# CONTINUITY OF HAUSDORFF MEASURE

# FOR

# CONFORMAL DYNAMICAL SYSTEMS

TOMASZ SZAREK, MARIUSZ URBAŃSKI, AND ANNA ZDUNIK

Abstract. Developing the pioneering work of Lars Olsen, we deal with the question of continuity of the numerical value of Hausdorff measures in topologized families of conformal dynamical systems. We prove such continuity for hyperbolic polynomials from the Mandelbrot set, and more genrally for the space of hyperbolic rational functions of a fixed degree. We go beyond hyperbolicity by proving continuity for maps including parabolic rational functions, for example that the parameter $1/4$ is such a continuity point for quadratic polynomials $z \mapsto z^2 + c$ for $c \in [0, 1/4]$. We prove the continuity of the numerical value of Hausdorff measures also for the spaces of conformal expanding repellers and parabolic ones, more generally for parabolic Walters conformal maps. We also prove some partial continuity results for all conformal Walters maps; these are in general of infinite degree. In order to do this, as one of our tools, we provide a detailed local analysis, uniform with respect to the parameter, of the behavior of conformal maps around parabolic fixed poins in any dimension. We also establish continuity of numerical values of Hausdorff measures for some families of infinite 1-dimensional iterated function systems.

## 1. Introduction

Let $\mathcal{K}(I)$ be the space of all non-empty compact subsets of the unit interval $I = [0, 1]$ topologized by the Hausdorff metric. As both the collection of all finite subsets and all finite unions of non-degenerate subintervals of $I$ are dense in $\mathcal{K}(I)$, the Hausdorff dimension function $\mathcal{K}(I) \ni F \mapsto \mathrm{HD}(F) \in \mathbb{R}$ is discountinuous at every point, the Hausdorff dimension behaves badly indeed with respect to Hausdorff metric. This firmly suggest that one should not expect too much from this function. It is therefore the more astonishing that the situation changes dramatically when a dynamics is involved. Indeed, David Ruelle has asserted in [Rue] that the function $c \mapsto J(z^2 + c)$, the latter being the Julia set of the quadratic polynomial $z^2+c$, is not only countinuous but even real-analytic. A variety generalization and extentions of Ruelle's result then followed (cf. for example [UZi], [UZd], [U3] or [AU]. These concerned hyperbolic systems, possiby being 1-to-infinity (iterated function systems and transcendental functions), systems allowing critical points in the Julia sets, and parabolic systems. All of them involved conformal measures. These are dynamically defined objects, frequently of a transparent geometric meaning. More precisely, normalized Hausdorff or packing measures. Having the machinery of thermodynamic formalism at

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hand (see [PU] or [MU3] for a contemporary exposition), it it not hard to prove that the in the context of the above systems conformal measures vary continuously in the topology of weak convergence.

A more subtle question is about regularity of the function ascribing to a system, or parameter, the numerical value of the Hausdorff measure of the corresponding Julia or limit set. The breakthrough came with the work of Lars Olsen ([Ol] who proved such continuity for finite iterated function systems consisting of similarities and satisfying the separation condition. In this paper we continue the direction of research originated by Olsen. We work with Walters conformal systems that comprise all: conformal expanding repellers, finite and infinite hyperbolic iterated function systems satisfying the separation condition, and hyperbolic and parabolic rational functions. Concerning hyperbolic systems, as the most transparent results we prove these.

**Theorem.** *If $(T_n : X_0^n \to X^n)_{n=1}^\infty$ is a sequence of conformal Walters maps converging sub-finely to a conformal Walters map $T : X_0 \to X$, then*

$$\limsup_{n\to\infty} H_{h_{T_n}}(X_\infty^n) \le H_{h_T}(X_\infty).$$

Here, $X_\infty^n$ and $X_\infty$ are, respectively, limit sets of these Walters maps, $h_{T_n} = \mathrm{HD}(X_\infty^n)$, and $h_T = \mathrm{HD}(X_\infty)$. The sub-fine convergence, and nice convergence are defined in section 9. Walters maps are defined in section 7.

**Theorem.** *With respect to the topology of sub-fine convergence, each conformal Walters map $S$ with $H_{h_T}(X_\infty(S)) = 0$ is a continuous point of the Hausdorff measure function $T \to H_{h_T}(X_\infty(T))$.*

In the context of conformal expanding repellers or finite iteration function system the most transparent results are these.

**Theorem.** *If $(T_n : X^n \to X^n)_1^\infty$ is a sequence of conformal expanding repellers in $\mathbb{R}^q$, converging sub–finely to some conformal Walters map $T : X \to X$, then*

$$\lim_{n\to\infty} H_{h_{T_n}}(X^n) = H_{h_T}(X).$$

and

**Theorem.** *If $E$ is a finite set, then each contracting conformal iterated function system $S \in \mathrm{CIFS}(X, E, A)$ satisfying the separation condition is a continuity point of the Hausdorff measure function $\mathrm{CIFS}(X, E) \ni \Phi \to H_{h_\Phi}(J_\Phi)$ with $\mathrm{CIFS}(X, E)$ endowed with the metric $d$ given by formula (15.1).*

The latter one is a fairly far going generalization of the original result of Olsen ([Ol]). As their immediate consequence, we have this.

**Corollary.** *For every $c \in \mathbb{C}$ let $J_c$ be the Julia set of the quadratic polynomial $\mathbb{C} \ni z \to z^2+c$ and let $h_c = \mathrm{HD}(J_c)$. Then the map $\mathbb{C} \ni c \to H_{h_c}(J_c)$ is continuous at each hyperbolic element $c \in \mathbb{C}$.*

In the parabolic case, which is technically much more complicated than the hyperbolic one, we proved the following.

**Theorem.** *If $(T_n : X^n \to X^n)_1^\infty$ is a sequence of conformal parabolic Walters maps converging nicely to some conformal parabolic Walters map $T : X \to X$ for which $\mathrm{HD}(X) > 1$, then $\lim_{n\to\infty} H^{h_{T_n}}(X^n) = H_{h_T}(X)$.*

and

**Theorem.** *If $(T_n : X^n \to X^n)_1^\infty$ is a sequence of finitely conformal Walters maps converging finely to some finitely conformal Walters map $T : X \to X$ for which $\mathrm{HD}(X) > 1$, and if its subsequence of all parabolic maps converges nicely to $T$, then $\lim_{n\to\infty} H^{h_{T_n}}(X^n) = H_{h_T}(X)$.*

As a consequence of the latter one, we get the following.

**Corollary.** *For every $c \in \mathbb{C}$ let $J_c$ be the Julia set of the quadratic polynomial $\mathbb{C} \ni z \to f_c(z) = z^2 + c$ and let $h_c = \mathrm{HD}(J_c)$. Then*

$$\lim_{\mathbb{R}\ni c\nearrow 1/4} H_{h_c}(J_c) = H_{h_{1/4}}(J_{1/4}).$$

**Corollary.** *For every $\lambda \in \mathbb{C} \setminus \{0\}$ let*

$$f_\lambda(z) = z(1 - z - \lambda^2 z).$$

*Let $J_\lambda := J(f_\lambda)$ be the Julia set of $f_\lambda$ and let $h_\lambda := \mathrm{HD}(J_{f_\lambda})$. Then for $R > 0$ sufficiently small, the function*

$$D_*(0, R) := \{\lambda \in \mathbb{C} \setminus \{0\} : |\lambda| < R\} \ni \lambda \to H_{h_\lambda}(J_\lambda)$$

*is continuous.*

Proving continuity of Hausdorff measures for infinite systems is also fairly involved. We devote to this end the last section of our paper and prove it for some selected subclass of linear infinite iterated function systems as Theorem 16.1.

Our general approach to the issue of continuity of Hausdorff measures is based on Olsen's intuitive formula expressing the value of Hausdorff measure in terms of normalized Hausdorff measure (conformal one) and diameters of the sets involved. The largest technical challenge in our paper is caused by parabolic systems as it is a central issue for our arguments to have a uniform behaviour of dynamics. In order to take rigorous care of this problem we analyze in detail in several first sections the local behaviour of parabolic conformal maps in all dimensions, 1, 2, and 3, all of them requiring different treatment. We then define Walters conformal maps, their various subclasses like expanding (hyperbolic) and parabolic ones. In the next section we define and study the modes of convergence of Walters maps. Having all this prepared, we prove in the following sections the actual continuity properties of Hausdorff measures as described above.

## 2. Conformal Maps Parabolic at Infinity; the case $q \geq 3$

Fix an integer $q \geq 3$. Given a linear isometry $D$, i.e. $D \in O(q)$, and given also $c \in \mathbb{R}^q$, let

$$\psi_{D,c} = D + c : \mathbb{R}^q \to \mathbb{R}^q.$$

Let $\mathrm{Par}_\infty(q)$ be the image of $O(q) \times \mathbb{R}^q$ under the map $(D, c) \to \psi_{D,c}$. We refer to $\mathrm{Par}_\infty(q)$ as the space of conformal maps parabolic at infinity on $\mathbb{R}^q$. Note that the map $O(q) \times \mathbb{R}^q \ni (D, c) \to \psi_{D,c}$ is 1–to–1. For every $\psi \in \mathrm{Par}_\infty(q)$ there are then uniquely defined $D_\psi \in O(q)$ and $c_\psi \in \mathbb{R}^q$ such that

$$\psi = D_\psi + c_\psi. \tag{2.1}$$

Since

$$\mathbb{R}^q = \mathrm{Fix}(D_\psi) \oplus \mathrm{Fix}^\perp(D_\psi),$$

we can uniquely write

$$c_\psi = b_\psi + a_\psi,$$

where $b_\psi \in \mathrm{Fix}(D_\psi)$ and $a_\psi \in \mathrm{Fix}^\perp(D_\psi)$. Iterating (2.1), we get for all $n \in \mathbb{Z}$ and all $z \in \mathbb{R}^q$ that

$$\psi^n(z) = \Big(\sum_{j=0}^{n-1} D_\psi^j\Big) a_\psi + n b_\psi + D_\psi^n z. \tag{2.2}$$

Since

$$(D_\psi - \mathrm{id})\Big(\sum_{j=0}^{n-1} D_\psi^j\Big) a_\psi = D_\psi^n a_\psi - a_\psi \tag{2.3}$$

and since $||D_\psi^n a_\psi - a_\psi|| \leq 2||a_\psi||$, we therefore conclude that for all $n \in \mathbb{Z}$,

$$\Big\|\sum_{j=0}^{n} D_\psi^j a_\psi\Big\| \leq 2||a_\psi|| \cdot ||(D_\psi - \mathrm{id})^{-1}|_{\mathrm{Fix}^\perp(\mathrm{D}_\psi)}||. \tag{2.4}$$

For every $R>0$ endow $\mathrm{Par}_\infty(q)$ with the pseudometric

$$\rho_R(S,T)=\|S_{|\overline{B}(0,R)}-T_{|\overline{B}(0,R)}\|_\infty=\sup\{\|S(x)-T(x)\|:x\in\overline{B}(0,R)\}.$$

Clearly $\rho_R$ is in fact a metric on $\mathrm{Par}_\infty(q)$. Endow $\overline{\mathbb{R}}^q$ with the spherical metric $\hat\rho_*$. Note that each member $T$ of $\mathrm{Par}_\infty(q)$ can be then treated as a continuous map from $\overline{\mathbb{R}}^q$ to $\overline{\mathbb{R}}^q$ by declaring that $T(\infty)=\infty$. Define the spherical metric $\rho_*$ on $\mathrm{Par}_\infty(q)$ by setting

$$\rho_*(T,S)=\sup\{\hat\rho_*(T(x),S(x)):x\in\overline{\mathbb{R}}^q\}.$$

We shall prove the following

**Proposition 2.1.** *All the metric $\rho_R$, $R>0$, and $\rho_*$ are equivalent in the sense that they induce the same topology on $\mathrm{Par}_\infty(q)$. With this topology, the map $O(q)\times\mathbb{R}^q\ni(D,c)\xrightarrow{\psi}\psi_{D,c}\in\mathrm{Par}_\infty(q)$ becomes a homeomorphism.*

*Proof.* Denote the topologies generated by $\rho_R$, $R>0$, respectively by $\tau_R$. Denote the topology generated by $\rho_*$ by $\tau_*$. Finally let $\tau$ be topology on $\mathrm{Par}_\infty(q)$ induced by the map $\psi$. Clearly, for every $R>0$ the maps $\mathrm{id}:(\mathrm{Par_\infty(q)},\tau)\to(\mathrm{Par_\infty(q)},\mathrm{\tau_R})$ and $\mathrm{id}:(\mathrm{Par_\infty(q)},\tau_*)\to(\mathrm{Par_\infty(q)},\mathrm{\tau_R})$ are continuous. Now, if $\lim_{n\to\infty}\psi_n=\psi$ in $\tau_R$, then

$$c_\psi=\psi(0)=\lim_{n\to\infty}\psi_n(0)=\lim_{n\to\infty}c_{\psi_n},$$

so the function $(\mathrm{Par}_\infty(q),\tau_R)\ni T\to c_T\in\mathbb{R}^q$ is continuous. Furthermore,

$$\begin{aligned}\limsup_{n\to\infty}\{\|(D_{\psi_n}-D_\psi)|_{\overline{B}(0,R)}\|_\infty&\le\limsup_{n\to\infty}\{\|\psi_n-\psi)|_{\overline{B}(0,R)}\|+\|c_{\psi_n}-c_\psi\|\}\\&\le\limsup_{n\to\infty}\{\|(\psi_n-\psi)|_{\overline{B}(0,R)}\|\}+\lim_{n\to\infty}\|c_{\psi_n}-c_\psi\|\\&=\lim_{n\to\infty}\rho_R(\psi_n,\psi)=0.\end{aligned}$$

This means that the function $(\mathrm{Par}_\infty(q),\tau_R)\ni\psi\to D_\psi\in O(q)$ is continuous. Along with the previous assertion this gives that the map $\mathrm{id}:(\mathrm{Par_\infty(q)},\mathrm{\tau_R})\to(\mathrm{Par_\infty(q)},\tau)$ is continuous. And along with an even earlier assertion the identity map $\mathrm{id}:(\mathrm{Par_\infty(q)},\tau)\to(\mathrm{Par_\infty(q)},\mathrm{\tau_R})$ is a homeomorphism, or equivalently the metrics $\tau$ and $\tau_R$ induce the same topology. Now assume that $\lim_{n\to\infty}\psi_n=\psi$ in $\tau$. Abbreviate

$$D:=D_\psi,\quad D_n:=D_{\psi_n},\quad c:=c_\psi,\quad\text{and}\quad c_n:=c_{\psi_n}.$$

We have for all $x\in\mathbb{R}^d$ that,

$$\begin{aligned}\hat\rho_*(\psi_n(x),\psi(x))&=\frac{2\|\psi_n(x)-\psi(x)\|}{1+\|\psi_n(x)\|^2)^{1/2}(1+\|\psi(x)\|^2)^{1/2}}\\&\le\frac{\|D_n-D\|\|x\|+\|c_n-c\|}{1+\|D_nx+c_n\|^2)^{1/2}(1+\|Dx+c\|^2)^{1/2}}.\end{aligned}$$

Disregarding finitely many terms we may assume without loss of generality that $\|c_n\|\le\|C\|+1$ for all $n\ge1$. If $\|x\|\le2(\|c\|+1)$, then we get

$$\hat\rho_*(\psi_n(x),\psi(x))\le4(\|c\|+1)(\|D_n-D\|+\|c_n-c\|).$$

If $\|x\| \geq 2(\|c\|+1)$, then

$$\begin{aligned}\hat{\rho}_*(\psi_n(x),\psi(x)) &\leq 2\frac{\|x\|(\|D_n-D\|+\|c_n-c\|)}{(\|D_nx\|-\|c_n\|)(\|Dx\|-\|c\|)} \leq 2\frac{\|x\|(\|D_n-D\|+\|c_n-c\|)}{(\|x\|-\|c\|-1)(\|x\|-\|c\|)}\\ &\leq 2\frac{\|x\|(\|D_n-D\|+\|c_n-c\|)}{\|x\|/2\|x\|/2} = \frac{8}{\|x\|}(\|D_n-D\|+\|c_n-c\|)\\ &\leq \frac{4}{(1+\|c\|)}(\|D_n-D\|+\|c_n-c\|).\end{aligned}$$

In either case, we get

$$\hat{\rho}_*(\psi_n(x),\psi(x)) \leq 4(1+\|c\|)(\|D_n-D\|+\|c_n-c\|). \tag{2.5}$$

Since $\psi_n(\infty)=\psi(\infty)=\infty$, the same estimate is true for $x=\infty$. It follows from (2.5) that $\hat{\rho}_*(\psi_n,\psi) \leq 4(1+\|c\|)(\|D_n-D\|+\|c_n-c\|)$. Therefore $\lim_{n\to\infty}\rho_*(\psi_n,\psi) \leq 4(1+\|c\|)(\lim_{n\to\infty}\|D_n-D\|+\lim_{n\to\infty}\|c_n-c\|)=0$. Thus, the identity map id : $(\mathrm{Par}_\infty(\mathrm{q}),\tau) \to (\mathrm{Par}_\infty(\mathrm{q}),\tau_*)$ is continuous, and with above proved, it is a homeomorphism. The proof is complete. □

From now on we consider $\mathrm{Par}_\infty(q)$ as a topological space with the topology established in Proposition 2.1. Let $G(q)$ be the set of all non–zero vector subspaces of $\mathbb{R}^q$. We endow $G(q)$ with the following metric

$$d_H(V,W) = d_H(V\cap\{x\in\mathbb{R}^q : \|x\|=1\}, W\cap\{x\in\mathbb{R}^q : \|x\|=1\}),$$

where the second $d_H$ is the Hausdorff metric on the collection of all non–empty compact subsets of $\mathbb{R}^q$. Given $0\leq l\leq q$ let

$$\mathrm{Par}_\infty(q,l) = \{\psi\in\mathrm{Par}_\infty(q) : \dim(\mathrm{Fix}(\psi)) = l\}.$$

We shall prove the following

**Lemma 2.2.** *Fix $0\leq l\leq q$. Then the following maps are continuous.*

(a) $\mathrm{Par}_\infty(q)\ni\psi\mapsto a_\psi\in\mathbb{R}^q$
(b) $\mathrm{Par}_\infty(q)\ni\psi\mapsto b_\psi\in\mathbb{R}^q$
(c) $\mathrm{Par}_\infty(q)\ni\psi\mapsto c_\psi\in\mathbb{R}^q$
(d) $\mathrm{Par}_\infty(q,l)\ni\psi\mapsto \mathrm{Fix}(D_\psi)$
(e) $\mathrm{Par}_\infty(q,l)\ni\psi\mapsto \mathrm{Fix}^\perp(D_\psi)$.

*Proof.* Since $c_\psi=\psi(0)$, the map $\psi\to c_\psi$ is continuous. So, (c) is proved. For every vector subspace $V$ of $\mathbb{R}^q$ put $V_1=\{x\in V : \|x\|=1\}$. Now we shall prove item(d). To do this it suffices to show that given $\varepsilon>0$ there exists $k\in\mathbb{N}$ such that

$$\mathrm{Fix}_1(D_{\psi_n}) \subset B(\mathrm{Fix}_1(D_\psi),\varepsilon) \tag{2.6}$$

and

$$\mathrm{Fix}_1(D_\psi) \subset B(\mathrm{Fix}_1(D_{\psi_n}),\varepsilon) \tag{2.7}$$

for all $n\geq k$, where $B(A,r)=\{x\in\mathbb{R}^q : \inf\{\|x-a\| : a\in A\}<r\}$. Assume contrary and suppose first that there is no $k\geq 1$ such that (2.6) holds. Then there exists an increasing

sequence $(n_j)_1^\infty$ such that $\mathrm{Fix}_1(D_{\psi_{n_j}}) \not\subseteq B(\mathrm{Fix}_1(D_\psi), \varepsilon)$. This means that for every $j \geq 1$ there exists $x_j \in \mathrm{Fix}_1(D_{\psi_{n_j}})$ such that

$$\|x_j - b\| \geq \varepsilon \tag{2.8}$$

for all $b \in \mathrm{Fix}_1(D_\psi)$. Passing to a subsequence, we may assume without loss of generality that $(x_j)_1^\infty$ converges to some vector $x \in \mathbb{R}^q$. Then $\|x\| = 1$ and

$$D_\psi x = \lim_{n\to\infty} D_{\psi_n} x_n = \lim_{n\to\infty} x_n = x.$$

So, $x \in \mathrm{Fix}_1(D_\psi)$ and, by (2.8), $\|x - b\| \geq \varepsilon$ for all $b \in \mathrm{Fix}_1(D_\psi)$. The contradiction shows that (2.6) holds. Assume in turn that (2.7) fails. Then there exists an increasing sequence $(m_j)_1^\infty$ such that $\mathrm{Fix}_1(D_\psi) \not\subseteq B(\mathrm{Fix}_1(D_{\psi_n}), \varepsilon)$. This means that for every $j \geq 1$ there exists $y_j \in \mathrm{Fix}_1(D\psi)$ such that

$$\|y_j - d\| \geq \varepsilon \tag{2.9}$$

for all $d \in \mathrm{Fix}_1(D_{\psi_{n_j}})$. Passing to a subsequence we may assume without loss of generality that $(y_j)_1^\infty$ converges to some vector $y \in \mathbb{R}^q$. Then $\|y\| = 1$ and $y \in \mathrm{Fix}_1(D_\psi)$. On the other hand, let for every $j \geq 1$, $\{v_1^{(j)}, \ldots, v_l^{(j)}\}$ be an orthonormal basis of $\mathrm{Fix}(\psi_{n_j})$. Passing yet to another subsequence we may assume without loss of generality that there are vectors $v_1, \ldots, v_l \in \mathbb{R}^q$ such that $\lim_{j\to\infty} v_i^l = v_i$ for all $1 \leq i \leq l$. Then $\{v_1, \ldots, v_l\}$ is an orthonormal set and

$$Dv_i = \lim_{j\to\infty} D_{\psi_{n_j}} v_i^{(j)} = \lim_{j\to\infty} v_i^j = v_i$$

for all $i = 1, \ldots, l$. Thus $\mathrm{Lin}(v_1, \ldots, v_l) \leq \mathrm{Fix}(D_\psi)$ and, as both the spaces have the same dimension, equal to $l$, they are equal That is

$$\mathrm{Lin}(v_1, \ldots, v_l) = \mathrm{Fix}(D_\psi). \tag{2.10}$$

In virtue of (2.9) fixing $a_1, \ldots, a_l \in \mathbb{R}$ such that $\sum_{i=1}^l a_i^2 = 1$ we have

$$\left\| y_j - \sum_{i=1}^l a_i v_i^{(j)} \right\| \geq \varepsilon$$

for all $j \geq 1$. Therefore

$$\left\| y_j - \sum_{i=1}^l a_i v_i \right\| \geq \varepsilon.$$

Along with (2.10) this means that $\|y - b\| \geq \varepsilon$ for all $b \in \mathrm{Fix}_1(D_\psi)$. Taking $b = y \in \mathrm{Fix}_1(D_\psi)$ we get a contradiction. This proves part (d). Part (e) immediately follows from (b). Parts (a) and (b) follow now from (c), (d) and (e) as the orthogonal projections from $\mathbb{R}^q$ onto $\mathrm{Fix}_1(D_\psi)$ and $\mathrm{Fix}^\perp(D_\psi)$ are continuous. $\square$

As a fairly straightforward consequence of this lemma, we shall prove the following.

