# ERGODIC THEORY FOR HOLOMORPHIC ENDOMORPHISMS OF COMPLEX PROJECTIVE SPACES

MARIUSZ URBAŃSKI AND ANNA ZDUNIK

ABSTRACT. For every holomorphic endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ of a complex projective space $\mathbb{P}^k, k \geq 1$, there exists a positive number $\kappa(f) > 0$ such that if $\phi : J \to \mathbb{R}$ is a Hölder continuous function with $\sup(\phi) - \inf(\phi) < \kappa(f)$, then $\phi$ admits a unique equilibrium state $\mu_\phi$ on $J$. This equilibrium state is equivalent to a fixed point of the normalized dual Perron-Frobenius operator. In addition, the dynamical system $(f, \mu_\phi)$ is K-mixing, whence ergodic. Proving almost periodicity of the corresponding Perron-Frobenius operator is the main technical task of the paper. It requires to produce sufficiently many "good" inverse branches and to control the distortion of the Birkhoff sums of the potential $\phi$. In the case when the Julia set $J$ does not intersect any periodic irreducible variety of the critical set of $f$, we have that $\kappa_f = \log d$.

## 1. INTRODUCTION

The thermodynamic formalism for holomorphic endomorphisms of the Riemann sphere $\hat{\mathbb{C}}$ and Hölder continuous potentials, with sufficiently small oscillation, was originated in [DU]. The existence and uniqueness of equilibrium states of all such potentials was proved there, see also [Pr]. The corresponding Perron-Frobenius operator was shown to be almost periodic and the equilibria were shown to be K-mixing. Later ([DPU], [Ha]) more refined mixing and stochastic properties of these equilibria were established. The natural question then arises about the existence and uniqueness of equilibria in the higher dimensional case, namely, for complex projective spaces of an arbitrary dimension. Up to our knowledge, so far, in such generality, only the case of the potential $\phi$ identically equal to zero has been treated. The existence and uniqueness of its equilibria (e. i. the measure of maximal entropy) were proved in [BD]. Some stochastic properties were established in [FS], [Br], and [Du]; for related

*Date*: May 25, 2009.

1991 *Mathematics Subject Classification.* Primary: 37D35; Secondary: 32H50, 37C40, 28D05.

*Key words and phrases.* Holomorphic dynamics, complex projective spaces, Julia sets, topological pressure, equilibrium states, Hölder continuous potentials, Perron-Frobenius operators.

The research of the first author supported in part by the NSF Grant DMS 0700831. The research of the second author supported in part by the Polish MNiSW Grant N N201 0234 33.

topics see also [D1] and [D2]; the expository paper [DS] contains a complete survey of up to date results.

Our goal in this paper is to build the thermodynamic formalism for all holomorphic endomorphisms $f : \mathbb{P}^k \to \mathbb{P}^k$ of all complex projective spaces of an arbitrary dimension $k \geq 1$, and, what we would like to emphasize equally strongly, for all Hölder continuous potentials $\phi : J \to \mathbb{R}$, with sufficiently small value (depending only on the endomorphisms $f : \mathbb{P}^k \to \mathbb{P}^k$ and denoted in the sequel by $\kappa_f$) of their oscillation $\sup(\phi) - \inf(\phi)$. $J = J(f)$ in here and in the sequel, throughout the paper, denotes the Julia set of the map $f : \mathbb{P}^k \to \mathbb{P}^k$, i. e. the topological support of the measure of maximal entropy.

Note that in the literature our set $J$ is usually denoted by $J_k$ and it may be essentially smaller than the set $J_1$, which is also frequently called a Julia set and which is defined with use of the standard normality condition.

Our class of potentials is large indeed. It contains the restrictions to $J$ of all Hölder continuous functions $\phi : \mathbb{P}^k \to \mathbb{P}^k$ such that $\sup(\phi) - \inf(\phi) < \kappa_f$. As a matter of fact, if the Julia set $J$ does not intersect periodic irreducible varieties of the critical set of $f$, then we can take $\kappa_f$ as large as possible, namely equal to $\log\deg(f)$. We observe (see Corollary 4.4) that if $k$, the dimension of the projective space, is equal to 2, then this intersection consists of finitely many critical periodic orbits only, whence $\kappa_f$ is easier to estimate.

In order to build the thermodynamic formalism for $f$ and $\phi$ we, apart from the methods of algebraic geometry, employ the techniques of ergodic theory, and we cope especially hard with estimating the distortion of the Birkhoff sums of the potential $\phi$. This task is entirely absent in the case of the measure of maximal entropy; the distortion is always zero. For more general potentials, the ones we are dealing with, the situation is just opposite; the issue of bounded distortion becomes the central issue of our approach. As a matter of fact, the bounded distortion for, in a sense, most inverse branches, and the existence of sufficiently many "good" inverse branches, are the two main tools used to produce upper and lower bounds of iterates of corresponding Perron-Frobenius operators. This is the main technical theme of the paper. We would like to mention that in the case of the measure of maximal entropy, this issue actually trivializes; the function identically equal to one is then a fixed point of the Perron-Frobenius operator for free.

A basic notion of ergodic theory is that of metric (Kolmogorov-Sinaj) entropy $\mathrm{h}_\mu(f)$ of a probability $f$-invariant measure. The basic notion of thermodynamic formalism is that of topological pressure $\mathrm{P}(\phi) = \mathrm{P}(f, \phi)$ introduced by D. Ruelle in [Ru1]. We define them both in Section 2. Their alternative definitions and properties can be found for instance in [Wa] and [PU]. The formula relating these two, seemingly independent, concepts is the celebrated Variational Principle stating that

$$\mathrm{P}(\phi) = \sup\left\{\mathrm{h}_\mu(f) + \int \phi \, d\mu\right\}, \tag{1.1}$$

where the supremum is taken over all Borel probability $f$-invariant measures $\mu$. The measures $\mu$ for which $\mathrm{h}_\mu(f) + \int \phi d\mu = \mathrm{P}(\phi)$ are called equilibrium states for the potential $\phi$. We prove the following.

**Theorem 1.1.** *For every holomorphic endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ of a complex projective space $\mathbb{P}^k$, $k \geq 1$, there exists a positive number $\kappa(f) > 0$ such that if $\phi : J(f) \to \mathbb{R}$ is a Hölder continuous function with $\sup(\phi) - \inf(\phi) < \kappa(f)$, then $\phi$ admits a unique equilibrium state $\mu_\phi$ on $J$. This equilibrium state is equivalent to a fixed point of the normalized dual Perron-Frobenius operator. In addition the dynamical system $(f, \mu_\phi)$ is K-mixing, whence ergodic. In the case when the Julia set $J$ does not intersect any periodic irreducible varieties of the critical set of $f$, we have that $\kappa_f = \log d$.*

Due to Yomdin's work [Yo], the existence of equilibria is true for all $C^\infty$ smooth endomorphisms of compact differentiable manifolds. In fact, Yomdin proved that the entropy function is upper semi-continuous. Our proof of existence of equilibria is entirely different; in particular we do not use upper semi-continuity of the entropy function. With this respect we prove more than merely the existence. We construct an equilibrium as a fixed point of the normalized dual Perron-Frobenius operator. This gives a piece of a valuable information about the structure of this equilibrium and, along with a detailed analysis of iterates of the Perron-Frobenius operator, allows us to conclude the uniqueness of the equilibrium. We do it by showing that the topological pressure function is differentiable. We would like to remark that uniqueness of equilibria is not in general true, and, without appropriate constrains on the potentials, fails even for so smooth maps as rational functions of the Riemann sphere. The K-mixing property is due to almost periodicity of the corresponding Perron-Frobenius operator. We provide a more detailed description of the allowed oscillation $\kappa(f)$ in Section 2. We also provide sufficient conditions for $\kappa(f)$ to be equal to $\log \deg(f)$, nearly as good as in [DU]. The proof of Theorem 1.1 contains two additional ingredients we would like to bring reader's attention to. Firstly, we prove a form of uniformly subexponentially slow increase of local degrees of iterates of the map $f : \mathbb{P}^k \to \mathbb{P}^k$. This fact, being crucial for our entire proof, is interesting itself, and is related to some results of Favre ([F1] and [F1]); in fact it generalizes and simplifies one of his main propositions. Secondly, motivated by the argument of M. Gromov from [Gr], we prove that the topological pressure is not larger than the logarithm of the eigenvalue of the dual to the Perron-Frobenius operator. The proof is based on Lelong's Theorem and makes extensive use of geometry of projective spaces.

Our paper provides a generalization of corresponding results for the dynamics of rational maps in $\mathbb{P}^1$. Although the formulation of our result is analogous to that for one-dimensional case, its proof required the development of a new approach, and several new major ideas appear in our arguments. Firstly, the problem of local degree is absent in the one dimensional case (it is evident that $\deg_z f^n$ is then bounded in $J$, independently of $n$). Secondly, we are unable to control the distortion along the branches of $f^{-n}$ defined on balls. Instead, for every pair of points the preimages are paired in a way which depends of the points chosen and a collection of "good" pairs of preimages is then distinguished. This is done by keeping track of connected components of the preimage of one-dimensional discs under iterates $f^n$ and a careful choice of "good" components. Surprisingly, in order to estimate the iterates of the Perron-Frobenius operator, we have to extend our potential $\phi$ to some neighbourhood of $J$ and to estimate the Perron-Frobenius operator in this neighbourhood. More surprisingly, in order to prove the crucial inequality $P(\phi) \leq \log\lambda$, we have to extend the potential properly to the entire projective space $\mathbb{P}^k$ and to prove the appropriate estimates for the iterates of the operator acting on the space $\mathbb{P}^k$ (see Section 7). Moreover, we have to cope with periodic varieties contained in the critical set which may intersect the Julia set, see Section 4. This is a phenomenon which has no counterpart in the dimension 1 and this is why we have to estimate separately the part of the Perron-Frobenius operator acting "along" such "critical periodic varieties". This may cause the maximal allowable oscillation $\kappa_f$ to be smaller than $\log d$.

Our paper is organized as follows. In Preliminaries, Section 2, we prove an appropriate Hölder continuous extension result, and we define topological pressure and metric entropy. In Section 3, Contracting Inverse Branches, we construct sufficiently many (as it turns out later in the course of the paper) exponentially shrinking inverse branches. We refer to them as good ones. In Section 4 we introduce and discuss the maximally allowed oscillation $\kappa(f)$ for the potential $\phi$, we introduce the corresponding Perron-Frobenius operator, we prove the existence of the "geometric" Gibbs state $m_\phi$ (formula 4.10), i.e. an eigenmeasure of the dual operator, and the corresponding positive eigenvalue $\lambda$. In Section 5, Uniform Bounds of Iterates of the Perron-Frobenius Operator, in a sense a central section of the paper, we establish upper and lower uniform bounds of iterates of the Perron-Frobenius operator. Section 6, Almost Periodicity of the Perron-Frobenius Operator, is devoted to proving almost periodicity of this operator, and its uniform version, needed for the proof of the uniqueness of the equilibrium state $\mu_\phi$. Consequently, we produce a continuous fixed point $\rho_\phi$ of the Perron-Frobenius operator, and the $f$-invariant measure $\mu_\phi = \rho_\phi m_\phi$ is our candidate for the only equilibrium state of the potential $\phi$. Based on almost periodicity of the Perron-Frobenius operator, we establish its spectral properties, and in particular, we show that its iterates converge uniformly. At the end of this section we deduce from this convergence of the Perron-Frobenius operator, K-mixing of the dynamical system $(f, \mu_\phi)$, whence its ergodicity. We also prove the decay of correlations. In Section 7, Pressure versus Eigenvalue, developing the idea of Gromov from [Gr], we prove equality of the topological pressure $\mathrm{P}(\phi)$ and the logarithm $\log\lambda$. Section 8, Existence and Uniqueness of Equilibrium States, is devoted to proving existence and uniqueness of equilibrium states. The existence part is done by showing that the measure $\mu_\phi$ is an equilibrium state. The idea of the proof of uniqueness is to show the differentiability of the pressure function and to use the uniform periodicity of the Perron-Frobenius operator established in Section 6. We may therefore state there the main result, existence and uniqueness of equilibrium states, of our paper. The last section of our paper, Local Degree, contains the proof of uniformly subexponentially slow grow of local degrees of iterates of the map $f : \mathbb{P}^k \to \mathbb{P}^k$.

## 2. Preliminaries

**Lemma 2.1.** *Suppose that $(X, \rho)$ is a compact metric space and $F$ is a closed subset of $X$. If $g : F \to \mathbb{R}$ is a Hölder continuous function with an exponent $\alpha \in (0, 1)$, then there exists a Hölder continuous function $\tilde{g} : X \to \mathbb{R}$ with the same exponent $\alpha$ and with the following two properties: $\tilde{g}|_F = g$, $\sup(\tilde{g}) = \sup(g)$ and $v_\alpha(\tilde{g}) \leq 2v_\alpha(g)$.*

*Proof.* Let $H \geq 0$ be a Hölder constant of the function $g$, i.e. $|g(y) - g(x)| \leq H\rho^\alpha(x, y)$ whenever $x, y \in F$. For every $x \in X$, put

$$g_*(x) = \inf_{a \in F}\{g(a) + H\rho^\alpha(x, a)\},$$

and set

$$\tilde{g}(x) = g_*(x) - H\rho^\alpha(x, F).$$

If $x \in F$, then $g_*(x) \leq g(x)$ and $\rho^\alpha(x, F) = 0$, and consequently, $\tilde{g}(x) \leq g(x)$. Also, for every $a \in F$, $g(a) + H\rho^\alpha(x, a) \geq g(x)$, and therefore, $g_*(x) \geq g(x)$. Thus $\tilde{g}(x) \geq g(x)$. Hence $\tilde{g}(x) = g(x)$, which means that $\tilde{g}|_F = g$. Now, fix two arbitrary points $x, y \in X$. Then

$$\begin{aligned} g_*(x) &\leq \inf_{a \in F}\{g(a) + H(\rho(a, y) + \rho(y, x))^\alpha\} \leq \inf_{a \in F}\{g(a) + H(\rho^\alpha(y, a) + \rho^\alpha(x, y))\} \\ &= g_*(y) + H\rho^\alpha(x, y). \end{aligned}$$

Hence $g_*(x) - g_*(y) \leq H\rho^\alpha(x, y)$, and changing the roles of $x$ and $y$, we get that $|g_*(x) - g_*(y)| \leq H\rho^\alpha(x, y)$. Now, $\rho(x, F) \leq \rho(x, y) + \rho(y, F)$, and consequently,

$$\rho^\alpha(x, F) \leq (\rho(x, y) + \rho(y, F))^{\leq}\rho^\alpha(x, y) + \rho^\alpha(y, F).$$

Thus, changing also the roles of $x$ and $y$), $|H\rho^\alpha(x, F) - H\rho^\alpha(y, F)| \leq H\rho^\alpha(x, y)$. So, $|\tilde{g}(x) - \tilde{g}(y)| \leq 2H\rho^\alpha(x, y)$, meaning that $\tilde{g}$ is Hölder continuous with the exponent $\alpha$. Obviously, $\sup(\tilde{g}) \geq \sup(g)$. To show the opposite inequality, take an arbitrary $x \in X$ and then take $b \in F$ such that $\rho(x, F) = \rho(x, b)$. Then $\tilde{g}(x) \leq g(b) + H\rho^\alpha(x, b) - H\rho^\alpha(x, F) = g(b) \leq \sup(g)$. Thus, $\sup(\tilde{g}) \leq \sup(g)$. We are done. $\square$

A dynamical system $T : X \to X$ is called topologically exact if and only if for every non-empty open set $U$ in $X$, there exists $j \geq 0$ such that $T^j(U) = X$. Since it is easy to see that the map $f : J \to J$ is locally eventually onto (the union of all forward iterates of every non-empty open subset of $J$ covers $J$), and since $J$ is contained in the closure of repelling periodic orbits, we have the following.

**Proposition 2.2.** *If $f : \mathbb{P}^k \to \mathbb{P}^k$ is a holomorphic endomorphism, then the dynamical system $f : J \to J$ is topologically exact.*

As we have already explained in the introduction, our main result is related to topological pressure and Variational Principle. We want to define and formulate them now. An interested reader is however encouraged to consult [Ru2], [Wa], [Bo] or [PU] for example, to find a comprehensive treatment of these concepts.
Let us begin with recalling the notion of entropy of a measure-theoretic dynamical system. Suppose that $T : X \to X$ is a continuous map of a compact metric space $(X, d)$ and that $\mu$ is a Borel $T$-invariant probability measure on $X$. $T$-invariance means that if $A$ is a Borel subset of $X$ then $\mu(f^{-1}(A)) = \mu(A)$. We call $\mu$ (or $T$) *ergodic* if the only Borel invariant subsets of $T$ (i.e. satisfying $T^{-1}(A) = A$) are either of measure 0 or 1. Given $n \geq 0$ we define the metric $d_n$ on $X$ by setting

$$d_n(x, y) = \max\{d(T^i(x), T^i(y)) : 0 \leq i \leq n\}.$$

Denote by $B_n(x, r)$ the open ball in the metric $d_n$ centered in $x$ and with radius $r$. If the measure $\mu$ is ergodic, then (see [BK]) for $\mu$-a.e. point $x \in X$ the limit

$$\mathrm{h}_\mu(T) = \lim_{r\to 0} \lim_{n\to\infty} -\frac{\log \mu(B_n(x, r))}{n}$$

exists, is independent of $x$, and this limit is called the (metric) entropy of the system $T$ with respect to the measure $\mu$. This entropy is denoted by $\mathrm{h}_\mu(T)$. Roughly speaking it measures the exponential rate of decay of the measure of points that stay $\varepsilon$-close to the point $x$ under forward iterates of $f$. Usually a different, more classical approach is undertaken to define the entropy $\mathrm{h}_\mu(T)$ (see [Bo], [Wa], or [PU] for example), the one chosen here is probably the fastest and, at the same time, it reflects in a better way, the nature of entropy.

In order to introduce topological pressure, we choose one of the fastest methods, and simultaneously the one, we will need in Section 7, Pressure Versus Eigenvalue. For alternative approaches better suited to derive various properties of topological pressure see also the positions quoted above. Consider a continuous mapping $T : X \to X$ of a compact metric space $(X, d)$ and a continuous function $\phi : X \to \mathbb{R}$, called, following physical tradition, a potential. Given $n \geq 0$ and $\varepsilon > 0$, we say that a subset $F$ of $X$ is $(n, \varepsilon)$-*separated* if it is separated with respect to the metric $d_n$, which means that if $x$ and $y$ are two distinct points of $X$, then $d_n(x, y) \geq \varepsilon$. Fixing now $\varepsilon > 0$ we consider an arbitrary sequence $F_n(\varepsilon)$, $n \geq 1$, of maximal (in the sense of inclusion) $(n, \varepsilon)$-separated sets. We then define the topological pressure of the function $\phi$ with respect to the mapping $T$ as follows

$$\mathrm{P}(T, \phi) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \left( \sum_{x\in F_n(\varepsilon)} \exp \sum_{j=0}^{n-1} \phi \circ T^j(x) \right).$$

Topological pressure belongs to topological dynamics, whereas metric entropy is a notion in ergodic theory. The link joining them is given by the following formula called the variational principle (see [Bo], [Ru2], [Wa], or [PU] for example), which we have already mentioned in the introduction.

$$\mathrm{P}(T,\phi)=\sup_{\mu}\{\mathrm{h}_\mu(T)+\int \phi d\mu\},$$

where the supremum is taken over all Borel probability $T$-invariant (ergodic) measures of $X$.

## 3. Contracting Inverse Branches

Keeping $f:\mathbb{P}^k\to\mathbb{P}^k$ a holomorphic endomorphism, let $\mathrm{Crit}(F)$ be the set of all critical points of $f$, i. e. the set of such points $z\in\mathbb{P}^k$ that $\deg_z f\geq 2$. We need the following definition.

**Definition 3.1.** *Given an integer $n\geq 1$ the periodic critical set $A_n$ is the union of orbits of all irreducible varieties, that are contained in the critical set and are periodic under an iterate $f^l$ with some $l\leq n$. In particular, an orbit of a critical periodic point of period $l\leq n$ is in the critical periodic set $A_n$.*

For all positive integers $n$ and $p$ such that $n>p$ let $E_n^p$ be the set defined as follows.

**Definition 3.2.** *$E_n^p$ is the set of all points $x\in\mathbb{P}^k$ for which there exists a non-negative integer $i\leq n-1$ such that $f^i(x)\in A_p$.*

Our main result concerning the behavior of the local degree is the following.

**Proposition 3.3.** *For every $\beta>0$ there exist $p=p(\beta)$ and $N=N(\beta)$ such that for every $n\geq N$ and for every $x\notin E_n^p$ we have*

$$\#\{j\leq n : f^j(x)\in C\}\leq \beta n.$$

Because of the very technical and combinatorial nature of the proof of this proposition, and to focus first on the main stream of arguments, we decided to move our presentation of this proof to Section 9, the last section of the paper.

Now, take $\gamma\in(0,1)$. It follows from Proposition 3.3 that there exist two least integers $q_1(\gamma)$ and $q_2(\gamma)$ such that if $q\geq q_2(g)$, $z\in\mathbb{P}^k$, $j\geq q_2(g)$, and $f^j(z)\notin A_{q_1(\gamma)}$, then

$$\deg_z(f^j)\leq\gamma^{-j}. \tag{3.1}$$

Let us record the following obvious observation.

**Lemma 3.4.** *The two functions $(0,1) \ni \gamma \mapsto q_1(\gamma)$, $q_2(\gamma)$ are weakly increasing, the function $(0,1) \ni \gamma \mapsto A_{q_1(\gamma)}$ is weakly ascending, and, consequently, there exists $\gamma_* \in (0,1)$ such that these three functions are constant throughout the interval $(0,\gamma_*)$.*

Put

$$A_\gamma := A_{q_1(\gamma)}.$$

We normalize the Fubini-Study metric $\rho$ on $\mathbb{P}^k$ so that the area $A$ of any ball of radius 1 on a projective line is equal to 1. We will need in the proof of Lemma 3.6 the fact easily following from Lelong's Theorem ([La], Theorem II.3.6 or [McM], Theorem 2.45) and homogenuity of complex projective spaces.

**Theorem 3.5.** *There exists a constant $c > 0$ such that if $x \in \mathbb{P}^k$, $0 < R \le 2\mathrm{diam}_\rho(\mathbb{P}^k)$, and $X$ is a 1-dimensional closed complex variety contained in $B(x,R)$, then*

$$\mathrm{Area}(X \cap B(x,R)) > c^{-1}r^2.$$

For any two distinct points $a, b \in \mathbb{P}^k$ denote by $\Gamma_{a,b}$ the projective line passing through $a$ and $b$. Our main (technical) result in this section is the following.

