(A_\gamma\big)^{-1} \le \alpha_k(A_\gamma)^{k-s} \le ||A_\gamma||^{k-s} \le \kappa^{(k-s)|\gamma|}.$$

Hence, we can continue (5.9) as follows.

$$I \le C\sum_{n=0}^{\infty} \kappa^{(k-s)n} \sum_{|\gamma|=n} \nu([\gamma]) = C\sum_{n=0}^{\infty} \kappa^{(k-s)n} = C(1-\kappa^{k-s})^{-1} < +\infty.$$

Hence, for m-a.e. $x \in X$ and λ_G^E-a.e. $a \in G^E$, we have that

$$I_{(x,a)} := \int_{E^{\mathbb{N}}} \int_{E^{\mathbb{N}}} \frac{d\nu(\omega)\, d\nu(\tau)}{||\pi_{(x,a)}(\omega) - \pi_{(x,a)}(\tau)||^s} < +\infty.$$

This means that

$$\int_{J_{(x,a)}}\int_{J_{(x,a)}} \frac{d(\nu\circ\pi_{(x,a)}^{-1})(z)\,d(\nu\circ\pi_{(x,a)}^{-1})(\xi)}{||z-\xi||^s} < +\infty,$$

and this in turn (see [2], comp. [Ma]) implies that $\mathrm{HD}(J_{(x,a)}) \ge s$. Thus, $\mathrm{HD}(J_{(x,a)}) \ge \min\{q, \mathrm{FD}(\mathcal{S})\}$, and the proof is finished. $\square$

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