**Lemma 2.3.** *For every $0 \le l \le q$ and every $\psi \in \mathrm{Par}_\infty(q,l)$ there exists a neighbourhood $V'_\psi$ of $\psi$ in $\mathrm{Par}_\infty(q,l)$ such that*

$$\sup\{\|(D_A-\mathrm{id})^{-1}|_{\mathrm{Fix}^\perp(\mathrm{D_A})}\| : \mathrm{A} \in \mathrm{V}'_\psi\} < +\infty.$$

*Proof.* Suppose for a contrary that there exists a sequence $(\psi_n)_1^\infty \subseteq \mathrm{Par}_\infty(q,l)$ such that

$$\lim_{n\to\infty} \|(D_{\psi_n}-\mathrm{id})^{-1}|_{\mathrm{Fix}^\perp(\mathrm{D}_{\psi_n})}\| = +\infty.$$

This means that there exists a sequence $(x_n)_1^\infty$ such that

$$x_n \in \mathrm{Fix}^\perp(D_{\psi_n}), \quad ||x_n|| = 1, \quad \text{and} \quad \lim_{n\to\infty} \|(D_{\psi_n}-\mathrm{id})^{-1}(\mathrm{x_n})\| = +\infty.$$

Put $y_n := (D_{\psi_n}-\mathrm{id})^{-1}(\mathrm{x_n}) \in \mathrm{Fix}^\perp(\mathrm{D}_{\psi_n})$. So $x_n = (D_{\psi_n}-\mathrm{id})\mathrm{y_n}$. Equivalently

$$\frac{x_n}{\|y_n\|} = (\mathrm{id}-\mathrm{D}_{\psi_n})\left(\frac{\mathrm{y_n}}{\|\mathrm{y_n}\|}\right). \tag{2.11}$$

Passing to a subsequence, we may assume without loss of generality that $\lim_{n\to\infty}(y_n/\|y_n\|) = y$ with some $y \in (\mathbb{R}^q)_1$. It then follows from Lemma 2.2 that $y \in \mathrm{Fix}_1^\perp(D_\psi)$. But on the other hand, follows from (2.11) that $0 = (D_\psi-\mathrm{id})(\mathrm{y})$, which means that $y \in \mathrm{Fix}(D_\psi)$. This contradiction finishes the proof. $\square$

As a direct consequence of this lemma and Lemma 2.2, we get the following.

**Lemma 2.4.** *For every $0 \le l \le q$ and every $\psi \in \mathrm{Par}_\infty(q,l)$ there exists a neighbourhood $V_\psi \subset V'_\psi$ of $\psi$ in $\mathrm{Par}_\infty(q,l)$ such that*

$$\kappa_\psi := 2\sup\{\|(D_A-\mathrm{id})^{-1}|_{\mathrm{Fix}^\perp(\mathrm{D_A})}\| \cdot \|\mathrm{a_A}\| : \mathrm{A} \in \mathrm{V}_\psi\} < +\infty.$$

As an immediate consequence of this lemma, (2.2), and (2.4), we get the following strengthening of Lemma 9.2.3 in [MU3].

**Lemma 2.5.** *For every $0 \le l \le q$ and every $\psi \in \mathrm{Par}_\infty(q,l)$ we have that*

$$\|A^n(z) - nb_A\| \le \|z\| + \kappa_\psi$$

*for all $A \in V_\psi$ ($V_\psi$ and $\kappa_\psi$ coming from Lemma 2.4), all $z \in \mathbb{R}^q$, and all $n \in \mathbb{Z}$.*

**Remark 2.6.** *Assume from now on that $l \ge 1$. In view of Lemma 2.2 we may and we do assume that*

$$0 < \beta_\psi^- := \inf\{\|b_A\| : A \in V_\psi\} \;\le\; \beta_\psi^+ := \sup\{\|b_A\| : A \in V_\psi\} < +\infty$$

*with $V_\psi$ being defined in Lemma 2.4 (and also appearing in Lemma 2.5).*

Given $\psi \in \mathrm{Par}_\infty(q,l)$, $R>0$, and $\gamma \in (0,\pi)$, let

$$S_\psi(R,\gamma) := \{z \in \mathbb{R}^q : \|z\| \ge R \text{ and } \angle(z,b_\psi) < \gamma\},$$
$$S_\psi^*(R,\gamma) := \{z \in \mathbb{R}^q : \|z\| \le R \text{ and } \angle(z,b_\psi) < \gamma\}.$$

and

$$S_\psi(r,R;\gamma) = S_\psi(r,\gamma) \cap S_\psi^*(R,\gamma).$$

**Lemma 2.7.** *With $1 \le l \le q$ and $\varphi \in \mathrm{Par}_\infty(q,l)$, for every $Q>0$, every $\beta \in (0,\pi)$ and every $\alpha \in [0,\beta)$ there exists $R>0$ such that*

$$\psi^n(S_\psi(R,\alpha)) \subset S_\psi(Q,\beta)$$

*for all $\psi \in V_\varphi$ and all $n \ge 0$.*

*Proof.* For every $\psi \in \mathrm{Par}_\infty(q,l)$ let $P_\psi : \mathbb{R}^q \to \mathbb{R}b_\psi$ be the orthogonal projection onto $\mathbb{R}b_\psi$ and let $P_\psi^\perp : \mathbb{R}^q \to (\mathbb{R}b_\psi)^\perp$ be the orthogonal projection onto $(\mathbb{R}b_\psi)^\perp$, the orthogonal complement of $\mathbb{R}b_\psi$. Formula (2.2) yields

$$P_\psi(\psi^n(z)) = nb_\psi + P_\psi(z) = nb_\psi + t_\psi(z)b_\psi,$$

and

$$P_\psi^\perp(\psi^n(z)) = \sum_{j=0}^n D_\psi^j a_\psi + D_\psi^n(P_\psi^\perp(z)),$$

where $t_\psi(z) \in \mathbb{R}$ is uniquely determined by the relation $P_\psi^\perp(z) = t_\psi(z)b_\psi$. Assuming from now on throughout the proof that $\psi \in V_\varphi$, we get from (2.4) and Lemma 2.4 that

$$\begin{aligned}\tan\angle(\psi^n(z),b_\psi) &= \frac{||P_\psi^\perp(\psi^n(z))||}{(t_\psi(z)+n)||b_\psi||} \le \frac{\kappa_\varphi + ||P_\psi^\perp(z)||}{t_\psi(z)||b_\psi||} \\ &= \tan\angle(z,b_\psi) + \frac{\kappa_\varphi}{t_\psi(z)||b_\psi||} \\ &\le \tan\alpha + \frac{\kappa_\varphi}{\beta_\varphi^- t_\psi(z)}.\end{aligned} \tag{2.12}$$

Assume first $\beta \in (0,\pi/2]$. Fix $\delta>0$ so small that if $\gamma \ge 0$ and $\tan\gamma < \tan\alpha + \delta$, then $\gamma < \beta$. Now, if $z \in S_\psi(R,\alpha)$ with some $R>0$, then $t_\psi(z)>0$ and

$$\begin{aligned}R^2 \le ||z||^2 &= t_\psi^2(z)||b_\psi||^2 + ||P_\psi^\perp(z)||^2 \\ &\le t_\psi^2(z)||b_\psi||^2 + \tan^2(\alpha)t_\psi^2(z)||b_\psi||^2 \\ &= (1+\tan^2\alpha)t_\psi^2(z)||b_\psi||^2 \\ &\le (1+\tan^2\alpha)\beta_\varphi^+||b_\psi||^2.\end{aligned}$$

So,

$$t_\psi(z) \ge (1+\tan^2\alpha)^{-\frac{1}{2}}(\beta_\varphi^+)^{-1}R.$$

Therefore, taking $R>0$ so large that $(1+\tan^2\alpha)^{-\frac{1}{2}}(\beta_\varphi^+)^{-1}R \ge Q$ and $\beta_\varphi^+(\beta_\varphi^-)^{-1}\kappa_\varphi R^{-1} < \delta$, we are done in this case. So, assume that $\beta \in (\pi/2,\pi)$. There then exists $\delta' > 1$ so close to 1 that if $\cot\gamma > \delta'\cot\alpha$, then $\gamma < \beta$. because of the previous case, taking $z \in S_\psi(R,\alpha)$ with some $R > 0$, we may assume without loss of generality that $\angle(z, b_\psi) > \pi/2$, i.e. $t_\psi(z) < 0$. Then

$$\begin{aligned}\cot\angle(\psi^n(z), b_\psi) &= \frac{(t_\psi(z)+n)||b_\psi||}{||P_\psi^\perp(\psi^n(z))||} \ge \frac{(t_\psi(z)+n)||b_\psi||}{||P_\psi^\perp(z)|| - \kappa_\varphi} \\ &= \frac{(t_\psi(z)+n)||b_\psi||}{||P_\psi^\perp(z)||\big(1-\kappa_\varphi/||P_\psi^\perp(z)||\big)} \\ &= \cot\angle(z,b_\psi)\big(1-\kappa_\varphi/||P_\psi^\perp(z)||\big)^{-1} \\ &> \cot\alpha\big(1-\kappa_\varphi/||P_\psi^\perp(z)||\big)^{-1}.\end{aligned}$$

But

$$R^2 \le ||z||^2 = t_\psi^2(z)||b_\psi||^2 + ||P_\psi^\perp(z)||^2 \le (1+\cot^2\alpha)||P_\psi^\perp(z)||^2.$$

So,

$$||P_\psi^\perp(z)|| \ge (1+\cot^2\alpha)^{-\frac{1}{2}}R^{-1}.$$

Therefore, taking $R > 0$ so large that $(1+\cot^2\alpha)^{-\frac{1}{2}}R > Q$ and $\big(1-\kappa_\varphi(1+\cot^2\alpha)R^{-1}\big)^{-1}\delta'$, we are done in this case too. The proof is complete. $\square$

A complementary statement to the above is the following.

**Lemma 2.8.** *With $1 \le l \le q$ and $\varphi \in \mathrm{Par}_\infty(q,l)$, for every $R > 0$ and every $\alpha \in [0,\pi)$, we have that*

$$\lim_{n\to\infty}\sup\{||\psi^n(z)|| : z \in S_\psi^*(R,\alpha),\ \psi \in V_\varphi\} = +\infty$$

*and*

$$\lim_{n\to\infty}\sup\{\angle(\psi^n(z), b_\psi) : z \in S_\psi^*(R,\alpha),\ \psi \in V_\varphi\} = 0$$

*Proof.* According to (2.12) and (2.2), for all $\psi \in V_\varphi$, all $z \in S_\psi^*(R,\alpha)$, and all $n > 2||z||/\beta_\varphi^-$, the following hold.

$$||\psi^n(z)|| \ge |t_\psi(z)+n|||b_\psi|| \ge \frac{1}{2}||b_\psi||n \ge \frac{1}{2}||\beta_\varphi^-||n$$

and

$$\tan\angle(\psi^n(z), b_\psi) = \frac{\Big\|\sum_{j=0}^n D_\psi^j a_\psi + D_\psi^n(P_\psi^\perp(z))\Big\|}{(t_\psi(z)+n)||b_\psi||} \le \frac{\kappa_\varphi + ||z||}{||b_\psi||}\cdot\frac{1}{n} \le \frac{\kappa_\varphi + R}{\beta_\varphi^-}\cdot\frac{1}{n}.$$

We are done. $\square$

## 3. Conformal Parabolic Maps in $\mathbb{R}^q$; the case $q \geq 3$

Let $U$ be an open subset of $\mathbb{R}^q$. We recall that a $C^1$ map $\varphi : U \to \mathbb{R}^q$ is called conformal if its derivative $\varphi'(x) : \mathbb{R}^q \to \mathbb{R}^q$ at every point $x \in U$ is a similarity map. We denote the corresponding similarity factor by $|\varphi'(x)|$. Of course, $|\varphi'(x)| = ||\varphi'(x)||$, the latter being the operator norm of $\varphi'(x)$. In this section we apply the results of the previous section to study parabolic conformal maps at any dimension $q \geq 3$. The parabolic fixed point need not be any longer infinity but one can concjugate such map by an inversion sending this fixed point to infinity. And at this moment the results of the previous section can be used. We begin our analysis with a fairly general distortion theorem that holds for all conformal maps in dimension $q \geq 3$. A proof the just announced distortion theorem can be extracted from the proof of Theorem 4.1.3 in [MU3]. However, because of its importance for the treatment of the parabolic case and its shorteness, we include it below.

**Theorem 3.1.** *Suppose $V$ is a non-empty open connected subset of $\mathbb{R}^q$, where $q \geq 3$ and $F \subset V$ is a bounded set such that $\overline{F} \subset V$. If $\varphi : V \to \mathbb{R}^q$ is a conformal map, then*

$$\frac{|\varphi'(x)|}{|\varphi'(y)|} \leq \left(1 + \frac{\operatorname{diam}(F)}{\operatorname{dist}(F, V^c)}\right)^2$$

*for all $x, y \in F$.*

*Proof.* In virtue of Liouville's Theorem there exist a real scalar $\lambda > 0$, a linear isometry $A : \mathbb{R}^q \to \mathbb{R}^q$ and points $a \in \mathbb{R}^q \cup \{\infty\}$, and $b \in \mathbb{R}^q$ such that

$$\varphi = \lambda A \circ i_a + b,$$

where $i_a : \overline{\mathbb{R}}^q \to \overline{\mathbb{R}}^q$ is the inversion with respect to the unite sphere centered at $a$ in the case when $a \in \mathbb{R}^q$ and $i_\infty = \mathrm{id}_{\mathbb{R}^q}$. In this latter case our theorem is trivially true since then the left hand side of its assertion is equal to 1. So, we may assume without loss of generality that $a \in \mathbb{R}^q$. By our hypothesis $a \notin V$. Hence, if $x, y \in F$, then

$$\frac{\|x-a\|}{\|y-a\|} \leq \frac{\|x-y\| + \|y-a\|}{\|y-a\|} = 1 + \frac{\|x-y\|}{\|y-a\|} \leq 1 + \frac{\operatorname{diam}(F)}{\operatorname{dist}(F, V^c)}.$$

Thus,

$$\frac{|\varphi'(y)|}{|\varphi'(x)|} = \frac{\lambda||y-a||^{-2}}{\lambda||x-a||^{-2}} = \frac{\|x-y\|^2}{\|y-a\|^2} \leq \left(1 + \frac{\operatorname{diam}(F)}{\operatorname{dist}(F, V^c)}\right)^2.$$

We are done. □

We now pass to actual analysis of parabolic conformal maps at dimension $q \geq 3$. Recall from Definition 9.2.1 in [MU3] that a conformal map $\psi : \overline{\mathbb{R}}^q \to \overline{\mathbb{R}}^q$, $q \geq 3$ is called parabolic if it has a fixed point $\omega_\psi \in \mathbb{R}^q$ and a point $\xi \in \mathbb{R}^q \setminus \{\omega\}$ such that $|\psi'(\omega)| = 1$ and $\lim_{n\to+\infty} \psi^n(\xi) = \omega_\psi$. Let

$$\tilde{\psi} = i_{\omega_\psi} \circ \psi \circ i_{\omega_\psi} = i_{\omega_\psi} \circ \psi \circ i_{\omega_\psi}^{-1}. \tag{3.1}$$

Then $\tilde{\psi}(\infty)=\infty$, and therefore, $\tilde{\psi}$ is conformal map parabolic at infinity, the class of maps studied in detail in the previous section. We put

$$D_\psi = D_{\tilde{\psi}}, \quad b_\psi = b_{\tilde{\psi}} \quad \text{and} \quad \kappa = \kappa_{\tilde{\psi}}.$$

It follows from (3.1) that $\tilde{\psi^{-1}} = \tilde{\psi}^{-1}$, and we conclude further that $\psi^{-1}$ is also a parabolic map, $D_{\psi^{-1}} = D_\psi^{-1}$ and $b_{\psi^{-1}} = -b_\psi$. Given $1 \le l \le q$. We denote by $\mathrm{Par}(q,l)$ the collection of all those parabolic self–maps $\psi$ of $\overline{\mathbb{R}}^q$ for which $\tilde{\psi} \in \mathrm{Par}_\infty(q,l)$. We shall prove the following.

**Proposition 3.2.** *Suppose $(\psi_n)_1^\infty$ is a sequence of parabolic self–maps of $\overline{\mathbb{R}}^q$ and $\psi : \overline{\mathbb{R}}^q \circlearrowleft$ is also a parabolic self–map. Then*

(1) *$\psi_n \to \psi$ with respect to the $\rho_*$ metric if and only if $\tilde{\psi}_n \to \tilde{\psi}$.*
(2) *If $\psi_n \to \psi$ (with respect to the $\rho_*$ metric), then $\psi_n \to \psi$ in $C^k$–norm, $k \ge 0$, on all compact subsets of $\mathbb{R}^q$.*
(3) *If $\psi_n \to \psi$, then $\omega_{\psi_n} \to \omega_\psi$.*

*Proof.* Item (3) is immediate. Indeed, if $\omega$ is any cluster point (in $\overline{\mathbb{R}}^q$) of the sequence $(\omega_{\psi_n})$, say $\omega = \lim_{j\to\infty} \omega_{\psi_{n_j}}$, then $\psi(\omega) = \lim_{j\to\infty} \psi_{n_j}(\omega_{\psi_{n_j}}) = \lim_{j\to\infty} \psi_{n_j}\omega_{\psi_{n_j}} = \omega$. Since $\psi$ has only one fixed point, namely $\omega_\psi$, we must have $\omega = \omega_\psi$ and item 3 is established. Since all the inversions $i_\omega \overline{\mathbb{R}}^q \circlearrowleft$, $\omega \in \mathbb{R}^q$, are uniformly bi–Lipschitz (with the same bi–Lipschitz constant) with respect to the spherical metric $\hat{\rho}_*$, we have $\psi_n \to \psi$ if and anly if $i_{\omega_\psi} \circ \psi_n \circ i_{\omega_\psi} \to \tilde{\psi}$. But because of (3) $i_{\omega_{\psi_n}} \to i_{\omega_\psi}$ with the metric $\rho_*$. Thus $i_{\omega_\psi} \circ \psi_n \circ i_{\omega_\psi} \to \tilde{\psi}$ iff $i_{\omega_{\psi_n}} \circ \psi_n \circ i_{\omega_{\psi_n}} \to \tilde{\psi}$. But this exactly means that $\tilde{\psi}_n \to \tilde{\psi}$. Item (1) is established. Dealing with (2) notice that if $\psi_n \to \psi$, then $D_{\psi_n} \to D_\psi$ by (1) and by Proposition 2.1. All higher order derivatives of $\tilde{\psi}_n$ and $\tilde{\psi}$ vanish. So $\tilde{\psi}_n \to \tilde{\psi}$ in $C^k$ for all $k \ge 0$. Since, by (3), $i_{\omega_{\psi_n}} \to i_{\omega_\psi}$ in $C^k$ on $\overline{\mathbb{R}}^q$ for all $k \ge 0$, and $\psi_n = i_{\omega_{\psi_n}} \circ \tilde{\psi}_n \circ i_{\omega_{\psi_n}}$ and $\psi = i_{\omega_\psi} \circ \tilde{\psi} \circ i_{\omega_\psi}$, we conclude that $\psi_n \to \psi$ in $C^k$, $k \ge 0$, on $\overline{\mathbb{R}}^q$. Part (2) follows. □

For all $\beta \in [0,\pi]$ and all $0 \le r < R \le +\infty$ put

$$S_\psi(r,R;\beta) = \{z \in \mathbb{R}^q : r \le ||z-\omega_\psi|| \le R \ \text{ and } \ |\measuredangle(z-\omega, b_\psi))| \le \beta\},$$
$$S_\psi(R,\beta) = S_{\omega_\psi}(0,R;\beta)$$

and

$$S_\psi^*(r,\beta) = S_{\omega_\psi}(r,+\infty;\beta).$$

Since $i_{\omega_\psi}(\mathbb{R}_+ b_\psi) = \mathbb{R}_+ b_\psi$ and $i_{\omega_\psi}(S_{\tilde{\psi}})(r,R;\beta) = S_\psi^*(R^{-1}, r^{-1};\beta)$, as an immediate consequence of Lemma 2.7 and Lemma 2.8, along with Proposition 3.2, we respectively get the following.

**Proposition 3.3.** *If $1 \le l \le q$ and $\varphi \in \mathrm{Par}(q,l)$, then there exists $W_\varphi$, a neighborhood of $\varphi \in \mathrm{Par}(q,l)$, such that for all $\varepsilon > 0$, all $\beta \in (0,\pi)$ and every $\alpha \in [0,\beta)$ there exist $\delta > 0$ such that if $\psi \in W_\varphi$ and $n \ge 0$, then*

$$\psi^n(S_\psi(\delta,\alpha)) \subset S_\psi(\varepsilon,\beta).$$

Also

**Proposition 3.4.** *If $1 \le l \le q$ and $\varphi \in \mathrm{Par}(q,l)$, then decreasing $W_\varphi$ from the previous proposition if necessary, the folloging hold. For every $R>0$ and every $\alpha \in (0,\pi)$,*

$$\lim_{n\to\infty} \sup\{||\psi^n(z)-\omega_\psi|| : z \in S_\psi(R,\alpha),\ \psi \in W_\varphi\} = 0$$

*and*

$$\lim_{n\to\infty} \sup\{\angle(\psi^n(z), b_\psi) : z \in S_\psi(R,\alpha),\ \psi \in W_\varphi\} = 0.$$

## 4. Conformal Parabolic Maps in $\mathbb{R}^2$

In this section we deal with conformal parabolic maps in $\mathbb{R}^2$. There is a bigger variety of them now than in the case of $q \ge 3$ since the Liouville's Theorem does not hold. However each conformal map in $\mathbb{C} = \mathbb{R}^2$ is either holomorphic or antiholomorphic and its second iterate is then holomorphic. This observation forms the starting point of our anlysis in this section.