**Lemma 3.6.** *For every $\gamma \in (0,1)$, every integer $s \ge 1$, every integer $q \ge q_2(\gamma)$, and every $\eta > 0$ there exists $R(\eta) = R(\gamma,s,q;\eta) \in (0,1)$ such that for every $z \in \mathbb{P}^k \setminus B(A_\gamma,\eta)$ there exists a dense subset $D(z) = D(\gamma,s,q,\eta;z)$ of $B(z,R(\eta)) \subset \mathbb{P}^k$ such that for every $\xi \in D(z)$ and for all $n \ge s$, there exists a family $W_n(q,\eta,z,\xi)$ of connected components of $f^{-qn}(B(z,R(\eta)) \cap \Gamma_{z,\xi})$ with the following properties.*

- $(a_n)$ *If $V \in W_{n+1}(q,\eta,z,\xi)$, then $f^q(V) \in W_n(q,\eta,z,\xi)$.*
- $(b_n)$ $\max\{\mathrm{diam}(V) : V \in W_n(q,\eta,z,\xi)\} \le \gamma^{n/2}$.
- $(c_n)$ *If $n \ge s$, then*

  $$\#\big(Z_n(q,\eta,z,\xi) \setminus W_n(q,z,\xi)\big) \le \gamma^{-sq}\big(4c\gamma^{-3qn} + (k-1)^{d+1}q\big)d^{(k-1)q(n+1)},$$

  *where $Z_n(q,\eta,z,\xi)$ is the family of all connected components of all the sets of the form $f^{-q}(V)$, where $V \in W_{n-1}(q,\eta,z,\xi)$ ($c > 0$ is the constant coming from Theorem 3.5).*
- $(d_n)$ *For every $n \ge s$ and every $V \in W_n(q,\eta,z,\xi)$, we have $V \cap f^q(Crit(f^q)) = \emptyset$, and $f^{qn}|_F$ is at most $\gamma^{-sq}$-to-1.*

*Proof.* In virtue of (3.1) there exists $\hat{R}(\eta) > 0$ so small that if $z \in \mathbb{P}^k \setminus B(A_q,\eta)$, $x \in f^{-sq}(z)$, and if $V'_x$ is the connected component of $f^{-sq}(B(z,\hat{R}(\eta)))$ containing $x$, then

$$\deg\big(f^{sq}|_{V'_x}\big) \le \gamma^{-sq} \tag{3.2}$$

and

$$\mathrm{diam}(V'_x) \le \gamma^{q/2} \;\text{ and }\; A(V'_x) \le (4c)^{-1}\gamma^{3q}. \tag{3.3}$$

Since $\dim(\mathrm{Crit}(f)) = k-1$, $\deg(\mathrm{Crit}(f)) = (k-1)^{d+1}$, and since for every projective line $\Gamma$ and every $l \geq 0$, $\deg(f^{-l}(\Gamma)) = d^{(k-1)l}$, using Bézout's Theorem, we deduce that there exists $D(z)$, a dense subset of $B(z, \hat{R}(\eta))$, such that for all $\xi \in D(z)$ and for all $l \geq 0$, we have

$$\#(\mathrm{Crit}(f) \cap f^{-l}(\Gamma_{z,\xi})) = (k-1)^{d+1} d^{(k-1)l}. \tag{3.4}$$

We shall first construct the sets $W'_n(q,\eta,z,\xi)$ and $Z'_n(q,\eta,z,\xi)$, $n \geq s$, recursively such that the conditions $(a_n)$, $(b'_n)$, $(c'_n)$, and $(d_n)$ are satisfied. The condition $(b'_n)$ is in here the following.

$(b'_n)$ If $V' \in W'_n(q,\eta,z,\xi)$, then $\mathrm{Area}(V') \leq \gamma^{n+2qs}/4c$,

and $(c'_n)$ is the same as $(c_n)$ with the number $\gamma^{-sq}$ disappear. We start the recursion by putting

$$W'_s(q,\eta,z,\xi) = Z'_s(q,\eta,z,\xi),$$

where $Z'_s(q,\eta,z,\xi)$ is defined to consist of all connected components of all sets of the form $f^{-sq}(B(z,\hat{R}(\eta)) \cap \Gamma_{z,\xi})$. Now assume that for some $n \geq s$ the family $W'_n(q,\eta,z,\xi)$ has been constructed so that condition $(b'_n)$ is satisfied. The inductive step is to construct the family $W'_{n+1}(q,\eta,z,\xi)$ so that the conditions $(a_n)$, $(b'_{n+1})$, $(c_{n+1})$, and $(d_{n+1})$ are satisfied. The family $W'_{n+1}(q,\eta,z,\xi)$ is defined to consist of all connected components $V'$ of all the sets $f^{-q}(G)$ with $G \in W'_n(q,\eta,z,\xi)$ for which

$$\mathrm{Area}(V') \leq (4c)^{-1}\gamma^{(n+1)+2qs} \quad \text{and} \quad V' \cap \bigcup_{j=1}^{q} f^j(\mathrm{Crit}(f)) = \emptyset. \tag{3.5}$$

Conditions $(a_n)$ and $(b'_{n+1})$ are then automatically satisfied as well as the first part of $(d_{n+1})$. If $n = s$, then $f^{qs}|_V$ is at most $\gamma^{-sq}$-to-1 by the choice of $\tilde{R}(\eta)$, so $(d_{n+1})$ is verified. In order to prove $(c'_{n+1})$, let us first estimate from above $\#(f^{-q(n+1)}(\Gamma_{z,\xi}) \cap f^j(\mathrm{Crit}(f))$ for all $j = 1,2,\ldots,q$. Indeed, if $x \in f^{-q(n+1)}(\Gamma_{z,\xi}) \cap f^j(\mathrm{Crit}(f))$, then $x = f^j(c_x)$ with some $c_x \in \mathrm{Crit}(f)$. Thus $c_x \in f^{-q((n+1)+j)}(\Gamma_{z,\xi})$, and since the function $x \mapsto c_x$ is 1-to-1, we conclude from (3.4), that

$$\#\big(f^{-q(n+1)}(\Gamma_{z,\xi}) \cap f^j(\mathrm{Crit}(f))\big) \leq (k-1)^{d+1} d^{(k-1)q(n+1)+j)} \leq (k-1)^{d+1} d^{(k-1)q(n+2)}.$$

Hence,

$$\#\big(f^{-q(n+1)}(\Gamma_{z,\xi}) \cap \bigcup_{j=1}^{q} f^j(\mathrm{Crit}(f))\big) \leq (k-1)^{d+1} q d^{(k-1)q(n+2)}.$$

Thus, the number of elements from $Z'_n(q,\eta,z,\xi)$ that fail to satisfy the second condition of (3.5), is bounded above by $(k-1)^{d+1} q d^{(k-1)q(n+2)}$. Since $\mathrm{Area}(f^{-q(n+1)}(\Gamma_{z,\xi})) = d^{(k-1)q(n+1)}$, the number of elements from $Z'_{n+1}(q,\eta,z,\xi)$ that fail to satisfy the first condition of (3.5), is bounded above by

$$4c\gamma^{-(n+1)-2q} d^{(k-1)q(n+1)} \leq 4c\gamma^{-3q(n+1)} d^{(k-1)q(n+1)}.$$

Thus the condition $(c'_{n+1})$ for $W'$ and $Z'$ is established, and the inductive construction of the families $W'_n(q,\eta,z,\xi)$ and $Z'_n(q,\eta,z,\xi)$ satisfying conditions $(a_n)$,$(b'_n)$, $(c_n)$, and $(d_n)$ is complete. Now, decreasing $\tilde{R}(\eta)$ appropriately (the smaller radius will be called $R(\eta)$) we shall check that condition $(b_n)$ also holds. Let $0 < R(\eta) \le \hat{R}(\eta)$ be sufficiently small as specified later in the course of the proof. For every $n \ge s$ define $W_n(q,\eta,z,\xi)$ to consist of all connected components $V$ of all elements of $W_n'(q,\eta,z,\xi)$ intersected with $f^{-qn}(B(z,R(\eta))\cap\Gamma_{z,\xi})$. Since each element of $W_n'(q,\eta,z,\xi)$ contains at least one and at most $\gamma^{-sq}$ elements of $W_n(q,\eta,z,\xi)$, item $(c_n)$ follows immediately from the corresponding statement for $W_n'$. Conditions $(a_n)$ and $(d_n)$ also follow from corresponding statements for families $W_n'$. We are to show that $(b_n)$ holds. We shall specify the value of $R(\eta)$ at this step. First, fix a positive integer

$$M > \#\left(\Gamma_{z,\xi}\cap\bigcup_{j=1}^{sq} f^j(\mathrm{Crit}(f))\right)$$

Then fix an integer $a > 1$ such that $\gamma^{sq}\log a > 1$, and let $0 < R(\eta) < \hat{R}(\eta)$ be so small that

$$0 < \frac{R(\eta)}{\hat{R}(\eta)} < a^{-(M+1)},$$

Now, for all $p = 0, 1, \dots, M$, consider the annuli

$$A_p = \big(B(z, a^{p+1}R(\eta)) \setminus B(z, a^p R(\eta))\big) \cap \Gamma_{z,\xi}.$$

By the choice of $M$, there exists at least one annulus in this collection, that does not intersect the set $\bigcup_{j=1}^q f^j(\mathrm{Crit}(f))$. Let us keep the notation $A_p$ for this specified annulus. Set

$$D' = B(z,\hat{R}(\eta))\cap\Gamma_{z,\xi},\ \ D = B(z,R(\eta))\cap\Gamma_{z,\xi},$$

$$D_1 = B(z,a^pR(\eta))\cap\Gamma_{z,\xi},\ \ D_2 = B(z,a^{p+1}R(\eta))\cap\Gamma_{z,\xi}$$

(so $D \subset D_1 \subset D_2 \subset D'$). As above, let $V \in W_n$ be a connected component of $f^{-qn}(D)$. Then, let $V_1$ be the connected component of $f^{-qn}(D_1)$ containing $V$, let $V_2$ be the connected component of $f^{-qn}(D_2)$ containing $V_1$, and, as above, let $V'$ be the connected component of $f^{-qn}(D')$ containing $V_2$ (thus $V \subset V_1 \subset V_2 \subset V'$). Clearly, $V' \cap f^{-qn}(A_p)$ is a union of at most $\gamma^{-qs}$ of some annuli, say, $A_j'$, the modulus of each annulus $A_j'$ is bounded below by $\gamma^{qs}\log a$, and, after appropriate rearrangement of indices $j$,

$$V_2 \setminus V_1 = \bigcup_{j=1}^m A_j',$$

with some $m \le \gamma^{-sq}$. Since the modulus of every annulus $A_j'$ in $V_2 \setminus V_1$ is larger than $\gamma^{qs}\log a > 1$, we have

$$1 > \frac{1}{\mathrm{mod}(A_j')} = \sup_\rho\left\{\frac{\inf_l^2\{\mathrm{length}_\rho(l)\}}{\mathrm{Area}_\rho(A_j')}\right\} \ge \frac{\mathrm{length}^2(l)}{\mathrm{Area}(A_j')},$$

where the supremum is taken over all measurable Riemannian metrics on $A_j'$ and the infimum is taken over all closed piecewise-smooth curves that separate both components of the boundary of $A_j'$. The values $\mathrm{length}(l)$ and $\mathrm{Area}(A_j')$ respectively denote the length and the area calculated with respect to the Fubini-Study metric. Thus for every annulus $A_j'$ there exists a curve $l_j$ in this family such that

$$\mathrm{length}(l_j) \le \sqrt{\mathrm{Area(A_j')}} \le \sqrt{\mathrm{Area}(V_2)} \le \sqrt{\mathrm{Area}(V')}.$$

We claim that this implies the following.

$$\mathrm{diam}(V) < 2\sqrt{c}\sqrt{(\mathrm{Area}(V'))}\ \gamma^{-qs}. \tag{3.6}$$

Indeed, one can enlarge $V_1$ so that the boundary of this modified domain is exactly the union of curves $l_1, \dots l_m$. Let us keep the notation $V_1$ for this modified domain. Put $Q = \sqrt{c}\sqrt{\mathrm{Area}(V')}$. Then consider the following two cases. Either

(a) there exists $x \in V_1$ such that $d(x, l_i) > Q$ for all $i = 1, \dots, m$.
or
(b) for every $x \in V_1$ there exists $l_x \in \{l_1, \dots, l_m\}$ such that $d(x, l_x) < Q$ .

In the case (a), take $x \in V_1$, satisfying (b), and let

$$U := V_1 \cap B(x, Q) \subset B(x, Q).$$

Then $U$ is a closed algebraic variety in $B(x, Q)$ and, using Theorem 3.5, we get

$$\mathrm{Area}(U \cap B(x, Q)) > \frac{1}{c} Q^2.$$

But $\mathrm{Area}(U \cap B(x, Q)) \le \mathrm{Area}(V') = \frac{1}{c}Q^2$. This is a contradiction implying that the case (a) never occurs. In the case (b) we get that

$$V_1 \subset \bigcup_{i=1}^{m} B(l_i, Q) = B(l_1, Q) \cup \bigcup_{i=2}^{m} B(l_i, Q).$$

Since the set $V_1$ is connected and both sets in the above union are open, they must intersect, say $B(l_2, Q) \cap B(l_1, Q) \ne \emptyset$. Thus, proceeding by induction and after permuting the sets $B(l_i, Q)$ if necessary, we can require that

$$B(l_{j+1}, Q) \cap \bigcup_{i=1}^{j} B(l_i, Q) \ne \emptyset$$

Therefore, if $x \in B(l_1, Q)$ and $y \in B(l_j, Q)$ then $\mathrm{dist}(x, y) \le jQ + j \sup_i\{\mathrm{length}(l_i)\}$. This implies that

$$\mathrm{diam}(V_1) \le mQ + m \sup_i\{\mathrm{length}(l_i)\} \le 2\sqrt{c}\sqrt{\mathrm{Area}(V')}\gamma^{-qs}.$$

As $V \subset V'$, the formula (3.6) is thus proved. But, by our condition $(b_n')$ on the area of $V'$, we can now write

$$\mathrm{diam}(V) \le 2\sqrt{c}\frac{\gamma^{\frac{n}{2}}\gamma^{qs}}{\sqrt{c}\cdot 2}\gamma^{-qs} = \gamma^{\frac{n}{2}}.$$

So, $(b_n)$ is established and we are done. $\square$

We would like to note that, although we do not use explicitly in the above proof the geometric distortion lemma from [BD], this lemma has motivated our approach here. We directly used Lelong's inequality instead.

## 4. Potential's Oscillation

For every $\gamma \in (0,1)$ let

$$A_{J,\gamma} = A_\gamma \cap J.$$

Since the Julia set $J$ is backward and forward invariant and since the set $A_\gamma$ fails to be backward invariant, there exists a least integer $q_3(\gamma) \geq 1$ such that

$$\deg\big(f^{q_3(\gamma)}|_{A_{J,\gamma}}\big) \leq d^{q_3(\gamma)k} - 1. \tag{4.1}$$

Note that like $q_1$ and $q_2$ the function $q_3 : (0,1) \to \mathbb{N}$ is weakly increasing and is constant throughout the interval $(0,\gamma_*)$. For every $\gamma \in (0,1)$ set,

$$g_\gamma := \max\left\{\frac{1}{q_3(\gamma)}\log_d\left(\deg\big(f^{q_3(\gamma)}|_{A_{J,\gamma}}\big)\right), k-1\right\} < k. \tag{4.2}$$

Now, for every $\kappa \in (0,\log d)$ let

$$\gamma_\kappa = \exp\left(\frac{1}{5}(\kappa - \log d)\right). \tag{4.3}$$

It then follows from above that the function $(0,\log d) \ni \kappa \mapsto g_{\gamma_\kappa}$ is weakly increasing and takes on a constant value (in $(0,k)$) throughout the interval $(0,\log(d\gamma_*^4))$. Consequently, the function $(0,\log d) \ni \kappa \mapsto k - g_{\gamma_\kappa}$ is weakly decreasing and takes on a constant value (in $(0,k)$) throughout the interval $(0,\log(d\gamma_*^4))$. Now we can define the maximal oscillation of our potentials. Namely,

$$\kappa_f := \sup\{\kappa \in (0,\log d) : (k - g_{\gamma_\kappa})\log d > \kappa\} \in (0,\log d] \tag{4.4}$$

and

$$g_f := k - \frac{\kappa_f}{\log d} \in [k-1,k). \tag{4.5}$$

Let us record now the following obvious observation.

**Lemma 4.1.** *If $A_q \cap J = \emptyset$ for all $q \geq 1$, then*

$$\kappa_f = \log d. \tag{4.6}$$

Sticking to this issue, we would like to discuss at this moment the case when the dimension $k = 2$. Although then the set $A_q \cap J$ does not have to be empty, nevertheless the task to estimate $\kappa_f$ reduces to looking at finitely many periodic points only. Indeed. we recall the following lemma from [FS] (Lemma 7.9).

**Lemma 4.2.** *Suppose that $g : \mathbb{P}^2 \to \mathbb{P}^2$ is a holomorphic map of degree $d$ and that $g$ maps a compact complex hypersurface $Z$ into itself and that $Z$ is contained in the critical set of $g$. Then*

$$\operatorname{dist}(f(z), Z) = o(\operatorname{dist}(z, Z)).$$

We shall prove the following.

**Lemma 4.3.** *Suppose that $g : \mathbb{P}^2 \to \mathbb{P}^2$ is a holomorphic map of degree $d$. If $D \subset C$ is an irreducible component of the critical set $C$, and $D$ is periodic under $f$ ($f^l D = D$ for some $l \geq 1$), then $D$ does not intersect the Julia set $J$.*

*Proof.* Indeed, let $z \in J$ and let $U$ be an arbitrary neighborhood of $z$. It then follows from the construction of the maximal measure in [BD] that $\bigcup_{n \geq 0} f^n(U) = \mathbb{P}^2 \backslash E$ where $E$ is a (possibly empty) exceptional set, i.e. the largest totally invariant algebraic subset of the critical set. Applying Lemma 4.3, we conclude that if $D$ is a periodic irreducible component of the critical set $C$ then there exists a neighborhood of $D$ which is mapped into itself under $f^l$. Therefore, $D \cap J = \emptyset$. We are done. $\square$

For every periodic point $z$ of $f$ let $p(z) \geq 1$ be the least integer such that $f^{p(z)}(z) = z$. Denote by $\operatorname{Per}(f)$ the set of all periodic points of $f$. As a corollary of Lemma 4.3 we get the following.

**Corollary 4.4.** *If $f : \mathbb{P}^2 \to \mathbb{P}^2$ is a holomorphic map of degree $d$, then the set*

$$W := \{z : \deg_z f > d\} = \{z : \deg_z f \geq d + 1\}$$

*is finite and*

$$\kappa_f \geq 2 \log d - \max \left\{ \log d, \max_{z \in W \cap \operatorname{Per}(f) \cap J} \left\{ \frac{1}{p(z)} \log \deg_z (f^{p(z)}) \right\} \right\}.$$

*In particular, if $W \cap J = \emptyset$, then $\kappa_f = \log d$.*

As was indicated in the introduction, our general assumption is that $\phi : J \to \mathbb{R}$ is a Hölder continuous function and

$$\sup(\phi) - \inf(\phi) < \kappa_f. \tag{4.7}$$

Let us take the first fruits of this assumption. First of all fix two positive numbers $\alpha$ and $\beta$ such that

$$\sup(\phi) - \inf(\phi) < \alpha < \beta < \kappa_f. \tag{4.8}$$

Set

$$\theta = \frac{\beta - \alpha}{2} > 0. \tag{4.9}$$

We consider the dynamical system $f : J \to J$. Let $C(J)$ denote the Banach space of all complex-valued continuous functions on $J$ endowed with the supremum norm. For every $g \in C(J)$ define $\mathcal{L}_\phi g$ by the formula

$$\mathcal{L}_\phi g(z) = \sum_{x \in f^{-1}(z)} e^{\phi(x)} g(x),$$

where the inverse images of critical values of $f$ are counted with multiplicities. Then $\mathcal{L}_\phi g \in C(J)$ and the linear operator $\mathcal{L}_\phi g : C(J) \to C(J)$ is bounded. $\mathcal{L}_\phi$ is called the Perron-Frobenius (transfer) operator associated to the potential $\phi$. Now consider the dual operator $\mathcal{L}_\phi^* : C^*(J) \to C^*(J)$ defined by the formula $\mathcal{L}_\phi^* \mu(g) = \mu(\mathcal{L}_\phi g)$. Let $M_J$ be the set of all Borel probability measures on $J$. The map

$$\mu \mapsto \frac{\mathcal{L}_\phi^* \mu}{\mathcal{L}_\phi^* \mu(1\!\!1)}, \ \mu \in M_g,$$

is well-defined and continuous. Since $M_J$ is convex and compact (in the weak-$*$ topology), this map has a fixed point in virtue of Schauder-Tichonov Theorem. Denote this fixed point by $m_\phi$ and set $\lambda = \mathcal{L}_\phi m_\phi(1\!\!1)$. Then

$$\mathcal{L}_\phi^* m_\phi = \lambda m_\phi. \tag{4.10}$$

Hence, using (4.7), we get

$$\begin{aligned} \lambda = \int \mathcal{L}_\phi 1\!\!1 dm_\phi = \int \sum_{x \in f^{-1}(z)} e^{\phi(x)} dm_\phi \geq \int d^k \exp(\inf(\phi)) dm_\phi = d^k \exp(\inf(\phi)) \\ = \exp(k \log d + \inf(\phi)) > \exp\bigl(\sup(\phi) - \alpha + k \log d\bigr). \end{aligned}$$

Equivalently,

$$\sup(\phi) - \log \lambda < \alpha - k \log d. \tag{4.11}$$

Therefore,

$$\beta - \log d + \log \lambda - \sup(\phi) - (k-1) \log d. > \beta - \alpha = 2\theta \tag{4.12}$$

Since the Julia set $J$ is completely invariant, iterating (4.10), and making use of the topological exactness of the map $f : J \to J$ (Theorem 2.2), we obtain the following.

**Proposition 4.5.** *The measure $m_\phi$ is positive on non-empty open subsets of $J$; in other words,* $\mathrm{supp}(m_\phi) = J$.