Let $\omega \in \mathbb{R}^q$. A conformal map $\varphi : V \to \mathbb{C}$, where $V$ is an open neighbourhood of $\omega$, is called *parabolic* if $\varphi(\omega) = \omega$ and $(\varphi^2)'(\omega)$ is a root of unity. Replacing $\varphi$ by its sufficiently high even iterate $\varphi^k$, $k \ge 1$, we will have

$$\varphi : V \to \mathbb{C} \ \text{ holomorphic, } \ \varphi(\omega) = \omega \ \text{ and } \ \varphi'(\omega) = 1.$$

We call such conformal parabolic maps *simple*. From now on, throughout this section $\varphi : V \to \mathbb{C}$ is assumed to be a simple parabolic conformal map. Represent locally, around $\omega$, the holomorphic map $\varphi$, in the form of its Taylor series:

$$\varphi(z) = z + a_\varphi(z-\omega_\varphi)^{p+1} + a_{\varphi,p+2}(z-\omega_\varphi)^{p+2} + \dots \tag{4.1}$$

with some integer $p \ge 1$ and $a_\varphi \in \mathbb{C} \setminus \{0\}$. The set $\{z \in \mathbb{C} : a(z-\omega_\varphi)^p < 0\}$ is a union of $p$ rays $l_+(\omega,1), \dots, l_+(\omega_\varphi,p)$ emanating from $\omega_\varphi$ and forming consecutive angles equal to $2\pi/p$. These are referred to as attracting directions of the map $\varphi$ at $\omega_\varphi$. Likewise, the set $\{z \in \mathbb{C} : a(z-\omega_\varphi)^p > 0\}$ is a union of $p$ rays $l_-(\omega,1), \dots, l_-(\omega,p)$ emanating from $\omega_\varphi$ and forming consecutive angles equal to $\pi/p$. These are referred to as repelling directions of the map $\varphi$ at $\omega_\varphi$. For all $j = 1, \dots, p$ and all $\eta, \delta, \alpha > 0$ let

$$S^{j,\pm}_{\omega_\varphi}(\eta,\delta;\alpha) := \{z \in \mathbb{C} : \eta \le |z-\omega_\varphi| \le \delta \ \text{ and } \ \measuredangle(z-\omega_\varphi, l_\pm(\omega_\varphi,j)) \le \alpha\}$$

and

$$S^{j,\pm}_{\omega_\varphi}(\delta;\alpha) := S^{j,\pm}_{\omega_\varphi}(0,\delta;\alpha).$$

We also put

$$V = V_\varphi.$$

and

$$R_\varphi(z) = \sum_{n=2}^{\infty} a_{\varphi,p+n}(z-\omega_\varphi)^{p+n}.$$

So that

$$\varphi(z) = z + a_\varphi(z-\omega_\varphi)^{p_\varphi+1} + R_\varphi(z).$$

We further assume that $\omega_\varphi$ is the only fixed point of $\varphi$ in $V_\varphi$. Given $p \geq 1$ we denote by $\mathrm{Par}(2,p)$ the class of all simple parabolic bounded maps $\varphi$ for which $p_\varphi = p$. We introduce a topology, called parabolic, on $\mathrm{Par}(2,p)$ by saying that a sequence $(\varphi_n)_1^\infty$ converges to $\varphi \in \mathrm{Par}(2,p)$ if and only if there exists an open ball $V \subset \mathbb{C}$ such that

(a) $V_\varphi \cap \bigcap_{n=1}^\infty V_{\varphi_n} \supseteq V$
(b) $\omega_\varphi, \omega_{\varphi_n} \in \frac{1}{2}\overline{V}$ for all $n \geq 1$
(c) $\varphi_n \to \varphi$ uniformly on $V$.
(d) $\lim_{n\to\infty} a_{\varphi_n} = a_\varphi$.

**Remark 4.1.** *Note that, actually, item (d) follows from the previous ones.*

We then also say that the sequence $(\varphi_n)_1^\infty$ converges to $\varphi$ parabolically.

**Observation 1.** *If $(\varphi_n)_1^\infty$ convereges to $\varphi$ parabolically in $\mathrm{Par}(2,p)$, then $\lim_{n\to\infty} \omega_{\varphi_n} = \omega_\varphi$.*

*Proof.* Let $\omega$ be an arbitrary cluster point of $(\omega_{\varphi_n})_1^\infty$. Then $\omega \in \frac{1}{2}\overline{V} \subset V$ and, as $\varphi_n \to \varphi$ uniformly, $\varphi(\omega) = \lim_{n\to\infty} \varphi_n(\omega_{\varphi_n}) = \lim_{n\to\infty} \omega_n = \omega$. Thus $\omega = \omega_\varphi$, and the proof is complete. □

With this observation, we readily get the following.

**Observation 2.** *Suppose that $\varphi$ and $\varphi_n$, $n \geq 1$, are all in $\mathrm{Par}(2,p)$. If (a), (b) and (c) hold, then*

(1) $\lim_{n\to\infty} a_{\varphi_n} = a_\varphi$
(2) *the sequence of maps $(V \ni z \mapsto a_{\varphi_n}(z-\omega_{\varphi_n})^{p+1})_1^\infty$ converges uniformly to the map $V \ni z \mapsto a_\varphi(z-\omega_\varphi)^{p+1}$.*
(3) *the sequence of maps $(V \ni z \to R_{\varphi_n}(z))_1^\infty$ converges uniformly to the map $V \ni z \mapsto R_\varphi(z)$*

Given an open ball $V \subset \mathbb{C}$, we put

$$\mathrm{Par}_V(2,p) = \{\varphi \in \mathrm{Par}(2,p) : V_\varphi \supset V \,\mathrm{and}\, \omega_\varphi \in \frac{1}{2}\overline{V}\}.$$

We introduce the metric $\rho_V$ on $\mathrm{Par}_V(2,p)$ as

$$\rho_V(\varphi,\psi) = \|(\varphi-\psi)_{|V}\|_\infty.$$

Clearly $(\mathrm{Par}_V(2,p), \rho_V)$ is a complete metric space and we have the following.

**Observation 3.** *A sequence $(\varphi_n)_1^\infty$ convereges to $\varphi$ parabolically in* $\mathrm{Par}(2,p)$ *if and only if there exists an open ball $V \subseteq \mathbb{C}$ such that*

$$V \subseteq V_\varphi \cap \bigcap_{n=1}^{\infty} V_{\varphi_n} \quad \text{and} \quad \lim_{n\to\infty} \rho_V(\varphi_n, \varphi) = 0.$$

Now for every open ball $V \subseteq \mathbb{C}$ and every $\varphi \in \mathrm{Par}_V(2,p)$ let $\varphi_0 : V - \omega_\varphi \to \mathbb{C}$ be defined by the formula

$$\varphi_0 = T_{\omega_\varphi}^{-1} \circ \varphi \circ T_{\omega_\varphi},$$

where, we recall $T_{\omega_\varphi} : \mathbb{C} \to \mathbb{C}$ is the translation about the vector $\omega_\varphi$. Then $\varphi_0 \in \mathrm{Par}_{V-\omega_\varphi}(2,p)$ and $\omega_{\varphi_0} = 0$. Moreover,

$$\varphi_0(z) = z + a_\varphi z^{p+1} + R_\varphi^0(z),$$

where $R_\varphi^0(z) = R_\varphi(z+\omega_\varphi) = \sum_{n=2}^\infty a_{\varphi,p+n} z^{p+n}$. In particular

$$a_{\varphi_0} = a_\varphi, \quad a_{\varphi_0,p+n} = a_{\varphi,p+n} \ (n \geq 2) \quad \text{and} \quad R_\varphi(z) = R_\varphi^0(z - \omega_\varphi).$$

Changing the system of coordinates by a rotation about the origin, we may assume without loss of generality that $a_\varphi \in \mathbb{R}$ and $a_\varphi > 0$. Now, let $\sqrt[p]{z}$ be the holomorphic branch of the $p$-th radical defined on $\mathbb{C} \setminus (-\infty, 0]$ and sending 1 to 1. Define then the holomorphic map $H : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ by the formula

$$H(z) = \frac{1}{\sqrt[p]{z}},$$

and consider the conjugate map

$$\tilde{\varphi}_0 = H^{-1} \circ \varphi_0 \circ H : U := \big(\mathbb{C} \setminus (-\infty, 0] \cap H^{-1}(V - \omega_\varphi)\big) \to \mathbb{C},$$

where $H^{-1}(w) = \frac{1}{w^p}$. For all $z \in U$ we have

$$\begin{aligned}
\tilde{\varphi}_0(z) &= H^{-1}\big(\varphi_0(H(z))\big) = H^{-1}\Big(H(z) + a_{\varphi,n} H(z)^{p+1} + \sum_{n=2}^\infty a_{\varphi,n} H(z)^{n+p}\Big) \\
&= H^{-1}\left(\frac{1}{\sqrt[p]{z}} - a_\varphi z^{-\frac{p+1}{p}} + \sum_{n=2}^\infty a_{\varphi,n} z^{-\frac{p+n}{p}}\right) \\
&= H^{-1}\left(\frac{1}{\sqrt[p]{z}}\left(1 + a_\varphi z^{-1} + \sum_{n=2}^\infty a_{\varphi,n} z^{-\frac{p+n-1}{p}}\right)\right) \\
&= \frac{z}{\big(1 + a_\varphi z^{-1} + \sum_{n=2}^\infty a_{\varphi,n} z^{-\frac{p+n-1}{p}}\big)^p}
\end{aligned} \tag{4.2}$$

Set $w = H(z) = z^{-\frac{1}{p}}$ and put

$$G_\varphi(w) = 1 + a_\varphi w^p + \sum_{n=2}^\infty a_{\varphi,n}(0) w^{p+n-1}, \quad w \in V - \omega_\varphi. \tag{4.3}$$

Keep $\varphi \in \mathrm{Par}_V(2,p)$. Take $W_\varphi$, the largest open ball centered at 0 and contained in $V_{\omega_\varphi}$ but so small that $\|(G_\varphi - 1)_{|W_\varphi}\|_\infty < 1/8$. Now notice that the function $B_V(\varphi, \eta_\varphi) \ni \psi \mapsto$

$(G_\psi - 1)|_{1/2W_\varphi}$ is continuous if $(G_\psi - 1)|_{1/2W_\varphi}$ is considered as an element of the Banach space of bounded holomorphic functions defined on an $1/2W_\varphi$ endowed with the supremum norm. Fix $\eta_\varphi > 0$ so small that if $\psi \in B_V(\varphi, \eta_\varphi)$, then

$$W_\psi \supseteq \frac{5}{8}W_\varphi, \quad \frac{1}{2}a_\varphi \le |a_\psi| \le 2a_\varphi \quad \text{and} \quad \frac{1}{2}a_\varphi \le \mathrm{Re}(a_\psi) \le 2a_\varphi. \tag{4.4}$$

Therefore there exists $\delta_\varphi \in (0, \eta_\varphi)$ so small that

$$\|(G_\psi - 1)|_{\frac{1}{2}W_\varphi}\|_\infty < \frac{1}{4} \tag{4.5}$$

for all $\psi \in B_V(\varphi, \delta_\varphi)$. Write

$$\hat{G}_\psi = G_\psi^{-p}|_{\frac{1}{2}W_\varphi}. \tag{4.6}$$

It then follows from (4.5) that $\hat{G}_\psi : \frac{1}{2}W_\varphi \to \mathbb{C}$ is a holomorphic function and

$$\|\hat{G}_\psi\|_\infty \le \left(\frac{4}{3}\right)^p. \tag{4.7}$$

From (4.3) we get that

$$\hat{G}_\psi(0) = 1, \quad \frac{\partial^k \hat{G}_\psi(w)}{\partial w^k}\Big|_0 = 0 \;\text{ for all } k = 1, 2, ..., p-1, \text{ and } \frac{\partial^p \hat{G}_\psi(w)}{\partial w^p}\Big|_0 = p!\, a_\psi.$$

$$\hat{G}_\psi(w) = 1 + pa_\psi w^p + \sum_{n=1}^\infty b_n(\psi) w^{p+n} \tag{4.8}$$

with some appropriate coefficients $b_n(\psi)$, $n \ge 1$. Write

$$B_\psi(w) = \sum_{n=1}^\infty b_n(\psi) w^n, \quad w \in \frac{1}{2}W_\varphi.$$

It follows from (4.8) that $B_\psi$ is a holomorphic function. Invoking (4.7) and (4.4), we get that

$$\begin{aligned} |B_\psi(w)| &\le \left|\frac{\hat{G}_\psi(w) - 1 - pa_\psi w^p}{w^p}\right| \le \frac{|\hat{G}_\psi(w)| + 1 + p|a_\psi||w|^p}{|w|^p} \\ &\le \frac{1 + (4/3)^p + p(R_\varphi^{(1)})^p|a_\psi|}{(R_\varphi^{(1)})^p} \\ &\le M_\varphi := (R_\varphi^{(1)})^{-p}(1 + (4/3)^p + 2p|a_\varphi|(R_\varphi^{(1)})^p) \end{aligned} \tag{4.9}$$

for all $w \in \partial B(0, R_\varphi^{(1)})$, where $R_\varphi^{(1)}$ is the radius of $\frac{1}{4}W_\varphi$. It then follows from the Maximum Modulus Theorem that

$$|B_\psi(w)| \le M_\varphi$$

for all $w \in \overline{B}(0, R_\varphi)$ and all $\psi \in B_V(\varphi, \delta_\varphi)$. Hence, it follows from Cauchy's Formula that

$$|B'_\psi(w)| \le M_\varphi(R_\varphi^{(1)})^{-2}. \tag{4.10}$$

For all $\psi \in B_V(\varphi,\delta_\varphi)$ and all $w \in \overline{B}(0,R_\varphi^{(1)})$. Hence (remember that $B_\psi(0)=0$) there exists $0<R_\varphi \le R_\varphi^{(1)}$ so small that

$$|B_\psi(w)| < \frac{1}{4}pa_\varphi \tag{4.11}$$

for all $\psi \in B_V(\varphi,\delta_\varphi)$ and all $w \in B(0,R_\varphi)$. Going back to the variable $z=w^{-p}$ we get from (4.2), (4.3), (4.6), and (4.8) that,

$$\begin{aligned}\tilde\psi_0(z) &= z\hat G_\psi(H(z)) = z(1+pa_\psi H(z)^p + \sum_{n=1}^\infty b_n(z)H(z)^{p+n})\\ &= z(1+pa_\psi\frac{1}{z}+\frac{1}{z}\sum_{n=1}^\infty b_n(\psi)H(z)^n)\\ &= z+pa_\psi + B_\psi(H(z)),\end{aligned} \tag{4.12}$$

for all $\psi \in B_V(\varphi,\delta_\varphi)$ and all $z \in B^c(0,R_\varphi^{-p})$. So, using also (4.11) we get that,

$$\mathrm{Re}(\tilde\psi_0(z)) - (z+pa_\psi)) = \mathrm{Re}(B_\psi(H(z))) > -\frac{1}{4}pa_\varphi.$$

Thus

$$\begin{aligned}\mathrm{Re}(\tilde\psi_0(z)) &> \mathrm{Re}(z)+\mathrm{Re}(pa_\psi) - \frac{1}{4}pa_\varphi > \mathrm{Re}(z)+\frac{1}{2}pa_\varphi - \frac{1}{4}pa_\varphi\\ &= \mathrm{Re}(z)+\frac{1}{4}pa_\varphi\end{aligned} \tag{4.13}$$

for all $\psi \in B_V(\varphi,\delta_\varphi)$ and all $z \in B^c(0,R_\varphi^{-p})$. Therefore, using (4.11) again,

$$|\tilde\psi_0(z)| \le |z|+|pa_\psi|+|B_\psi(H(z))| \le |z|+2pa_\varphi+\frac{1}{4}paa_\varphi \le |z|+3pa_\varphi.$$

By an obvious induction we get from this and (4.13) that

$$\mathrm{Re}(z)+\frac{1}{4}pa_\varphi n \le \mathrm{Re}(\tilde\psi_0^n(z)) \le |\tilde\psi_0^n(z)| \le |z|+3pa_\varphi n \tag{4.14}$$

for all $\psi \in B(\varphi,\delta_\varphi)$ and all $z\in\mathbb{C}$ with $\mathrm{Re}(z) > R_\varphi^{-p}$.

Now given $\alpha \in (0,\pi)$ and $t>0$ let

$$\Delta(\alpha,t) = \{z\in\mathbb{C} : \angle([z,t],[t,+\infty)) \le \alpha\}.$$

We shall prove the following.

**Lemma 4.2.** *If $\varphi \in \mathrm{Par}_V(2,p)$ and $\alpha \in (0,\pi)$, then there exists $t_\alpha > R_\varphi^{-p}$ such that for all $\psi \in B_V(\varphi,\delta_\varphi)$, we have that*

$$\tilde\psi_0(\Delta(\alpha,t_\alpha)) \subset \Delta(\alpha,t_\alpha).$$

*Proof.* Assume first that $0<\alpha<\pi/2$. In view of (4.10), there exists $0<R_\alpha \le R_\varphi$ so small that

$$|B_\psi(w)| < \frac{1}{2}pa_\varphi \tan\alpha \tag{4.15}$$

for all $\psi \in B_V(\varphi, \delta_\varphi)$ and all $w \in B(0, R_\alpha)$. Take $t_\alpha$ to be an arbitrary real number larger than $R_\alpha^{-p}$. Fix $z \in \Delta(\alpha, t_\alpha)$. This means that

$$|\mathrm{Im}(z)| \le \tan\alpha(\mathrm{Re}(z) - t_\alpha).$$

It then follows from (4.12) and (4.13) that,

$$\begin{aligned}\frac{\mathrm{Re}(\tilde\psi_0(z)) - t_\alpha}{|\mathrm{Im}(\tilde\psi_0(z))|} &\le \frac{|\mathrm{Im}(z) + B_\psi(H(z))|}{\mathrm{Re}(z) + \frac{1}{2}pa_\varphi - t_\alpha} \\ &\le \frac{|\mathrm{Im}(z)| + |B_\psi(H(z))|}{\cot\alpha|\mathrm{Im}(z)| + \frac{1}{2}pa_\varphi} \\ &\le \tan\alpha \frac{|\mathrm{Im}(z)| + \frac{1}{2}pa_\varphi \tan\alpha}{|\mathrm{Im}(z)| + \frac{1}{2}pa_\varphi \tan\alpha} \\ &= \tan\alpha.\end{aligned}$$

So, we are done in this case and we may assume that $\alpha > \pi/2$. First take $\kappa_\alpha > 0$ so small that

$$(1 + |\cot\alpha|)\kappa_\alpha < \frac{1}{4}pa_\varphi. \tag{4.16}$$

Then, in view of (4.10), there exists $0 < R_\alpha \le R_\varphi$ so small that

$$|B_\psi(w)| < \kappa_\alpha \tag{4.17}$$

for all $\psi \in B_V(\varphi, \delta_\varphi)$ and all $w \in B(0, R_\alpha)$. Now, take $t_\alpha > 0$ so large that

$$B(0, R_\alpha^{-p}) \subset \mathbb{C} \setminus \Delta(\alpha, t_\alpha). \tag{4.18}$$

This equivalently means that $t_\alpha > R_\alpha^{-p}$; the latter number is larger than $R_\varphi^{-p}$. It follows from (4.13) that if $z \in \Delta(\alpha, t_\alpha)$ and $\mathrm{Re}(z) > t_\alpha$, then $\tilde\psi_0(z) \in \Delta(\alpha, t_\alpha)$. So, suppose that $z \in \Delta(\alpha, t_\alpha)$ and $\mathrm{Re}(z) \le t_\alpha$. Seeking contradiction, assume that $\tilde\psi_0(z) \notin \Delta(\alpha, t_\alpha)$ and $|\mathrm{Im}(z)| \le \kappa_\alpha$. Then, $\kappa_\alpha \ge |\mathrm{Im}(z)| \ge |\tan\alpha|(t_\alpha - \mathrm{Re}(z))$. Equivalently, $\mathrm{Re}(z) - t_\alpha \ge -|\cot\alpha|\kappa_\alpha$. So, as $z \in \Delta(\alpha, t_\alpha)$, by (4.13), (4.18), (4.17), and (4.16), we get that,

$$\mathrm{Re}(\tilde\psi_0(z)) - t_\alpha > \mathrm{Re}(z) - t_\alpha + \frac{1}{4}pa_\varphi - \kappa_\alpha \ge \frac{1}{4}pa_\varphi - (1 + |\cot\alpha|)\kappa_\alpha > 0.$$

So, $\tilde\psi_0(z) \in \Delta(\alpha, t_\alpha)$. This contradiction shows that $|\mathrm{Im}(z)| > \kappa_\alpha$. Also, applying (4.12) and using (4.17), we get that

$$|\mathrm{Im}(\tilde\psi_0(z))| = |\mathrm{Im}(z) + B_\psi(H(z))| \ge |\mathrm{Im}(z)| - |B_\psi(H(z))| \ge |\mathrm{Im}(z)| - \kappa_\alpha.$$

Then, by the same token as above, we get

$$\begin{aligned}
\frac{t_\alpha - \mathrm{Re}(\tilde{\psi}_0(z))}{|\mathrm{Im}(\tilde{\psi}_0(z))|} &\le \frac{t_\alpha - \mathrm{Re}(\tilde{\psi}_0(z))}{|\mathrm{Im}(z)| - \kappa_\alpha} \\
&\le \frac{(t_\alpha - \mathrm{Re}(z)) - pa_\psi + \kappa_\alpha}{|\mathrm{Im}(z)| - \kappa_\alpha} \le \frac{|\cot\alpha||\mathrm{Im}(z)| - pa_\psi + \kappa_\alpha}{|\mathrm{Im}(z)| - \kappa_\alpha} \\
&= \frac{|\cot\alpha|(|\mathrm{Im}(z)| - \kappa_\alpha) - (pa_\psi - |\cot\alpha|\kappa_\alpha) + \kappa_\alpha}{|\mathrm{Im}(z)| - \kappa_\alpha} \\
&= |\cot\alpha| - \frac{pa_\psi - (1 - |\cot\alpha|)\kappa_\alpha}{|\mathrm{Im}(z)| - \kappa_\alpha} \\
&\le |\cot\alpha| - \frac{\frac{1}{2}pa_\varphi - (1 - |\cot\alpha|)\kappa_\alpha}{|\mathrm{Im}(z)| - \kappa_\alpha} \\
&< |\cot\alpha|.
\end{aligned}$$

This shows that $\tilde{\psi}_0(z) \in \Delta(\alpha, t_\alpha)$, and the proof is complete. $\square$

Now we shall prove the following.

**Lemma 4.3.** *If $\varphi \in \mathrm{Par}_V(2,p)$, then for every $\alpha \in (0,\pi)$ and every $R > 0$, we have*

$$\lim_{n\to\infty} \sup\{\angle(\tilde{\psi}_0^n(z), [0,+\infty)) : \psi \in B_V(\varphi, \delta_\varphi) \text{ and } z \in B(0,R) \cap \Delta(\alpha, t_\alpha)\} = 0,$$

*where $t_\alpha > R_\varphi^{-p}$ is the number produced in Lemma 4.2.*

*Proof.* Since $t_\alpha > R_\varphi^{-p}$, looking at Lemma 4.2 and applying sufficiently many times (4.13), we conclude that there exists an integer $n_1 \ge 1$ such that $\mathrm{Re}(\tilde{\psi}_0^{n_1}(z)) > R_\varphi^{-p}$ for all $\psi \in B_V(\varphi, \delta_\varphi)$ and all $z \in B(0,R) \cap \Delta(\alpha, t_\alpha)$. It therefore follows from the left–hand side of (4.14) that with some integer $n_2 \ge n_1$, we have for all $\psi \in B_V(\varphi, \delta_\varphi)$, all $z \in B(0,R) \cap \Delta(\alpha, t_\alpha)$, and all $n \ge n_2$ that

$$|\tilde{\psi}_0^n(z)| \ge \mathrm{Re}(\tilde{\psi}_0^n(z)) > \frac{1}{8} pa_\varphi n.$$

It then follows from (4.12) that for all $n \ge n_2$, we have

$$\begin{aligned}
|\mathrm{Im}(\tilde{\psi}_0^{n+1}(z))| &= |\mathrm{Im}(\tilde{\psi}_0^n(z)) + B_\psi(H(\tilde{\psi}_0^n(z)))| \\
&\le |\mathrm{Im}(\tilde{\psi}_0^n(z))| + |B_\psi(H(\tilde{\psi}_0^n(z)))| \\
&\le |\mathrm{Im}(\tilde{\psi}_0^n(z))| + M_\varphi (R_\varphi^{(1)})^{-2} |H(\tilde{\psi}_0^n(z)| \\
&\le |\mathrm{Im}(\tilde{\psi}_0^n(z))| + M_\varphi (R_\varphi^{(1)})^{-2} |\tilde{\psi}_0^n(z)|^{-\frac{1}{p}} \\
&\le |\mathrm{Im}(\tilde{\psi}_0^n(z))| + 8^p M_\varphi (R_\varphi^{(1)})^{-2} (pa_\varphi)^{-\frac{1}{p}} n^{-\frac{1}{p}}.
\end{aligned}$$

Therefore, using the right–hand side of (4.14), for every $n \ge n_2$ we get that

$$|\mathrm{Im}(\tilde{\psi}_0^n(z))| \le |\mathrm{Im}(\tilde{\psi}_0^{n_2}(z))| + C_\varphi^{(1)} \sum_{k=n_1}^{n-1} k^{-\frac{1}{p}} \le C_\varphi^{(2)} + C_\varphi^{(3)} n^{1-\frac{1}{p}},$$

with some positive constants $C_\varphi^{(1)}$, $C_\varphi^{(2)}$ and $C_\varphi^{(3)}$. Combining this with the left hand side of (4.14), our lemma follows. $\square$

Passing from $\tilde{\psi}_0$ back to $\psi$, as an immediate consequence of the last two lemmas and formulas (4.13) and (4.14), we get the following.