## 5. Uniform Bounds of Iterates of the Perron-Frobenius Operator

Consider the number $\gamma_\beta$ as defined by formula (4.3). In view of (4.12) and (4.3) there exists $q_* \geq \max\{q_1(\gamma_\beta), q_2(\gamma_\beta), q_3(\gamma_\beta)\}$ so large that for all $q \geq q_*$ and all $n \geq 1$, we have

(5.1)

$$\gamma_\beta^{-sq}\bigl(4c\gamma_\beta^{-3qn} + (k-1)^{d+1} q\bigr) \exp\left( qn\Bigl((k-1)\log d + \frac{k-1}{qn} \log d + \sup(\phi) - \log \lambda\Bigr)\right) \leq e^{-\theta qn}.$$

We also assume $q_*$ to be so large that $q \geq q_*$ so large that

$$(1-e^{-q\theta})^{-1}\big(1+(1-e^{-q\theta})^{-1}\lambda^{-q}e^{q\sup(\phi)}d^{qk}\big)e^{-\theta sq} \leq \frac{1}{4}. \tag{5.2}$$

Now set

$$A_* := A_{\gamma_\beta},$$
$$A_J^* = A_{\gamma_\beta} \cap J.$$

and

$$g_* = g_{\gamma_\beta}.$$

Now, assume in addition that $q$ is an integral multiple of $q_3(\gamma_\beta)$. Apply Lemma 3.6 with this $q$ and $\gamma := \gamma_\beta$. By (4.2) there exits $\Delta_q^* > 0$ so small that

$$\deg\big(f^q : B(A_J^*, \Delta_q^*) \cap f^{-q}(B(A_J^*, \Delta_q^*)) \to B(A_J^*, \Delta_q^*)\big) \leq d^{g_* q}. \tag{5.3}$$

Since the Julia set $J$ is backward and forward invariant, there exists $\Delta_q^{(1)} \in (0, \Delta_q^*/2)$ so small that

$$f^{-q}(B(A_J^*, \Delta_q^{(1)})) \cap B(A_*, \Delta_q^*) \subset B(A_J^*, \Delta_q^*), \tag{5.4}$$

and then $\Delta_q^{(2)} \in (0, \Delta_q^{(1)}/2)$ so small that

$$f^{-q}(B(A_* \setminus B(A_J^*, \Delta_q^{(1)}), \Delta_q^{(2)})) \cap B(A_*, \Delta_q^*) \subset B(A_* \setminus B(A_J^*, \Delta_q^{(1)}), \Delta_q^{(1)}/2). \tag{5.5}$$

Since $\{B(A_J^*, \Delta_q^{(1)}), B(A_* \setminus B(A_J^*, \Delta_q^{(1)}), \Delta_q^{(2)})\}$ is an open cover of the compact set $A_*$, there exists $\Delta_q \in (0, \Delta_q^{(2)})$ such that

$$B(A_*, 2\Delta_q) \subset B(A_J^*, \Delta_q^{(1)}) \cup B(A_* \setminus B(A_*, \Delta_q^{(1)}), \Delta_q^{(2)}). \tag{5.6}$$

Fix $\tau \in \mathbb{R}$ so small that $\tau < \sup(\phi)$ and

$$\lambda^{-1}d^k e^\tau \leq e^{-\theta}. \tag{5.7}$$

Define the function $\phi_q : J \cup \overline{B}(A_* \setminus B(A_J^*, \Delta_q^{(1)}), \Delta_q^{(1)}/2) \to \mathbb{R}$ by the following formula.

$$\phi_q(z) = \begin{cases} \sum_{j=0}^{q-1} \phi(f^j((z))) & \text{if } z \in J, \\ q\tau & \text{if } z \in \overline{B}(A_* \setminus B(A_J^*, \Delta_q^{(1)}), \Delta_q^{(1)}/2) \end{cases}$$

This function is well defined since $J \cap \overline{B}(A_* \setminus B(A_J^*, \Delta_q^{(1)}), \Delta_q^{(1)}/2) = \emptyset$. Clearly, it is Hölder continuous and $\sup(\phi_q) \leq q\sup(\phi)$. Let $\tilde{\phi}_q : \mathbb{P}^k \to \mathbb{R}$ be the Hölder continuous extension, produced in Lemma 2.1, of the function $\phi_q : J \cup \overline{B}(A_* \setminus B(A_J^*, \Delta_q^{(1)}), \Delta_q^{(1)}/2) \to \mathbb{R}$. Remember that $\sup(\tilde{\phi}_q) = \sup(\phi_q)$, and that $\tilde{\phi}_q$ has the same Hölder exponent as $\phi_q$ and $\phi$. Denote this exponent by $\omega$ and the $\omega$-variation of $\tilde{\phi}_q$ by $H_q$, which, by Lemma 2.1, is bounded by the double $\omega$-variation of $\phi_q$. for every $g : \mathbb{P}^k \to \mathbb{R}$ let

$$S_n g = \sum_{j=0}^{n-1} g \circ f^{qj}.$$

It then follows from Lemma 3.6($b_n$) that for every $n \geq 1$, every $V \in W_n(q,\Delta,z,\xi)$, and all $x,y \in V$, we have

$$
\begin{aligned}
|S_n\tilde{\phi}_q(x) - S_n\tilde{\phi}_q(y)| &\leq \sum_{j=0}^{n-1} |\tilde{\phi}_q(f^{qj}(x)) - \tilde{\phi}_q(f^{qj}(y))| \\
&\leq \sum_{j=0}^{n-1} H_q\rho^\omega(f^{qj}(x), f^{qj}(y)) \leq H_q \sum_{j=0}^{n-1} H_q\gamma^{\frac{n-j}{2}\omega} \\
&\leq H_q \sum_{j=0}^{\infty} \gamma^{\frac{\omega}{2}j} \\
&= H_q(1-\gamma^{\omega/2})^{-1}.
\end{aligned} \tag{5.8}
$$

Hence, for all $n \geq q$,

$$
\tilde{C}_q^{-1} \leq \frac{\lambda^{-qn}\exp\big(S_n\tilde{\phi}(x))\big)}{\lambda^{-qn}\exp\big(S_n\tilde{\phi}(y))\big)} \leq \tilde{C}_q, \tag{5.9}
$$

where $\tilde{C}_q = \exp\big(H_q(1-\gamma^{\omega/2})^{-1}\big)$. In this section we will need more auxiliary Perron-Frobenius operators. First, define $\mathcal{L}_{\tilde{\phi}_q} : C(\mathbb{P}^k) \to C(\mathbb{P}^k)$ by the formula

$$
\mathcal{L}_{\tilde{\phi}_q} g(z) = \sum_{x \in f^{-q}(z)} e^{\tilde{\phi}_q(x)} g(x),
$$

where the summation is taken over all the points of $f^{-q}(z)$ counted with multiplicities. Similarly, as in Preliminaries, $\mathcal{L}_{\tilde{\phi}_q} : C(\mathbb{P}^k) \to C(\mathbb{P}^k)$, is a bounded linear operator acting on $C(\mathbb{P}^k)$. It is also called the Perron-Frobenius (transfer) operator associated to the potential $\tilde{\phi}$. Define the operators $\hat{\mathcal{L}}_{\tilde{\phi}_q} : C(\mathbb{P}^k) \to C(\mathbb{P}^k)$ and $\hat{\mathcal{L}}_\phi : C(J) \to C(J)$ by the formulas

$$
\hat{\mathcal{L}}_\phi = \lambda^{-1}\mathcal{L}_\phi \ \text{ and } \hat{\mathcal{L}}_{\tilde{\phi}_q} = \lambda^{-q}\mathcal{L}_{\tilde{\phi}_q},
$$

Our goal now is to prove some sufficiently good uniform upper and lower bounds on the iterates $\hat{\mathcal{L}}_\phi^n$, $n \geq 0$. This will be done inductively. Fix

$$
0 < \eta \leq \Delta_q.
$$

For every $n \geq 0$, set

$$
\hat{\mathcal{L}}_q^n 1\!\!1 = \hat{\mathcal{L}}_{\tilde{\phi}_q}^n 1\!\!1|_{B^c(A_*,\Delta_q)} \ \text{ and } \hat{\mathcal{L}}_{J,q}^n 1\!\!1 = \hat{\mathcal{L}}_\phi^{qn} 1\!\!1|_{B^c(A_J^*,\Delta_q)}.
$$

Set $R_q = R(\eta) > 0$. For every $n \geq q$, every $z \in \mathbb{P}^k \setminus B(A_*,\eta)$, every $\xi \in D(z)$, and every $w \in \Gamma_{z,\xi} \cap B(z,R_q)$, set

$$
G_{\tilde{\phi}_q,z,\xi}^{(n)}(w) = \sum_{x \in f^{-qn}(w) \cap \cup W_n(q,\Delta_q,z,\xi)} \lambda^{-qn}\exp\big(S_n\tilde{\phi}_q(x)\big),
$$

and

$$
B_{\tilde{\phi}_q,z,\xi}^{(n)}(w) = \sum_{x \in f^{-qn}(w) \cap \cup (Z_n(q,\Delta_q,z,\xi) \setminus W_n(q,\Delta_q,z,\xi))} \lambda^{-qn}\exp\big(S_n\tilde{\phi}_q(x)\big).
$$

It follows from (5.9) that

$$\tilde{C}_q \le \frac{G^{(n)}_{\tilde{\phi}_q,z,\xi}(w)}{G^{(n)}_{\tilde{\phi}_q,z,\xi}(z)} \le \tilde{C}_q. \tag{5.10}$$

It also follows from Lemma 3.6($c_n$) and ($d_n$), and from (5.1) that
(5.11)

$$\begin{aligned}
B^{(n)}_{\tilde{\phi}_q,z,\xi}(w) &\le \lambda^{-qn}\exp(\sup(\tilde{\phi}_q))\#(Z_n(q,\Delta_q,z,\xi)\setminus W_n(q,\Delta_q,z,\xi)) \\
&\le \gamma_\beta^{-sq}\exp(qn(\sup(\phi)-\log\lambda))\big(4c\gamma_\beta^{-3qn}+(k-1)^{d+1}q\big)d^{(k-1)q(n+1)} \\
&= \gamma_\beta^{-sq}\big(4c\gamma_\beta^{-3qn}+(k-1)^{d+1}q\big)\exp\Bigg(qn((k-1)\log d+\frac{k-1}{qn}\log d+ \\
&\quad +\sup(\phi)-\log\lambda)\Bigg) \\
&\le e^{-\theta qn}.
\end{aligned}$$

for every $n \ge s$, every $z \in \mathbb{P}^k \setminus B(A_*,\eta)$, every $\xi \in D(z)$, and every $w \in \Gamma_{z,\xi} \cap B(z,R_q)$. Now, let $L_\infty(\overline{B}(A_*,\Delta_q))$ be the Banach space of all real-valued bounded functions on $\overline{B}(A_*,\Delta_q)$. For every $h \in L_\infty(\overline{B}(A_*,\Delta_q))$ and every $z \in \overline{B}(A_*,\Delta_q)$, let

$$\hat{\mathcal{L}}_* g(z) = \sum_{x\in f^{-q}(z)\cap\overline{B}(A_*,\Delta_q))} \lambda^{-q}\exp\big(\tilde{\phi}_q(x)\big)g(x). \tag{5.12}$$

Obviously, $\hat{\mathcal{L}}_* g(z)$ is a linear operator acting on $L_\infty(\overline{B}(A_*,\Delta_q))$. If $z \in \overline{B}(A_J^*,\Delta_q))$, then it follows from (5.3), (5.4), (4.4), and (4.12) that,

$$\begin{aligned}
\hat{\mathcal{L}}_*1\!\!1(z) &\le \lambda^{-q}d^{g_*q}e^{\sup(\tilde{\phi}_q)} \le \exp\big(q(\sup(\phi)-\log\lambda+g_*\log d)\big) \\
&< \exp\big(\sup(\phi)-\log\lambda+k\log d-\beta\big) \\
&< e^{-2\theta q} < e^{-\theta q}.
\end{aligned} \tag{5.13}$$

If, on the other hand, $z \in \overline{B}(A_*\setminus B(A_J^*,\Delta_q^{(1)}),\Delta_q^{(2)})$, then it follows from (5.7), and the definitions of $\phi_q$ and $\tilde{\phi}_q$, that

$$\hat{\mathcal{L}}_*1\!\!1(z) \le \lambda^{-q}d^{kq}e^{q\tau} \le e^{-\theta q}. \tag{5.14}$$

Both (5.13) and (5.14) along with (5.6), imply that

$$||\hat{\mathcal{L}}_*|| = ||\hat{\mathcal{L}}_*1\!\!1||_\infty \le e^{-\theta q}.$$

Consequently, for all $n \ge 0$ and all $z \in \overline{B}(A_*,\Delta_q)$,

$$\hat{\mathcal{L}}_*^n 1\!\!1(z) \le e^{-\theta qn}. \tag{5.15}$$

Now, fix $z \in \mathbb{P}^k \setminus B(A_*,\eta)$, $\xi \in D(z)$, and $w \in \Gamma_{z,\xi}\cap B(z,R_q)$. Set

$$Z_j = Z_j(q,\eta,z,\xi) \ \text{ and } W_j = W_j(q,\eta,z,\xi).$$

We then have

$$
\begin{aligned}
\hat{\mathcal{L}}^n_{\tilde{\phi}_q}1\!\!1(w) = &\sum_{j=s}^{n} \sum_{x\in f^{-qj}(w)\cap B^c(A_*,\Delta_q)\cap(\cup(Z_j\setminus W_j))} \lambda^{-qj}\exp(S_j\tilde{\phi}_q(x))\hat{\mathcal{L}}_q^{n-j}1\!\!1(x)+\\
&+\sum_{j=s}^{n}\sum_{x_1\in\Lambda_1(w)}\lambda^{-qj}\exp(S_j\tilde{\phi}_q(x_1))\sum_{i=0}^{n-j}\sum_{x_2\in\Lambda_2(x_1)}\lambda^{-qi}\exp(S_i\tilde{\phi}_q(x_2))\,\cdot\\
&\cdot\sum_{x_3\in\Lambda_2(x_2)}\lambda^{-q}e^{\tilde{\phi}_q(x_3)}\hat{\mathcal{L}}_q^{n-(j+i+1)}1\!\!1(x_3)+G^{(n)}_{\tilde{\phi}_q,z,\xi}(w),
\end{aligned}
\tag{5.16}
$$

where

$$
\begin{aligned}
\Lambda_1(w) &= f^{-qj}(w)\cap B(A_*,\Delta_q)\cap(\cup(Z_j\setminus W_j)),\\
\Lambda_2(x_1) &= f^{-qi}(x_1)\cap\bigcap_{l=0}^{i} f^{-ql}(B(A_*,\Delta_q)),\\
\Lambda_3(x_2) &= f^{-q}(x_2)\cap B^c(A_*,\Delta_q).
\end{aligned}
$$

Denote the first summand in (5.16) by $\Sigma_1^{(n)}(w)$ and the second one by $\Sigma_2^{(n)}(w)$. We will estimate each of them separately. Set for all $l\ge s$,

$$
M_l^*(\tilde{\phi}_q) = \max\left\{\left\|\hat{\mathcal{L}}_q^j1\!\!1\right\|_\infty : s\le j\le l\right\}
$$

and

$$
M_l^*(\phi) = \max\left\{\left\|\hat{\mathcal{L}}_\phi^{qj}(1\!\!1)\right\|_\infty : s\le j\le l\right\}.
$$

Because of (5.11), we have

$$
\begin{aligned}
\Sigma_1^{(n)}(w) &\le \sum_{j=s}^{n}\sum_{x\in f^{-qj}(w)\cap B^c(A_*,\Delta_q)\cap\cup(Z_j\setminus W_j))}\lambda^{-qj}\exp(S_j\tilde{\phi}_q(x))M^*_{n-1}(\tilde{\phi}_q)\\
&\le M^*_{n-1}(\tilde{\phi}_q)\sum_{j=s}^{n}B^{(j)}_{\tilde{\phi}_q,z,\xi}(w)\\
&\le M^*_{n-1}(\tilde{\phi}_q)\sum_{j=s}^{n}e^{-\theta qj}\\
&\le (1-e^{-q\theta})^{-1}e^{-\theta sq}M^*_{n-1}(\tilde{\phi}_q).
\end{aligned}
\tag{5.17}
$$

Because of (5.11) and (5.15) we have
(5.18)

$$
\begin{aligned}
\Sigma_2^{(n)}(w) \le & \sum_{j=s}^{n} \sum_{x_1\in\Lambda_1(w)} \lambda^{-qj}\exp(S_j\tilde{\phi}_q(x_1)) \sum_{i=0}^{n-j} \sum_{x_2\in\Lambda_2(x_1)} \lambda^{-qi}\exp(S_i\tilde{\phi}_q(x_2)) \sum_{x_3\in\Lambda_3(x_2)} \lambda^{-q} e^{\tilde{\phi}_q(x_3)} M_{n-1}^*(\tilde{\phi}_q) \\
&\le M_{n-1}^*(\tilde{\phi}_q) \sum_{j=s}^{n} \sum_{x_1\in\Lambda_1(w)} \lambda^{-qj}\exp(S_j\tilde{\phi}_q(x_1)) \sum_{i=0}^{n-j} \sum_{x_2\in\Lambda_2(x_1)} \lambda^{-qi}\exp(S_i\tilde{\phi}_q(x_2)) \hat{\mathcal{L}}_{\phi_q}\mathbb{1}(x_2) \\
&\le M_{n-1}^*(\tilde{\phi}_q) \left\|\hat{\mathcal{L}}_{\phi_q}\right\| \sum_{j=s}^{n} \sum_{x_1\in\Lambda_1(w)} \lambda^{-qj}\exp(S_j\tilde{\phi}_q(x_1)) \sum_{i=0}^{n-j} \sum_{x_2\in\Lambda_2(x_1)} \lambda^{-qi}\exp(S_i\tilde{\phi}_q(x_2)) \\
&\le M_{n-1}^*(\tilde{\phi}_q) \left\|\hat{\mathcal{L}}_{\phi_q}\right\| \sum_{j=s}^{n} \sum_{x_1\in\Lambda_1(w)} \lambda^{-qj}\exp(S_j\tilde{\phi}_q(x_1)) \sum_{i=0}^{n-j} \tilde{\mathcal{L}}_*^i \mathbb{1}(x_1) \\
&\le M_{n-1}^*(\tilde{\phi}_q) \left\|\hat{\mathcal{L}}_{\phi_q}\right\| \sum_{j=s}^{n} \sum_{x_1\in\Lambda_1(w)} \lambda^{-qj}\exp(S_j\tilde{\phi}_q(x_1)) \sum_{i=0}^{n-j} e^{-\theta q i} \\
&\le M_{n-1}^*(\tilde{\phi}_q) \left\|\hat{\mathcal{L}}_{\phi_q}\right\| \sum_{j=s}^{n} \sum_{x_1\in\Lambda_1(w)} \lambda^{-qj}\exp(S_j\tilde{\phi}_q(x_1)) (1-e^{-q\theta})^{-1} \\
&\le M_{n-1}^*(\tilde{\phi}_q) \left\|\hat{\mathcal{L}}_{\phi_q}\right\| (1-e^{-q\theta})^{-1} \sum_{j=s}^{n} B_{\tilde{\phi}_q,z,\xi}^{(j)}(w) \\
&\le M_{n-1}^*(\tilde{\phi}_q) \left\|\hat{\mathcal{L}}_{\phi_q}\right\| (1-e^{-q\theta})^{-1} \sum_{j=s}^{\infty} e^{-\theta q j} \\
&\le (1-e^{-q\theta})^{-2} \left\|\hat{\mathcal{L}}_{\phi_q}\right\| e^{-\theta q s} M_{n-1}^*(\tilde{\phi}_q) \\
&\le \lambda^{-q} e^{\sup(\phi_q)} d^{qk} (1-e^{-q\theta})^{-2} e^{-\theta q s} M_{n-1}^*(\tilde{\phi}_q) \\
&\le \big(\lambda^{-1} e^{\sup(\phi)} d^k\big)^q (1-e^{-q\theta})^{-2} e^{-\theta q s} M_{n-1}^*(\tilde{\phi}_q)
\end{aligned}
$$

Combining (5.16), (5.17) and (5.18) together, and then making use of (5.2), we get for all $n \ge 1$, all $z \in \mathbb{P}^k \setminus B(A_*, \Delta_q)$, all $\xi \in D(z)$, and all $w \in \Gamma_{z,\xi} \cap B(z, R_q)$ that,
(5.19)

$$
\begin{aligned}
\hat{\mathcal{L}}_{\tilde{\phi}_q}^n \mathbb{1}(w) \le & (1-e^{-q\theta})^{-1} e^{-\theta s q} M_{n-1}^*(\tilde{\phi}_q) + \big(\lambda^{-1} e^{\sup(\phi)} d^k (1-e^{-q\theta})^{-2}\big) e^{-\theta s q} M_{n-1}^*(\tilde{\phi}_q) + \\
& \qquad + G_{\tilde{\phi}_q,z,\xi}^{(n)}(w) \le \\
&\le (1-e^{-q\theta})^{-1} \big(1 + (1-e^{-q\theta})^{-1} \lambda^{-q} e^{q\sup(\phi)} d^{qk}\big) e^{-\theta s q} M_{n-1}^*(\tilde{\phi}_q) + G_{\tilde{\phi}_q,z,\xi}^{(n)}(w) \\
&\le G_{\tilde{\phi}_q,z,\xi}^{(n)}(w) + \frac{1}{4} M_{n-1}^*(\tilde{\phi}_q).
\end{aligned}
$$

If $z \in J \setminus B(A_*, \eta)$ and $w = z$, we thus get

$$\hat{\mathcal{L}}_\phi^{qn}\mathbb{1}(w) \le G^{(n)}_{\tilde{\phi}_q,w,\xi}(w) + \frac{1}{4}M^*_{n-1}(\phi) \tag{5.20}$$

for all $n \ge s$ and all $\xi \in D(z)$. Now, it follows from (5.10) that

$$\tilde{C}_q^{-1} \le \frac{G^{(n)}_{\tilde{\phi}_q,w,\xi}(w)}{G^{(n)}_{\tilde{\phi}_q,w,\xi}(z)} \le \tilde{C}_q. \tag{5.21}$$

In view of (4.10), we get for every $z \in J$ that

$$1 = \int \hat{\mathcal{L}}_\phi^q \mathbb{1} dm_\phi \ge \int_{B(z,R_q)} \hat{\mathcal{L}}_\phi^q \mathbb{1} dm_\phi \ge \hat{C}_q^{-1}\hat{\mathcal{L}}_\phi^q \mathbb{1}(x)$$

with some $x \in J \cap B(z, R_q)$, where $\hat{C}_q^{-1} = \inf\{m_\phi(B(y, R_q/2)) : y \in J\}$ is positive in virtue of Proposition 4.5. Hence $\hat{\mathcal{L}}_\phi^q \le \hat{C}_q$. Since the function $\hat{\mathcal{L}}^n_{\tilde{\phi}_q}\mathbb{1} : \mathbb{P}^k \to \mathbb{R}$ is continuous, there exists $T \in (0, R_q - \rho(z, x))$ such that $\hat{\mathcal{L}}^n_{\tilde{\phi}_q}(\mathbb{1})(w) \le 2\hat{C}_q$ for all $w \in B(x, T) \subset B(z, R_q)$. Now assume that $z \in J \setminus B(A^*_J, \Delta_q)$. Since the set $D(z)$ is dense in $B(z, R_q)$, there thus exists $y \in D(z) \cap B(x, T)$ such that $\hat{\mathcal{L}}^n_{\tilde{\phi}}(\mathbb{1})(y) \le 3\hat{C}_q$. So, $G^{(n)}_{\tilde{\phi}_q,z,y}(y) \le 3\hat{C}_q$. Along with (5.21), this implies that

$$G^{(n)}_{\tilde{\phi}_q,z,y}(z) \le \tilde{C}_q G^{(n)}_{\tilde{\phi}_q,z,y}(y) \le C_q := 3\hat{C}_q\tilde{C}_q.$$

Inserting this into (5.20), we get

$$\hat{\mathcal{L}}_\phi^n\mathbb{1}(z) \le C_q + \frac{1}{4}M^*_{n-1}(\phi) \tag{5.22}$$

for all $n \ge s$ and all $z \in J \setminus B(A_*, \eta)$. Thus, (remember that $\eta \le \Delta_q$),

$$M^*_n(\phi) \le C_q + \frac{1}{4}M^*_{n-1}(\phi). \tag{5.23}$$

Now, we can prove by induction the following.

**Lemma 5.1.** *There exists a constant $Q_q^+ > 0$ such that $||\hat{\mathcal{L}}_\phi^{qn}||_\infty \le Q_q^+$ for all $n \ge 0$.*

*Proof.* Put $Q_q^* = \max\left\{\frac{4}{3}C_q, M^*_{s-1}(\phi)\right\}$. We shall show first by induction that $M^*_n(\phi) \le Q^*_q$ for every $n \ge s-1$. The case $n = s-1$ is obvious. So, suppose that $n \ge s$ and $M^*_{n-1}(\phi) \le Q^*_q$. We then get by (5.23) that

$$M^*_n(\phi) \le C_q + \frac{1}{4}Q^*_q \le \frac{3}{4}Q^*_q + \frac{1}{4}Q^*_q = Q^*_q. \tag{5.24}$$

The inductive proof is complete, and in fact (5.24) holds for all $n \geq 0$. Now, fix $n \geq 0$ and $w \in B(A_*, \Delta_q)$. It then follows from (5.15) that

(5.25)

$$
\begin{aligned}
\hat{\mathcal{L}}^n_{\tilde{\phi}_q}(w) &= \sum_{j=0}^{n} \sum_{x \in f^{-qj}(w) \cap \cap_{l=0}^{j} f^{-ql}(B(A_*,\Delta_q))} \lambda^{-j} \exp(S_j \tilde{\phi}_q(x)) \sum_{y \in f^{-q}(x_2) \cap B^c(A_*,\Delta_q)} \lambda^{-q} e^{\tilde{\phi}_q(y)} \hat{\mathcal{L}}_q^{n-j-1} 1\!\!1(y) \\
&\leq M^*_{n-1}(\tilde{\phi}_q) \sum_{j=0}^{n} \sum_{x \in f^{-qj}(w) \cap \cap_{l=0}^{j} f^{-ql}(B(A_*,\Delta_q))} \lambda^{-qj} \exp(S_j \tilde{\phi}_q(x)) \sum_{y \in f^{-q}(x_2) \cap B^c(A_*,\Delta_q)} \lambda^{-q} e^{\tilde{\phi}_q(y)} \\
&\leq M^*_{n-1}(\tilde{\phi}_q) ||\tilde{\mathcal{L}}_{\phi_q} 1\!\!1||_\infty \sum_{j=0}^{n} \sum_{x \in f^{-qj}(w) \cap \cap_{l=0}^{j} f^{-qj}(B(A_*,\Delta_q))} \lambda^{-qj} \exp(S_j \tilde{\phi}_q(x)) \\
&\leq M^*_{n-1}(\tilde{\phi}_q) \big(\lambda^{-1} d^k e^{\sup(\phi)}\big)^q \sum_{j=0}^{n} \hat{\mathcal{L}}^j_*(w) \\
&\leq M^*_{n-1}(\tilde{\phi}_q) \big(\lambda^{-1} d^k e^{\sup(\phi)}\big)^q \sum_{j=0}^{n} e^{-\theta qj} \\
&\leq \big(\lambda^{-1} d^k e^{\sup(\phi)}\big)^q (1 - e^{-q\theta})^{-1} M^*_{n-1}(\tilde{\phi}_q).
\end{aligned}
$$

Moreover, if $w \in B(A_*, \Delta_q) \cap J$, then $M^*_{n-1}(\tilde{\phi}_q)$ can be replaced by $M^*_{n-1}(\phi)$, and then along with (5.24), this estimate gives,

$$
\hat{\mathcal{L}}^{qn}_{\phi}(w) \leq (\lambda^{-1} d^k e^{\sup(\phi)}\big)^q (1 - e^{-q\theta})^{-1} Q^*_q.
$$

We are therefore done by setting

$$
Q^+_q = Q^*_q \max\Big\{1, \big(\lambda^{-1} d^k e^{\sup(\phi)}\big)^q (1 - e^{-q\theta})^{-1}\Big\}.
$$

□

**Lemma 5.2.** *There exists a constant $Q^-_q > 0$ such that $\hat{\mathcal{L}}^{qn}_{\phi} 1\!\!1(z) \geq Q^-_q$ for all $n \geq 0$ and all $z \in J$.*

*Proof.* Fix $n \geq s$. Take $z_n \in J \setminus B(A_*, \Delta_q)$ such that $\hat{\mathcal{L}}^n_q 1\!\!1(z_n) = M^*_n(\phi)$. It then follows from (5.20) that

$$
M^*_n(\phi) \leq G^{(n)}_{\tilde{\phi}_q, z_n, \xi}(z_n) + \frac{1}{4} M^*_{n-1}(\phi) \leq G^{(n)}_{\tilde{\phi}_q, z_n, \xi}(z_n) + \frac{1}{4} M^*_n(\phi).
$$

for every $\xi \in D_q(z_n)$. Therefore,

$$
G^{(n)}_{\tilde{\phi}_q, z_n, \xi}(z_n) \geq \frac{3}{4} M^*_n(\phi) \tag{5.26}
$$

But $\int \hat{\mathcal{L}}^{qn}_{\phi} 1\!\!1 dm_\phi = \int 1\!\!1 dm_\phi = 1$, and so, there exists a point $y_n \in J$ such that $\hat{\mathcal{L}}^{qn}_{\phi} 1\!\!1(y_n) \geq 1$. If $y_n \in J \setminus B(A_*, \Delta_q)$, the we get that

$$
M^*_n(\phi) \geq \hat{\mathcal{L}}^{qn}_{\phi} 1\!\!1(y_n) \geq 1.
$$

Otherwise, it follows from (5.25) that

$$\begin{aligned} 1 &\le \hat{\mathcal{L}}_{\phi}^{qn}1\!\!1(y_n) \\ &\le \big(\lambda^{-1}d^k e^{\sup(\phi)}\big)^q(1-e^{-q\theta})^{-1}M_{n-1}^*(\tilde{\phi}_q). \\ &\le \big(\lambda^{-1}d^k e^{\sup(\phi)}\big)^q(1-e^{-q\theta})^{-1}M_n^*(\tilde{\phi}_q) \end{aligned}$$

Thus,

$$M_n^*(\phi) \ge \big(\lambda d^{-k}e^{-\sup(\phi)}\big)^q(1-e^{-q\theta}).$$

In either case,

$$M_n^*(\phi) \ge M := \min\Big\{1, \big(\lambda d^{-k}e^{-\sup(\phi)}\big)^q(1-e^{-q\theta})\Big\}.$$

Hence, by (5.26),

$$G_{\tilde{\phi}_q,z_n,\xi}^{(n)}(z_n) \ge \frac{3}{8}M$$

Thus, using (5.10) we obtain for every $\xi \in D_q(z_n)$, that

$$G_{\tilde{\phi}_q,z_n,\xi}^{(n)}(\xi) \ge 3(8\hat{C}_q)^{-1}M.$$

Consequently, we get for every $\xi \in D_q(z_n)$, that

$$\hat{\mathcal{L}}_{\tilde{\phi}_q}^{n}1\!\!1(\xi) \ge G_{\tilde{\phi}_q,z_n,\xi}^{(n)}(\xi) \ge 3(8\hat{C}_q)^{-1}M.$$

Since $\hat{\mathcal{L}}_{\tilde{\phi}_q}^{n}1\!\!1$ is continuous and $D_q(z_n)$ is dense in $B(z_n,R_q)$, this inequality extends to all $\xi \in B(z_n,R_q)$. Since, by Proposition 2.2, the map $f^q : J \to J$ is topologically exact, there exists $l \ge 1$ such that $f^{ql}(B(z_n,R_q)\cap J) = J$ for all $n \ge 1$. Hence, for every $x \in J$ and every $n \ge l+s$, there exists $\xi \in B(z_n,R_q)\cap J$ such that $f^{ql}(\xi) = x$. Therefore,

$$\hat{\mathcal{L}}_{\phi}^{qn}1\!\!1(x) \ge \lambda^{-ql}\exp(ql\inf(\phi))\hat{\mathcal{L}}_{\phi}^{q(n-l)}1\!\!1(\xi) \ge 3(4\hat{C}_q)^{-1}M\lambda^{-ql}\exp(ql\inf(\phi)).$$

We are therefore done. □

## 6. Almost Periodicity of the Perron-Frobenius Operator

As an immediate consequence of (5.21) and Lemma 5.1,

$$G_{\tilde{\phi}_q,z,\xi}^{(n)}(w) \le \tilde{C}_q Q_q^+ \tag{6.1}$$

for all $z \in J \setminus B(A_*,\eta)$, $\xi \in D(z)$ and all $w \in B(z,R(\eta))\cap\Gamma_{z,\xi}$. Now, we shall prove the following.

**Lemma 6.1.** *There exists a Hölder continuous function $\hat{\phi}_q : \mathbb{P}^k \to \mathbb{R}$ with the following properties.*

(a) *There exists a neighborhood $U \subset \mathbb{P}^k$ of $J$ such that $\hat{\phi}_q|_U = \tilde{\phi}_q$. In particular, $\hat{\phi}_q|_J = \sum_{j=0}^{q-1}\phi\circ f^j$.*

(b) $\hat{\phi}_q \le \tilde{\phi}_q$ *throughout* $\mathbb{P}^k$.
(c) $\hat{Q}_q := \sup_{n\ge 0}\{||\hat{\mathcal{L}}^n_{\hat{\phi}_q}1\!\!1||_\infty\} < +\infty$.

*Proof.* Assume without loss of generality that

$$R_q \le \frac{1}{2}\Delta_q.$$

Consider two sets

$$B_q = B(J \setminus B(A_*, \Delta_q/2), R_q)$$

and

$$(\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q).$$

We shall show that

$$B(J, R_q) \cap \big((\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q)\big) = \emptyset. \tag{6.2}$$

Indeed, suppose for the contrary that there is some $z \in B(J, R_q) \cap \big((\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q)\big)$. Then $z \in B(J, R_q)$ and $z \notin B(J \setminus B(A_*, \Delta_q/2), R_q)$. Hence, $z \in B(J \cap B(A_*, \Delta_q/2), R_q)$. So, $z \in B(A_*, \frac{1}{2}\Delta_q + R_q) \subset B(A_*, \Delta_q)$. This contradiction finishes the proof of (6.2).

Since $f^q(J) = J$, there exists $\varepsilon_q \in (0, R_q/2)$ such that

$$\overline{B}(J, \varepsilon_q) \cap \Big((\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q)\big) \cup f^{-q}\big((\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q)\big)\Big) = \emptyset.$$

Fix $t > 0$ so large that

$$e^{-t}(\lambda^{-2}d^{2k}e^{\sup(\phi)})^q(1 + e^{-q\theta}(1 - e^{-q\theta})^{-1}) \le \frac{1}{4}. \tag{6.3}$$

and

$$-t \le \inf\Big\{\tilde{\phi}_q(z) : z \in (\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q)\big) \cup f^{-q}\big((\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q)\big)\Big\}.$$

So, by Lemma 2.1 there exists a Hölder continuous function $\hat{\phi}_q : \mathbb{P}^k \to \mathbb{R}$ such that $\hat{\phi}_q|_{\overline{B}(J,\varepsilon_q)} = \tilde{\phi}_q$, $\hat{\phi}_q$ restricted to $(\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q)\big) \cup f^{-q}\big((\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q)\big)$ is equal to $-t$ and $\sup(\hat{\phi}_q) \le \max\Big\{\sup\big(\tilde{\phi}_q|_{\overline{B}(J,\varepsilon_q)}\big), -t\Big\} \le \sup(\tilde{\phi}_q)$. Conditions (a) and (b) are satisfied by the very definition of $\hat{\phi}$ with $U = B(J, \varepsilon_q)$. Put

$$M_n^{(1)} = \sup\{\hat{\mathcal{L}}^n_{\hat{\phi}_q}1\!\!1(z) : z \in B_q)\}, \quad M_n^{(2)} = \sup\{\hat{\mathcal{L}}^n_{\hat{\phi}_q}1\!\!1(z) : z \in (\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q)\}.$$

and

$$M_n^*(\hat{\phi}_q) = \sup\{\hat{\mathcal{L}}^n_{\hat{\phi}_q}1\!\!1(z) : z \in \mathbb{P}^k \setminus B(A_*, \Delta_q)\} = \max\{M_n^{(1)}, M_n^{(2)}\}.$$

Fix now an arbitrary $z \in J \setminus B(A_*, \Delta_q/2)$, $\xi \in D(z)$, and $w \in \Gamma_{z,\xi} \cap B(z, R_q)$. Since $\hat{\phi}_q \le \tilde{\phi}_q$, it then follows from (5.19), applied with $\eta = \Delta_q/2$), that

$$\hat{\mathcal{L}}^n_{\hat{\phi}_q}(1\!\!1)(w) \le G^{(n)}_{\tilde{\phi}_q,z,\xi}(w) + \frac{1}{4}M^*_{n-1}(\hat{\phi}_q).$$

Applying (6.1) we thus further get

$$\hat{\mathcal{L}}^n_{\hat{\phi}_q}(1\!\!1)(w) \le \tilde{C}_q Q_q^+ + \frac{1}{4} M^*_{n-1}(\hat{\phi}_q).$$

Since the union

$$\bigcup \{\Gamma_{z,\xi} \cap B(z, R_q) : z \in z \in J \setminus B(A_*, \Delta_q/2), \xi \in D(z)\}$$

is dense in $B(J \setminus B(A_*, \Delta_q/2), R_q) = B_q$, we thus get that

$$M_n^{(1)} \le \tilde{C}_q Q_q^+ + \frac{1}{4} M^*_{n-1}(\hat{\phi}_q). \tag{6.4}$$

For every $j \ge 0$ put

$$\Lambda_j = \bigcap_{i=0}^{j} f^{-qi}(B(A_*, \Delta_q)).$$

Using the definition of $\hat{\phi}_q$ and (5.15), we get for all $w \in \mathbb{P}^k$ that,

(6.5)

$$
\begin{aligned}
&\hat{\mathcal{L}}^n_{\hat{\phi}_q}(1\!\!1)(w) = \\
&\quad = \sum_{x \in f^{-q}(w) \cap B^c(A_*, \Delta_q)} \lambda^{-q} e^{\hat{\phi}_q(x)} \hat{\mathcal{L}}^{n-1}_{\hat{\phi}_q}(1\!\!1)(x) + \\
&\qquad\qquad + \sum_{j=1}^{n} \sum_{y \in f^{-qj}(w) \cap \Lambda_j} \lambda^{-qj} \exp(S_j \hat{\phi}_q(y)) \sum_{x \in f^{-q}(y) \cap B^c(A_*, \Delta_q)} \lambda^{-q} e^{\hat{\phi}_q(x)} \hat{\mathcal{L}}^{n-(j+1)}_{\hat{\phi}_q}(1\!\!1)(x) + \\
&\qquad\qquad + \sum_{x \in f^{-n}(w) \cap \Lambda_{n-1}} \lambda^{-qn} \exp(S_n \hat{\phi}_q(x)) \\
&\quad \le \sum_{x \in f^{-q}(w) \cap B^c(A_*, \Delta_q)} \lambda^{-q} e^{\sup(\hat{\phi}_q)} \hat{\mathcal{L}}^{n-1}_{\hat{\phi}_q}(1\!\!1)(x) + \\
&\qquad\qquad + M^*_{n-1}(\hat{\phi}_q) \sum_{j=1}^{n} \sum_{y \in f^{-qj}(w) \cap \Lambda_j} \lambda^{-qj} \exp(S_j \hat{\phi}_q(y)) \sum_{x \in f^{-q}(y) \cap B^c(A_*, \Delta_q)} \lambda^{-q} e^{\hat{\phi}_q(x)} + \\
&\qquad\qquad + \sum_{x \in f^{-q}(w)} \lambda^{-q} e^{\hat{\phi}_q(x)} \hat{\mathcal{L}}^{n-1}_{*} 1\!\!1(x) \\
&\quad \le (\lambda^{-1} d^k e^{\sup(\phi)})^q M^*_{n-1}(\hat{\phi}_q) + \\
&\qquad\qquad + M^*_{n-1}(\hat{\phi}_q) \sum_{j=1}^{n} \sum_{y \in f^{-qj}(w) \cap \Lambda_j} \lambda^{-qj} \exp(S_j \hat{\phi}_q(y)) (\lambda^{-1} d^k e^{\sup(\phi)})^q + \\
&\qquad\qquad + (\lambda^{-1} d^k e^{\sup(\phi)})^q e^{-\theta q(n-1)} \\
&\quad \le (\lambda^{-1} d^k e^{\sup(\phi)})^q M^*_{n-1}(\hat{\phi}_q) + M^*_{n-1}(\hat{\phi}_q) (\lambda^{-1} d^k e^{\sup(\phi)})^q \sum_{j=1}^{n} \hat{\mathcal{L}}^j_* 1\!\!1(w) \\
&\qquad\qquad + (\lambda^{-1} d^k e^{\sup(\phi)})^q e^{-\theta q(n-1)} \\
&\quad \le (\lambda^{-1} d^k e^{\sup(\phi)})^q M^*_{n-1}(\hat{\phi}_q) + + M^*_{n-1}(\hat{\phi}_q) (\lambda^{-1} d^k e^{\sup(\phi)})^q \sum_{j=1}^{n} e^{-\theta q j} + \\
&\qquad\qquad + (\lambda^{-1} d^k e^{\sup(\phi)})^q e^{-\theta q(n-1)} \\
&\quad \le (\lambda^{-1} d^k e^{\sup(\phi)})^q M^*_{n-1}(\hat{\phi}_q) + M^*_{n-1}(\hat{\phi}_q) (\lambda^{-1} d^k e^{\sup(\phi)})^q e^{-q\theta} (1 - e^{-q\theta})^{-1} + \\
&\qquad\qquad + (\lambda^{-1} d^k e^{\sup(\phi)})^q \\
&\quad \le T M^*_{n-1}(\hat{\phi}_q) + (\lambda^{-1} d^k e^{\sup(\phi)})^q,
\end{aligned}
$$

where $T = (\lambda^{-1}d^k e^{\sup(\phi)})^q(1+e^{-q\theta}(1-e^{-q\theta})^{-1})$. Therefore, using the definition of $\hat{\phi}_q$ and (6.3), we get for every $w \in (\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q)$, that

$$\begin{aligned}
\hat{\mathcal{L}}^n_{\hat{\phi}_q}(1\!\!1)(w) &= \sum_{x \in f^{-q}(w)} \lambda^{-q} e^{\hat{\phi}_q(x)} \hat{\mathcal{L}}^{n-1}_{\hat{\phi}_q}(1\!\!1)(x) \\
&\le \sum_{x \in f^{-q}(w)} \lambda^{-q} e^{\hat{\phi}_q(x)} \big(T M^*_{n-1}(\hat{\phi}_q) + (\lambda^{-1}d^k e^{\sup(\phi)})^q\big) \\
&= \big(T M^*_{n-1}(\hat{\phi}_q) + (\lambda^{-1}d^k e^{\sup(\phi)})^q\big) \sum_{x \in f^{-q}(w)} \lambda^{-q} e^{-s} \\
&= (\lambda^{-1}d^k)^q e^{-s}\big(T M^*_{n-1}(\hat{\phi}_q) + (\lambda^{-1}d^k e^{\sup(\phi)})^q\big) \\
&\le \frac{1}{4} M^*_{n-1}(\hat{\phi}_q) + \frac{1}{4}.
\end{aligned}$$

Thus,

$$M_n^{(2)} \le \frac{1}{4} + \frac{1}{4} M^*_{n-1}(\hat{\phi}_q).$$

Combining this with (6.4), we get

$$M_n^*(\hat{\phi}_q) \le \max\left\{\frac{1}{4}, \tilde{C}_q Q_q^+\right\} + \frac{1}{4} M^*_{n-1}(\hat{\phi}_q).$$

Now we can prove in the same standard way as in the proof of Lemma 5.1 that

$$M^* := \sup_{n \ge 1}\{M_n^*(\hat{\phi}_q)\} < +\infty.$$

So, applying (6.5). we get for every $n \ge 1$ that

$$||\hat{\mathcal{L}}^n_{\hat{\phi}_q} 1\!\!1||_\infty \le T M^* + (\lambda^{-1}d^k e^{\sup(\phi)})^q.$$

We are done. □

We will need the following strengthening of the distortion property (5.8).