**Proposition 4.4.** *If $\varphi \in \mathrm{Par}_V(z,p)$, then there exists $\delta_\varphi > 0$ such that for all $\varepsilon > 0$ and all $\alpha \in (0, \pi/p)$ there exists $\delta > 0$ such that*

(a) *For all $\psi \in B_V(\varphi, \delta_\varphi)$, all the iterates $\psi^n$, $n \geq 0$, are well defined and injective on the set $S^{j,+}_{\omega_\psi}(\delta;\alpha)$ for all $j = 1, \dots, p$ and*
$$\psi^n(S^{j,+}_{\omega_\psi}(\delta;\alpha)) \subseteq S^{j,+}_{\omega_\psi}(\varepsilon;\alpha)$$
*for all $n \geq 0$.*

(b) *For every $\eta \in (0, \delta)$,*
$$\lim_{n\to\infty} \sup\{|\psi^n(z) - \omega_\psi| : \psi \in B_V(\varphi, \delta_\varphi),\ z \in S^{j,+}_{\omega_\psi}(\eta, \delta, \alpha)\} = 0.$$

(c) *For every $0 < \eta < \delta$ there exists a constant $\hat{K}_\eta \geq 1$ such that*
$$\frac{|(\psi^n)'(\zeta)|}{|(\psi^n)'(\xi)|} \leq \hat{K}_\eta$$
*for all $\psi \in B_V(\varphi, \delta_\varphi)$, all $j = 1, \dots, p$, all $\xi, \zeta \in S^{j,+}_{\omega_\psi}(\eta, \delta, \alpha)$, and all $n \geq 0$.*

(d) *For all $\psi \in B_V(\varphi, \delta_\varphi)$ and for every $0 < \eta < \delta$,*
$$\lim_{n\to+\infty} \sup\{\angle([\psi^n(z), \omega_\psi], l_+(\omega_\varphi, j)) : \psi \in B_V(\varphi, \delta_\varphi),\ z \in S^{j,+}_{\omega_\psi}(\eta, \delta, \alpha)\} = 0$$

## 5. Parabolic Maps in $\mathbb{R}$

Finally, let us briefly consider the case when $q = 1$. Let $V \subseteq \mathbb{R}$ be a bounded open interval. We fix a point $\omega \in V$. We call a $C^{1+\varepsilon}$ map $\varphi : V \to \mathbb{R}$ *parabolic* if (with $\omega_\varphi = \omega$)
$$\varphi(\omega_\varphi) = \omega_\varphi, \quad \varphi'(\omega_\varphi) \in \{-1, 1\}$$
and
$$\varphi(x) = \omega_\varphi + \varphi'(\omega_\varphi)(x - \omega_\varphi) + a_\varphi|x - \omega_\varphi|^{p+1} + o(|x - \omega_\varphi|^{p+1}) \tag{5.1}$$
for all $x \in V$ close enough to $\omega_\varphi$, where $a_\varphi \neq 0$ and $p > 0$ are arbitrary real numbers. We set $V = V_\varphi$. Write uniquely $V_\varphi = V_\varphi^+ \cup V_\varphi^-$, where $V_\varphi^+$ is a subinterval of $V_\varphi$ having $\omega_\varphi$ as its left-hand endpoint and $V_\varphi^-$ is a subinterval of $V_\varphi$ having $\omega_\varphi$ as its right-hand endpoint. We put $\mathrm{sgn}\,(V_\varphi^+) = 1$ and $\mathrm{sgn}\,(V_\varphi^-) = -1$. We define $V_\varphi^a$ to be $V_\varphi^+$ if $\mathrm{sgn}\,(a_\varphi)\varphi'(\omega_\varphi)\mathrm{sgn}\,(V_\varphi^+) < 0$ and to be $V_\varphi^-$ if $\mathrm{sgn}\,(a_\varphi)\varphi'(\omega_\varphi)\mathrm{sgn}\,(V_\varphi^)) < 0$. A comprehensive treatment of the local behavior of one-dimensional parabolic maps can be found in [U2]. Given a real number $p > 0$, we denote the collection of all corresponding 1-dimensional parabolic maps by $\mathrm{Par}(1,p)$. As

in the case of $q = 2$, we introduce a topology, called parabolic, on $\mathrm{Par}(1, p)$ by saying that a sequence $(\varphi_n)_1^\infty$ converges to $\varphi \in \mathrm{Par}(1, p)$ if and only if there exists an open interval $V \subset \mathbb{R}$ such that

(a) $V_\varphi \cap \bigcap_{n=1}^\infty V_{\varphi_n} \supseteq V$
(b) $\omega_\varphi, \omega_{\varphi_n} \in \frac{1}{2}\overline{V}$ for all $n \geq 1$
(c) $\varphi_n \to \varphi$ uniformly on $V$.
(d) $\lim_{n\to\infty} a_{\varphi_n} = a_\varphi$.

We then also say that the sequence $(\varphi_n)_1^\infty$ converges to $\varphi$ parabolically.
As in the case $q = 2$, given an open interval $V \subset \mathbb{R}$, we put

$$\mathrm{Par}_V(1, p) = \{\varphi \in \mathrm{Par}(1, p) : V_\varphi \supset V \,\mathrm{and}\, \omega_\varphi \in \frac{1}{2}\overline{V}\}.$$

We introduce the metric $\rho_V$ on $\mathrm{Par}_V(1, p)$ as

$$\rho_V(\varphi, \psi) = \|(\varphi - \psi)_{|V}\|_\infty + |a_\varphi - a_\Psi|.$$

Clearly $(\mathrm{Par}_V(1, p), \rho_V)$ is a complete metric space and we have the following.

**Observation 4.** *A sequence $(\varphi_n)_1^\infty$ converges to $\varphi$ parabolically in $\mathrm{Par}(1, p)$ if and only if there exists an open interval $V \subseteq \mathbb{R}$ such that*

$$V \subseteq V_\varphi \cap \bigcap_{n=1}^\infty V_{\varphi_n} \quad \textit{and} \quad \lim_{n\to\infty} \rho_V(\varphi_n, \varphi) = 0.$$

With the techniques and tools developed in [U2], we can prove in the case $q = 1$ the following result forming a 1-dimensional counterpart of Proposition 4.4.

**Proposition 5.1.** *If $\varphi \in \mathrm{Par}_V(z, p)$, then there exists $\delta_\varphi > 0$ such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that*

(a) *For all $\psi \in B_V(\varphi, \delta_\varphi)$, all the iterates $\psi^n$, $n \geq 0$, are well defined on the set $V_\psi^a \cap (\omega_\psi - \delta, \omega_\psi + \delta)$ and*

$$\psi^n\big(V_\psi^a \cap (\omega_\psi - \delta, \omega_\psi + \delta)\big) \subseteq B(\omega_\psi, \varepsilon)$$

*for all $n \geq 0$.*

(b)

$$\lim_{n\to\infty} \sup\{|\psi^n(z) - \omega_\psi| : \psi \in B_V(\varphi, \delta_\varphi), z \in V_\psi^a \cap (\omega_\psi - \delta, \omega_\psi + \delta)\} = 0.$$

(c) *For every $0 < \eta < \delta$ there exists a constant $\hat{K}_\eta \geq 1$ such that*

$$\frac{|(\psi^n)'(\zeta)|}{|(\psi^n)'(\xi)|} \leq \hat{K}_\eta$$

*for all $\psi \in B_V(\varphi, \delta_\varphi)$, all $\zeta, \xi \in V_\psi^a \cap \{t \in \mathbb{R} : \eta \leq |t - \omega_\psi| < \delta\}$, and all $n \geq 0$.*

(d) *Furthermore, there exists a monotone increasing function* $K : [0,1) \to [1,+\infty)$ *such that* $\lim_{t\to 0} K(t) = K(0) = 1$ *and*
$$\frac{|(\psi^n)'(y)|}{|(\psi^n)'(x)|} \le K(t)$$
*for all* $\psi \in B_V(\varphi,\delta_\varphi)$*, all* $z \in (\omega_\psi - \delta, \omega_\psi + \delta) \setminus \{\omega_\psi\}$*, and all* $x, y \in (z - t|z - \omega_\psi, z + t|z - \omega_\psi)$*, and all* $n \ge 0$*.*

## 6. Parabolic Maps; Local Behavior III.

As a fairly straightforward consequences of Section 3, Section 4, and Section 5 one can prove (see Section 9.2 and 9.3 of [MU3] for the cases $q \ge 3$ and $q = 2$ respectively, and [U2] for the case $q = 1$) the following.

**Proposition 6.1.** *Suppose that* $\varphi$ *is a simple parabolic conformal map acting in an arbitrary phase space* $\mathbb{R}^q$*,* $q \ge 1$*. Let* $p = 1$ *if* $q \ge 3$*, let* $p \ge 1$ *be the integer coming from* (4.1) *if* $q = 2$ *and let* $p > 0$ *be the real number coming from* (4.1) *if* $q = 1$*. Let* $V$ *be an open ball contained in* $V_\varphi$ *and contaning* $\omega_\varphi$ *if* $q = 1, 2$*. Let* $W_\varphi$ *come from Proposition* 3.3 *if* $q \ge 3$ *and let* $\delta_\varphi$ *come form Proposition* 4.4 *and Proposition* 5.1 *if* $q = 2$ *and* $q = 1$ *respectively. Then for every* $\varepsilon > 0$ *and every* $\alpha \in (0, \pi/p)$ *there exists* $\delta > 0$ *such that for every* $\eta \in (0,\delta)$ *there are constants* $A_\varphi > 1$ *and an integer* $s_\varphi \ge 1$ *such that for all* $\psi \in W_\varphi$ *if* $q \ge 3$ *or* $\psi \in B_V(\varphi,\delta_\varphi)$ *if* $q = 1, 2$*, (call jointly such a neighborhood of* $\varphi$ *by* $Z(\varphi)$*) the following hold. For all* $j = 1, \dots, p$*, all* $z \in S^{j,+}_{\omega_\psi}(\eta,\delta;\alpha)$ *(with appriopriately adjusted meaning if* $q = 1$ *or* $q \ge 3$*) and all* $n \ge 1$ *we have,*

(a)
$$A_\varphi^{-1} n^{-1/p} \le \|\psi^n(z) - \omega_\psi\| \le A_\varphi n^{-1/p}$$
*and*

(b)
$$A_\varphi^{-1} n^{-(p+1)/p} \le |(\psi^n(z))'| \le A_\varphi n^{-(p+1)/p}$$

(d)
$$\|\psi^n(z) - \psi^k(z)\| \le A_\varphi (k^{-1/p} - n^{-1/p})$$
*if* $1 \le k < n$ *and*

(e)
$$\|\psi^n(z) - \psi^k(z)\| \ge A_\varphi^{-1} (k^{-1/p} - n^{-1/p})$$
*if, in addition,* $n - k \ge s_\varphi$*.*

Let
$$A(w; r, R) := \{z \in \mathbb{C} : r < |z - w| < R\}$$
be the annulus centered at a point $w \in \mathbb{C}$ and with the inner radius $r$ and outer radius $R$. As an immediate consequence of this proposition we get the following.

**Proposition 6.2.** *With the setting of Proposition* 6.1 *we have*

(a)

$$\psi^n(S_\omega^{j,+}(\eta,\delta;\alpha)) \subseteq A(\omega_\psi; A_\varphi^{-1}n^{-1/p}, A_\varphi n^{-1/p})$$

*and*

(b)

$$A_\varphi^{-1}n^{-(p+1)/p} \le \operatorname{diam}(\psi^n(S_{\omega_\psi}^{j,+}(\eta,\delta;\alpha))) \le A_\varphi n^{-(p+1)/p}$$

*perhaps with a larger constant $A_\varphi$ in (b) than its counterpart in item (b) of Proposition* 6.1.

## 7. Walters Maps

Let $X_0$ be an open and dense subset of a compact metric space $X$ endowed with a metric $\rho$. Let $\Omega$ be a finite subset of $X_0$. We call a continuous map $T : X_0 \to X$ *Walters* if the following conditions are satisfied:

(1) $\Omega$ is a union of periodic orbits of $T$.

(2) The set $T^{-1}(x)$ is countable for each $x \in X$ and there exists $\xi = \xi_T > 0$ such that $T^{-1}(B(x,2\xi))$ can be written uniquely as a disjoint union of open sets $\{B_y(x) : y \in T^{-1}(x)\}$ such that $y \in B_y(x)$ and $T : B_y(x) \to B(x,2\xi)$ is a homeomorphism from $B_y(x)$ onto $B(x,2\xi)$. The corresponding inverse map from $B(x,2\xi)$ to $B_y(x)$, $y \in T^{-1}(x)$, will be denoted by $T_y^{-1}$.

(3) If $x \in X_\infty(T) := \bigcap_{n=0}^\infty T^{-n}(X)$ and $\rho(T^n(x),\Omega) \le 2\xi$ for all $n \ge 0$, then $x \in \Omega$.

(4) For every $\theta > 0$ there exists $\delta = \delta_\theta \in (0,\xi)$ such that for every $x \in X \setminus B(\Omega,\theta)$ and every $n \ge 0$, the set $T^{-n}(B(x,2\delta))$ can be written uniquely as a disjoint union of open sets $\{B_y(n,x) : y \in T^{-n}(x)\}$ such that $y \in B_y(n,x)$ and the map $T^n : B_y(n,x) \to B(x,2\delta)$ is a homeomorphism from $B_y(n,x)$ onto $B(x,2\delta)$. The corresponding inverse map from $B(x,2\xi)$ to $B_y(n,x)$ will be denoted by $T_y^{-n}$.

(5) For every $\theta > 0$ there exists $n_\theta \ge 1$ such that if $x \in X$ and

$$B(\Omega,\theta) \cap \{T^k(x) : 0 \le k \le n_\theta\} = \emptyset,$$

then

$$\rho(T^{n_\theta}(w), T^{n_\theta}(z)) \ge 4\rho(w,z)$$

for all $w,z \in T_x^{-n_\theta}(B(T^{n_\theta},2\delta_\theta))$.

(6) $\forall\, \varepsilon > 0\ \exists\, s > 0\ \forall\, x \in X\ \ T^{-s}(x)\ $ is $\varepsilon$–dense in $X$.

(7)

$$T_y^{-1}(B(\omega,2\xi)) \cap B(\Omega,\xi) = \emptyset$$

for all $\omega \in \Omega$ and all $y \in T^{-1}(\omega) \setminus \{\omega\}$.

(8) For every $\theta > 0$ there exists an integer $k_\theta \ge 1$ such that if $x \in X \setminus \Omega$, $T^j(x) \in X \cap B(\Omega,\theta)$ for all $j = 0,1,...,k_\theta - 1$, and $T^{k_\theta}(x) \notin B(\Omega,\theta)$, then the map $T^{k_\theta} : B_x(k_\theta, T^{k_\theta}(x)) \to B(T^{k_\theta}(x),2\delta)$ is Lipschitz continuous with some Lipschitz constant $\ge 2$.

A Walters map is referred to as *finitely Walters map* if $X_0 = X$. Note that then each inverse image $T^{-1}(x)$, $x \in X$, is finite, in fact $\sup\{\#T^{-1}(x) : x \in X\} < +\infty$.

A Walters map is called *conformal* if $X \subset \mathbb{R}^d$, $d \geq 1$, and if the following hold.

(8) For every $x \in X$ and every $y \in T^{-1}(x)$, the inverse map

$$T_y^{-1} : B(x, 2\xi) \to B_y(x)$$

has a unique conformal $C^1$–extension to the open ball $B_d(x, 2\xi) \subset \mathbb{R}^d$. We assume in addition that the functions

$$B_d(x, 2\xi) \ni z \to \|D_z T_y^{-1}\|$$

are all Hölder continuous with the same Hölder exponent $\alpha$ and Hölder constants bounded above by $C\|D_x T_y^{-1}\|$ with some constant $C$ independent of $x$ and $y$. In the case when $q \geq 2$, this is automatically satisfied with $\alpha = 1$.

(9) For every $\theta > 0$, every $x \in X \setminus B(\Omega, \theta)$, every $n \geq 0$ and every $y \in T^{-n}(x)$, the map $T_y^{-n} : B(x, 2\delta_\theta) \to X$ has a unique conformal extension to the open ball $B_d(x, 2\delta_\theta) \subset \mathbb{R}^d$.

A Walters map is called *expanding* if $\Omega = \emptyset$. A finitely Walters conformal map is called *parabolic* if $\Omega \neq \emptyset$ and for any $\omega \in \Omega$ (with period $n(\omega) \geq 1$) $T^{n(\omega)}$ is a parabolic conformal map on some neighborhood of $\omega$. A conformal expanding finitely Walters map is referred to as conformal expanding repeller.

The following bounded distortion properties follow from (8) and (9) along with Theorem 3.1 if $q \geq 3$, from Koebe's Distortion Theorem if $q = 2$, and considerations following closely [U2] if $q = 1$.

Since we will be interested in the value of the Hausdorff measure $H_h(X_\infty)$, we may, without loss of generality, pass to so high iterate of $T$ that all parabolic point, i.e. members of $\Omega$, become simple.

As an immediate consequence of Theorem 3.1 and Koebe's Distortion Theorem, we get the following.

**Fact 7.1.** *(***Bounded Distortion Property I***) If $d \geq 2$, the the following holds. For every $\beta > 0$ there exists $\beta_* \in (0, 1]$ such that if $T : X_0 \to X$ is a Walters conformal map, $\theta > 0$, $x \in X \setminus B(\Omega, \theta)$, $n \geq 0$, $y \in T_y^{-n}(x)$ and $w, z \in \bar{B}(x, \beta_* \delta_\theta)$, then*

$$(1 + \beta)^{-1} \leq \frac{|(T_y^{-n})'(w)|}{|(T_y^{-n})'(z)|} \leq (1 + \beta).$$

*Also,*

$$K := K_1 = \sup\left\{\frac{|(T_y^{-n})'(w)|}{|(T_y^{-n})'(z)|} : w, z \in \bar{B}(x, \delta_\theta)\right\} < +\infty,$$

*with the supremum being taken also over all $\theta > 0$, $x \in X \setminus B(\Omega, \theta)$, $n \geq 0$ and $y \in T^{-n}(x)$.*

and

**Fact 7.2.** *Assume $d \geq 2$. Fix $\beta > 0$. Then the the following holds. Let $T : X_0 \to X$ be a Walters conformal map. If $x \in X$, $w, z \in \overline{B}(x, \beta_* \xi_T)$, and $y \in T^{-1}(x)$, then*

$$(1+\beta)^{-1} \leq \frac{|(T_y^{-1})'(w)|}{|(T_y^{-1})'(z)|} \leq (1+\beta).$$

As fairly straightforward consequence of the definition of a Walters map, particularly of property (5) and (8), we get the following.

**Lemma 7.3.** *Le $T : X_0 \to X$ be a Walters map, and fix $\theta > 0$. If $x \in X$ and $(n_j)_1^\infty$ is an incereasing sequence of iterates of $T$ such that $T^{n_j}(x) \notin B(\Omega, \theta)$ for all $j \geq 1$, then the Lipschitz constants of the maps $T^{n_j} : B_x(n_j, T^{n_j}(x)) \to B(T^{n_j}(x), 2\delta)$ diverge to infinity.*

We recall that rational function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is called *parabolic* if its Julia set contains no critical points but contains at least one parabolic periodic point and it is called *hyperbolic* if its Julia set contains no critical points nor parabolic periodic periodic points. If the map $f$ is hyperbolic, then its restriction to its Julia set $J(f)$ forms a conformal expanding repeller. If, on the other hand, the map $f$ is parabolic, then each parabolic point must be necessairly rationally indifferent, meaning that its multiplier is a root of unity, and the function $f$ restricted to its Julia set $J(f)$ forms a finite parabolic conformal Walters map

## 8. Walters Conformal Maps; Conformal Meaures

Throughout this section $T : X_0 \to X$ is assumed to be an arbitrary conformal Walters map. We recall that

$$X_\infty = \bigcap_{n=0}^{\infty} T^{-n}(X).$$

We say that a Borel finite measure $m_t$, $t \geq 0$, supported on $X_\infty$, is $t$–conformal if

$$m_t(T_y^{-1}(A)) = \int_A |(T_y^{-1})'(z)|^t dm_t(z)$$

for all $x \in X$, all $y \in T^{-1}(x)$, and all Borel sets $A \subset B(x, 2\xi)$. Note that then

$$m_t(T_w^{-n}(A)) = \int_A |(T_w^{-n})'(z)|^t dm_t(z)$$

for all $\theta > 0$, all $n \geq 0$, all $x \in X \setminus B(\Omega, \theta)$, all $\omega \in T^{-n}(x)$, and all Borel sets $A \subset B(x, 2\xi)$. Observe that if $h = \mathrm{HD}(X_\infty)$ and $H_h$ is the $h$–dimensional Hausdorff measure on $X_\infty$, then $H_h$ is $h$–conformal, and, if $H_h(X_\infty) > 0$, then so is also its normalized version $H_h/H_h(X)$. We denote it by

$$H_h^1|_{X_\infty}.$$

We shall prove the following.

**Lemma 8.1.** *Let $T : X_0 \to X$ be a conformal Walters map. If $m_t$ is an atomless $t$–conformal measure on $X_0$ for the map $T$, then $m_t(X_0 \setminus U_\infty) = 0$, for every non-empty open set $U \subset X$, where*

$$U_\infty = \{x \in X_\infty : T^n(x) \in U \textit{ for infinitely many } n \geq 0\}.$$

*Proof.* Since $T^{-1}(U)$ is an open subset of $X$, it suffices to show that $m_t(X_\infty \setminus U_1) = 0$, where

$$U_1 = \{x \in X_\infty : T^n(x) \in U \text{ for at least one } n \geq 1\}.$$

In view of condition (6) it suffices to prove the lemma for every non-empty open subset $U \subset X$ which is $\delta_\xi/8$–dense in $X$. Select a finite set $F \subset U$ which is $\delta_\xi/8$–dense in $X$, and let $\gamma \in (0, (2K^2)^{-1}\delta_\xi)$ be so small that

$$\bigcup_{z \in F} B(z, \gamma) \subset U.$$

Fix an arbitrary point $x \in X_\infty \setminus \bigcup_{n=0}^\infty T^{-n}(\Omega)$ and a real number $s > 0$. In view of (2a) and (4) there exists $n \geq 1$ so large that

$$T^n(x) \in X \setminus B(\Omega, \xi) \text{ and } K^{-1}|(T^n)'(x)|^{-1}\delta_\xi < s.$$

Now, by the definition of the set $F$, there exists $y \in F$ such that $\rho(T^n(x), y) < \gamma$. So, $B(y, \gamma) \subset U \cap B(T^n(x), 2\gamma) \subset U \cap B(T^n(x), \delta_\xi)$. Hence

$$T_x^{-n}(B(y, \gamma)) \subseteq U_1, \tag{8.1}$$

and, by Bounded Distortion Property (Fact 7.1),

$$m_t(T_x^{-n}(B(y, \gamma))) \geq K^{-t}|(T^n)'(x)|^{-t} m_t(B(y, \gamma)) \geq M_\gamma K^{-t}|(T^n)'(x)|^{-t}, \tag{8.2}$$

where $M_\gamma = \inf\{m_t(B(z, \gamma)) : z \in X\} > 0$ since $\mathrm{supp}(m_t) = X$. Also, by the same Bounded Distortion Property (Fact 7.1)

$$\begin{aligned} T_x^{-n}(B(y, \gamma)) &\subseteq T_x^{-n}(B(T^n(x), K^{-2}\delta_\xi)) \subseteq B(x, K^{-1}|(T^n)'(x)|^{-1}\delta_\xi) \\ &\subseteq T_x^{-n}(B(T^n(x), \delta_\xi)) \end{aligned} \tag{8.3}$$

and

$$\begin{aligned} m_t(B(x, K^{-1}|(T^n)'(x)|^{-1}\delta_\xi)) &\leq m_t(T_x^{-n}(B(T^n(x), \delta_\xi)) \\ &\leq K^t m_t(B(T^n(x), \delta_\xi))|(T^n)'(x)|^{-t} \\ &\leq K^t m_t(X)|(T^n)'(x)|^{-t}. \end{aligned}$$

Combining this, (8.3), (8.1), and (8.2), we get

$$\begin{aligned} \frac{m_t(B(x, K^{-1}|(T^n)'(x)|^{-1}\delta_\xi) \cap U_1)}{m_t(B(x, K^{-1}|(T^n)'(x)|^{-1}\delta_\xi))} &\geq \frac{m_t(T_x^{-n}(B(y, \gamma)))}{m_t(B(x, K^{-1}|(T^n)'(x)|^{-1}\delta_\xi))} \\ &\geq \frac{M_\gamma K^{-t}|(T^n)'(x)|^{-t}}{K^t m_t(X)|(T^n)'(x)|^{-t}} \\ &= M_\gamma (m_t(X)K^{2t})^{-1} \\ &> 0. \end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{m_t((X_\infty \setminus U_1) \cap B(x, K^{-1}|(T^n)'(x)|^{-1}\delta_\xi))}{m_t(B(x, K^{-1}|(T^n)'(x)|^{-1}\delta_\xi))} = \\
&= \frac{m_t(B(x, K^{-1}|(T^n)'(x)|^{-1}\delta_\xi)) - m_t(B(x, K^{-1}|(T^n)'(x)|^{-1}\delta_\xi) \cap U_1)}{m_t(B(x, K^{-1}|(T^n)'(x)|^{-1}\delta_\xi))} \\
&= 1 - \frac{m_t(B(x, K^{-1}|(T^n)'(x)|^{-1}\delta_\xi) \cap U_1)}{m_t(B(x, K^{-1}|(T^n)'(x)|^{-1}\delta_\xi))} \le 1 - M_\gamma(m_t(X)K^{2t})^{-1} < 1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\liminf_{r\to 0} \frac{m_t(B(x,r) \cap (X_\infty \setminus U_1))}{m_t(B(x,r))} &\le \liminf_{n\to\infty} \frac{m_t((X_\infty \setminus U_1) \cap B(x, K^{-1}|(T^n)'(x)|^{-1}\delta_\xi))}{m_t(B(x, K^{-1}|(T^n)'(x)|^{-1}\delta_\xi))} \\
&\le 1 - M_\gamma(m_t(X)K^{2t})^{-1} \\
&< 1.
\end{aligned}$$

Hence, it follows from Lebesgue's Density Theorem that $m_t((X_\infty \setminus U_1) \setminus \bigcup_{n=0}^\infty T^{-n}(\Omega)) = 0$. Since the conformal measure is assumed to be atomless, this yields $m_t(X_\infty \setminus U_1) = 0$. We are done. $\square$

**Definition 8.2.** *Let $T : X_0 \to X$ be a conformal Walters map and put $h = \mathrm{HD}(X_\infty)$. The map $T : X_0 \to X$ is said to be regular if there exists an $h$–conformal Borel probability measure $m_h$ on $X_\infty$ such that $m_h(\Omega) = 0$; otherwise the map $T$ is called irregular.*

The following theorem collects the basic properties of regular and irregular conformal Walters maps.

**Theorem 8.3.** *Let $T : X_0 \to X$ be a conformal Walters map. Then the following hold.*

- (a) *If $T$ is regular, then*
  - (a1) *The conformal measure $m_h$ is atomless, unique up to a uniformly bounded, above and away from zero, Radon-Nikodym derivative, and*
  - (a2) $H_h|_{X_\infty} << m_h$.
  - (a3) *The Radon–Nikodym derivative $\frac{\mathrm{d}H_h}{\mathrm{d}m_h}$ is uniformly bounded above.*
- (b) *If $T$ is a conformal expanding repeller, then*
  - (b1) $0 < H_h(X_\infty = X) < +\infty$,
  - (b2) *The conformal measure $m_h$ is unique and equal to $H_h^1|_X$.*
  - (b3) *In particular the map $T$ is regular.*
- (c) *If $T$ is parabolic then $T$ is regular and*
  - (c1) *If $h < 1$, then $H_h(X_\infty = X) = 0$.*
  - (c2) *If $h \ge 1$, then $0 < H_h(X_\infty = X) < +\infty$.*
  - (c3) *The conformal measure $m_h$ is unique and equal to $H_h^1|_X$.*

*Proof.* The proof of item (a) is standard. It employes the reasoning from [DU1] or Theorem 2.13 in [KU] for example. Item (c) is essentially known since the work [Bo] of R.

Bowen. Its complete proof can be found in [PU]. Finally, to prove item (c), notice first that it can be easily seen that the map $T : X \to X$ is expansive. Therefore, see for example [PU], it has Markov partitions of arbitrariy small diameters. This permits us to associate with $T$ its jump transformation $T^*$ as in [DU2]. This jump transformation is a jump-like conformal map in the sense of [KU]. We are then done by invoking Theorem 3.3 from [KU]. $\square$

If $H_t(Y)$, a $t$-dimensional Hausdorff measure of a Borel subset $Y$ of some metric space is positive and finite, then we denote by $H_t^1|_Y$, the normalized $t$-dimensional Hausdorff measure restricted to $Y$. Precisely,

$$H_t^1(A) = \frac{H_t(A)}{H_t(Y)}$$

for every Borel set $A \subset Y$. We have already observed that in the context of Theorem 8.3, if $0 < H_h(X_\infty) < +\infty$, then

$$m_h = H_h^1|_{X_\infty}.$$

## 9. Modes of convergence

Since our ultimate results will be about continuity of Hausdorff measure, we need a right notion of convergence of conformal Walters maps. We say that a sequence $(T_n : X_0^n \to X^n)_{n=1}^\infty$ of conformal Walters maps converges strongly to a conformal map $T : X_0^0 \to X = X^0$ if the following conditions are satisfied

(1) There exists a bounded open neighborhood $U$ of $X$ in $\mathbb{R}^q$ such that $X^n \subseteq U$ for all $n \geq 0$ and each map $T_n$, $n \geq 1$, and $T$ extend conformally to a map from $U$ to $\mathbb{R}^q$. We keep for these extensions the same symbols $T_n$ and $T$ respectively.
(2) $X_n \to X$ in $\mathcal{K}(U)$, the space of all non–empty compact subsets of $U$ endowed with the Hausdorff metric
(3) $\xi_- := \liminf_{n\to\infty} \xi_n > 0$
(4) $\liminf_{n\to\infty} \delta_{n,\theta} > 0$ for all $\theta > 0$.
(5) $\lim_{n\to\infty} \inf\{|T_n'(z)| : z \in U\} > 0$
(6) $\lim_{n\to\infty} \|T_n'\|_\infty < +\infty$,

where $\delta_{n,\theta}$ and $\xi_n$ are respective constants for the map $T_n$.

We say that a sequence $(T_n : X_0^n \to X^n)_{n=1}^\infty$ of conformal Walters maps converges sub–finely to a conformal Walters map $T : X_0 \to X$ if it converges strongly and the following conditions are satisfied

(7) $\lim_{n\to\infty} h_n = h$, where $h_n = \mathrm{HD}(X_\infty^n)$, $n \geq 1$, and $h = \mathrm{HD}(X_\infty)$.
(8) $H_{h_n}(X_\infty^n) > 0$ for all $n \geq 1$ and $T : X_\infty \to X$ is regular.
(9) $(H_{h_n}^1|X_\infty^n)_1^\infty$ converges weakly to $m_h$, all measures treated as Borel probability measures on $\overline{U}$.

We say that a sequence $(T_n : X_0^n \to X^n)_{n=1}^\infty$ of finitely Walters conformal maps converges finely to a finitely Walters conformal map $T : X_0 \to X$ if it converges sub–finely and the following holds. All the maps $T_n, T$, $n \geq 1$, extend conformally to all of $U$ and

$$\lim_{n\to\infty} T_n = T$$

in the space $C^1(U)$.

We say that a sequence $(T_n)_1^\infty$ of conformal expanding repellers converges $C^1$–uniformly to an expanding conformal repeller $T$ if (1) is satisfied and $T_n \to T$ in $C^1(U)$. We reword the following well-known fact.

**Proposition 9.1.** *If a sequence $(T_n)_1^\infty$ of conformal expanding repellers converges $C^1$–uniformly to an expanding conformal repeller $T$, then $(T_n)_1^\infty$ converges finely.*

We say that a sequence $(T_n)_1^\infty$ of parabolic Walters conformal maps converges nicely to a parabolic Walters conformal map $T$ if it converges to $T$ strongly, (8) holds, and the following conditions are satisfied.

(10) $T_n \to T$ in $C^1(U)$

(11) For every parabolic point $\omega$ of $T$, there exists $\theta > 0$ such that $(T_n^u|_{B(\omega,\theta)})_1^\infty$ converges to $T^u|_{B(\omega,\theta)}$ in $\mathrm{Par}_{B(\omega,\theta)}(q, p(\omega))$, where $u \geq 1$ is so large that each parabolic point of $T$ is simple for $T^u$.

(12) For all but finitely many $n \geq 1$, $\Omega(T_n) \subseteq B(\Omega(T), \theta)$.

The method of the proof of the main result in [UZd2] gives the following.

**Proposition 9.2.** *If a sequence $(T_n)_1^\infty$ of parabolic Walters conformal maps converges nicely to a parabolic conformal map $T$, then this sequence $(T_n)_1^\infty$ convereges finely to $T$.*

We shall now describe the structure of parabolic Walters conformal maps in sufficiently small neighborhoods of their parabolic points.

**Lemma 9.3.** *Let $T : X_0 \to X$ be a parabolic Walters conformal map. Assume without loss of generality that the positive integer $u$ resulting from item (11) is equal to 1. Let $\theta$ also come from item (11). Fix an open neighborhood $W$ of $T$ in the topology of nice convergence, so small that if $Q \in W$, then for every parabolic point $\omega$ of $T$ there exists a unique parabolic point $\omega_Q \in B(\omega,\theta)$ of $Q$, and $Q|_{B(\omega,\theta)} \in Z(T|_{B(\omega,\theta)})$, where the neighborhood $Z(T|_{B(\omega,\theta)})$ of $T|_{B(\omega,\theta)}$ comes from Proposition 6.1. Then for every parabolic point $\omega$ of $T$ and every $\alpha \in (0, \pi/p(\omega))$ there exists $\eta > 0$ such that*

$$X_\infty(Q) \cap B(\omega_Q, \eta) \subset \bigcup_{j=1}^{p(\omega_Q)} S_{\omega_Q}^{j,-}(\eta,\alpha) \quad \textit{and} \quad X(Q) \cap B(\omega_Q, \eta) \subset \bigcup_{j=1}^{p(\omega_Q)} S_{\omega_Q}^{j,-}(\eta,\alpha),$$

*the latter resulting from the former since $X(Q) = \bar{X}_\infty(Q)$.*

*Proof.* Ascribe $\delta > 0$ to $\epsilon = 2\xi$ according to Propositions 3.3, 4.4, and 5.1. Seeking contradiction suppose that there exist a sequence $(T_n)_{n=1}^\infty$ of Walters conformal maps in $W$ converging nicely to the Walters conformal map $T$, and a sequence $(x_n)_{n=1}^\infty$ of points respectively in $X_\infty(T_n)\cap B(\omega_Q,\delta)\setminus\bigcup_{j=1}^{p(\omega_n)} S_{\omega_n}^{j,-}(\delta,\alpha)$ such that $\lim_{n\to+\infty}||x_n-\omega_n||=0$, where $\omega_n = \omega_{T_n}$. Since $p(\omega_n) = p(\omega)$ for all $n \geq 1$, passing to a subsequence, we may further assume without loss of generality that $x_n \in S_{\omega_n}^{i,+}(\eta,(\pi/p)-\alpha)$ for all $n \geq 1$ and some $1 \leq i \leq p(\omega)$. But then, by Propositions 3.3, 4.4(a), and 5.1(a), $T_1^k(x_1) \in X_\infty(T_1)\cap B(\omega_1, 2\xi)$ for all $k \geq 0$, and it thus follows from item (3) of the definition of Walters maps that $x_1 = \omega_1$. This contradiction finishes the proof. □

As an immediate consequence of this lemma, looking at Proposition 3.3 and Theorem 3.1 in the case when $q \geq 3$, in Proposition 4.4(a) and Koebe's Distortion Theorem in the case when $q = 2$, and in Proposition 5.1 in the case when $q = 1$, we get the following.

**Lemma 9.4.** *With the settings of Lemma 9.3, there exists $\xi > 0$ sufficiently small that there exists a monotone increasing function $K : [0,1) \to [1,+\infty)$ such that $\lim_{t\to0} K(t) = K(0) = 1$ and*

$$\frac{|(Q_z^{-n})'(y)|}{|(Q_z^{-n})'(x)|} \leq K(t)$$

*for all $q \in W$, all $\omega \in \Omega$, all $n \geq 0$, all $z \in \bigcap_{j=0}^n Q^{-j}(X(Q))$ such that $Q^n(z) \in B(\omega_Q,\xi)$ and all $x, y \in B(Q^n(z), t\|Q^n(z)-\omega\|)$.*

In the case when a conformal Walters map is generated by a conformal GDMS (Graph Directed Markov System) $S = \{\varphi_h : X_{t(k)} \to X_{i(k)}\}_{k\in\mathbb{N}}$ satisfying the separation condition, we also consider the subsystems $S_n = \{\varphi_k : X_{t(k)} \to X_{i(k)}\}_{k\in\mathbb{N}_n}$, where $\mathbb{N}_n = \{1, 2, \dots, n\}$ and are interested in the problem of whether

$$\lim_{n\to\infty} H_{h_n}(J_{S_n}) = H_h(J_S),$$

where $h_n = \mathrm{HD}(J_{S_n})$ and $h = \mathrm{HD}(J_S)$. It is by the way known (see [MU3]) that $\lim_{n\to\infty} h_{S_n} = h_S$.

## 10. First continuity related technical results

Given an integer $q \geq 1$, $\xi, \delta, \gamma > 0$ let $W_q(\xi,\delta,\gamma)$ be the collection of all conformal Walters maps for which

$$\xi_T \geq \xi, \quad \delta_{\xi/8} \geq \delta \quad \text{and} \quad \inf\{|T'(x)| : x \in X_0(T)\} \geq \gamma^{-1}.$$

In this section we shall prove the following technical lemma.

**Lemma 10.1.** *Given $\xi, \delta, \gamma > 0$ we have*

$$\lim_{r\to0} \sup\left\{\frac{H_{h_T}(A\cap X_\infty(T))}{\mathrm{diam}^{h_T}(A)} : 0 < \mathrm{diam}(A) \leq r, T \in W_q(\xi,\delta,\gamma)\right\} \leq 1.$$

*Proof.* First, note that the limit exists, since the above function is increasing (as a function of $r$).

Let

$$W_q^+\xi,\delta) = \{T \in W_q(\xi,\delta,\gamma) : H_h(X_\infty) > 0\}.$$

Clearly, for every $r > 0$, the following two suprema are equal.

$$\sup\left\{\frac{H_{h_T}(A \cap X_\infty(T))}{\operatorname{diam}^{h_T}(A)} : A \text{ is a Borel set, } 0 < \operatorname{diam}(A) \le r,\ T \in W_q(\xi,\delta,\gamma)\right\}$$

and

$$\sup\left\{\frac{H_{h_T}(A \cap X_\infty(T))}{\operatorname{diam}^{h_T}(A)} : A = \overline{A},\ A \text{ is convex } 0 < \operatorname{diam}(A) \le r,\ T \in W_q^+(\xi,\delta,\gamma)\right\}.$$

We shall prove that the upper limit, as $r \to 0$, of the latter suprem is $\le 1$. Fix $\kappa > 0$. Take

$$r = \frac{1}{2}\{\delta, \xi/8\} \min\{\kappa_*, (6K^2)^{-1}\} \min\{1, \gamma^{-1}\}.$$

Consider an arbitrary closed set contained in $\mathbb{R}^q$ such that $0 < \operatorname{diam}(A) \le r$. Having $T \in W_q^+(\xi,\delta,\gamma)$ we may assume without loss of generality that

$$A \subseteq B(X, r).$$

If $A \cap B(\Omega_T, \xi/9) \ne \emptyset$, then $A \subseteq B(\omega, \xi/4)$ for some $\omega \in \Omega_T$. Fix an arbitrary $x \in A$ and $z \in T^{-1}(\omega) \setminus \{\omega\}$. Let $y = T_z^{-1}(x)$. Then

$$T_y^{-1}(A) \subseteq T_z^{-1}(B(\omega, \xi/4)),$$

and, according to item (7) of the definition of Walter maps,

$$T_y^{-1}(A) \cap B(\Omega_T, \xi) = \emptyset.$$

Since $\operatorname{diam}(A) < \kappa_*\xi$ and $A \subseteq B(x, \operatorname{diam}(A)) \subseteq B(x, 2\xi)$ we may apply Fact 7.2 to get

$$\begin{aligned}\frac{H_{h_T}(T_y^{-1}(A \cap X_\infty(T)))}{\operatorname{diam}(T_y^{-1}(A))} &\ge \frac{(1+\kappa)^{-h}|(T_y^{-1})'(x)|^{h_T} H_{h_T}(A \cap X_\infty(T))}{(1+\kappa)^{h}|(T_y^{-1})'(x)|^{h_T} H_{h_T}(A \cap X_\infty(T))} \\ &= (1+\kappa)^{-2h}\frac{H_{h_T}(A \cap X_\infty(T))}{\operatorname{diam}^{h_T}(A)}.\end{aligned} \tag{10.1}$$

So, our task is to estimate from above the quotient

$$\frac{H_{h_T}(\Gamma' \cap X_\infty(T))}{\operatorname{diam}^{h_T}(\Gamma')}$$

for every closed set $\Gamma' \subset B(X, \gamma r)$ such that

$$\Gamma' \cap B(\Omega_T, \xi/8) = \emptyset \quad \text{and} \quad \operatorname{diam}(\Gamma') < \gamma r.$$

Add to $\Gamma'$ one suffiiently small ball so that the resulting set $\Gamma$ has the following properties

(1) $\operatorname{Int}\Gamma \ne \emptyset$
(2) $\Gamma \subseteq B(X, \gamma r)$
(3) $\operatorname{diam}(\Gamma) < \gamma r$
(4) $\Gamma \cap B(\Omega_T, \xi/8) = \emptyset$
(5) $\frac{H_{h_T}(\Gamma \cap X_\infty(T))}{\operatorname{diam}^{h_T}(\Gamma)} \ge (1+\kappa)^{-h}\frac{H_{h_T}(\Gamma' \cap X_\infty(T))}{\operatorname{diam}^{h_T}(\Gamma')}$.

Fix a point $w \in X_\infty \cap \mathrm{Int}(\Gamma)$ such that with some $\eta > 0$,

$$B(w, 2\eta) \subset \mathrm{Int}(\Gamma). \tag{10.2}$$

Lemma 1.7 from [Fa] tells us that there exists $R > 0$ so small that

$$\sum_{i=1}^{\infty} \mathrm{diam}^{h_T}(U_i) \geq (1+\kappa)^{-h_T} H_{h_T}\left(X_\infty(T) \cap \bigcup_{i=1}^{\infty} U_i\right) \tag{10.3}$$

for any countable collection $\{U_i\}_{i=1}^{\infty}$ of sets with diameters $\leq R$. For every $x \in B(w,\eta)_\infty$, let $(n_j(x))_{j=1}^{\infty}$ be the increasing sequence of all positive integers $n$ such that $T^n(x) \in \mathrm{Int}(\Gamma)$, in particular $T^n(x) \notin B(\Omega, \xi/8)$. Consider the family

$$\mathcal{G} = \{T_x^{-n_j(x)}(\Gamma) : x \in B(w,\eta)_\infty \text{ and } j \geq 1\}.$$

We shall prove the following.

**Claim 1.** *The family $\mathcal{G}$ is a Vitali relation (in the sense of Federer (see p. 51 in* [Fe]*) for the measure $H_{h_T}$ restricted to the set $B(w,\eta)_\infty$.*

**Proof.** Fix $x \in B(w,\eta)_\infty$. It follows from Lemma 7.3 and Bounded Distortion that

$$\begin{aligned}\limsup_{j\to\infty} \mathrm{diam}(T_x^{-n_j(x)}(\Gamma)) &\leq \limsup_{j\to\infty} \mathrm{diam}(T_x^{-n_j(x)}(B(T^{n_j(x)}(x), \delta/4K^2)) \\ &\leq \limsup_{j\to\infty} \frac{1}{2K}\delta |(T^{n_j(x)})'(x)|^{-1} \\ &= 0.\end{aligned} \tag{10.4}$$

This means, in Federer's terminology that the relation $\mathcal{G}$ is fine at the point $x$. Aiming to apply Theorem 2.8.17 from [Fe], we set

$$B_j(x) = T_x^{-n_j(x)}(\Gamma),$$

and

$$\delta(B_j(x)) = \mathrm{diam}(T_x^{-n_j(x)}(\Gamma)).$$

Fix $\tau \in (1,2)$. With the notation from page 144 in [Fe], we get by the Bounded Distortion Property (Fact 7.1), (2), (3), (4), and the choice of $r$ that

$$\begin{aligned}\hat{B}_j(x) &= \bigcup\{B : B \in \mathcal{G}, B \cap B_j(x) \neq \emptyset,\ \delta(B) \leq \tau\delta(B_j(x))\} \\ &\subset B\left(x, (1+\tau)2K\frac{1}{6}\delta K^{-2}|(T^{n_j(x)})'(x)|^{-1}\right) \\ &\subset B\left(x, K^{-1}\delta_\theta|(T^{n_j(x)})'(x)|^{-1}\right) \\ &\subset T_x^{-n_j(x)}(B(T^{n_j(x)}(x), \delta)).\end{aligned}$$

Hence, putting $r_j(x) = |(T^{n_j(x)})'(x)|^{-1}$, we get from Fact 7.1, Lemma 8.1 with $m_t = H_{h_T}|_{X_\infty(T)}$ and from (10.2) that

$$
\begin{aligned}
\delta(B_j(x)) + \frac{H_{h_T}(B(w,\eta)_\infty \cap \hat{B}_j(x))}{H_{h_T}(B(w,\eta)_\infty \cap B_j(x))} &= \\
&= \delta(B_j(x)) + \frac{H_{h_T}(\hat{B}_j(x) \cap X_\infty(T))}{H_{h_T}(B_j(x) \cap X_\infty(T))} \\
&\le 2K\delta r_j(x) + \frac{H_{h_T}(T_x^{-n_j(x)}(B(T^{n_j(x)}(x),\delta) \cap X_\infty(T)))}{H_{h_T}(T_x^{-n_j(x)}(B(T^{n_j(x)}(x),\eta) \cap X_\infty(T)))} \\
&\le 2K\delta r_j(x) + \frac{K^{h_T}|(T^{n_j(x)})'(x)|^{-h_T} H_{h_T}(B(T^{n_j(x)}(x),\delta) \cap X_\infty(T))}{K^{-h_T}|(T^{n_j(x)})'(x)|^{-h_T} H_{h_T}(B(T^{n_j(x)}(x),\eta) \cap X_\infty(T))} \\
&\le 2K\delta r_j(x) + K^{2h_T} M_\eta^{-1},
\end{aligned}
$$

where $M_\eta = \inf\{H_{h_T}(B(z,\eta) \cap X_\infty(T)) : z \in X\} > 0$. Hence, using (10.4), we get

$$
\limsup_{j\to\infty} \left( \delta(B_j(x)) + \frac{H_{h_T}(\hat{B}_j(x) \cap X_\infty(T))}{H_{h_T}(B_j(x) \cap X_\infty(T))} \right) \le K^{2h_T} M_\eta^{-1} < +\infty.
$$

Thus, all the hypothesis of Theorem 2.8.17 in [Fe] are verified and the proof of Claim 1 is complete. □

In virtue of Claim 1 there exists a countable $\Lambda \subseteq \mathbb{N} \times B(w,\eta)_\infty$ such that

(a) The family $\{B_{\gamma_1}(\gamma_2) \cap B(w,\eta)_\infty : (\gamma_1,\gamma_2) \in \Lambda\}$ consist of mutually disjoint sets.
(b) $H_h\left(B(w,\eta)_\infty \setminus \bigcup_{(\gamma_1,\gamma_2)\in\Lambda} B_{\gamma_1}(\gamma_2)\right) = 0$, so $H_h\left(X_\infty(T) \setminus \bigcup_{(\gamma_1,\gamma_2)\in\Lambda}, B_{\gamma_1}(\gamma_2)\right) = 0$.
(c) $\mathrm{diam}(B_{\gamma_1}(\gamma_2)) < R$ for all $(\gamma_1,\gamma_2) \in \Lambda$.