**Lemma 6.2.** *For every $\varepsilon > 0$ and $\eta \in (0, \Delta_q]$ there exists $\delta_1 > 0$ such that*

$$|S_n\hat{\phi}_q(x) - S_n\hat{\phi}_q(y)| \le \varepsilon$$

*for all $n \ge s$, $z \in \mathbb{P}^k \setminus B(A_*, \eta)$, $\xi \in D(z)$, $V \in W_n(q, \eta, z, \xi)$ and all $x, y \in V$ with $\rho(f^{qn}(x), f^{qn}(y)) \le \delta_1$.*

*Proof.* Let $\hat{H} > 0$ be the Hölder constant of the Hölder continuous function $\hat{\phi}_q : \mathbb{P}^k \to \mathbb{R}$ produced in Lemma 6.1. Take $k_s \ge 1$ so large that $\hat{H}\sum_{j=k_s+1}^{\infty} \gamma^{\frac{\alpha}{2}j} \le \varepsilon/2$. Since all the functions $S_j\hat{\phi}_q$, $j = s, s+1, \dots, s+k_s-1$, are continuous (and there are only finitely many of them) it suffices to prove the lemma for all $n \ge s+k_s$. Take $\delta_2 > 0$ so small that

$$|\hat{\phi}_q(b) - \hat{\phi}_q(a)| \le \frac{\varepsilon}{2k_s},$$

whenever $\rho(a,b) \leq \delta_2$. By Lemma 3.6($b_n$) there exists $\delta_1 > 0$ so small that for all $n \geq s + k_s$ and every $n - k_s \leq j \leq n-1$, we have

$$\rho(f^{qj}(x), f^{qj}(y)) \leq \delta_2$$

whenever $z, \xi, V, x, y$ are as in the hypothesis of the lemma. Applying Lemma 3.6($b_n$) again, with $n, z, \xi, V, x, y$ as in the hypothesis of the lemma, we get

$$\begin{aligned}
&|S_n\hat{\phi}_q(x) - S_n\hat{\phi}_q(y)| \leq \\
&\quad \leq \sum_{j=0}^{n-(k_s+1)} |\hat{\phi}_q(f^{qj}(x)) - \hat{\phi}_q(f^{qj}(y))| + \sum_{j=n-k_s}^{n-1} |\hat{\phi}_q(f^{qj}(x)) - \hat{\phi}_q(f^{qj}(y))| \\
&\quad \leq \sum_{j=0}^{n-(k_s+1)} \hat{H}\gamma^{\frac{n-j}{2}\alpha} + \sum_{j=n-k_s}^{n-1} \frac{\varepsilon}{2k_s} \\
&\quad \leq \hat{H} \sum_{i=k_q+1}^{\infty} \gamma^{\frac{\alpha}{2}i} + \frac{\varepsilon}{2} \leq \varepsilon.
\end{aligned}$$

We are done. □

Recall that a bounded linear operator $L : B \to B$ acting on a Banach space $B$ is called almost periodic if and only if for every $x \in B$, the closure $\overline{\{L^n(x) : n \geq 0\}}$ is compact in $B$. We shall prove the following.

**Proposition 6.3.** *The Perron-Frobenius operator $\hat{\mathcal{L}}_{\hat{\phi}_q} : C(\mathbb{P}^k) \to C(\mathbb{P}^k)$ is almost periodic.*

*Proof.* Fix $\varepsilon > 0$. Then take $s \geq 1$ so large that

$$2(1-e^{-q\theta})^{-2}\big(1 - e^{-q\theta} + (\lambda^{-1}e^{\sup(\phi)}d^k)^q\big)e^{-\theta sq} < \varepsilon. \tag{6.6}$$

Fix a function $g \in C(\mathbb{P}^k)$. Generalizing the functions $\Sigma_1^{(n)}(w)$ and $\Sigma_2^{(n)}(w)$, for every $n \geq s$, every $z \in \mathbb{P}^k \setminus B(A_*, \Delta_q)$, every $\xi \in D(z)$, and every $w \in \Gamma_{z,\xi} \cap B(z, R_q)$, set

$$\Sigma_1^{(n)}(g)(w) = \sum_{j=s}^{n} \sum_{x \in f^{-qj}(w) \cap B^c(A_*,\Delta_q) \cap (\cup(Z_j \setminus W_j))} \lambda^{-qj} \exp(S_j\hat{\phi}_q(x))\hat{\mathcal{L}}_q^{n-j}(g)(x)$$

and

$$\begin{aligned}
&\Sigma_2^{(n)}(g)(w) = \\
&= \sum_{j=s}^{n} \sum_{x_1 \in \Lambda(w)} \lambda^{-qj} \exp(S_j\hat{\phi}_q(x_1)) \sum_{i=0}^{n-j} \sum_{x_2 \in \Lambda_2(x_1)} \lambda^{-qi} \exp(S_i\hat{\phi}_q(x_2)) \sum_{x_3 \in \Lambda_2(x_2)} \lambda^{-q} e^{\hat{\phi}_q(x_3)} \hat{\mathcal{L}}_q^{n-(j+i+1)}(g)(x_3).
\end{aligned}$$

We then have by (5.16), (5.17), (5.18), and (6.6), that

(6.7)

$$
\begin{aligned}
&|\hat{\mathcal{L}}^n_{\hat\phi_q} g(w) - \hat{\mathcal{L}}^n_{\hat\phi_q} g(z)| = \\
&\qquad = |(\Sigma_1^{(n)}(g)(w) - (\Sigma_1^{(n)}(g)(z)) + (\Sigma_2^{(n)}(g)(w) - (\Sigma_2^{(n)}(g)(z))+ \\
&\qquad\qquad + (G^{(n)}_{\hat\phi_q,z,\xi}(g)(w) - G^{(n)}_{\hat\phi_q,z,\xi}(g)(z))|+ \\
&\qquad \le |\Sigma_1^{(n)}(g)(w) - (\Sigma_1^{(n)}(g)(z)| + |\Sigma_2^{(n)}(g)(w) - (\Sigma_2^{(n)}(g)(z)| \qquad\qquad +| \\
&\qquad \le |\Sigma_1^{(n)}(g)(w)| + |\Sigma_1^{(n)}(g)(z)| + |\Sigma_2^{(n)}(g)(w)| + |\Sigma_2^{(n)}(g)(z)|+ \\
&\qquad\qquad + |G^{(n)}_{\hat\phi_q,z,\xi}(g)(w) - G^{(n)}_{\hat\phi_q,z,\xi}(g)(z)| \\
&\qquad \le |\Sigma_1^{(n)}(w)|||g||_\infty| + |\Sigma_1^{(n)}(z)|||g||_\infty + |\Sigma_2^{(n)}(w)|||g||_\infty + |\Sigma_2^{(n)}(z)|||g||_\infty+ \\
&\qquad\qquad + |G^{(n)}_{\hat\phi_q,z,\xi}(g)(w) - G^{(n)}_{\hat\phi_q,z,\xi}(g)(z)| \\
&\qquad \le 2(1-e^{-q\theta})^{-1} e^{-\theta sq}\hat{Q}_q||g||_\infty| + 2(\lambda^{-1}d^k e^{\sup(\phi)})^q(1-e^{-q\theta})^{-2}e^{-\theta sq}\hat{Q}_q||g||_\infty|+ \\
&\qquad\qquad + |G^{(n)}_{\hat\phi_q,z,\xi}(g)(w) - G^{(n)}_{\hat\phi_q,z,\xi}(g)(z)| \\
&\qquad \le 2(1-e^{-q\theta})^{-2}\big(1-e^{-q\theta} + (\lambda^{-1}e^{\sup(\phi)}d^k)^q\big)\hat{Q}_q e^{-\theta sq}||g||_\infty|+ \\
&\qquad\qquad + |G^{(n)}_{\hat\phi_q,z,\xi}(g)(w) - G^{(n)}_{\hat\phi_q,z,\xi}(g)(z)| \\
&\qquad \le \hat{Q}_q||g||_\infty\varepsilon + |G^{(n)}_{\hat\phi_q,z,\xi}(g)(w) - G^{(n)}_{\hat\phi_q,z,\xi}(g)(z)|.
\end{aligned}
$$

Denote $G^{(n)}_{\hat\phi_q,z,\xi}(g)$ by $G_n(g)$. As before, set

$$Z_j = Z_j(q,\eta,z,\xi) \text{ and } W_j = W_j(q,\eta,z,\xi).$$

Fix $\xi \in D(z)$, $n \ge 1$, and $V$, an arbitrary connected component in $W_n$. Since the map $f^{qn}|_V : V \to \Gamma_{z,\xi} \cap B(z,R_q)$ is proper, the degree of this map is well defined and is constant throughout $V$. By Lemma 3.6($d_n$) this degree is bounded above by $\gamma^{-sq}$. Let $\hat{f}_\xi^{-qn}(w)$ be the collection of all points $x$ from $f^{-qn}(w) \cap \cup W_n)$, each repeated according to the local degrees. Let $\sigma$ be an arbitrary bijection from $f_\xi^{-qn}(z)$ to $f_\xi^{-qn}(w)$ respecting all components $V \in W_n$. Put $\delta_3 = \min\{R_q, \delta_1, \delta_2\}$, where $\delta_1$ and $\delta_2$ come from Lemma 6.2. Using Lemma 6.2, and Lemma 6.1, we thus get for all

$n \geq s$, all $z \in \mathbb{P}^k \setminus B(A_*, \Delta_q)$, all $\xi \in D(z)$, and all $w \in \Gamma_{z,\xi} \cap B(z, \delta_3)$, that

$$
\begin{aligned}
&|G_n(g)(w) - G_n(g)(z)| = \\
&\quad = \Bigg| \sum_{x \in \hat{f}_\xi^{-qn}(z) \cap \cup W_n} \Big( \lambda^{-qn} \exp\big(S_n \hat{\phi}_q(\sigma(x))\big) g(\sigma(x)) - \lambda^{-qn} \exp\big(S_n \hat{\phi}_q(x)\big) g(x) \Big) \Bigg| \\
&\quad \leq \sum_{x \in \hat{f}_\xi^{-qn}(z) \cap \cup W_n} \lambda^{-qn} \Big| \exp\big(S_n \tilde{\phi}(\sigma(x))\big) - \exp\big(S_n \tilde{\phi}(x)\big) \Big| \, |g(\sigma(x))| + \\
&\qquad\qquad + \sum_{x \in \hat{f}_\xi^{-qn}(z) \cap \cup W_n} \lambda^{-qn} \exp\big(S_n \tilde{\phi}(x)\big) |g(\sigma(x)) - g(x)| \\
&\quad \leq ||g||_\infty \sum_{x \in \hat{f}_\xi^{-qn}(z) \cap \cup W_n} \lambda^{-qn} \max\Big\{ \exp\big(S_n \tilde{\phi}(\sigma(x))\big), \exp\big(S_n \tilde{\phi}(x)\big) \Big\} \, |S_n \tilde{\phi}(\sigma(x)) - S_n \tilde{\phi}(x)| + \\
&\qquad\qquad + \varepsilon ||g||_\infty \sum_{x \in \hat{f}_\xi^{-qn}(z) \cap \cup W_n} \lambda^{-qn} \exp\big(S_n \hat{\phi}_q(x)\big) + \\
&\quad \leq ||g||_\infty \Bigg( \varepsilon \sum_{x \in \hat{f}_\xi^{-qn}(z) \cap \cup W_n} \lambda^{-qn} \max\Big\{ \exp\big(S_n \tilde{\phi}(\sigma(x))\big), \exp\big(S_n \tilde{\phi}(x)\big) \Big\} + \varepsilon \hat{\mathcal{L}}_{\hat{\phi}_q} 1\!\!1(z) \Bigg) \\
&\quad \leq \varepsilon ||g||_\infty \big( \max\{ \hat{\mathcal{L}}_{\hat{\phi}_q} 1\!\!1(z), \hat{\mathcal{L}}_{\hat{\phi}_q} 1\!\!1(w) \} + \hat{Q}_q \big) \\
&\quad \leq 2 \hat{Q}_q ||g||_\infty \varepsilon
\end{aligned}
$$

Combining this with (6.7) we get that,

$$
|\hat{\mathcal{L}}_{\hat{\phi}_q}^n g(w) - \hat{\mathcal{L}}_{\hat{\phi}_q}^n g(z)| \leq \hat{Q}_q ||g||_\infty \varepsilon + 2\hat{Q}_q ||g||_\infty \varepsilon = 3\hat{Q}_q ||g||_\infty \varepsilon \tag{6.8}
$$

for all $n \geq s$, all $z \in \mathbb{P}^k \setminus B(A_*, \Delta_q)$, all $\xi \in D(z)$, and all $w \in \Gamma_{z,\xi} \cap B(z, \delta_3)$. Since the set $\bigcup_{\xi \in D(z)} \Gamma_{z,\xi} \cap B(z, \delta)$ is dense in $B(z, \delta)$ and since the functions $\hat{\mathcal{L}}_{\hat{\phi}_q}^n g$ are continuous, we conclude that

$$
|\hat{\mathcal{L}}_{\hat{\phi}_q}^n g(w) - \hat{\mathcal{L}}_{\hat{\phi}_q}^n g(z)| \leq 4\hat{Q}_q \varepsilon ||g||_\infty \tag{6.9}
$$

for all $n \geq s$, all $z \in \mathbb{P}^k \setminus B(A_*, \Delta_q)$, and all $w \in B(z, \delta_3)$.

Now suppose that $z \in B(A_*, \Delta_q)$. Recall that given $j \geq 0$ we denoted

$$
\Lambda_j = \bigcap_{l=0}^{j} f^{-ql}(B(A_*, \Delta_q)).
$$

According to (5.25), we have
(6.10)

$$\hat{\mathcal{L}}^n_{\hat{\phi}_q} g(z) = \sum_{j=0}^{s-1} \sum_{x\in f^{-qj}(z)\cap\cap\Lambda_j} \lambda^{-qj}\exp(S_j\hat{\phi}_q(x)) \sum_{y\in f^{-q}(x)\cap B^c(A_*,\Delta_q)} \lambda^{-q} e^{\hat{\phi}_q(y)} \hat{\mathcal{L}}_q^{n-j-1} g(y) +$$
$$+ \sum_{j=s}^{n} \sum_{x\in f^{-qj}(z)\cap\Lambda_j} \lambda^{-qj}\exp(S_j\hat{\phi}_q(x)) \sum_{y\in f^{-q}(x)\cap B^c(A_*,\Delta_q)} \lambda^{-q} e^{\hat{\phi}_q(y)} \hat{\mathcal{L}}_q^{n-j-1} g(y).$$

Now assume that $n \geq 2s$. Applying (6.9) we see that there exists $\delta_4 \in (0, \delta_3]$ so small that if $x, x' \in \mathbb{P}^k$ and $\rho(x', x) < \delta_4$, then
(6.11)

$$\Bigg| \sum_{j=0}^{s-1} \sum_{x\in f^{-qj}(z)\cap\cap\Lambda_j} \sum_{y\in f^{-1}(x)\cap B^c(A_*,\Delta_q)} \Big(\lambda^{-qj}\exp(S_j\hat{\phi}_q(x))\lambda^{-1} e^{\hat{\phi}_q(y)} \hat{\mathcal{L}}_q^{n-j-1} g(y) -$$
$$- \lambda^{-qj}\exp(S_j\hat{\phi}_q(x'))\lambda^{-1} e^{\hat{\phi}_q(y')} \hat{\mathcal{L}}_q^{n-j-1} g(y')\Big)\Bigg| \leq$$
$$\leq \hat{Q}_q ||g||_\infty \varepsilon.$$

Now, take $\delta_5 \in (0, \delta_4)$ so small that for all $j = 0, 1, \ldots, s-1$ and all $a, b \in \mathbb{P}^k$ with $\rho(a, b) < \delta$ there exists a bijection $\tau^j_{a,b} : f^{-qj}(a) \to f^{-qj}(b)$ such that $\tau^j_{a,b} \circ \tau^j_{b,a} = \mathrm{Id}$ and $\rho(\tau^j_{a,b}(x), x) < \delta_4$ for all $x \in f^{-qj}(a)$. If now $z \in B(A_*, \Delta_q)$ and $w \in B(z, \delta_5)$,

then looking at (6.10) we can write,
(6.12)

$$\hat{\mathcal{L}}^n_{\hat{\phi}_q} g(w) - \hat{\mathcal{L}}^n_{\hat{\phi}_q} g(z) =$$

$$= \sum_{\substack{0\le j\le s-1 \\ x\in f^{-qj}(z)\cap\cap\Lambda_j \\ y\in f^{-q}(x)\cap B^c(A_*,\Delta_q)}} \Bigg( \lambda^{-qj}\exp\big(S_j\hat{\phi}_q(\tau^j_{z,w}(x))\big)\lambda^{-q}\exp\big(\hat{\phi}_q(\tau^1_{x,\tau^j_{z,w}(x)}(y))\big)\hat{\mathcal{L}}^{n-j-1}_q g(\hat{\phi}_q(\tau^1_{x,\tau^j_{z,w}(x)}(y))-$$

$$- \lambda^{-qj}\exp(S_j\hat{\phi}_q(x))\lambda^{-q}\exp\big(\hat{\phi}_q(y)\big)\hat{\mathcal{L}}^{n-j-1}_q g(y)\Bigg)$$

$$+ \sum_{\substack{0\le j\le s-1 \\ x\in f^{-qj}(z)\cap\Lambda_j \\ y\in f^{-q}\left(\tau^j_{z,w}(x))\setminus \tau^1_{x,\tau^j_{z,w}}\left(f^{-q}(x)\cap B^c(A_*,\Delta_q)\right)\right)}} \Bigg( \lambda^{-qj}\exp\big(S_j\hat{\phi}_q(\tau^j_{z,w}(x))\big)\lambda^{-q}\exp\big(\hat{\phi}_q(y)\big)\hat{\mathcal{L}}^{n-j-1}_q g(y)-$$

$$- \lambda^{-qj}\exp(S_j\hat{\phi}_q(x))\lambda^{-q}\exp\big(\hat{\phi}_q(\tau^1_{\tau^j_{z,w}(x),x}(y))\big)\hat{\mathcal{L}}^{n-j-1}_q g\big(\tau^1_{\tau^j_{z,w}(x),x}(y))\Bigg)$$

$$+ \sum_{\substack{0\le s-1 \\ x\in(f^{-qj}(w)\cap\Lambda_j)\setminus\tau^j_{z,w}\left(f^{-qj}(z)\cap\cap\Lambda_j\right) \\ y\in f^{-q}(x)\cap B^c(A_*,\Delta_q)}} \Bigg( \lambda^{-qj}\exp(S_j\hat{\phi}_q(x))\lambda^{-q}\exp\big(\hat{\phi}_q(y)\big)\hat{\mathcal{L}}^{n-j-1}_q g(y)-$$

$$- \lambda^{-qj}\exp\big(S_j\hat{\phi}_q(\tau^j_{w,z}(x))\big)\lambda^{-q}\exp\big(\hat{\phi}_q(\tau^1_{x,\tau^j_{w,z}(x)}(y))\big)\hat{\mathcal{L}}^{n-j-1}_q g\big(\tau^1_{x,\tau^j_{z,w}(x)}(y))\Bigg)$$

$$+ \Sigma^*_2(g)(w) - \Sigma^*_2(g)(z),$$

where $\Sigma^*_2(g)(z)$ is a subsum of

$$\Sigma^*_3(g)(z) := \sum_{j=s}^n \sum_{x\in f^{-qj}(z)\cap\Lambda_j} \lambda^{-qj}\exp(S_j\hat{\phi}_q(x)) \sum_{y\in f^{-q}(x_2)\cap B^c(A_*,\Delta_q)} \lambda^{-q}e^{\hat{\phi}_q(y)}\hat{\mathcal{L}}^{n-j-1}_q g(y),$$

and likewise, $\Sigma^*_2(g)(w)$ is a subsum of

$$\Sigma^*_3(g)(w) := \sum_{j=s}^n \sum_{x\in f^{-qj}(w)\cap\Lambda_j} \lambda^{-qj}\exp(S_j\hat{\phi}_q(x)) \sum_{y\in f^{-q}(x_2)\cap B^c(A_*,\Delta_q)} \lambda^{-q}e^{\hat{\phi}_q(y)}\hat{\mathcal{L}}^{n-j-1}_q g(y).$$

The same calculation as in (5.25) shows that

$$|\Sigma^*_2(g)(w)| \le |\Sigma^*_2(|g|)(w)| \le |\Sigma^*_3(|g|)(w)| \le \hat{Q}_q||g||_\infty\varepsilon, \tag{6.13}$$

and similarly,

$$|\Sigma^*_2(g)(z)| \le |\Sigma^*_2(|g|)(z)| \le |\Sigma^*_3(|g|)(z)| \le \hat{Q}_q||g||_\infty\varepsilon, \tag{6.14}$$

Denote the first the first three differences in (6.12) by $\Sigma_1^*(g)(w) - \Sigma_1^*(g)(z)$. In view of (6.11) we have

$$|\Sigma_1^*(g)(w) - \Sigma_1^*(g)(z)| \le 3\hat{Q}_q||g||_\infty\varepsilon$$

Combining this along with (6.13), (6.14), and applying (6.12), we thus get that

$$|\hat{\mathcal{L}}_{\hat{\phi}_q}^n g(w) - \hat{\mathcal{L}}_{\hat{\phi}_q}^n g(z)| \le 5\hat{Q}_q||g||_\infty\varepsilon$$

for all $n \ge 2s$, all $z \in B(A_*, \Delta_q)$, and all $w \in B(z, \delta_5)$. In turn, combining this with (6.9), we obtain

$$|\hat{\mathcal{L}}_{\hat{\phi}_q}^n g(w) - \hat{\mathcal{L}}_{\hat{\phi}_q}^g n(z)| \le 5\hat{Q}_q||g||_\infty\varepsilon \tag{6.15}$$

for all $n \ge 2s$ and all $z, w \in \mathbb{P}^k$ with $\rho(z, w) < \delta_5$. Clearly, there exists $\delta \in (0, \delta_5)$ (independent of $g$) so small that

$$|\hat{\mathcal{L}}_{\hat{\phi}_q}^n g(w) - \hat{\mathcal{L}}_{\hat{\phi}_q}^n g(z)| \le \hat{Q}_q||g||_\infty\varepsilon$$

for all $n = 0, 1, \dots, 2s - 1$ and all $z, w \in \mathbb{P}^k$ with $\rho(z, w) < \delta$. Along with (6.15). Thus, the family $\big(\hat{\mathcal{L}}_{\hat{\phi}_q}^n g\big)_{n=0}^\infty$ is equicountinuous. Hence, invoking also Lemma 6.1(c), it follows from Arzela-Ascoli's theorem that the family $\big(\hat{\mathcal{L}}_{\hat{\phi}_q}^n g\big)_{n=0}^\infty$ is relatively compact. We are done. □

As an immediate consequence of this proposition and the fact that every function $g \in C(J)$ extends continuously to a $\mathbb{P}^k$ with the same supremum norm, we obtain the following.

**Proposition 6.4.** *The Perron-Frobenius operator $\hat{\mathcal{L}}_{\phi_q} : C(J) \to C(J)$ is almost periodic.*

As a direct consequence of Lemma 5.1 and Lemma 5.2, we get the following.

**Lemma 6.5.** *There exist constants $Q_+ > 0$ and $Q_- > 0$ such that $||\hat{\mathcal{L}}_\phi^n||_\infty \le Q_+$ for all $n \ge 0$, and $\hat{\mathcal{L}}_\phi^{qn}\mathbb{1}(z) \ge Q_-$ for all $n \ge 0$ and all $z \in J$.*

and

**Proposition 6.6.** *The Perron-Frobenius operator $\hat{\mathcal{L}}_\phi : C(J) \to C(J)$ is almost periodic.*

It follows from this proposition that the sequence $\big(\frac{1}{n}\sum_{j=0}^{n-1} \hat{\mathcal{L}}_\phi^j\mathbb{1}\big)_0^\infty$ is pre-compact, and it is easy to see that any of its limit points $\rho_\phi$ is a fixed point of the operator $\hat{\mathcal{L}}_\phi$ and its integral against the measure $m_\phi$ is equal to 1. Therefore, we have the following.