It then follows from Lemma 8.1, (2), (3), (4), the Bounded Distortion Property (Fact 7.1), and the choice of $R$ that
(10.5)

$$
\begin{aligned}
H_h(X_\infty(T)) = H_h(B(w,\eta)_\infty) &= H_h\left( B(\omega,\eta)_\infty \cap \bigcup_{(\gamma_1,\gamma_2)\in\Lambda} B_{\gamma_1}(\gamma_2) \right) \\
&= \sum_{(\gamma_1,\gamma_2)\in\Lambda} H_{h_T}\left(B(w,\eta)_\infty \cap B_{\gamma_1}(\gamma_2)\right) = \sum_{(\gamma_1,\gamma_2)\in\Lambda} H_{h_T}\big(B_{\gamma_1}(\gamma_2) \cap X_\infty(T)\big) \\
&\ge (1+\kappa)^{-h_T} \sum_{(\gamma_1,\gamma_2)\in\Lambda} |(T^{\gamma_1})'(\gamma_2)|^{-h} H_{h_T}\big(\Gamma \cap X_\infty(T)\big) \\
&= (1+\kappa)^{-h_T} \sum_{(\gamma_1,\gamma_2)\in\Lambda} |(T^{\gamma_1})'(\gamma_2)|^{-h_T} \mathrm{diam}^{h_T}(\Gamma) \frac{H_{h_T}\big(\Gamma \cap X_\infty(T)\big)}{\mathrm{diam}^{h_T}(\Gamma)} \\
&\ge (1+\kappa)^{-2h_T} \frac{H_{h_T}\big(\Gamma \cap X_\infty(T)\big)}{\mathrm{diam}^{h_T}(\Gamma)} \sum_{(\gamma_1,\gamma_2)\in\Lambda} \mathrm{diam}^{h_T}(B_{\gamma_1}(\gamma_2)).
\end{aligned}
$$

Using further (c), (b), and (10.3), we now continue as follows.

$$
\begin{aligned}
H_{h_T}(X_\infty(T)) &\ge (1+\kappa)^{-3h_T}\frac{H_{h_T}\big(\Gamma\cap X_\infty(T)\big)}{\mathrm{diam}^{h_T}(\Gamma)}H_{h_T}\left(X_\infty(T)\cap\bigcup_{(\gamma_1,\gamma_2)\in\Lambda}B_{\gamma_1}(\gamma_2)\right)\\
&=(1+\kappa)^{-3h_T}\frac{H_{h_T}\big(\Gamma\cap X_\infty(T)\big)}{\mathrm{diam}^{h_T}(\Gamma)}H_{h_T}\big(X_\infty(T)\big).
\end{aligned}
$$

Hence

$$
\frac{H_{h_T}\big(\Gamma\cap X_\infty(T)\big)}{\mathrm{diam}^{h_T}(\Gamma)}\le(1+\kappa)^{3h_T}.
$$

Along with (5) and (10.1) this gives

$$
\frac{H_{h_T}\big(A\cap X_\infty(T)\big)}{\mathrm{diam}^{h_T}(A)}\le(1+\kappa)^{6h_T}\le(1+\kappa)^{6q}.
$$

As $\kappa>0$ was arbitrary, we are done. □

## 11. Upper semi–continuity of Hausdorff measure.

We recall first the following "density" theorem for Hausdorff measures (see [Ma] for example).

**Fact 11.1.** *Let $X$ be a metric space, with* $\mathrm{HD}(X)=h$, *such that $H_h(X)<+\infty$. Then (see p. 91 in* [Ma]*),*

$$
\lim_{r\to0}\sup\left\{\frac{H_h(F)}{\mathrm{diam}^h(F)}:x\in F,\ \overline{F}=F,\ \mathrm{diam}(F)\le r\right\}=1
$$

*for $H_h$–a.e. $x\in X$.*

This fact yields the following strengthening of Lemma 10.1.

**Corollary 11.2.** *Given $\xi,\delta,\gamma>0$ we have*

$$
\lim_{r\to0}\sup\left\{\frac{H_{h_T}(A\cap X_\infty(T))}{\mathrm{diam}^{h_T}(A)}:0<\mathrm{diam}(A)\le r, T\in W_q(\xi,\delta,\gamma)\right\}=1.
$$

Recall that if in addition $H_h(X)>0$, then we denote by $H_h^1$ the normalized $h$–dimensional Hausdorff measure on $X$, i.e.

$$
H_h^1(A)=\frac{H_h(A)}{H_h(X)}
$$

for every Borel set $A\subseteq X$. As an immediate consequence of the above fact we get the following.

**Corollary 11.3.** *If $X$ is a metric space and $0<H_h(X)<+\infty$, then*

$$
H_h(X)=\lim_{r\to0}\inf\left\{\frac{\mathrm{diam}^h(F)}{H_h^1(F)}:x\in F,\ \overline{F}=F,\ \mathrm{diam}(F)\le r\right\}
$$

*for $H_h$–a.e. $x\in X$.*

Because of the Converse Frostman Lemma, we also have the following.

**Lemma 11.4.** *If $X$ is a metric space, $H_h(X) = 0$ and $\mu$ is an arbitrary locally finite Borel measure on $X$, then*

$$\liminf_{r\to 0}\left\{\frac{\operatorname{diam}^h(F)}{\mu(F)} : x \in F,\ \overline{F} = F,\ \operatorname{diam}(F) \le r\right\} = 0$$

*for $\mu$–a.e. $x \in X$.*

In virtue of Lemma 10.1, in the context of conformal Walters maps, we get the following "one sided" improvement of Corollary 11.3.

**Corollary 11.5.** *Given $\xi, \delta, \gamma > 0$ for every $\kappa > 0$ there exists $r > 0$ such that*

$$\inf\left\{\frac{\operatorname{diam}^{h_T}(F)}{H^1_{h_T}(F)} : F \text{ Borel},\ 0 < \operatorname{diam}(F) \le r\right\} \ge (1+\kappa)^{-1} H_{h_T}(X_\infty(T))$$

*for all $T \in W_+(\xi, \delta, \gamma)$.*

Our first continuity result is this.

**Theorem 11.6.** *If $(T_n : X^n_0 \to X^n)_{n=1}^\infty$ is a sequence of conformal Walters maps converging sub-finely to a conformal Walters map $T : X_0 \to X$, then*

$$\limsup_{n\to\infty} H_{h_n}(X^n_\infty) \le H_h(X_\infty),$$

*where we put $h_n = \operatorname{HD}(X^n_\infty)$ and $h = \operatorname{HD}(X_\infty)$ .*

*Proof.* Fix $\varepsilon > 0$. Because of the sub-fine convergence of $(T_n)_1^\infty$ to $T$, Corollary 11.5 yields a number $\alpha > 0$ and an integer $q_1 \ge 1$ such that

$$\frac{\operatorname{diam}^{h_n}(F)}{H^1_{h_n}(F \cap X^n_\infty)} \ge (1+\varepsilon)^{-5} H_{h_n}(X^n_\infty) \tag{11.1}$$

for all $n \ge q_1$ and all Borel sets $F \subseteq U$ with $0 < \operatorname{diam}(F \cap X^n_\infty) \le \alpha$. Consider first the case when $H_h(X^0_\infty) > 0$. In virtue of Corollary 11.3 there exists a closed set $E \subseteq \overline{U}$ such that $0 < \operatorname{diam}(E \cap X^0_\infty) < \alpha/2$ and

$$H_h(X^0_\infty) \ge (1+\varepsilon)^{-1}\frac{\operatorname{diam}^h(E \cap X^0_\infty)}{H^1_h(E \cap X^0_\infty)}.$$

Consequently,

$$H_h(X^0_\infty) \ge (1+\varepsilon)^{-2}\frac{\operatorname{diam}^h(B(E \cap X^0_\infty, r))}{H^1_h(E \cap X^0_\infty)}$$

for all $r > 0$ small enough, say $0 < r \le \eta_1 < \alpha/2$. Since

$$\lim_{n\to\infty} h_n = h \ \text{ and } \ \lim_{r\to 0} \operatorname{diam}(B(E\cap X_\infty^0, r)) = \operatorname{diam}(E\cap X_\infty^0),$$

this implies that

$$H_h(X_\infty^0) \ge (1+\varepsilon)^{-3} \frac{\operatorname{diam}^{h_n}(B(E\cap X_\infty^0), r)}{H_h^1(E\cap X_\infty^0)} \tag{11.2}$$

for al $r > 0$ sufficiently small, say $0 < r \le \eta_2 \le \eta_1$ and all $n \ge 1$ large enough, say $n \ge q_2 \ge q_1$. Now, since the probability measures $H_{h_n}^1|_{X_\infty^n}$ regarded as Borel probability measures on $\overline{U}$ converge weakly to the probability measure $H_h^1|_{X_\infty}$ also regarded as a Borel probability measure on $\overline{U}$, we get that

$$H_h^1(E\cap X_\infty^0) \le (1+\varepsilon) H_{h_n}^1(B(E\cap X_\infty^0, \eta_2)\cap X_\infty^n)$$

for all $n \ge 1$, say $n \ge q_4 \ge q_3$. Inserting this to (11.2), we get that

$$H_h(X_\infty^0) \ge (1+\varepsilon)^{-4} \frac{\operatorname{diam}^{h_n}(B(E\cap X_\infty^0, \eta_2)}{H_{h_n}^2(B(E\cap X_\infty^0, \eta_2)\cap X_\infty^n)}$$

for all $n \ge q_4$. Combining this with (11.1), we obtain

$$H_{h_n}(X_\infty^n) \le (1+\varepsilon)^{10} H_h(X_\infty^0)$$

for all $n \ge q_4$. Hence, we are done in the case when $H_h(X_\infty^0) > 0$. So, suppose that $H_h(X_\infty) = 0$. Then in view of Lemma 11.4 there exists a closed set $F \subseteq X_\infty^0$ such that $0 < \operatorname{diam}(F) < \alpha/2$ and

$$\frac{\operatorname{diam}^h(F)}{m_h(F)} \le \varepsilon(1+\varepsilon)^{-8},$$

where $m_h$ is an $h$–conformal measure on $X_\infty^0$ for the map $T$. Hence with $s \in (0, \alpha/2)$ sufficiently small

$$\frac{\operatorname{diam}^h(B(F,s))}{m_h(F)} \le \varepsilon(1+\varepsilon)^{-7}.$$

Since $\lim_{n\to\infty} h_n = h$, this implies that

$$\frac{\operatorname{diam}^{h_n}(B(F,s))}{m_h(F)} \le \varepsilon(1+\varepsilon)^{-6} \tag{11.3}$$

for all $n \ge 1$ large enough, say $n \ge n_1 \ge q_1$. Since the sequence $(H_{h_n}^1)_1^\infty$ converges weakly to $m_h$, for all $n \ge 1$ large enough, say $n \ge n_2 \ge n_1$ we have $m_h(F) \le (1+\varepsilon)H_{h_n}^1(B(F,s))$. Along with (11.3) this yields

$$\frac{\operatorname{diam}^{h_n}(B(F,s_1))}{H_{h_n}^1(F)} \le \varepsilon(1+\varepsilon)^{-5}.$$

Since $\operatorname{diam}(B(F,s)) < \alpha$ and $n \ge q_1$, we may insert this inequality to (11.1) to get $H_h(X_\infty^n) \le \varepsilon$. We are done. $\square$

As immediate consequences of Theorem 11.6 we get the following.

**Corollary 11.7.** *With respect to the topology of sub-fine convergence, each conformal Walters map $S$ with $H_{h_T}(X_\infty(S)) = 0$ is a continuity point of the Hausdorff measure function $T \to H_{h_T}(X_\infty(T))$.*

## 12. Conformal Expanding Repellers

Given $\delta, \gamma > 0$ let $\mathrm{CER}_q(\delta, \gamma)$ be the collection of all conformal expending repellers $T$ in $\mathbb{R}^q$ for which $\delta_{\xi_T} \ge \delta$ and $\|T'\|_\infty \le \gamma$; we do not impose any a priori condition on $\xi_T$. We shall prove the following.

**Lemma 12.1.** *Fix $\delta, \gamma > 0$. Then for all $\beta > 0$ and all $\varepsilon > 0$ there exists $\alpha > 0$ such that*

$$\inf\left\{\frac{\mathrm{diam}^{h_T}(E)}{H^1_{h_T}(E)} : \alpha \le \mathrm{diam}(F) \le \beta\right\} \le (1+\varepsilon) H_h(X_\infty(T))$$

*for all $T \in CER_q(\delta, \gamma)$.*

*Proof.* We may assume without loss of generality that $\beta \in (0, \delta/4)$. Fix $\eta > 0$ so small that $(1+\eta)^{2q+1} < \varepsilon$. Let $\eta_* > 0$ be associated to $\eta > 0$ according to Fact 7.1 (Bounded Distortion Property). In virtue of Corollary 11.3 for every $T \in \mathrm{CER}_q(\delta, \gamma)$ there exist $x \in X$ and a closed convex set $F \subseteq X$ with $x \in F$, $\mathrm{diam}(F) \le \beta$ and such that

$$\frac{\mathrm{diam}^{h_T}(F)}{H^1_{h_T}(F)} \le (1+\eta) H_{h_T}(X). \tag{12.1}$$

Let $n \ge 0$ be the largest integer such that

$$\mathrm{diam}(T^n(F)) \le \eta_* \beta < \beta. \tag{12.2}$$

It then follows from the definition of $\beta$ (as less than $\delta/4$) and Fact 7.1 (Bounded Distortion Property) applied to an appropriate continuous inverse branch of $T^n$ mapping $T^n(F)$ onto $F$. that

$$\mathrm{diam}(T^n(F)) \le (1+\eta)|(T^n)'(x)|\,\mathrm{diam}(F)$$

and

$$H^1_{h_T}(T^n(F)) \ge (1+\eta)^{-h_T}|(T^n)'(x)|^{h_T} H^1_{h_T}(F).$$

Hence, using (12.1), we get that

$$\begin{aligned}\frac{\mathrm{diam}^{h_T}(T^n(F))}{H^1_{h_T}(T^n(F))} &\le (1+\eta)^{2h}\frac{\mathrm{diam}^h(F)}{H^1_{h_T}(F)} \le (1+\eta)^{1+2h_T} H_{h_T}(X)\\ &\le (1+\eta)^{2q+1} H_{h_T}(X)\\ &< (1+\varepsilon) H_{h_T}(X).\end{aligned} \tag{12.3}$$

As $T \in \mathrm{CER}_q(\delta, \gamma)$,

$$\eta_*\beta < \mathrm{diam}(T^{n+1}(F)) \le \|T'\|_\infty \,\mathrm{diam}(T^n(F)) \le \gamma\,\mathrm{diam}(T^n(F))$$

Hence $\mathrm{diam}(T^n(F)) \ge \gamma^{-1}\eta_*\beta$. Put $\alpha = \gamma^{-1}\eta_*\beta$. Along with (12.2) and (12.3) this completes the proof. $\square$

Our two main results in this section are the following.

**Theorem 12.2.** *If $(T_n : X^n \to X^n)_1^\infty$ is a sequence of conformal expanding repellers in $\mathbb{R}^q$, converging sub–finely to some conformal Walters map $T : X \to X$, then*

$$\lim_{n\to\infty} H_{h_{T_n}}(X^n) = H_{h_T}(X).$$

*Proof.* Put

$$h_n := h_{T_n} \quad \text{and} \quad h := h_T.$$

First note that there exists a bounded open set $U$ such that $X^n \subset U$ for all $n \geq 1$. In view of Theorem 11.6 , we then only need to show that

$$\liminf_{n\to\infty} H_{h_n}(X^n) \geq H_h(X).$$

If $H_h(X) = 0$, we are obviously done. So, suppose that $H_h(X) > 0$. Denote the normalized Hausdoff measures $H^1_{h_n}$ and $H^1_h$ respectively by $m_{h_n}$ and $m_h$. Because of sub–fine convergence there exist $\delta > 0$ and $\gamma > 0$ such that $T_n \in \mathrm{CER}_q(\delta, \gamma)$ for all $n \geq 1$ sufficiently large, disregarding finitely many of them, we may assume without loss of generality that for all $n \geq 1$. Fix an arbitrary $\varepsilon \in (0, 1)$. In virtue of Corollary 11.5 there exists $\beta \in (0, 1/4)$ so small that

$$\frac{\mathrm{diam}^h(F)}{m_h(F)} \geq (1+\varepsilon)^{-1} H_h(F), \tag{12.4}$$

whenever $F \subseteq X^\infty$ and $\mathrm{diam}(F) \leq 2\beta$. Associate to $\varepsilon$ and $\beta$ the number $\alpha > 0$ according to Lemma 12.1. In view of this lemma and regularity of measure $m_{h_n}$, for every $n \geq 1$ there exists a compact set $E_n \subseteq X^n$ such that

$$\alpha \leq \mathrm{diam}(E_n) \leq \beta$$

and

$$\frac{\mathrm{diam}^{h_n}(E_n)}{m_{h_n}(E_n)} \leq (1+\varepsilon) H_{h_n}(X^n). \tag{12.5}$$

Passing to a subsequence if necessary, we may assume that the limit $\lim_{n\to\infty} H_{h_n}(X^n)$ exists. Then, passing further to a subsequence, we may assume without loss of generality that the sequence $(E_n)_1^\infty$ converges in $\mathcal{K}(\overline{U})$. Let $E = \lim_{n\to\infty} E_n \subset X$. Then $\alpha \leq \mathrm{diam}(E) \leq \beta$. Fix $r \in (0, \beta/4)$ so small that

$$\mathrm{diam}(B(E, 2r)) \leq (1+\varepsilon)\,\mathrm{diam}(E).$$

Then, take $k_1 \geq 1$ so large that

$$\mathrm{diam}(E) \leq (1+\varepsilon)\,\mathrm{diam}(E_n) \quad \text{and} \quad E_n \subseteq B(E, r)$$

for all $n \geq k_1$. We then have for all $n \geq k_1$ that

$$\frac{\mathrm{diam}^{h_n}(E_n)}{m_{h_n}(E_n)} \geq (1+\varepsilon)^{-2h_n} \frac{\mathrm{diam}^{h_n}(B(E, 2r))}{m_{h_n}(B(E, r))}. \tag{12.6}$$

Since $\lim_{n\to\infty} h_n = h$, we have for all $n \geq 1$ large enough, say $n \geq k_2 \geq k_1$, that

$$\mathrm{diam}^{h_n}(B(E,2r)) \geq (1+\varepsilon)^{-1}\mathrm{diam}^h(B(E,2r))$$

and

$$(1+\varepsilon)^{-2h_n} \geq (1+\varepsilon)^{-3h} \geq (1+\varepsilon)^{-3q}.$$

Inserting this into (12.6) and using (12.5), we get that

$$H_{h_n}(X^n) \geq (1+\varepsilon)^{-(3q+2)}\frac{\mathrm{diam}^h(B(E,2r)}{m_{h_n}(B(E,r))}.$$

Now, since the sequence $(m_{h_n})_1^\infty$ converges weakly to the measure $m_{h_\infty}$, there exists $k_3 \geq k_2$ such that $m_{h_n}(B(E,r)) \leq m_h(B(E,2r))$ for all $n \geq k_3$. Hence

$$H_{h_n}(X^n) \geq (1+\varepsilon)^{-(3q+2)}\frac{\mathrm{diam}^h(B(E,2r))}{m_h(B(E,2r))}, \qquad k \geq k_3.$$

Noting that $\mathrm{diam}(B(E,2r)) \leq 2\beta$ and employing finally (12.4), we get that

$$H_{h_n}(X^n) \geq (1+\varepsilon)^{-(3q+3)}H_h(X^\infty)$$

for all $n \geq k_3$. Letting $\varepsilon \to 0$, we thus get that $\liminf_{n\to\infty} H_{h_n}(X^n) \geq H_h(X^\infty)$. This finishes the proof. $\square$

As an immediate consequence of this theorem and Proposition 9.1, we get the following.

**Theorem 12.3.** *The map $T \mapsto H_{h_T}(X(T))$ is continuous on the space of conformal expanding repellers endowed with the topology of $C^2$-uniform convergence.*

There are several immediate consequences of this theorem.

**Corollary 12.4.** *For every $c \in \mathbb{C}$ let $J_c$ be the Julia set of the quadratic polynomial $\mathbb{C} \ni z \to z^2 + c$ and let $h_c = \mathrm{HD}(J_c)$. Then the map $\mathbb{C} \ni c \to H_{h_c}(J_c)$ is continuous at each hyperbolic element $c \in \mathbb{C}$.*

## 13. Parabolic Walters conformal maps

Recall that a finitely Walters conformal map $T : X \to X$ is called parabolic if $\Omega(T) \neq \emptyset$. We then have

$$\#_T := \sup\{\#T^{-1}(z) : z \in X\} < +\infty.$$

For every $\omega \in \Omega(T)$ and $0 < \eta < \theta$ put

$$L_\theta^T(\omega) = \bigcup_{j=1}^{p(\omega)} S_\omega^{j,-}\left(\theta, \theta/2, \frac{\pi}{4p(\omega)}\right)$$

and, for all $n \geq 1$,

$$L^T_{\theta,n}(\omega) := \bigcap_{j=0}^{n} T^{-j}(B(\omega,\theta)) \cap T^{-n}(L^T_\theta(\omega))$$

The choice of the angle $\frac{\pi}{4p(\omega)}$ is entirely arbitrary. In fact any angle in $(0, \frac{\pi}{4p(\omega)})$ would be equally good. Because of requirement (11) of nice convergence, looking at Lemma 9.3, we deduce from Proposition 6.1 the following.