**Proposition 6.7.** *There exists a continuous function $\rho_\phi : J \to [0,+\infty)$ with the following properties: $\hat{\mathcal{L}}_\phi \rho_\phi = \rho_\phi$, $\int \rho_\phi dm_\phi = 1$, $Q_- \le \inf(\rho_\phi) \le \sup(\rho_\phi) \le Q_+$. In particular, $\mu_\phi = \rho_\phi m_\phi$ is a Borel probability $f$-invariant measure equivalent to $m_\phi$.*

An important ingredient in the proof that $\mu_\phi$ is a unique equilibrium state of the potential $\phi : J \to \mathbb{R}$ is the fact that for every continuous function $g : J \to \mathbb{C}$, the iterates $\hat{\mathcal{L}}_\phi^n g$ converge uniformly to $(\int g dm_\phi)\rho_\phi$. We are going to prove it now. The proof requires some preparations. We start with the following.

**Lemma 6.8.** *Let $\mathcal{F}$ be a $\sigma$-algebra of subsets of some set $X$. Suppose that $\mu_1$ and $\mu_2$ are some equivalent probability measures on $\mathcal{F}$ with uniformly bounded Radon-Nikodym derivatives. If $\mathcal{B}$ is a sub $\sigma$-algebra of $\mathcal{F}$ and $E_{\mu_2}(g|\mathcal{B}) = E_{\mu_1}(g|\mathcal{B})$ for every $\mathcal{F}$-measurable function $g \in L^1(\mu_1) = L^1(\mu_2)$, then the Radon-Nikodym derivative $\frac{d\mu_2}{d\mu_1}$ is $\mathcal{B}$-measurable.*

*Proof.* Put $\rho = \frac{d\mu_2}{d\mu_1}$. Seeking contradiction suppose that $\rho$ is not $\mathcal{B}$-measurable. Then at least one of the following two sets has a positive measure $\mu_1$:

$$A = \{x \in X : E_{\mu_1}(\rho|\mathcal{B})(x) < \rho(x)\} \ \text{ and } \ A^c = \{x \in X : E_{\mu_1}(\rho|\mathcal{B})(x) > \rho(x)\}.$$

Assume without loss of generality that $\mu_1(A) > 0$. Then, on the one hand,

$$\begin{aligned}
\mu_2(A) &= \int_X 1\!\!1_A d\mu_2 = \int_X E_{\mu_2}(1\!\!1_A|\mathcal{B})\, d\mu_2 = \int_X E_{\mu_2}(1\!\!1_A|\mathcal{B})\rho\, d\mu_1 \\
&= \int_X E_{\mu_1}(1\!\!1_A|\mathcal{B})\rho\, d\mu_1 = \int_X E_{\mu_1}\big(E_{\mu_1}(1\!\!1_A|\mathcal{B})\rho\big)\, d\mu_1 \\
&= \int_X E_{\mu_1}(1\!\!1_A|\mathcal{B})\rho E_{\mu_1}(\rho|\mathcal{B})\, d\mu_1,
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
\mu_2(A) &= \int_X \rho 1\!\!1_A d\mu_1 = \int_X E_{\mu_1}(\rho 1\!\!1_A|\mathcal{B})\, d\mu_1 \\
&> \int_X E_{\mu_1}\big(E_{\mu_1}(\rho|\mathcal{B})1\!\!1_A\big)\, d\mu_1 \\
&= \int_X E_{\mu_1}(\rho|\mathcal{B})E_{\mu_1}(1\!\!1_A|\mathcal{B})\, d\mu_1.
\end{aligned}$$

So, $\mu_2(A) > \mu_2(A)$, and this contradiction finishes the proof. □

Our second auxiliary fact, interesting itself, is this.

**Proposition 6.9.** *The number 1 is the only unitary eigenvalue of the Perron-Frobenius operator $\hat{\mathcal{L}}_\phi : C(J) \to C(J)$ and the corresponding eigenspace is equal to $\mathbb{C}\rho_\phi$.*

*Proof.* Suppose that

$$\hat{\mathcal{L}}_\phi g = \xi g$$

with some $\xi \in \mathbb{C}$ of modulus one and some non-zero function $g \in C(J)$. Having Lemma 6.5, it follows from Theorem 4.9 and Exercise 2 (p. 326/327) in [Sch] that the unitary eigenvalues of the operator $\hat{\mathcal{L}}_\phi : C(J) \to C(J)$ form a finite cyclic group. There thus exists $l \geq 1$ such that $\xi^l = 1$. We then have

$$\hat{\mathcal{L}}_\phi^l g = g.$$

Since $\hat{\mathcal{L}}_\phi$ preserves the class of real-valued functions, the same is true for Re$g$ and also for Im$g$. Hence, it is sufficient to consider the case when $g : J(f) \to \mathbb{R}$. Denote $g_+ = \max\{0, g\}$ and $g_- = \min\{0, g\}$. Since the operator $\hat{\mathcal{L}}_\phi$ is positive, we have $g_1^+ = \hat{\mathcal{L}}_\phi^l g_0^+ \geq 0$, $g_1^- = \hat{\mathcal{L}}_\phi^l g_0^- \leq 0$ and $g = \hat{\mathcal{L}}_\phi^l g = g_1^+ + g_1^-$. Clearly there is not a unique decomposition of $g$ into a positive and negative function. However, the functions $g_+$ and $g_-$ are extremal in the sense that they are the smallest functions that have this property. Consequently $g_1^+ \geq g_+$ and $g_1^- \leq g_-$. Since these functions are continuous and since $\int g_1^+ \, dm_\phi = \int g_+ \, dm_\phi$, we have $g_1^+ = g_+$ and, for the same reasons, $g_1^- = g_-$. Therefore,

$$\hat{\mathcal{L}}_\phi^l g_+ = g_+ \quad \text{and} \quad \hat{\mathcal{L}}_\phi^l g_- = g_-.$$

Suppose that $g_+$ does not vanish identically. $\hat{g} = m_\phi(g_+)^{-1} g_+$. Then

$$\hat{\mathcal{L}}_\phi^l \hat{g} = \hat{g}, \quad \hat{g} \geq 0, \quad \text{and} \quad \int \hat{g} dm_\phi = 1.$$

Since the map $f^l : J \to J$ is topologically exact and since $\hat{g}$ is a non-negative not identically vanishing continuous function, we thus conclude that $\hat{g}$ is strictly positive everywhere throughout $J$. Hence, the two invariant measures $\mu_\phi = \rho_\phi m_\phi$ and $\hat{\mu} = \hat{g} m_\phi$ are equivalent with uniformly bounded Radon-Nikodym derivatives. Therefore, all versions of expected values $E_{\hat{\mu}}(u|\mathcal{I})$ and $E_{\mu_\phi}(u|\mathcal{I})$ are well defined on a set of common measure 1 for $\hat{\mu}$ and $\mu_\phi$, where $u : J \to \mathbb{R}$ is any $\hat{\mu}$ (equivalently $\mu_\phi$) integrable function and $\mathcal{I}$ is the $\sigma$-algebra of all $f^l$-invariant subsets of $J$. Since at almost every point $z \in J$ both $E_{\hat{\mu}}(u|\mathcal{I})$ and $E_{\mu_\phi}(u|\mathcal{I})$ are equal to the Birkhoff's ergodic averages

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} u \circ f^{lj}(z),$$

we therefore conclude that

$$E_{\hat{\mu}}(u|\mathcal{I}) = E_{\mu_\phi}(u|\mathcal{I}) \text{ a.e.}$$

Hence, it follows from Lemma 6.8 that the Radon-Nikodym derivative is measurable with respect to the $\sigma$-algebra $\mathcal{I}$. This means that the ratio $\hat{g}/\rho_\phi$ is constant on grand orbits of almost all points in $J$. Since by topological exactness of the map $f^l : J \to J$, the grand orbit of every point in $J$ is dense in $J$, and since the function $\hat{g}/\rho_\phi$ is continuous, we conclude that the function $\hat{g}/\rho_\phi$ is constant throughout $J$. As both measures $\hat{\mu}$ and $\mu_\phi$ are probabilistic, this implies that $\hat{g}/\rho_\phi$ is equal to 1, i.e.

$\hat{g} = \rho_\phi$. So, $g_+ = a\rho_\phi$ with some $a \in \mathbb{C}$, and likewise, $g_- = b\rho\phi$ with some $b \in \mathbb{C}$. Consequently, $g = g_+ - g_- = (a-b)\rho_\phi$. We are done. □

For every bounded operator $A : B \to B$ of a Banach space $B$, let $B_u$ be the closure of the linear span of unitary eigenvectors of $A$, and let $B_0 = \{g \in B : \lim_{n\to\infty} A^n g = 0\}$. M. Lyubich proved in [Ly] that if $A : B \to B$ is an almost periodic operator, then $B = B_u \bigoplus B_0$, the direct sum of closed vector subspaces. We shall prove the following.

**Theorem 6.10.** *We have*

$$C(J)_u = \mathbb{C}\rho_\phi, \quad C(J)_0 = \left\{ g \in C(J) : \int g dm_\phi = 0 \right\}.$$

*In addition, if $g = g_u + g_0$ with $g_u \in C(J)_u$ and $g_0 \in C(J)_0$, then $g_u = (\int g dm_\phi)\rho_\phi$ and the sequence $\big(\hat{\mathcal{L}}_\phi^n g\big)_0^\infty$ converges to $(\int g dm_\phi)\rho_\phi$ uniformly on $J$. In particular, $m_\phi$ is the only Borel probability measure on $J$ satisfying (4.10) and $\rho_\phi$ is the only non-negative fixed point of the operator $\hat{\mathcal{L}}_\phi$ such that $\int \rho_\phi dm_\phi = 1$.*

*Proof.* The fact that $C(J)_u = \mathbb{C}\rho_\varphi$ is the content of Lemma 6.9. If $g \in C(J)_0$, then $\int g dm_\phi = \lim_{n\to\infty} \int \hat{\mathcal{L}}_\phi^n g dm_\phi = 0$. so,

$$C(J)_0 \subset \left\{ g \in C(J) : \int g dm_\phi = 0 \right\}. \tag{6.16}$$

If, on the other hand, $\int g dm_\phi = 0$, write uniquely $g = g_u + g_0$ where $g_u \in C(J)_u$ and $g_0 \in C(J)_0$. Then $\int g_u dm_\phi = \int g_0 dm_\phi - \int g dm_\phi = 0 - 0 = 0$, whence $g = (\int g_0 dm_\phi)\rho_\phi = 0\rho_\phi = 0$. Hence, $g = g_0 \in C(J)_0$. The inclusion $\{g \in C(J) : \int g dm_\phi = 0\} \subset C(J)_0$ is proved, and together with (6.16) it yields

$$C(J)_0 = \left\{ g \in C(J) : \int g dm_\phi = 0 \right\}.$$

The second assertion of our theorem is now obvious and the third assertion follows from the first one too since we know that $g - (\int g dm_\phi)\rho_\phi \in C(J)_0$, and therefore, because of Lyubich's Theorem, $\lim_{n\to\infty}\big(g - (\int g dm_\phi)\rho_\phi\big) = 0$, the limit considered in the Banach space $C(J)$. We are done. □

We shall now record two mixing properties resulting from this theorem. The proof of the first one is the same as the proof of Corollary 37 in [DU], the second property is also its straightforward consequence.

**Theorem 6.11.** *The dynamical system $(J, f, \mu_\phi)$ is metrically exact. This means that $\bigcap_{n=0}^\infty f^{-n}(\mathcal{B})$ consists only of sets of measure 0 and 1, where $\mathcal{B}$ denotes the $\sigma$-algebra of Borel sets of $J$. In particular, Rokhlin's natural extension of $(J, f, \mu_\phi)$ is a $K$-system and the dynamical system $(J, f, \mu_\phi)$ is mixing of any order. In particular, it is ergodic.*

**Theorem 6.12.** *(Decay of Correlations) If $g \in C(J)$ and $h \in L^1(m_\phi) = \mathcal{L}^1(\mu_\phi)$, then*

$$\lim_{n\to\infty} \int h \circ f^n \cdot g d\mu_\phi = \int h d\mu_\phi \int g d\mu_\phi.$$

Given $H \geq 0$ and $0 \leq t < \kappa_f$ let $\mathcal{P}_H^t(f)$ denote the class of all Hölder continuous potentials $\phi : J \to \mathbb{R}$ such that $||\phi||_\alpha \leq H$ and $\sup(\phi) - \inf(\phi) \leq t$. Call all such potentials $(H,t)$-admissible. By Arzela-Ascoli theorem, $\mathcal{P}_H^t(f)$ is a compact subset of $C(J)$. Another crucial technical fact for the uniqueness of equilibrium states is the following refinement of Proposition 6.3.

**Lemma 6.13.** *For every $H \geq 0$, $0 \leq t < \kappa_f$, and every relatively compact set $K \subset C(\mathbb{P}^k)$, the set $\{\hat{\mathcal{L}}^n_{\hat{\phi}_q}(g) : \phi \in \mathcal{P}_H^t(f), g \in K, n \geq 0\}$ is relatively compact.*

*Proof.* Suppose first that $K$ is a singleton, i.e. $K = \{g\}$ for some $g \in C(J)$. Observe that the number $\theta > 0$ coming from (4.9) can be taken the same for all $\phi \in \mathcal{P}_H^t(f)$. The constructions of $Q_q^+$ and then of $\hat{Q}_q$ lead to a number $\hat{Q} > 0$ such that $\hat{Q}_q \leq \hat{Q}$ for all $\phi \in \mathcal{P}_H^t(f)$. Having this, one verifies that given $\varepsilon > 0$ the numbers, $\delta_1$ through $\delta_5$ and $\delta$ appearing in the proof of Proposition 6.3 can be taken the same for all $\phi \in \mathcal{P}_H^t(f)$. Then the proof of Proposition 6.3 shows that the set $\{\hat{\mathcal{L}}^n_{\phi_q}(g) : \phi \in \mathcal{P}_H^t(f), n \geq 0\}$ is relatively compact. Now consider the general case. Take a sequence $\big(\hat{\mathcal{L}}^{n_j}_{(\hat{\phi}_j)_q}(g_j)\big)_{j=1}^\infty$. Since the set $K$ is relatively compact and $\mathcal{P}_H^t(f)$ is compact, passing to a subsequence, we may assume without loss of generality that $(g_j)_{j=1}^\infty$ converges uniformly to some $g \in C(J)$ and $((\hat{\phi}_j)_q)_{j=1}^\infty$ converges uniformly to some $\phi \in \mathcal{P}_H^t(f)$. Then

$$\Big|\Big|\hat{\mathcal{L}}^{n_j}_{(\hat{\phi}_j)_q}(g_j) - \hat{\mathcal{L}}^{n_j}_{(\hat{\phi}_j)_q}(g)\Big|\Big|_\infty = \Big|\Big|\hat{\mathcal{L}}^{n_j}_{(\hat{\phi}_j)_q}(g_j - g)\Big|\Big|_\infty \leq \hat{Q}||g_j - g||_\infty.$$

Since $\lim_{j\to\infty} \hat{Q}||g_j - g||_\infty = 0$ and since, in view of the already proved singleton part from our lemma, the sequence $\big(\hat{\mathcal{L}}^{n_j}_{(\hat{\phi}_j)_q}(g)\big)_{j=1}^\infty$ is relatively compact, we conclude that $\big(\hat{\mathcal{L}}^{n_j}_{(\hat{\phi}_j)_q}(g_j)\big)_{j=1}^\infty$ contains a converging subsequence. We are done. □

As an immediate consequence of this lemma we get the following.

**Lemma 6.14.** *For every $H \geq 0$, $0 \leq t < \kappa_f$, and every relatively compact set $K \subset C(J)$, the set $\{\hat{\mathcal{L}}^n_\phi(g) : \phi \in \mathcal{P}_H^t(f), g \in K, n \geq 0\}$ is relatively compact.*

## 7. Pressure versus Eigenvalue

In this section, developing the idea of Gromov from [Gr], we prove equality of the topological pressure $\mathrm{P}(\phi)$ and the logarithm $\log\lambda$. This major part of our argument is done in the following.

**Theorem 7.1.** *Let $\psi$ be a continuous function $\psi : \mathbb{P}^k \to \mathbb{R}$. Assume that there exists $\lambda > 0$ and $Q > 0$ such that*

$$\sup\psi + (k-1)\log d \leq \log\lambda$$

*and, for all integers $n \geq 1$ the following inequality*

$$\mathcal{L}_\psi^n(\mathbb{1})(x) \leq Q\lambda^n$$

*holds. Then $\mathrm{P}(\psi) \leq \log\lambda$.*

*Proof.* We shall follow the idea of the proof of the inequality

$$\mathrm{h_{top}}(f) \leq \log(\mathrm{deg_{top}f}) = k\log d$$

which is due to M. Gromov, ([Gr]) Thus, Gromov's inequality corresponds to the case $\phi = 0$. Let us consider the following integral

$$\int_{\mathbb{P}^k} \exp(S_n\psi)(\omega + f^*\omega + \cdots + (f^{n-1})^*\omega)^k.$$

As in [Gr] we consider the embedding, a generalized graph,

$$f_n : \mathbb{P}^k \to X_n = (\mathbb{P}^k)^n$$

given by the formula

$$f_n(x) = (x, f(x), \ldots, f^{n-1}(x)).$$

Let $\pi_i : X_n \to X^{(i)} = \mathbb{P}^k$ be the projection to the $i$-th coordinate. We endow the space $X_n$ with a Kähler form $\eta$ by putting

$$\omega_i = \pi_i^*\omega \quad \text{and} \quad \eta = \omega_1 + \cdots + \omega_n.$$

Now, let $E$ be an $(n, 2\varepsilon)$-separated set in $\mathbb{P}^k$, i.e. $d_n(x, y) > 2\varepsilon$ for $x, y \in E, x \neq y$, where $d_n$ is a metric in $\mathbb{P}^k$ given by $d_n(x, y) = \max_{0\leq i<n}\{d(f^ix, f^iy)\}$. Then

$$d_\eta(f_n(x), f_n(y)) = (\sum_{i=0}^{n-1} d(f^ix, f^iy)^2)^{\frac{1}{2}} \geq \max_{0\leq i<n}\{d(f^ix, f^iy)\} > 2\varepsilon. \tag{7.1}$$

Thus, the balls $B(f_n(x), \varepsilon), x \in E$, (with respect to the metric $d_\eta$) are mutually disjoint. Now, we use Lelong's Theorem ([La], [McM]) for the form $\eta$ and the embedded complex analytic variety $f_n(\mathbb{P}^k) \subset X_n$ and conclude that the $\eta$-volume of $f_n(\mathbb{P}^k) \cap B(p, \varepsilon)$, i.e

$$\int_{f_n(\mathbb{P}^k)\cap B(p,\varepsilon)} \eta^k$$

is bounded below by a constant $c_\varepsilon$, depending on $\varepsilon$ only. Now, fix an arbitrary $\delta > 0$. Since the function $\psi$ is uniformly continuous, there exists $\varepsilon > 0$ such that, if $d(x,y) < \varepsilon$ then $|\psi(x) - \psi(y)| < \delta$ and, consequently,

$$e^{-\delta} < \frac{e^{\psi(x)}}{e^{\psi(y)}} < e^{\delta}.$$

Let $E$ be an $(n, 2\varepsilon)$-separated set. Then we can write

$$\begin{aligned}
\sum_{x\in E} \exp(S_n\psi(x)) &\le e^{\delta n} \sum_{x\in E} \inf_{y\in B_{d_n}(x,\varepsilon)} \{\exp(S_n\psi(y))\} \\
&\le e^{\delta n} \frac{1}{c_\varepsilon} \sum_{x\in E} \inf_{y\in B_{d_n}(x,\varepsilon)} \{\exp(S_n\psi)\}\eta^k(B(f_n(x),\varepsilon)) \\
&= e^{\delta n} \frac{1}{c_\varepsilon} \sum_{x\in E} \inf_{y\in B_{d_n}(x,\varepsilon)} \{\exp(S_n\psi(y))\}(f^*\eta^k)(f_n^{-1} B_{d_\eta}(f_n(x),\varepsilon)) \\
&\le e^{\delta n} \frac{1}{c_\varepsilon} \sum_{x\in E} \inf_{y\in B_{d_n}(x,\varepsilon)} \{\exp(S_n\psi(y))\}(f^*\eta^k)(B_{d_n}(x,\varepsilon)) \\
&\le e^{\delta n} \frac{1}{c_\varepsilon} \int_{\mathbb{P}^k} \exp(S_n\psi(f_n^*\eta^k))
\end{aligned} \tag{7.2}$$

where we have used (7.1) in the second inequality. Since $f_n^*(\omega_i) = (f^i)^*\omega$, the last integral takes on the form

$$\int_{\mathbb{P}^k} \exp(S_n\psi)(\omega + f^*\omega + \cdots + (f^{(n-1)})^*\omega)^k \tag{7.3}$$

We shall estimate this integral from above. First, notice that, if, instead of the above integral, we had

$$\int_{\mathbb{P}^k} \exp(S_n\psi)(f^{(n-1)})^*\omega^k,$$

then the integral would transform immediately to

$$\int_{\mathbb{P}^k} \mathcal{L}_0^n(1\!\!1)d\omega^k$$

since the operator $f^*$ acts on measures on $\mathbb{P}^k$ as a conjugate to the operator

$$f_*g(x) = \sum_{y\in f^{-1}(x)} g(y). \tag{7.4}$$

In our case, we have to write the integral (7.3) as a sum of integrals, and then to use the above observation.

(7.5)

$$\int_{\mathbb{P}^k} \exp(S_n\psi)\left(\omega + f^*\omega + \cdots + (f^{(n-1)})^*\omega)^k\right) =$$

$$= \int_{\mathbb{P}^k} \exp(S_n\psi)\left(\sum_{0\le i_1,\dots,i_n\le n-1} (f^{i_i})^*\omega \wedge (f^{i_2})^*\omega \wedge \cdots \wedge (f^{i_k})^*\omega\right).$$

Since all forms $(f^{i_i})^*\omega \wedge (f^{i_2})^*\omega \wedge \cdots \wedge (f^{i_k})^*\omega$ are positive, we can treat them as measures and estimate

(7.6)

$$\int_{\mathbb{P}^k} \exp(S_n\psi)\left(\omega + f^*\omega + \cdots + (f^{(n-1)})^*\omega\right)^k \le$$

$$\le k! \int_{\mathbb{P}^k} \exp(S_n\psi) \sum_{i_1\le i_2\le\cdots\le i_k\le n-1} (f^{i_1})^*\omega \wedge \cdots \wedge (f^{i_k})^*\omega$$

$$= k! \sum_{i=0}^{n-k} \int_{\mathbb{P}^k} \exp(S_n\psi) \sum_{j_2\le\cdots\le j_k\le n-i} (f^i)^*\left(\omega \wedge (f^{j_2})^*\omega \wedge \cdots \wedge (f^{j_k})^*\omega\right).$$

Using the observation 7.4 again, one can rewrite the above sum as

$$(7.7)\quad k! \sum_{i=0}^{n-k} \int_{\mathbb{P}^k} \mathcal{L}^i_\psi(\mathbb{1})(x)\exp(S_{n-i}\psi(x)) \left[\sum_{j_2\le\cdots\le j_k\le n-i} \omega \wedge (f^{j_2})^*\omega \wedge \cdots \wedge (f^{j_k})^*\omega\right].$$

Recall that, according to our assumptions, we have

$$\mathcal{L}^i_\psi(\mathbb{1})(x) \le Q\lambda^n$$

for all $x \in \mathbb{P}^k$. By our assumption on $\psi$ we can estimate the above sum by

$$(7.8)\qquad k!Q\sum_{i=0}^{n-k} \lambda^i \exp((n-i)\alpha) \int_{\mathbb{P}^k} \sum_{j_2\le\cdots\le j_k\le n-i} \omega \wedge (f^{j_2})^*\omega \wedge \cdots \wedge (f^{j_k})^*\omega$$

It remains to calculate the total mass of each measure $\omega \wedge (f^{j_2})^*\omega \wedge \cdots \wedge (f^{j_k})^*\omega$. Recall that $(f^j)^*\omega = d^j\omega$ in the de Rham cohomology group $H^2(\mathbb{P}^k)$. It is then straightforward to check that

$$\int_{\mathbb{P}^k} \omega \wedge (f^{j_2})^*\omega \wedge \cdots \wedge (f^{j_k})^*\omega = \int_{\mathbb{P}^k} d^{j_2+\cdots+j_k}\omega^k = d^{j_2+\cdots+j_k} \le dd^{(k-1)(n-i)}$$

since $j_m \le n-i$ for all $2 \le m \le k$. Now, the number of all possible choices of $j_2, \dots, j_k$ can be estimated above by $n^k$. Finally, we can estimate (7.8) by

$$
\begin{aligned}
c_k n^k \sum_{i=0}^{n-k} \lambda^i \exp((n-i)\sup(\psi)) d^{(k-1)(n-i)} &= \\
&= c_k n^k \lambda^n \sum_{i=0}^{n-k} [\exp(-\log\lambda + \sup(\psi) + (k-1)\log d)]^{n-i} \qquad (7.9)\\
&= c_k n^k \lambda^n \sum_{i=0}^{n-k} [\exp(-\log\lambda + \sup(\psi) + (k-1)\log d)]^{(n-i)}
\end{aligned}
$$

The last sum is bounded by a constant depending on $k$ but independent of $n$ since $-\log\lambda + \sup(\psi) + (k-1)\log d < 0$. Therefore, we obtain the following. For every $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\delta)$ such that for every $\varepsilon < \varepsilon_0$ and every $(n, 2\varepsilon)$-separated set $E$ we have

$$\sum_{x\in E} \exp S_n\psi(x) \le e^{\delta n} C(\varepsilon, k) n^k \lambda^n,$$

where $C(\varepsilon, k)$ is a constant depending on $\varepsilon$ and $k$. This gives immediately

$$\mathrm{P}(\psi) \le \log\lambda + \delta$$

and, as $\delta$ was arbitrarily small,

$$\mathrm{P}(\psi) \le \log\lambda.$$

We are done □

## 8. Existence and Uniqueness of Equilibrium States

As we show in the following proposition the measure $\mu_\phi$ turns out to be an equilibrium state for the potential $\phi : J \to \mathbb{R}$.

**Proposition 8.1.** *The invariant measure $\mu_\phi = \rho_\phi m_\phi$ is an equilibrium state for the potential $\phi : J \to \mathbb{R}$. In addition,* $\mathrm{P}(\phi) = \log\lambda$.

*Proof.* Let $\hat{\phi}_q : \mathbb{P}^k \to \mathbb{R}$ be the extension of $\varphi_q : J \to \mathbb{R}$ produced in Lemma 6.1. It follows from this lemma that the function $\psi = \hat{\phi}_q$ satisfies the assumptions of Theorem 7.1 for the the dynamical system $f^q : \mathbb{P}^k \to \mathbb{P}^k$. Applying this theorem we get that $\mathrm{P}(\phi) = \frac{1}{q}\mathrm{P}(\hat{\phi}_q) \le \frac{1}{q}\mathrm{P}(\hat{\phi}_q) \le \log\lambda$. Therefore,

$$
\begin{aligned}
\mathrm{h}\mu_\phi + \int \phi d\mu_\phi &\ge \int \log J_{\mu_\phi} + \int \phi d\mu_\phi \\
&= \int \log\rho_\phi d\mu_\phi - \int \log\rho_\phi \circ f d\mu_\phi + \log\lambda + \int \phi d\mu_\phi - \int \phi d\mu_\phi \\
&= \log\lambda \ge \mathrm{P}(\phi)
\end{aligned}
$$

Invoking the Variational Principle, i. e. formula (1.1), finishes the proof. □

In order to demonstrate the uniqueness of equilibrium states we shall prove an appropriate version of differentiability of topological pressure. The proof is based on Lemma 6.14 and two facts proved below. It goes along the general scheme presented in [PU].

**Proposition 8.2.** *For every $H \geq 0$ and every $0 \leq t < \kappa_f$ the function*

$$C(J) \supset \mathcal{P}_H^t(f) \ni \phi \mapsto \rho_\phi \in C(J)$$

*is continuous with respect to the topology of uniform convergence on $C(J)$.*

*Proof.* Suppose the contrary. Then there exist $\delta > 0$ and a sequence $(\phi_k)_{k=1}^\infty$ of functions in $\mathcal{P}_H^t(f)$ that converge uniformly to some function $\phi \in \mathcal{P}_H^t(f)$ and

$$||\rho_{\phi_k} - \rho_\phi||_\infty > \delta \tag{8.1}$$

for all $k \geq 1$. Since, by Lemma 6.14, the family $(\rho_{\phi_k})_{k=1}^\infty$ is equicontinuous, passing to a susequence, we may assume that the sequence $(\rho_{\phi_k})_{k=1}^\infty$ converges uniformly to some function $\rho \in C(J)$. By (8.1), we have that

$$\rho \neq \rho_\phi. \tag{8.2}$$

For every $k \geq 1$ let $m_k$ be the measure produced in (4.10) for the potential $\phi_k$. By Proposition 8.1, we have

$$\mathcal{L}_{\phi_k}^*(m_k) = e^{\mathrm{P}(\phi_k)} m_k.$$

Passing to a subsequence again, we may assume that the sequence $(m_k)_{k=1}^\infty$ converges weakly to a Borel probability measure $m$ on $J$. Since the topological pressure function $C(J) \ni g \mapsto \mathrm{P}(g)$ is continuous (in fact Lipschitz continuous with the Lipschitz constant equal to 1), we have that $\lim_{k\to\infty} \mathrm{P}(\phi_k) = \mathrm{P}(\phi)$. Thus, the limit measure $m$ satisfies the equation $\mathcal{L}_\phi^*(m) = e^{\mathrm{P}(\phi)} m$, and, because of Theorem 6.10, $m$ is the only Borel probability measure on $J$ satisfying this equation, i.e. $m = m_\phi$. Now, fix an arbitrary function $g \in C(J)$. Since the sequence $(\phi_k)_{k=1}^\infty$ converges uniformly to $\phi \in C(J)$, and since $(m_k)_{k=1}^\infty$ converges weakly to a $m$, we get

$$\begin{aligned}
\lim_{k\to\infty} \Big( \int g\rho_{\phi_k} dm_k - \int g\rho dm \Big) = \\
&= \lim_{k\to\infty} \left( \int (g\rho_{\phi_k} - h\rho) dm_k + \left( \int g\rho dm_k - \int g\rho dm \right) \right) \\
&= \lim_{k\to\infty} \left( \int (g\rho_{\phi_k} - h\rho) dm_k \right) + \lim_{k\to\infty} \left( \int g\rho dm_k - \int g\rho dm \right) \\
&= 0 + 0 = 0.
\end{aligned}$$

Therefore, the sequence $(\mu_k)_{k=1}^\infty$ converges weakly to the Borel probability $f$-invariant measure $\mu = \rho m$. Thus, $\rho$ is a non-negative fixed point of the normalized Perron-Frobenius operator $\hat{\mathcal{L}}_\phi$ with $\int \rho dm = 1$. However, in view of Theorem 6.10, $\rho_\phi$ is a unique fixed point of $\hat{\mathcal{L}}_\phi$ in $C(J)$ with $\int \rho_\phi dm_\phi = 1$. As $m = m_\phi$, we thus get that $\rho = \rho_\phi$, contrary to (8.2). $\square$

In this proof we have also estblished the following.

**Proposition 8.3.** *For every $H \geq 0$ and every $0 \leq t < \kappa_f$ the function*

$$C(J) \supset \mathcal{P}_H^t(f) \ni \phi \mapsto m_\phi \in$$

*is continuous with respect to the weak topology on the space of Borel probability measures on $J$.*

**Lemma 8.4.** *Fix $H \geq 0$ and $0 \leq t < \kappa_f$. If $\phi \in \mathcal{P}_H^t(f)$ and $g \in C(J)$, then*

$$\lim_{n\to\infty} \left\| \frac{\frac{1}{n}\hat{\mathcal{L}}_{\phi_q}^n(S_n g)}{\hat{\mathcal{L}}_{\phi_q}^n(1\!\!1)} - \Big(\int g d\mu_\phi\Big) 1\!\!1 \right\|_\infty = 0.$$

*Proof.* We have

$$\frac{1}{n}\frac{\hat{\mathcal{L}}_{\phi_q}^n(S_n g)}{\hat{\mathcal{L}}_{\phi_q}^n(1\!\!1)} = \frac{\frac{1}{n}\sum_{j=0}^{n-1}\hat{\mathcal{L}}_{\phi_q}^n(g\circ f^j)}{\hat{\mathcal{L}}_{\phi_q}^n(1\!\!1)} = \frac{\frac{1}{n}\sum_{j=0}^{n-1}\hat{\mathcal{L}}_{\phi_q}^{n-j}(g\mathcal{L}_{\phi_q}^j(1\!\!1))}{\hat{\mathcal{L}}_{\phi_q}^n(1\!\!1)}. \tag{8.3}$$

It follows from Lemma 6.14 that

$$\text{the set } \{g\mathcal{L}_{\phi_q}^j(1\!\!1) : j \geq 0\} \text{ is relatively compact.} \tag{8.4}$$

Applying Lemma 6.14 again we thus conclude that

$$\text{the set } \{\hat{\mathcal{L}}_{\phi_q}^{n-j}(g\mathcal{L}_{\phi_q}^j(1\!\!1)) : n \geq 0,\ 0 \leq j \leq n-1\} \text{ is relatively compact.} \tag{8.5}$$

Therefore

$$\text{the set } \left\{\frac{1}{n}\sum_{j=0}^{n-1}\hat{\mathcal{L}}_{\phi_q}^{n-j}(g\mathcal{L}_{\phi_q}^j(1\!\!1)) : n \geq 1\right\} \text{ is relatively compact.} \tag{8.6}$$

By Theorem 6.10,

$$\lim_{n\to\infty} ||\hat{\mathcal{L}}_{\phi_q}^n(1\!\!1) - \rho_\phi||_\infty = 0. \tag{8.7}$$

By the definition of $\hat{Q}_q$ we have,

$$\begin{aligned}
\left\|\hat{\mathcal{L}}_{\phi_q}^{n-j}(g\mathcal{L}_{\phi_q}^j(1\!\!1)) - \Big(\int g d\mu_\phi\Big)\rho_\phi\right\|_\infty &= \\
&= \left\|\hat{\mathcal{L}}_{\phi_q}^{n-j}\big(g\mathcal{L}_{\phi_q}^j(1\!\!1) - g\rho_\phi\big) + \hat{\mathcal{L}}_{\phi_q}^{n-j}\big(g\rho_\phi - \Big(\int g d\mu_\phi\Big)\rho_\phi\big)\right\|_\infty \\
&\leq ||\hat{\mathcal{L}}_{\phi_q}^{n-j}||_\infty ||g(\mathcal{L}_{\phi_q}^j(1\!\!1) - \rho_\phi)||_\infty + ||\hat{\mathcal{L}}_{\phi_q}^{n-j}(g\rho_\phi) - \Big(\int g d\mu_\phi\Big)\rho_\phi||_\infty \\
&\leq ||\hat{Q}||g||_\infty ||\mathcal{L}_{\phi_q}^j(1\!\!1) - \rho_\phi||_\infty + ||\hat{\mathcal{L}}_{\phi_q}^{n-j}(g\rho_\phi) - \Big(\int g d\mu_\phi\Big)\rho_\phi||_\infty.
\end{aligned}$$

Since in addition, in view of Theorem 6.10, $\lim_{n-j\to\infty} ||\hat{\mathcal{L}}_{\phi_q}^{n-j}(g\rho_\phi) - (\int g d\mu_\phi)\rho_\phi||_\infty = 0$, invoking (8.7), we thus see that

$$\lim_{\substack{j\to\infty \\ n-j\to\infty}} ||\hat{\mathcal{L}}_{\phi_q}^{n-j}(g\mathcal{L}_{\phi_q}^{j}(1\!\!1)) - (\int g d\mu_\phi)\rho_\phi||_\infty = 0.$$

Consequently,

$$\lim_{n\to\infty} \left|\left| \frac{1}{n}\sum_{j=0}^{n-1} \hat{\mathcal{L}}_{\phi_q}^{n-j}\big(g\mathcal{L}_{\phi_q}^{j}(1\!\!1)\big) - (\int g d\mu_\phi)\rho_\phi) \right|\right|_\infty = 0.$$

It follows from this and (8.7) that

$$\lim_{n\to\infty} \left|\left| \frac{\frac{1}{n}\sum_{j=0}^{n-1} \hat{\mathcal{L}}_{\phi_q}^{n-j}(g\mathcal{L}_{\phi_q}^{j}(1\!\!1))}{\hat{\mathcal{L}}_{\phi_q}^{n}(1\!\!1)} - (\int g d\mu_\phi)1\!\!1 \right|\right|_\infty = 0.$$

We now are done by invoking (8.3). □

**Proposition 8.5.** *Suppose that $\phi : J \to \mathbb{R}$ is a Hölder continuous potential with $\sup(\phi) - \inf(\phi) < \kappa_f$ and that $g : J \to \mathbb{R}$ is a Hölder continuous function. Then the function $\mathbb{R} \ni t \mapsto \mathrm{P}(\phi + tg) \in \mathbb{R}$ is differentiable on a sufficiently small open neighborhood of zero and*

$$\frac{d}{dt}\mathrm{P}(\phi + tg) = \int g d\mu_{\phi+tg}.$$

*Proof.* Put

$$\phi_t = \phi + tg.$$

Clearly, there are $H > 0$, $0 \le s < \kappa_f$ and $\delta > 0$ such that $\phi_t \in \mathcal{P}_H^s(f)$ for all $t \in [-\delta, \delta]$. It then follows from Proposition 5.1, Lemma 5.2, and Proposition 8.1 that for every $x \in J$,

$$\begin{aligned} \mathrm{P}(\phi_t) &= \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_{\phi_t}^n(1\!\!1)(x) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{y\in f^{-n}(x)} \exp(S_n\phi(y)) \left(\exp(S_n g(y))\right)^t . \end{aligned} \tag{8.8}$$

Fix an arbitrary $x \in J$ and for every $n \ge 1$ put,

$$F_n(t) = \frac{1}{n} \log \mathcal{L}_{\phi_t}^n(1\!\!1)(x).$$

Then

$$P(\phi_t) = \lim_{n\to\infty} F_n(t). \tag{8.9}$$

We shall prove the following.

**Claim:** For every $n \ge 1$ the function $[-\delta, \delta] \ni t \mapsto F_n(t)$ is convex.

Indeed, for all $t_1, t_2 \in [-\delta, \delta]$ and all $\gamma \in [0,1]$, using Minkowski's inequality, we get

$$
\begin{aligned}
F_n(\gamma t_1 + (1-\gamma)t_2) &= \\
&= \frac{1}{n}\log\left(\sum_{y\in f^{-n}(x)} \exp(S_n\phi(y)) + \gamma t_1 S_n g(y) + (1-\gamma)t_2 S_n g(y))\right) \\
&= \frac{1}{n}\log\left(\sum_{y\in f^{-n}(x)} \exp(\gamma(S_n\phi(y) + t_1 S_n g(y)) + (1-\gamma)(S_n\phi(y)) + t_2 S_n g(y))\right) \\
&\le \frac{1}{n}\gamma\log\left(\sum_{y\in f^{-n}(x)} \exp(S_n\phi(y) + t_1 S_n g(y))\right) + \\
&\qquad + \frac{1}{n}(1-\gamma)\log\left(\sum_{y\in f^{-n}(x)} \exp((S_n\phi(y)) + t_2 S_n g(y))\right).
\end{aligned}
$$

We are done. □

But, since

$$
F_n'(t) = \frac{\frac{1}{n}\mathcal{L}_{\phi_t}^n(S_n(g)(x)}{\mathcal{L}_{\phi_t}^n(\mathbb{1})(x)} = \frac{\frac{1}{n}\hat{\mathcal{L}}_{\phi_t}^n(S_n(g)(x)}{\hat{\mathcal{L}}_{\phi_t}^n(\mathbb{1})(x)},
$$

Lemma 8.4 yields that the sequence of functions $[-\delta,\delta] \ni t \mapsto F_n'(t)$ converges pointwise to the integral to $\int g d\mu_{\phi_t}$. Therefore, the functions $t \mapsto F_n'(t)$ are monotone (by the Claim) and converge pointwise to the function $t \mapsto \int g d\mu_{\phi_t}$, which is continuous in view of Proposition 8.2 and Proposition 8.3. Hence, this convergence is uniform. Combining this with (8.8), we thus conclude that the function $(-\delta,\delta) \ni t \mapsto \mathrm{P}(\phi+tg) \in \mathbb{R}$ is differentiable and the derivative $\frac{d}{dt}\mathrm{P}(\phi+tg)$ is equal to $\int g d\mu_{\phi+tg}$, the limit of derivatives. We are done. □

Now, after all this preparation the proof of the main result of our paper is ready.

**Theorem 8.6.** *If $f : \mathbb{P}^k \to \mathbb{P}^k$ is a holomorphic endomorphism of a complex projective space $\mathbb{P}^k$, and $\phi : J \to \mathbb{R}$ is a Hölder continuous potential satisfying* $\sup(\phi) - \inf(\phi) < \kappa_f$*, then there exists exactly one equilibrium state for $\phi$. This equilibrium state is equal to $\mu_\phi = \rho_\phi m_\phi$ and it is metrically exact.*

*Proof.* The fact that $\mu_\phi = \rho_\phi m_\phi$ is an equilibrium state for $\phi$ was established in Proposition 8.1. Its uniqueness follows directly from Proposition 8.5 and Corollary 2.6.7 from [PU]. Metrical exactness of the dynamical system $(f, \mu_\phi)$ coincides with Theorem 6.11. We are done. □

## 9. Local Degree

Let us start by looking at the following result of Charles Favre (see [F1]).

**Proposition 9.1.** *For all $x \in \mathrm{P}^k$ the limit*

$$d(x) = \lim_{n\to\infty} (\deg_x f^n)^{\frac{1}{n}}$$

*exists. If $d(x) > 1$ then there exists an irreducible complex subspace $V$ and integers $k, l$ such that $f^k(x) \in V$ and $f^l(V) = V$.*

The proof of this theorem in [F1] is based on Szemeredi's Theorem, and the thesis [F1] provides a way of avoiding it. Favre's statement, however, is not sufficient for us; we need a uniform estimate of degrees of $n'$th iterates. So, our main result of this section, Proposition 3.3, may be understood as a strengthening of Favre's result. In addition, as a byproduct, we obtain also an elementary proof of Favre's Proposition. For the convenience of the reader we begin with two short definitions that have been already formulated in Section 2, Preliminaries.

**Definition 9.2.** *Given an integer $n \geq 1$ the periodic critical set $A_n$ is the union of orbits of all irreducible varieties, that are contained in the critical set and are periodic under an iterate $f^l$ with some $l \leq n$. In particular, an orbit of a critical periodic point of period $l \leq n$ is in the critical periodic set $A_n$.*

For all positive integers $n$ and $p$ such that $n > p$ let $E_n^p$ be the set defined as follows.

**Definition 9.3.** *$E_n^p$ is the set of all points $x \in \mathbb{P}^k$ for which there exists a non-negative integer $i \leq n-1$ such that $f^i(x) \in A_p$.*

Our main result in this section is the following stated in Preliminaries as proposition 3.3.

**Proposition 9.4.** *For every $\beta > 0$ there exist $p = p(\beta)$ and $N = N(\beta)$ such that for every $n > N$ and for every $x \notin E_n^p$ we have*

$$\#\{j \leq n : f^j(x) \in C\} \leq n\beta.$$

For its proof we need the following three simple lemmas.