**Lemma 13.1.** *If a sequence $(T_i)_{i=1}^\infty$ of parabolic Walters maps converges nicely to a parabolic Walters map $T$, then for every $\theta > 0$ small enough there exist integers $i_\theta, k_\theta \geq 1$, real numbers $\eta_\theta \in (0,\theta)$, $A_\theta > 1$ and an integer $s = s_\theta \geq 1$ with the following properties. For all $n \geq 1$ put $L^{T_i}_{\theta,n}(\omega) := L^{T_i}_{\eta_\theta,\theta,n}(\omega)$. If $i \geq i_\theta$ and $\omega \in \Omega(T_i)$, then*

(a)

$$X(T_i) \cap B(\omega,\theta) \subseteq \bigcup_{n=0}^{\infty} L^{T_i}_{\theta,n}(\omega).$$

(b) *If $x \in L^{T_i}_{\theta,n}(\omega)$, then*

$$A_\theta^{-1} n^{-\frac{1}{p}} \leq \|x-\omega\| \leq A_\theta n^{-\frac{1}{p}} \tag{13.1}$$

*and*

$$A_\theta^{-1} n^{\frac{p}{p+1}} \leq |(T_i^n)'(x)| \leq A_\theta n^{\frac{p}{p+1}}. \tag{13.2}$$

(c) *If in addition $y \in L^{T_i}_{\theta,k}(\omega)$ and $1 \leq k < n$, then*

$$A_\theta^{-1}(k^{-\frac{1}{p}} - n^{-\frac{1}{p}}) \leq \|y-x\| \leq A_\theta(k^{-\frac{1}{p}} - n^{-\frac{1}{p}}), \tag{13.3}$$

*where the first inequality holds assuming in addition that $n-k \geq s_\theta$.*

Since this is crucial for the subsequent proofs, we want to stress at this moment that constant $A_\theta$ above is indeed independent of $n$. Increasing this constant if necessary, we get from (13.2) and (13.1) that for all $i \geq i_\theta$ and all $n \geq 1$,

$$A_\theta^{-1} n^{-\frac{p+1}{p}} \leq \operatorname{diam}(L^{T_i}_{\theta,n}(\omega)) \leq A_\theta n^{-\frac{p+1}{p}}, \tag{13.4}$$

$$L^{T_i}_{\theta,n}(\omega) \subseteq A(\omega, A_\theta^{-1} n^{-\frac{1}{p}}, A_\theta n^{-\frac{1}{p}}). \tag{13.5}$$

Since the maps $T_i^n|_{L^{T_i}_{\theta,n}}(\omega)$ are *bounded*–to–1 independently of $i$ and $n$, it follows from conformality of the measure $m_{h_{T_i}}$ and from (13.2) that, with possibly larger $A_\theta$,

$$A_\theta^{h_{T_i}} n^{-\frac{p+1}{p}h_{T_i}} \leq m_{h_{T_i}}(L^{T_i}_{\theta,n}(\omega)) \leq A_\theta^{h_{T_i}} n^{-\frac{p+1}{p}h_{T_i}} \tag{13.6}$$

for all $n \geq 1$. On the other hand, it follows from (13.3) that

$$\operatorname{diam}(F) \geq A_\theta^{-1} |n^{-\frac{1}{p}} - k^{-\frac{1}{p}}| \tag{13.7}$$

for every set $F \subseteq \mathbb{R}^q$ such that $F \cap L^{T_i}_{\theta,k}(\omega) \neq \emptyset$, $F \cap L^{T_i}_{\theta,n}(\omega) \neq \emptyset$ and $|n-k| \geq s_\theta$. Now, set

$$Q_{\theta,i} := \left( A_\theta^{2h_{T_i}} \left( \frac{p+1}{p} h_{T_i} - 1 \right) \right)^{-1} \cdot \min \left\{ 2^{-\frac{p+1}{p} h_{T_i}}, (1 - 2^{-\frac{1}{p}})^{h_{T_i}} \right\}.$$

Since $h_T > \frac{p}{p+1}$ (see [ADU], where this was proved for parabolic rational functions, and the same argument continues to hold for parabolic Walters maps), after dropping off finitely many $i$'s if necessary, it follows from condition (7) of sub-fine convergence, which in turn results from nice convergence by virtue of Proposition 9.2, that

$$Q_\theta := \inf_{i \geq 1} \{Q_{\theta,i}\} > 0.$$

**Lemma 13.2.** *Assume that a sequence $(T_i)_{i=1}^\infty$ of parabolic Walters maps converges nicely to a parabolic Walters map $T$ and $h_{T_i} \geq 1$ for all $i \geq 1$. If $\omega \in \Omega(T_i)$ and $F \subseteq \bigcup_{j=k}^n L^{T_i}_{\theta,j}(\omega)$ is an arbitrary set such that $F \cap L^{T_i}_{\theta,k}(\omega) \neq \emptyset$ and $F \cap L^{T_i}_{\theta,n}(\omega) \neq \emptyset$ with $n - k \geq s_\theta$, then*

$$\frac{\operatorname{diam}^{h_i}(F)}{m_{h_{T_i}}(F)} \geq Q_\theta \min{}^{h_i - 1}\{k, n-k\},$$

*where $h_i := h_{T_i}$.*

*Proof.* In view of (13.6) and (13.7) along with the integral comparison test, we have

$$\begin{aligned}
\frac{\operatorname{diam}^{h_i}(F)}{m_{h_i}(F)} &\geq \frac{A_\theta^{-h_i}(k^{-\frac{1}{p}} - n^{-\frac{1}{p}})^{h_i}}{\sum_{j=k}^n m_{h_i}(L_j(\omega))} \geq \frac{A_\theta^{-h_i}(k^{-\frac{1}{p}} - n^{-\frac{1}{p}})^{h_i}}{\sum_{j=k}^n A_\theta^{h_i} j^{-\frac{p+1}{p}h_i}} \\
&\geq A_\theta^{-2h_i} \frac{(k^{-\frac{1}{p}} - n^{-\frac{1}{p}})^{h_i}}{\sum_{j=k}^n j^{-\frac{p+1}{p}h_i}} \qquad (13.8) \\
&\geq A_\theta^{-2h_i} \left( \frac{p+1}{p} h_i - 1 \right) \frac{(k^{-\frac{1}{p}} - n^{-\frac{1}{p}})^{h_i}}{(k^{1-\frac{p+1}{p}h_i} - n^{1-\frac{p+1}{p}h_i})}.
\end{aligned}$$

Now, let us estimate the quantity

$$\frac{(\frac{1}{k^{1/p}} - \frac{1}{n^{1/p}})^{h_i}}{(k^{1-\frac{p+1}{p}h_i} - n^{1-\frac{p+1}{p}h_i})}$$

in the following two ways.

**Case 1.** $n \leq 2k$. Applying the Mean Value Theorem, we get two real numbers $k \leq a, b \leq n$ such that

$$k^{-\frac{1}{p}} - n^{-\frac{1}{p}} = \frac{1}{p} a^{-\frac{1}{p}-1}(n-k) \tag{13.9}$$

and

$$k^{1-\frac{p+1}{p}h_i} - n^{1-\frac{p+1}{p}h_i} = \left( \frac{p+1}{p} h_i - 1 \right) b^{-\frac{p+1}{p}h_i}(n-k). \tag{13.10}$$

Hence

$$\begin{aligned}\frac{(k^{-\frac{1}{p}}-n^{-\frac{1}{p}})^{h_i}}{k^{1-\frac{p+1}{p}h_i}-n^{1-\frac{p+1}{p}h_i}} &= \left(p^{h_i}\left(\frac{p+1}{p}h_i-1\right)\right)^{-1}\frac{b^{\frac{p+1}{p}h_i}}{a^{(\frac{1}{p}+1)h_i}}(n-k)^{h_i-1}\\ &= \left(p^{h_i}\left(\frac{p+1}{p}h_i-1\right)\right)^{-1}\left(\frac{b}{a}\right)^{\frac{p+1}{p}h_i}(n-k)^{h_i-1}\\ &\geq 2^{-\frac{p+1}{p}h_i}\left(p^{h_i}\left(\frac{p+1}{p}h_i-1\right)\right)^{-1}(n-k)^{h_i-1}.\end{aligned}$$

**Case 2.** $n \geq 2k$.

$$\begin{aligned}\frac{(k^{-\frac{1}{p}}-n^{-\frac{1}{p}})^{h_i}}{k^{1-\frac{p+1}{p}h_i}-n^{1-\frac{p+1}{p}h_i}} = \frac{k^{-\frac{h_i}{p}}(1-(\frac{k}{n})^{\frac{1}{p}})^{h_i}}{k^{1-\frac{p+1}{p}h_i}(1-(\frac{k}{n})^{\frac{p+1}{p}h_i-1})} &\geq k^{h_i-1}\left(1-\left(\frac{k}{n}\right)^{\frac{1}{p}}\right)^{h_i}\\ &\geq (1-2^{-\frac{1}{p}})^{h_i}k^{h_i-1}.\end{aligned}$$

These two cases along with (13.8) yield

$$\frac{\mathrm{diam}^{h_i}(F)}{m_{h_i}(F)} \geq Q_\theta \mathrm{min}^{h_i-1}\{k, n-k\}.$$

The proof is complete. □

Having this lemma proved, we can establish a parabolic counterpart of Lemma 12.1.

**Lemma 13.3.** *Assume that $(T_n : X^n \to X^n)_1^\infty$ is a sequence of parabolic Walters conformal maps converging nicely to some parabolic Walters conformal map $T : X \to X$. If $h = h_T > 1$, then $\forall\, \varepsilon > 0\, \forall\, \beta > 0\, \exists\, \alpha > 0\, \exists\, j \geq 1\, \forall\, n \geq j$*

$$\inf\left\{\frac{\mathrm{diam}^{h_n}(E)}{m_{h_n}(E)} : \alpha \leq \mathrm{diam}(E) \leq \beta\right\} \leq (1+\varepsilon)H_{h_n}(X^n).$$

*where $h_n := h_{T_n}$.*

*Proof.* Assume without loss of generality that $\varepsilon < 1$. Take $\eta > 0$ so small that $(1+\eta)^{2q+1} < 1+\varepsilon$. Let $\eta_* > 0$ be associated to $\eta > 0$ according to Fact 7.1 (Bounded Distortion Property). Since $(T_n)_1^\infty$ converges strongly to $T$, we have

$$\xi := \inf\{\xi_n : n \geq 1\} > 0, \quad \delta := \inf\{\delta_n : n \geq 1\} > 0, \text{ and } \gamma := \sup\{||T_n'||_\infty : n \geq 1\} < +\infty.$$

Fix $\theta \in (0, \xi/2)$. We may assume without loss of generality that $\beta < \frac{1}{4}\{1, \delta_\theta\}$. In virtue of Corollary 11.3 for every $n \geq 1$ there exist $x_n \in X^n$ and closed convex set $F_n \subseteq X^n$ such that $x_n \in F_n$, $\mathrm{diam}(F_n) \leq \beta$ and

$$\frac{\mathrm{diam}^{h_n}(F_n)}{m_{h_n}(F_n)} \leq (1+\eta)H_{h_n}(X^n). \tag{13.11}$$

On the other hand, in virtue of Theorem 11.6 there exists $n_1 \geq 1$ such that

$$H_{h_n}(X_n) \leq (1+\eta)H_h(X) \tag{13.12}$$

for all $n \geq n_1$. Let

$$\kappa := (h_T - 1)/2 > 0,$$

let

$$Z_\theta := (Q_\theta^{-1} 4^{q+1} H_{h_\infty}(X))^{1/\kappa}, \quad u := \max\{s_\theta, Z_\theta\}, \quad \text{and} \quad \hat{\eta} := K^{-1}(1+\eta),$$

where the function $K : [0,1) \to [1, \infty)$ comes from Lemma 9.4 (distortion estimate) and $h = \mathrm{HD}(X)$. Finally, let $\tilde{\eta} \in (0, \eta_*]$ be so small that

$$(A_\theta \beta)^{-p} \left(\frac{\hat{\eta}}{\tilde{\eta}}\right)^p - u > A_\theta^2 \hat{\eta}^{-1} (1+u)^{1/p}. \tag{13.13}$$

Given $n \geq n_1$, let $j_n \geq 0$ be the largest integer such that

$$\mathrm{diam}(T_n^k(F_n)) \leq \tilde{\eta}\beta \tag{13.14}$$

for all $0 \leq k \leq j_n$. We then have,

$$\eta_* \beta < \mathrm{diam}(T_n^{j_n+1}(F_n)) \leq \|T_n'\|_\infty \mathrm{diam}(T_n^{j_n}(F_n)) \leq 2\|T'\|_\infty \mathrm{diam}(T_n^{j_n}(F_n)).$$

Hence,

$$\mathrm{diam}(T_n^{j_n}(F_n)) \geq (2\|T'\|_\infty)^{-1} \tilde{\eta}\beta. \tag{13.15}$$

Consider now two cases. Assume first that $T_n^{j_n}(F) \cap B^c(\Omega(T_n), \theta) \neq \emptyset$. It then follows from (13.14), the fact that $\tilde{\eta} \leq \eta_*$, and the choice of $\beta$, that Fact 7.1 applies to the inverse branch $T_{n,x_n}^{-j_n} : B(T_n^{j_n}(x_n), \tilde{\eta}\delta) \to \mathbb{R}^q$, to give

$$\frac{\mathrm{diam}^{h_n}(T_n^{j_n}(F_n))}{m_{h_n}(T_n^{j_n}(F_n))} \leq \frac{(1+\eta)^{h_n} \mathrm{diam}^{h_n}(F_n)}{(1+\varepsilon)^{-h_n} m_{h_n}(F_n)} = (1+\eta)^{2h_n} \frac{\mathrm{diam}^{h_n}(F_n)}{m_{h_n}(F_n)} \leq (1+\eta)^{2q} \frac{\mathrm{diam}^{h_n}(F_n)}{m_{h_n}(F_n)}.$$

Along with (13.11) this gives that

$$\frac{\mathrm{diam}^{h_n}(T_n^{j_n}(F_n))}{m_{h_n}(T_n^{j_n}(F_n))} \leq (1+\eta)^{2q+1} H_{h_n}(X^n) < (1+\varepsilon) H_{h_n}(X^n). \tag{13.16}$$

Together with (13.15), which determines $\alpha$, this finishes the proof in our first case. So, suppose on the other hand that

$$T_n^{j_n}(F_n) \subseteq B(\omega, \theta)$$

with some $\omega \in \Omega(T_n)$. Let $0 \leq i_n \leq j_n$ be the least integer such that

$$T_n^k(F_n) \subseteq B(\omega, \theta)$$

for all $i_n \leq k \leq j_n$. Then $T_n^{i_n-1}(F_n) \cap B^c(\Omega(T_n), \theta) \neq \emptyset$, and as $\tilde{\eta} \leq \eta_*$ it follows from (13.14) that Fact 7.1 applies for the inverse branch $T_{n,x_n}^{-(i_n-1)} : B(T_n^{i_n-1}(x_n), \eta_*\delta) \to \mathbb{R}^q$, to give (comparing it with $T_n$ once)

$$(1+\eta)^{-2q} \frac{\mathrm{diam}^{h_n}(T_n^{i_n}(F_n))}{m_{h_n}(T_n^{i_n}(F_n))} \leq (1+\eta)^{-2h_n} \frac{\mathrm{diam}^{h_n}(T_n^{i_n}(F_n))}{m_{h_n}(T_n^{i_n}(F_n))} \leq \frac{\mathrm{diam}^{h_n}(F_n)}{m_{h_n}(F_n)} \tag{13.17}$$

and, in view of (13.11) and (13.12),

$$\frac{\mathrm{diam}^{h_n}(T_n^{i_n}(F_n))}{m_{h_n}(T_n^{i_n}(F_n))} \leq (1+\eta)^{2q+1} H_{h_n}(X^n) \leq (1+\eta)^{2(q+1)} H_h(X). \tag{13.18}$$

Let

$$k_n = \min\{j \geq 1 : L_{\theta,j}^{T_n}(\omega) \cap T_n^{i_n}(F_n) \neq \emptyset\} \quad \text{and} \quad l_n = \sup\{j \geq 1 : L_{\theta,j}^{T_n}(\omega) \cap T_n^{i_n}(F_n) \neq \emptyset\}.$$

Consider two cases. First assume that

$$l_n - k_n \geq s_\theta. \tag{13.19}$$

It then follows from Lemma 13.2 that

$$\frac{\operatorname{diam}^{h_n}(T_n^{i_n}(F_n))}{m_{h_n}(T_n^{i_n}(F_n))} \geq Q_\theta \min{}^{h_n-1}\{k_n, l_n - k_n\} \geq Q_\theta \min{}^{\kappa}\{k_n, l_n - k_n\}$$

for all $n \geq n_2 \geq n_1$ so large that $h_n - 1 > \kappa > 0$. Applying (13.18), we thus get

$$Q_\theta \min{}^{\kappa}\{k_n, l_n - k_n\} \leq (1+\eta)^{2(q+1)} H_h(X) \leq 4^{q+1} H_h(X),$$

or equivalently

$$\min\{k_n, l_n - k_n\} \leq Z_\theta.$$

Considering two subcases assume first that $k_n \leq Z_\theta$. It then follows from (13.19), (13.7), and (13.9), that

$$\operatorname{diam}(T_n^{i_n}(F_n)) \geq A_\theta^{-1}(k_n^{-1/p} l_n^{-1/p}) \geq \frac{1}{p} Z_\theta^{-\frac{p+1}{p}}(l_n - k_n) \geq \frac{s_\theta}{p} Z_\theta^{-\frac{p+1}{p}}.$$

Thus, invoking (13.18), we are done in this case too. Now, consider jointly the, whatsoever, remaining case

$$l_n - k_n \leq u := \max\{s_\theta, Z_\theta\}. \tag{13.20}$$

Let

$$k_n^* := \min\{a \geq 1 : L_{\theta,a}^{T_n}(\omega) \cap T_n^{j_n}(F_n) \neq \emptyset\}$$

and

$$l_n^* := \sup\{a \geq 1 : L_{\theta,a}^{T_n}(\omega) \cap T_n^{j_n}(F_n) \neq \emptyset\}.$$

Then $k_n^* = k_n + (j_n - i_n)$, $l_n^* = l_n + (j_n - i_n)$, and it follows from (13.20) that

$$l_n^* - k_n^* \leq u. \tag{13.21}$$

Now, if

$$\operatorname{diam}(T_n^{j_n}(F_n)) \leq \hat{\eta}\operatorname{dist}(\omega, T_n^{j_n}(F_n)), \tag{13.22}$$

It then follows from Lemma 9.4 that

$$\begin{aligned}\frac{\operatorname{diam}^{h_n}(T_n^{j_n}(F_n))}{m_{h_n}(T_n^{j_n}(F_n))} &\leq \frac{(1+\eta)^{h_n}|(T_n^{j_n})'(x_n)|^{h_n}\operatorname{diam}^{h_n}(F_n)}{(1+\eta)^{-h_n}|(T_n^{j_n})'(x_n)|^{h_n} m_{h_n}(F_n)} = (1+\eta)^{2h_n}\frac{\operatorname{diam}^{h_n}(F_n)}{m_{h_n}(F_n)} \\ &\leq (1+\eta)^{2q}\frac{\operatorname{diam}^{h_n}(F_n)}{m_{h_n}(F_n)}.\end{aligned}$$

Inserting this to (13.11) we thus get that

$$\frac{\operatorname{diam}^{h_n}(T_n^{j_n}(F_n))}{m_{h_n}(T_n^{j_n}(F_n))} \leq (1+\varepsilon)^{2q+1} H_{h_n}(X^n) < (1+\varepsilon) H_{h_n}(X^n).$$

Along with (13.15) this finishes the proof if (13.22) holds. So, assume finally that

$$\operatorname{diam}(T_n^{j_n}(F_n)) \geq \hat{\eta}\operatorname{dist}(\omega, T_n^{j_n}(F_n)). \tag{13.23}$$

Cover $T_n^{j_n}(F_n)$ by the sets $L_{\theta,k_n^*}^{T_n}(\omega), L_{\theta,k_n^*+1}^{T_n}(\omega), \ldots, L_{\theta,l_n^*}^{T_n}(\omega)$, i.e. write

$$T_n^{j_n}(F_n) = \bigcup_{a=k_n^*}^{l_n^*} (T_n^{j_n}(F_n) \cap L_{\theta,a}^{T_n}(\omega)). \tag{13.24}$$

Since $L_{\theta,b}(\omega) \cap L_{\theta,b+1}(\omega) \neq \emptyset$ for all $b \geq 0$ for all $b \geq 0$ (this can be assured by taking $0 < \eta < \theta$ sufficiently small), we get from (13.24), (13.4) and (13.21), that

$$\operatorname{diam}(T_n^{j_n}(F_n)) \leq \sum_{a=k_n^*}^{l_n^*} \operatorname{diam}(L_{\theta,a}^{T_n}(\omega)) \leq \sum_{a=k_n^*}^{l_n^*} A_\theta a^{-\frac{p+1}{p}} \leq A_\theta u (k_n^*)^{-\frac{p+1}{p}}. \tag{13.25}$$

On the other hand, it follows from (13.23), (13.24) and (13.27) that

$$\operatorname{diam}(T_n^{j_n}(F_n)) > \hat{\eta} A_\theta^{-1} (l_n^*)^{-\frac{1}{p}} \geq \hat{\eta} A_\theta^{-1} (k_n^* + u)^{-\frac{1}{p}}. \tag{13.26}$$

Combining this with (13.25), we obtain $A_\theta^{-1} \hat{\eta} (k_n^* + U)^{-\frac{1}{p}} \leq A_\theta u (k_n^*)^{-\frac{p+1}{p}}$ or equivalently

$$k_n^* \leq A_\theta^2 \hat{\eta}^{-1} (1 + u/k_n^*)^{\frac{1}{p}}. \tag{13.27}$$

On the other hand, combining (13.26) with (13.14), we get that $\hat{\eta} A_\theta^{-1} (k_n^* + u)^{-\frac{1}{p}} \leq \tilde{\eta}\beta$, or equivalently,

$$k_n^* \geq (A_\theta \beta)^{-p} \left( \frac{\hat{\eta}}{\tilde{\eta}} \right)^p - u.$$

However, along with (13.27) this contradicts (13.13) ruling out the case under consideration and finishing the proof. $\square$

Having this lemma we can repeat the proof of Theorem 12.2 verbatim to get the following.

**Theorem 13.4.** *If $(T_n : X^n \to X^n)_1^\infty$ is a sequence of conformal parabolic Walters maps converging nicely to some conformal parabolic Walters map $T : X \to X$ for which $\mathrm{HD}(X) > 1$, then $\lim_{n\to\infty} H_{h_{T_n}}(X^n) = H_{h_T}(X)$.*

Combining this theorem with Theorem 12.2, we get this.

**Theorem 13.5.** *If $(T_n : X^n \to X^n)_1^\infty$ is a sequence of finitely conformal Walters maps converging finely to some finitely conformal Walters map $T : X \to X$ for which $\mathrm{HD}(X) > 1$, and if its subsequence of all parabolic maps converges nicely to $T$, then $\lim_{n\to\infty} H_{h_{T_n}}(X^n) = H_{h_T}(X)$.*

As a fairly immediate consequences of this theorem, we get the following.