**Lemma 9.5.** *Let $h : X \to X$ be an arbitrary map and let $F \subset X$ be an arbitrary subset of $X$. Fix $\alpha > 0$ and consider the union*

$$F_\alpha = \bigcup_{i \le [\frac{1}{\alpha}]} F \cap h^{-i}(F).$$

*Next, consider a trajectory $x, h(x), \dots, h^{M-1}(x)$ of length $M > [\frac{1}{\alpha}]$. If*

$$\#\{s < M : h^s(x) \in F_\alpha\} \le \alpha M,$$

*then*

$$\#\{s < M : h^s(x) \in F\} \le 3\alpha M.$$

*Proof.* Divide the trajectory $x, h(x), \dots h^{M-1}(x)$ into blocks, ending at consecutive points in the trajectory, which are in $F$, i.e: $B_1 = [x, h(x), \dots h^{s_1}(x)]$ where $s_1$ is the smallest iterate of $x$ which falls to $F$, and, inductively, $B_{m+1} = [h^{s_m+1}x, \dots, h^{s_{m+1}}(x)]$ while the last block has the form $[h^{s_r}(x), h^{M-1}(x)]$. Let us choose all blocks $B_m$, $m < r$ of length $\le [\frac{1}{\alpha}] - 1$. Notice that then $h^{s_m}(x) \in F_\alpha$ since the distance $s_{m+1} - s_m$ is not larger than $[\frac{1}{\alpha}]$. Consequently, by our assumption, the number of such blocks is not larger than $\alpha M$.Moreover, in the remaining blocks $B_m$,$m < r$ every appearance of an element of $F$ is followed by at least $[\frac{1}{\alpha}]$ elements which are not in $F$. Consequently, the total number of elements of $F$ in the trajectory can be bounded from above by $\alpha M + \alpha M + 1 = 2\alpha M + 1 < 3\alpha M$ (since $M > [\frac{1}{\alpha}]$). □

**Lemma 9.6.** *Let $h : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic map and let $\mathcal{D} = \{D_1, \dots, D_t\}$ be a collection of irreducible varieties of the same codimension $p$. If, for some $i$, $h^{-i}(D_s) \cap D_r$ has an irreducible component of the same codimension $p$, then $h^i(D_r) = D_s$.*

*Proof.* Let $V$ be a component of $h^{-i}D_s \cap D_r$ of the same codimension. Since $V \subset D_r$ and $D_r$ is irreducible, we have $V = D_r$ and, consequently, $D_r \subset h^{-i}(D_s)$. Thus, $h^i(D_r) \subset D_s$. Next, since $D_s$ is also irreducible and $\dim h(D_s) = \dim D_r$, we get $h^i(D_r) = D_s$. □

**Lemma 9.7.** *Under the assumptions of Lemma 9.6, there exists an integer $l \ge 1$ such that, for $H = h^l$, if $H^i(D_r) = D_s$ for some $i \in \mathbb{N}$ and some $D_r, D_s \in \mathcal{D}$, then $H(D_s) = D_s$.*

*Proof.* We build a natural graph with vertices $D_1, \dots D_t$. We put an arrow from $D_s$ to $D_r$ if $D_s$ is mapped under some iterate $h^i$ of $h$ onto $D_r$ and, for all $1 \le j < i$, the image $h^j(D_s)$ is not a variety in our family $\mathcal{D}$. Associate to this arrow the weight $i$. To every maximal path in this graph, which is not eventually a loop, we associate its weight defined to be the product of weights of all arrows forming this path. Let $l$ be a multiple of the lengths of all simple loops and, which in addition, is larger than the weights of all maximal paths in the graph that do not contain loos. Then every variety $D_s$ is mapped by $H = h^l$ either on a variety which is not in our family $\mathcal{D}$, or onto a variety $D_r$ which belongs to a loop; thus fixed by $H$. □

We now pass to the proof of the Proposition 9.4. The proof will be performed in two steps. In the first, preparatory Step I, we construct recursively appropriate families of irreducible varieties contained in the critical set. They are then used in the inductive proof of Proposition 9.4 in Step II.

Step I. Construction of families of irreducible varieties $\mathcal{D}_m^{(1,2,j_3,\ldots j_m)}$.

Given $\beta > 0$, let $\beta_0 = \beta$, $\beta_1 = \frac{\beta}{3}$, and $\beta_m = \frac{\beta_{m-1}}{3k}$ for all $2 \leq m \leq k$. This choice of the sequence $\beta_m$ will become clear in the second step of the proof. First, let us use lemma 9.7 for the map $f$ and for the collection $\mathcal{D}_1^{(1)}$ of irreducible components $C_1, \ldots C_t$ of the critical set $C$. The superscript (1) stands here for the codimension 1 of all varieties in the family. We then replace the original map $f$ by its iterate $g_1 = f^{l_1}$, such that for which the statement of Lemma 9.7 is satisfied for the family of irreducible varieties $\mathcal{D}_1^{(1)}$.Next we define the family of varieties $\mathcal{D}_2^{(1,2)}$. This family consists of all irreducible components of the intersections $D_r \cap g_1^{-i} D_s$ of codimension 2, where $i \leq [\frac{3}{\beta}]$. Notice, that, for $k = 2$ the varieties in the family $\mathcal{D}_2^{(1,2)}$ are just points. For $k = 2$ our construction ends at this step.

For $k > 2$ we proceed as follows. Using Lemma 9.7 again, we find an iterate $g_2 = g_1^{l_2}$ such that the statement of this lemma is satisfied, i.e. for each $D_r, D_s \in \mathcal{D}_2^{(1,2)}$, if $g_2^i(D_r) = D_s$ for some $i \geq 1$, then $g_2(D_s) = D_s$. Now, for every $3 \leq j_3 \leq k$ let $\mathcal{D}_3^{(1,2,j_3)}$ be the family of all irreducible components of all intersections of the form $D_r \cap g_2^{-i} D_s$, $D_r, D_s \in \mathcal{D}_2$, that have codimension $3 \leq j_3 \leq k$ and $1 \leq i \leq [\frac{3}{\beta_1}]$. Let

$$\mathcal{D}_3 = \bigcup_{j_3=3}^{k} \mathcal{D}_3^{(1,2,j_3)}$$

Again, notice that if $k = 3$ then only $j_3 = 3$ is possible and the procedure ends at this point.

For a general $k$, proceeding by induction, fix $m \geq 3$ and assume that for every $1 \leq j_m \leq k$ the family $\mathcal{D}_m^{(1,2,j_3,\ldots j_m)}$ of irreducible varieties of codimension $j_m$ has been defined. Also, assume that the map $g_m$ (as an appropriate iterate of $f$) has been defined. We assume by induction that the map $g_m$ has the following property. If, for some $i \geq 1$, and $D_r, D_s \in \mathcal{D}_m^{(1,2,j_3,\ldots j_m)}$, $g_m^i(D_r) = D_s$, then $g_m(D_s) = D_s$. Fix $1 \leq j_m \leq k$. If $j_m = k$ then the procedure ends at this moment (note that the elements of $\mathcal{D}_m^{(1,2,j_3,\ldots j_m)}$ are then just points). If $j_m < k$, we define the families $\mathcal{D}_{m+1}^{(1,2,j_3,\ldots j_m,j_{m+1})}$, where $j_{m+1} > j_m$ as follows. The family $\mathcal{D}_{m+1}^{(1,2,j_3,\ldots j_m,j_{m+1})}$ consists of all irreducible components of all intersections $D_r \cap g_m^{-i} D_s$, $i \leq [\frac{3}{\beta_{m-1}}]$ that have codimension equal to $j_{m+1}$. Note that the range of admissible $j_{m+1}$'s is $\{i_m + 1, \ldots k\}$ but some families $\mathcal{D}_{m+1}^{(1,2,j_3,\ldots j_m,j_{m+1})}$ may be empty. Finally, for all families $\mathcal{D}_{m+1}^{(1,2,j_3,\ldots j_{m+1})}$, defined in this way, we find, using Lemma 9.7, a common value $l_{m+1}$ and an iterate $g_{m+1} = g_m^{l_{m+1}}$ of

$g_m$ such that if $D_s = g^i_{m+1}(D_r)$ for some $D_r, D_s \in \mathcal{D}^{(1,2,j_3,\dots j_{m+1})}_{m+1}$, then $g_{m+1}(D_s) = D_s$.

Step II. Fix $N = N_0 = l_1 \cdot l_2 \cdot \dots \cdot l_k N_k$, where $N_k > N^0_k$ and $N^0_k > [\frac{1}{\beta_k}]$ so that the statement of Lemma 9.5 is satisfied for all $M \geq N^0_k$ and all $\alpha \geq \beta_k$. Let $N_1 = l_2 \cdot l_3 \cdot \dots \cdot N_k$, $N_m = l_{m+1} \dots N_k$. (so that $N_{m-1} = l_m N_m$). Take now an arbitrary point $x \in \mathbb{P}^k$ and assume that the trajectory of $x$, up to $f^N(x)$ visits the critical set with a frequency larger than $\beta$:

$$\#\{n < N : f^n x \in C\} > \beta N.$$

Since $N = l_1 \cdot N_1$, the whole trajectory $x, f(x), \dots f^{N-1}(x)$ can be split in a natural way into $l_1$ trajectories of the points $f^i(x), i \leq l_1$, under $g_1 :\ f^i(x), g_1(f^i(x)) \dots g^{N_1-1}(f^i(x))$, and it is evident that there exists a point $\tilde{x} = f^j(x)$, $j < l_1$ such that

$$\#\{n < N_1 : g^n_1(\tilde{x}) \in C\} > \beta N_1.$$

Let us consider two cases. Either

(1) there exists $n < N_1$, $i \leq [\frac{1}{\beta}]$, such that $g^n_1(\tilde{x}) \in C_r$, $g^{n+i}_1(\tilde{x}) \in C_s$ and the component of $C_r \cap g^{-i}_1(C_s)$ containing $\tilde{x}$ has codimension 1,

or else, the following holds:

(2) if $g^n_1(\tilde{x}) \in C_r$ and $g^{n+i}(\tilde{x}) \in C_s$ for some $n \leq N_1$ $i \leq [\frac{3}{\beta}]$ then the component of $C_r \cap g^{-i}_1(C_s)$ containing $g^n_1(\tilde{x})$ has codimension 2.

If the case (1) occurs, we conclude that the point $g^{n+i}_1(\tilde{x})$, where $n + i < N_1, i \leq [\frac{1}{\beta}]$, lands in a critical variety $V$, which is fixed by $g_1$ thus periodic for $f$ with period $l_1$. This implies that $f^m(x) \in V$, where $m \leq N$, $f^{l_1}(V) = V$, and $x \in E^p_N$. The proof is then finished.

In the case (2) we proceed as follows. Using Lemma 9.5, we conclude that

$$\#\left\{0 \leq n \leq N_1 : g^n_1(\tilde{x}) \in \bigcup_{i \leq [\frac{3}{\beta}]} \left(C \cap g^{-i} C\right)\right\} > \beta/3 N_1.$$

This means that the trajectory $g_1(\tilde{x}), g^2_1(\tilde{x}), \dots, g^{N_1}_1(\tilde{x})$ of $\tilde{x}$, under $g_1$ visits the varieties $D_r$ from the family $\mathcal{D}^{(1,2)}_2$ with frequency larger than $\beta/3$, This property is referred to as $M_1(\tilde{x})$ and reads as follows.

$$\#\left\{0 \leq n \leq N_1 : g^n_1(\tilde{x}) \in \bigcup_{D_r \in \mathcal{D}^{(1,2)}_2} D_r\right\} > \beta/3 N_1 = \beta_1 N_1 \tag{9.1}$$

It is easy now to conclude the proof if $k = 2$. Indeed, then the varieties in $\mathcal{D}^{(1,2)}$ are just points, and we can proceed as follows. Let $R = R(\beta) = \#\mathcal{D}^{(1,2)}$. It is evident that, if $N_1$ is large enough and

$$\#\left\{0 \le n < N_1 : g_1^n(\tilde{x}) \in \bigcup_{D_r \in \mathcal{D}_2^{1,2}} D_r\right\} \ge \beta/3N_1$$

then there exist $m, m+i < N_1$ and a point $z$ in the family $\mathcal{D}_2^{(1,2)}$ such that $g_1^m(\tilde{x}) = g_1^{m+i}(\tilde{x}) = z$. Hence, there are $n < N$, and $j \le [\frac{3}{\beta}]$ such that $f^n(x) = f^{n+j}(x) = z$ and $z$ is a critical periodic point. thus $x \in E_N^p$, where $p = [\frac{3}{\beta}]$. This concludes the proof if $k = 2$.

For an arbitrary $k$ we use the following inductive procedure. Put $p = l_1 l_2 \dots l_k$. Recall that $\beta_1 = \beta/3$. Let $m \le k$ and assume that for the point $x$ the following property $M_{m-1}(x)$ holds.

$$\#\left\{0 \le n < N_{m-1} : g_{m-1}^n(x) \in \bigcup_{D \in \mathcal{D}_m^{(1,2,j_3 \dots j_m)}} D\right\} > \beta_{m-1} N_{m-1},$$

where $D \in \mathcal{D}_m^{(1,2,j_3 \dots j_m)}$ for some sequence $(1, 2, j_3 \dots j_m)$ which depends on $x$. Note that for $m = 2$ this is precisely the formula (9.1). Let $\mathcal{D}_m$ be the union of all families of the form $\mathcal{D}_m^{(1,2,j_3 \dots j_m)}$. Recall that $g_m = g_{m-1}^{l_m}$ is the iterate of $f$ such that, (see Step 1), if for some $D_r, D_s \in \mathcal{D}_m$, $g^i(D_r) = D_s$ then $g_m(D_s) = D_s$. Since the trajectory

$$\{x, g_{m-1}(x), g_{m-1}^2(x), \dots, g_{m-1}^{N_{m-1}}(x)\}$$

of the point $x$ under $g_{m-1}$ can be split into $l_m$ trajectories of the points $g_{m-1}^j(x)$, $0 \le j \le l_m - 1$, under $g_m$, it is evident that there exists a point $\tilde{x} = g_{m-1}^j(x), i < l_m$ such that

$$\#\left\{n \le N_m : g_m^n(\tilde{x}) \in \bigcup_{D \in \mathcal{D}_m^{(1,2,j_3 \dots, j_m)}} D\right\} \ge \beta_{m-1} N_m.$$

Recall that $N_{m-1} = l_m N_m$. Now, as in the case of $m = 1$, we consider two possibilities.

(Ind.1) There exist $n < N_m$, $i \le [\frac{3}{\beta_{m-1}}]$ and $D_r, D_s \in \mathcal{D}_m$ of the same codimension, say $j_m$, such that $g_m^n(\tilde{x}) \in D_r$, $g_m^{n+i}(\tilde{x}) \in D_s$, and the component of $D_r \cap g_m^{-i}(D_s)$ containing $\tilde{x}$ has codimension $j_m$.

(Ind.2) If $g_m^n(\tilde{x}) \in D_r$ and $g^{n+i}(\tilde{x}) \in D_s$ for some $n < N_m$,$i \le [\frac{3}{\beta_{m-1}}]$, then the component of $D_r \cap g_m^{-i}(D_s)$ containing $g_m^n(\tilde{x})$ has codimension larger than $j_m$.

If (Ind. 1) occurs then, by Lemma 9.6 and Lemma 9.7, we get that $g_m(D_s) = D_s$ and $g_m^{n+i}(\tilde{x})$ lands in an irreducible variety $D_s$, which is fixed by $g_m$. In particular, this case always occurs when $j_m = k$, i.e the varieties $D_r, D_s$ are just points. Since $g_m^{n+i}(\tilde{x})$ lands in the variety $D_s$, fixed by $g_m$, we see that $g_{m-1}^{j+l_m(n+i)}(x) = g_m^{n+i}(\tilde{x})$ lands in the component $D_s$. Note that $j + l_m(n+i) < N_{m-1}$ and $g_{m-1}^{l_m}(D_s) = g_m(D_s) = D_s$.

Thus, we conclude that for some $r < N_{m-1}$ the point $g_{m-1}^r(x)$ lands in the variety $D_s$, which is contained in the critical set and which is periodic under $g_{m-1}$ with period $l_m$. We are then done.

In the case (Ind. 2) we proceed as follows. Using Lemma 9.5, for $\alpha = \frac{\beta_{m-1}}{3}$, we can write

$$\#\left\{ n \le N_m : g_m^n(\tilde{x}) \in \bigcup_{j \le [\frac{3}{\beta_{m-1}}]} \bigcup_{D_r, D_s \in \mathcal{D}_m^{(1,2,j_3 \dots, j_m)}} (D_r \cap g_m^{-j} D_s) \right\} \ge \frac{\beta_{m-1}}{3} N_m.$$

Recall that the family $\mathcal{D}_{m+1}^{(1,2,j_3 \dots j_m, j_{m+1})}$ consists of all intersections $D_r \cap g_m^{-j} D_s$, $1 \le j \le [\frac{3}{\beta_{m-1}}]$, $D_r, D_s \in \mathcal{D}_m^{(1,2,j_3 \dots, j_m)}$, which have the same codimension $j_{m+1}$.

$$\{D_r \cap g_m^{-i} D_s : \; D_r, D_s \in \mathcal{D}_m^{(1,2,j_3 \dots j_m)}\} = \bigcup_{j_{m+1}} \mathcal{D}_{m+1}^{(1,2,j_3 \dots j_m, j_{m+1})}$$

and, since the range of all possible $j_{m+1}$'s is less than $k$, we can choose one value of $j_{m+1}$ such that the following property $M_m(\tilde{x})$ holds.

$$\#\left\{ n \le N_m : g_m^n(\tilde{x}) \in \bigcup_{D_r \in \mathcal{D}_{m+1}^{(1,2,j_3 \dots, j_{m+1})}} D_r \right\} \ge \frac{\beta_{m-1}}{3k} N_m = \beta_m N_m.$$

Therefore, we have checked the following. If the condition $M_{m-1}(x)$ is satisfied for $x$ then either there exists $r < N_{m-1}$ so that $g_{m-1}^r(x)$ falls into a variety which is contained in $C$ and periodic under $g_{m-1}$ with period $l_m$, or else, the condition $M_m(\tilde{x})$ is satisfied for some $\tilde{x} = f^j(x)$, $j \le l_m$, and some family $\mathcal{D}_m^{1,2,j_3,\dots,j_{m+1}}$. This ends the inductive step.

Now, take an arbitrary point $x \in \mathbb{P}^k$ and consider its trajectory $x, f(x), \dots, f^N(x)$. where $N = p \cdot N_k$, $N_k \ge N_k^0$. Let us turn again to the beginning of the Step II. Since the case (1) ends the proof, we are left to consider case (2). This leads to the condition $M_1(\tilde{x})$ (see (9.1)). We then apply the above inductive procedure, starting from the point $\tilde{x}$ and $m = 1$. If, at some step the case (Ind.1) occurs then the induction stops. It is evident that the number of inductive steps is at most $k$. Assume that the procedure ends for some $m = m_0 \le k$. It then follows that for some $r < N_{m_0-1}$ the point $g_{m_0-1}^r(\hat{x})$ lands in a variety $D_s$, which is contained in the critical set and which is periodic under $g_{m_0-1}$ with period $l_{m_0}$, thus periodic under $f$ with period $l_{m_0} \cdot l_{m_0-1} \cdot \dots \cdot l_1 \le p$. Observe that $\hat{x}$ is a point in the trajectory of $\tilde{x}$ and, in fact it is easy to see that $\hat{x} = f^t(x)$ for some $t < l_1 + l_1 l_2 + \dots + l_1 l_2 \dots l_{m_0-1}$. Since $g_{m_0-1} = f^{l_1 l_2 \dots l_{m_0-1}}$, we get $f^t(x) \in D_s$ for some $t < N$. Finally, take an arbitrary $n > l_1 l_2 \dots l_k N_k^0 = p N_k^0$. Then $n = pM + r$ for some $M \ge N_k^0$ and $0 \le r < p$. It is evident that, if $\#\{i \le n : f^i(x) \in C\} > 2n\beta$ then $\#\{i \le pM : f^i(x) \in C\} > n\beta$ if $M$ is large enough. Thus, the statement follows from the proven part for $N = pM$. $\square$

## References


[Bo] R. Bowen, Equilibrium states and the ergodic theory for Anosov diffeomorphisms. LectṄotes in Math4̇70, Springer, 1975. 2

[Br] J.-Y. Briend, La propriété de Bernoulli pour les endomorphismes de $\mathbb{P}^k(\mathbb{C})$. Ergod. Th. Dynam. Sys. 22 (2002), 323-327. 1

[BD] J.-Y. Briend, J. Duval, Deux caractÃ©risations de la mesure d'Ã©quilibre d'un endomorphisme de P$k$(**C**), Publ. Math. Inst. Hautes Ã93 (2001), 145-159. 1, 3, 4

[DPU] M. Denker, F. Przytycki, M. Urbański, On the transfer operator for rational functions on the Riemann sphere, Ergod. Th. and Dynam. Sys. 16 (1996), 255-266. 1

[DU] M. Denker, M. Urbański, Ergodic theory of equilibrium states for rational maps, Nonlinearity 4 (1991), 103-134. 1, 1, 6

[D1] T.-C. Dinh, Exponential estimates for plurisubharmonic functions and stochastic dynamics, Preprint 2008. 1

[D2] T.-C. Dinh, Equidistribution towards the Green current for holomorphic maps, Ann. Sci. École Norm. Sup. 41 (2208), 307-336. 1

[DS] T.-C. Dinh, N. Sibony, Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings, Preprint 2009. 1

[Du] C. Dupont, Bernoulli coding map and almost-sure invariance principle for endomorphisms of $CP(k)$, to appear Probability Theory and Related Fields 1

[F1] C. Favre, Multiplicty of holomorphic functions, Math. Ann. 316 (2000), 355-378. 1, 9, 9

[F1] C. Favre, Dynamique des applictiones rationelles, PhD thesis 2002. 1, 9

[FS] J.E. Fornaess and N. Sibony, Complex dynamics in higher dimension I, Astérisque 222 (1994), 201 - 231. 1, 4

[Gr] M. Gromov, On the entropy of holomorphic maps, Preprint IHES. 1, 7, 7

[Ha] N. Haydn, Convergence of the transfer operator for rational maps, Ergodic Theory Dynam. Systems 19 (1999), 657-669. 1

[La] S. Lang, *Introduction to Complex Hyperbolic Spaces,* Springer 1987. 3, 7

[Ly] M. Lyubich, Entropy properties of rational endomorphisms of the Riemann sphere. ErgodṪhḊynamṠys3̇ (1983), 351-386. 6

[McM] C. McMullen, Algebra and Dynamics, Lect. Notes 2004. 3, 7

[Pa] William Parry, *Entropy and Generators in Ergodic Theory,* Benjamin 1969.

[Pr] F. Przytycki, On the Perron-Frobenius-Ruelle operator for rational maps on the Riemann sphere and for Hölder continuous functions. BolṠocḂrasṀat2̇0 (1990), 95-125. 1

[PU] Feliks Przytycki, Mariusz Urbański, *Fractals in the Plane - Ergodic Theory Methods,* to appear Cambridge Univ. Press. 1, 2, 8, 8

[Ru1] D. Ruelle, Statistical Mechanics on a compact set with $\mathbb{Z}^\nu$ action satisfying expansiveness and specification, Trans. Amer. Math. Soc. 185 (1973), 237-251. 1

[Ru2] D. Ruelle, *Thermodynamic formalism, Encyclopedia of Mathȧnd Appl.,* vol. 5, Addison - Wesley 1976. 2

[Sch] H. Schaefer, *Banach lattices and positive operators,* (1974) Springer. 6

[Wa] P. Walters, An introduction to ergodic theory, Springer-Verlag, 1982. 1, 2

[Yo] Y. Yomdin, Volume growth and entropy, Israel J. Math. 57 (1987), 285-300. 1

Mariusz Urbański, Department of Mathematics, University of North Texas, Denton, TX 76203-1430, USA
*E-mail address*: urbanskiunt.edu
Web: www.math.unt.edu/~urbanski

Anna Zdunik, Institute of Mathematics, Warsaw University, ul. Banacha 2, 02-097 Warszawa, Poland
*E-mail address*: A.Zdunik@mimuw.edu.pl