**Corollary 13.6.** *For every $c \in \mathbb{C}$ let $J_c$ be the Julia set of the quadratic polynomial $\mathbb{C} \ni z \to f_c(z) = z^2 + c$ and let $h_c = \mathrm{HD}(J_c)$. Then*

$$\lim_{\mathbb{R}\ni c\nearrow 1/4} H_{h_c}(J_c) = H_{h_{1/4}}(J_{1/4}).$$

*Proof.* Indeed, all the maps $f_c$ with $c \in [0, 1/4)$ are conformal expanding repellers when restricted to their Julia sets and, it was established in [BZ] (comp. Proposition 9.2) that $(f_{c_n} : J_{c_n} \to J_{c_n})_1^\infty$ converges finely to $f_{1/4} : J_{1/4} \to J_{1/4}$ if $\mathbb{R} \ni c_n \nearrow 1/4$. It was proved in [Z] and [U1] that $\mathrm{HD}(J_{1/4}) > 1$. So, a direct application of Theorem 12.2 finishes the proof. $\square$

**Corollary 13.7.** *For every $\lambda \in \mathbb{C} \setminus \{0\}$ let*

$$f_\lambda(z) = z(1 - z - \lambda^2 z).$$

*Let $J_\lambda := J(f_\lambda)$ be the Julia set of $f_\lambda$ and let $h_\lambda := \mathrm{HD}(J_{f_\lambda})$. Then for $R > 0$ sufficiently small, the function*

$$D_*(0, R) := \{\lambda \in \mathbb{C} \setminus \{0\} : |\lambda| < R\} \ni \lambda \to H_{h_\lambda}(J_\lambda)$$

*is continuous.*

*Proof.* The fact that the map $D_*(0, R) \ni \lambda \to f_\lambda$ is continuous with respect to the nice convergence topology was essentially proved in [AU]. The fact $\mathrm{HD}(J_\lambda) > 1$ follows from [U1]. So, the direct application of Theorem 13.4 finishes the proof. $\square$

## 14. Graph Directed Markov Systems; Preliminaries

Suppose we are given an oriented multigraph $< E, V >$ consisting of countably many edges $E$ and finitely many vertices $V$. Suppose also that an incidence matrix $A : E\times E \to \{0, 1\}$ is given. Any finite word $\omega \in E^* = \bigcup_{n=0}^\infty E^n$ is called $A$-admissible provided that $A_{\omega_i\omega_{i+1}} = 1$ for all $1 \le i \le |\omega| - 1$, where $|\omega|$ is the length of $\omega$. The set of all finite $A$-admissible words is denoted by $E_A^*$ and the set of all words of some length $0 \le n \le \infty$ is denoted by $E_A^n$. The matrix $A$ is called finitely irreducible if there exists a finite set $\lambda \subseteq E_A^*$ such that for all $\alpha, \beta \in E_A^*$ there exists $\gamma \in \lambda$ such that $\alpha\beta\gamma \in E_A^*$. The matrix $A$ is called finitely primitive if the set $\lambda$ can be chosen to consist of the words with the same length. Assume further that an integer number $d \ge 1$ is fixed and for every $v \in V$ a compact connected set $X_v \subseteq \mathbb{R}^d$ is given, and an open connected set $W_v \supseteq X_v$ is also given. Assume also that two functions $i, t : E \to V$ are given with the property that $A_{ab} = 1$ whenever $t(a) = i(b)$. In most known natural examples this implication goes in fact in both directions, but we do not assume this. Assume lastly that for every $e \in E$ a continuous injective map $\varphi_e : W_{t(e)} \to \mathbb{R}^d$ is given. Fix also a finite set $\Omega \subseteq E$ such that $t(e) = i(e)$ for all $e \in \Omega$. Call a word $\omega \in E_A^*$ hyperbolic if either $\omega_{|\omega|} \notin \Omega$ or $\omega_{|\omega|-1} \neq \omega_{|\omega|}$ and $\omega_{|\omega|} \in \Omega$. All the objects introduced above are required to satisfy the following conditions.

(2a) $X_v = \overline{\operatorname{Int} X_v}$ for all $v \in V$.

(2b) $\varphi_e(X_{t(e)}) \subseteq X_{i(e)}$ for all $e \in E$. this enables us to define for every $\omega \in E_A^*$, say $\omega \in E_A^n$, the map $\varphi_\omega := \varphi_{\omega_1} \circ \varphi_{\omega_2} \circ \ldots \circ \varphi_{\omega_n} : X_{t(\omega_n)} \to X_{i(\omega_1)}$. Put also $t(\omega) = t(\omega_n)$ and $i(\omega) = i(\omega_1)$.

(2c) (Open Set Condition) $\varphi_a(\operatorname{Int} X_{t(a)}) \bigcap \varphi_b(\operatorname{Int} X_{t(b)}) = \emptyset$ whenever $a, b \in E$ and $a \neq b$.

(2d) (Cone Property) There exists $\gamma > 0$ such that for every $v \in V$ and for every $x \in X_v$ there exists an open cone $Cone(x, \gamma) \subseteq \operatorname{Int} X_v$ with vertex $x$, central angle $\gamma$ and some altitude $l$ which may depend on $x$.

(2e) If $\omega \in E_A^*$ is a hyperbolic word, then $\varphi_\omega : X_{t(\omega)} \to X_{i(\omega)}$ extends to a $C^2$-conformal map from $W_{t(\omega)}$ to $W_{i(\omega)}$. This conformal map is defined by the same symbol $\varphi_\omega$.

(2f) (Bounded Distortion Property) There exists $K \geq 1$ such that for every hyperbolic word $\omega \in E_A^*$ and all $x, y \in W_{t(\omega)}$,

$$\frac{|\varphi_\omega'(y)|}{|\varphi_\omega'(x)|} \leq K.$$

Here and in the sequel for any conformal mapping $\varphi$, $|\varphi'(z)|$ denotes the similarity factor (equivalently its norm as a linear map from $\mathbb{R}^d$ into $\mathbb{R}^d$) of the differential $\varphi'(z) : \mathbb{R}^d \to \mathbb{R}^d$. In addition, if $\varphi : W_v \to \mathbb{R}^d$ for some $v \in V$, then

$$||\varphi'|| := \sup\{|\varphi'(x)| : x \in W_{t(\omega)}.$$

(2g) There are constants $\alpha > 0$ and $L \geq 1$ such that

$$\big||\varphi_e'(y)| - |\varphi_e'(x)|\big| \leq ||\varphi_e'||||y - x||^\alpha$$

for all $e \in E$ and all $x, y \in W_{t(e)}$.

(2h) For every hyperbolic word $\omega \in E_A^*$, $||\varphi_\omega'|| < 1$.

(2i) For every $e \in \Omega$, $t(e) = i(e)$ and there exists a unique fixed point $x_e$ of the map $\varphi_e : X_{t(e)} \to X_{i(e)}$. In addition, $|\varphi_e'(x_e)| = 1$.

(2j) For every $e \in \Omega$,

$$\lim_{n\to\infty} \operatorname{diam}(\varphi_{e^n}(X_{t(e)})) = 0.$$

This implies that

$$\bigcap_{n=0}^{\infty} \varphi_{e^n}(X_{t(e)}) = \{x_e\}.$$

Any system $S$ satisfying the above conditions is called a conformal graph directed Markov system. If $\Omega = \emptyset$ the system $S$ is called hyperbolic and if $\Omega \neq \emptyset$, it is called parabolic. The set $\Omega$ is referred to as the set of parabolic vertices, the maps $\varphi_e$, $e \in \Omega$, are called parabolic maps, and $x_e$, $e \in \Omega$, are called parabolic fixed points. We could have in principle provided a somewhat less restrictive definition of a parabolic graph directed Markov system allowing finitely many parabolic periodic points (fixed points of $\varphi_\omega$, $\omega \in E_A^*$) that are not necessarily fixed points, but then passing to a sufficiently large iterate $S^n = \{\varphi_\omega : \omega \in E_A^*\}$ we would end up in a parabolic system as described above. Notice also that our assumptions imply each map $\varphi_\omega : X_{t(\omega)} \to X_{i(\omega)}$ such that $i(\omega) = t(\omega)$ to have a unique fixed point, call it $x_\omega$, and that the diameters $\operatorname{diam}(\varphi_\omega^n(X_{t(\omega)}))$ converge to zero exponentially fast unless $\omega \in \Omega^*$.

It is not difficult to prove (the same argument as in the proof of Lemma 8.1.2 in [MU3] goes through) that

$$\lim_{n\to\infty}\sup_{\omega\in E_A^n}\{\mathrm{diam}(\varphi_\omega(X_{t(\omega)}))\}=0. \tag{14.1}$$

Since for every $\omega\in E_A^\infty$, $\{\varphi_{\omega|_n}(X_{t(\omega)})\}_{n=1}^\infty$ is a descending sequence of compact sets, this implies that the intersection $\bigcap_{n=1}^\infty\varphi_{\omega|_n}(X_{t(\omega_n)})$ is a singleton. Call its only element $\pi(\omega)$. We thus have a well-defined map

$$\pi:E_A^\infty\to X:=\bigcup_{v\in V}X_v.$$

Fixing $s>0$ and endowing $E_A^\infty$ with the metric $d_s(\omega,\tau)=\exp(-s|\omega\wedge\tau|)$, where $\omega\wedge\tau$ is the longest common initial subword of $\omega$ and $\tau$, the map $\pi:E_A^\infty\to X$ becomes uniformly continuous. Its image, $\pi(E_A^\infty)$, is called the limit set of the attractor of the graph directed Markov system $S$, and is denoted by $J_s$ or simply by $J$ if only one system is under consideration. It satisfies the equation

$$J=\bigcup_{e\in E}\varphi_e(J\cap X_{t(e)}).$$

A conformal graph directed Markov system $S$ is referred to as a conformal iterated function system if the set of vertices $V$ is a singleton and the incidence matrix $A$ consists of $1s$ only.

A conformal graph directed system $S=\{\varphi_e\}_{e\in E}$ is said to satisfy the separation condition if

$$\varphi_a(X)\cap\overline{\bigcup_{b\in E\setminus\{a\}}\varphi_b(X)}=\emptyset$$

for all $a\in E$. Then a global map $T:\bigcup_{e\in E}\varphi_e(X)\to X$ is well-defined, given by the formula

$$T(\varphi_e(x))=x,\qquad x\in X.$$

Since $T(J_S)=J_S$, we have also $T(\overline{J}_S)=\overline{J}_S$. By the separation condition, $T^{-1}(\overline{J}_S)=\bigcup_{e\in E}\varphi_e(\overline{J}_S)\subset\overline{J}_S$ is an open set of $\overline{J}_S$. Observe also that

$$\overline{T^{-1}(\overline{J}_S)}=\overline{\bigcup_{e\in E}\varphi_e(\overline{J_S})}\supseteq\overline{\bigcup_{e\in E}\varphi_e(J_S)}=\overline{J}_S.$$

So,

$$\overline{T^{-1}(\overline{J}_S)}=\overline{J}_S,$$

and we may regard the transformation $T:\overline{T^{-1}(\overline{J}_S)}\to\overline{J}_S$ as a conformal Walters map with $X=\overline{J}_S$ and $X_0=T^{-1}(\overline{J}_S)$. The Walters conformal map $T:\overline{T^{-1}(\overline{J}_S)}\to\overline{J}_S$ is then expanding if the original system $S$ was hyperbolic and it is parabolic if the original system $S$ was parabolic.

## 15. Finite Graph Directed Markov Systems; Continuity of Hausdorff Measure

Let $E$ be a finite set, let $A : E \times E \to \{0,1\}$ be a primitive incidence matrix, and let $X$ be a compact connected subset of $\mathbb{R}^q$ such that $\overline{\operatorname{Int} X} = X$. Let $\mathrm{CGDMS}(X,E,A)$ be the collection of all contracting conformal graph directed Markov systems modelled on the alphabet $E$ with the incidence matrix $A$ and the phase space $X$. The Walters conformal map associated to each member of $\mathrm{CGDMS}(X,E,A)$ is a conformal expanding repeller. The space $\mathrm{CGDMS}(X,E,A)$ is endowed with metric $d$ given by the following form

$$d(\Phi,\Psi) = \sum_{e\in E}(\|\varphi_e - \psi_e\|_\infty + \|\varphi_e' - \psi_e'\|_\infty) \tag{15.1}$$

The topology induced by the metric $d$ is called the topology of uniform convergence. The subset of $\mathrm{CGDMS}(X,E,A)$ consisting of all its elements satisfying the separation condition is an open set. As was proved in [1] and [2], the convergence, with respect to this topology, of elements of $\mathrm{CGDMS}(X,E,A)$ to an element satisfying the separation condition, entails the sub-fine convergence of Walters expanding maps associated with them. So, as an immediate consequence of Theorem 12.2 we get the following.

**Theorem 15.1.** *If $E$ is a finite set and $A : E \times E \to \{0,1\}$ is a primitive incidence matrix, then each contracting conformal graph directed Markov system $S \in \mathrm{CGDMS}(X,E,A)$ satisfying the separation condition (if $E$ is finite this simply means that if $a,b \in E$ and $a \neq b$, then $\varphi_a(X) \cap \varphi_b(X) = \emptyset$) is a continuity point of the Hausdorff measure function $\mathrm{CIFS}(X,E,A) \ni \Phi \to H_{h_\Phi}(J_\Phi)$ with $\mathrm{CIFS}(X,E,A)$ endowed with the metric $d$ given by formula (15.1).*

Let $\mathrm{SGDMS}(X,E,A)$ denote the subspace of $\mathrm{CGDMS}(X,E,A)$ consisting of all similarities. As a direct application of Theorem 15.1, we get the following.

**Corollary 15.2.** *If $E$ is a finite set and $A : E\times E \to \{0,1\}$ is a primitive incidence matrix, then each contracting graph directed Markov system $S \in \mathrm{SGDMS}(X,E,A)$, so consisting of similarities, satisfying separation condition is a continuity point of the Hausdorff measure function $\mathrm{SGDMS}(X,E,A) \ni \Phi \to H_{h_\Phi}(J_\Phi)$.*

If all entries of $A$ are 1s only (the case of iterated function systems) we write $\mathrm{CIFS}(X,E)$ for $\mathrm{CGDMS}(X,E,A)$ and $\mathrm{SIFS}(X,E)$ for $\mathrm{SGDMS}(X,E,A)$. The two special but very important special cases respectively of Theorem 15.1 and Corollary 15.2 are these.

**Theorem 15.3.** *If $E$ is a finite set, then each contracting conformal iterated function system $S \in \mathrm{CIFS}(X,E,A)$ satisfying the separation condition is a continuity point of the Hausdorff measure function $\mathrm{CIFS}(X,E) \ni \Phi \to H_{h_\Phi}(J_\Phi)$ with $\mathrm{CIFS}(X,E)$ endowed with the metric $d$ given by formula (15.1).*

and

**Corollary 15.4.** *If $E$ is a finite set, then each contracting iterated function system $S \in$ SIFS$(X,E)$, so consisting of similarities, satisfying the separation condition is a continuity point of the Hausdorff measure function* SIFS$(X,E) \ni \varPhi \to H_{h_\varPhi}(J_\varPhi)$.

This corollary is the result proved by L. Olsen in [Ol].

## 16. Infinite Iterated Function Systems

In this section we shall describe a class of conformal infinite Walters expanding maps the Hausdorff measure function restricted to which is continuous. It will be more convenient for us to use the language of iterated function systems. Let $X \subseteq \mathbb{R}$ be a closed bounded interval. Given $\kappa \in (0,1)$, $\gamma > 1$, an integer $l \geq 0$ and $\xi \in (0,1]$ such that $(1-\kappa^\xi)\gamma^\xi > 1$ let SIFS$(X;\kappa,\gamma,l,\xi)$ be the collection of all conformal hyperbolic iterated function systems $\mathcal{S} = \{\varphi_i\}_{i\in\mathbb{N}}$ acting on $X$ and consisting of similarities with the following properties.

(a) $\mathrm{diam}(\varphi_{n+1}(X)) \leq \kappa\,\mathrm{diam}(\varphi_n(X))$ for all $n \geq l$.
(b) $\max(\varphi_{n+1}(X)) < \min(\varphi_n(X))$ for all $n \geq l$.
(c) $(\max(\varphi_{n+1}(X)), \min(\varphi_n(X)) \cap \bigcup_{j\in\mathbb{N}\setminus\{n,n+1\}} \varphi_j(X) = \emptyset$ for all $n \geq l$.
(d) $\min(\varphi_n(X)) - \max(\varphi_{n+1}(X)) \geq \gamma\,\mathrm{diam}(\varphi_n(X))$ for all $n \geq l$.
(e) $\mathrm{HD}(J_\mathcal{S}) \geq \xi$.

The main result of this section is the following.

**Theorem 16.1.** *Consider $\kappa,\gamma,l,\xi$ with $(1-\kappa^\xi)\gamma^\xi > 1$. Then the function*

$$\mathrm{SIFS}(X;\kappa,\gamma,l,\xi) \ni S \mapsto H_{h_\mathcal{S}}(J_S)$$

*is continuous with the topology of fine convergence on* SIFS$(X;\kappa,\gamma,l,\xi)$.

*Proof.* Suppose that $(S_n)_1^\infty$ converges finely to $S_\infty$ in SIFS$(X;\kappa,\gamma,l,\xi)$. Apply Corollary 11.3 for the set $X$ becoming $J_{\mathcal{S}_h}$. It yields that for every $n \geq 1$ there exists a finite word $\omega^{(n)} \in \mathbb{N}^*$ long enough so that

$$\frac{\mathrm{diam}^{h_n}(\varphi^{(n)}_{\omega^{(n)}}(X))}{m_{h_n}(\varphi^{(n)}_{\omega^{(n)}}(X))} > ((1-\kappa^\xi)\gamma^\xi)^{-1} H_{h_n}(J_{\mathcal{S}_n}).$$

Since the all the maps $\varphi_{\omega^{(n)}}$ are similarities, we therefore get that

$$\frac{\mathrm{diam}^{h_n}(X)}{m_{h_n}(X)} = \frac{\mathrm{diam}^{h_n}(\varphi^{(n)}_{\omega^{(n)}}(X))}{m_{h_n}(\varphi^{(n)}_{\omega^{(n)}}(X))} > ((1-\kappa^\xi)\gamma^\xi)^{-1/2} H_{h_n}(J_{\mathcal{S}_n}). \tag{16.1}$$

Consider now an arbitrary set $F \subseteq X$ such that

$$F \cap \bigcup_{j=1}^{l-1} \varphi_j^{(n)}(X) = \emptyset \quad \text{and} \quad \#\{j \in \mathbb{N} : \varphi_j^{(n)}(X) \cap F \neq \emptyset\} \geq 2.$$

Let $p_n = \min\{j : \varphi_j^{(n)}(X) \cap F \neq \emptyset\} \geq 2$ and let $q_n = \sup\{j : \varphi_j^{(n)}(X) \cap F \neq \emptyset\}$. Then $q_n > p_n$ and, using (a), we get that

(16.2)

$$\begin{aligned} m_{h_n}(F) &\leq \sum_{j=p_n}^{q_n} m_{h_n}(\varphi_j^{(n)}(X)) = \sum_{j=p_n}^{q_n} \|(\varphi_j^{(n)})'\|^{h_n} = (\operatorname{diam}(X))^{-h_n} \sum_{j=p_n}^{q_n} \operatorname{diam}^{h_n}(\varphi_j^{(n)}(X)) \\ &\leq (\operatorname{diam}(X))^{-h_n} \sum_{j=0}^{q_n-p_n} \kappa^{jh_n} \operatorname{diam}^{h_n}(\varphi_{p_n}^{(n)}(X)) \\ &\leq (\operatorname{diam}(X))^{-h_n}(1-\kappa^{h_n})^{-1} \operatorname{diam}^{h_n}(\varphi_{p_n}^{(n)}(X)). \end{aligned}$$

On the other hand, since the gap between $\varphi_{p_n}^{(n)}(X)$ and $\varphi_{p_n+1}^{(n)}(X)$ lies between the endpoints of $F$, we get from (d) that $\operatorname{diam}(F) \geq \gamma \operatorname{diam}(\varphi_{p_n}^{(n)}(X))$. Combining this with (16.2) and (16.1) we obtain

$$\begin{aligned} \frac{\operatorname{diam}^{h_n}(F)}{m_{h_n}(F)} &\geq (1-\kappa^{h_n})\gamma^{h_n} \operatorname{diam}^{h_n}(X) \geq (1-\kappa^{\xi})\gamma^{\xi} \frac{\operatorname{diam}^{h_n}(X)}{m_{h_n}(X)} \\ &\geq ((1-\kappa^{\xi})\gamma^{\xi})^{1/2} H_{h_n}(J_{S_n}), \end{aligned} \tag{16.3}$$

where the last inequality holds for all $n \geq 1$ large enough. Now fix a non-empty closed set $F_n \subseteq J_{s_n}$ containing at least two points and such that

$$\frac{\operatorname{diam}^{h_n}(F_n)}{m_{h_n}(F_n)} < ((1-\kappa^{\xi})\gamma^{\xi})^{1/2} H_{h_n}(J_{S_n}). \tag{16.4}$$

Let $k \geq 0$ be the the least integer such that $F_n \subseteq \varphi_\omega^{(n)}(J_{S_n})$ with some word $\omega$ of length $k$. Then $E_n := (\varphi_\omega^{(n)})^{-1}(F_n) \subseteq J_{S_n}$ and it intersects at least two distinct sets of the form $\varphi_e^{(n)}(X)$, $e \in \mathbb{N}$. Since $(\varphi_\omega^{(n)})^{-1}$ is a similarity and using (16.4), we get

$$\frac{\operatorname{diam}^{h_n}(E_n)}{m_{h_n}(E_n)} = \frac{\operatorname{diam}^{h_n}(F_n)}{m_{h_n}(F_n)} < ((1-\kappa^{\xi})\gamma^{\xi})^{1/2} H_{h_n}(J_{S_n}). \tag{16.5}$$

Combining this with (16.3), we see that $E_n \cap \bigcup_{j=0}^{l-1} \varphi_j^{(n)}(X) \neq \emptyset$ and therefore

$$\begin{aligned} \operatorname{diam}(E_n) &\geq \min\{\operatorname{dist}(\varphi_a^{(n)}(X), \varphi_b^{(n)}(X)) : 1 \leq a < b \leq l\} \\ &\geq \frac{1}{2} \min\{\operatorname{dist}(\varphi_a^{(\infty)}(X), \varphi_b^{(\infty)}(X)) : 1 \leq a < b \leq l\} := \Delta, \end{aligned}$$

where the last inequality holds for all $n \geq 1$ large enough. Thus, it follows from this, Corollary 11.3, (16.4), and the equality of (16.3) that

$$\inf\left\{\frac{\operatorname{diam}^{h_n}(E)}{m_{h_n}(E)} : E \subseteq X, \operatorname{diam}(E) \geq \Delta\right\} \leq H_{h_n}(X^n). \tag{16.6}$$

Now the proof goes in the same way as the proof of Theorem 12.2 with (12.5) replaced by (16.6). We are done. □

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Tomasz Szarek, Institute of Mathematics, University of Gdańsk, ul.Wita Stwosza 57, 80-952 Gdańsk, Poland

*E-mail address*: szarek@intertele.pl

Mariusz Urbański, Department of Mathematics, University of North Texas, Denton, TX 76203-1430, USA

*E-mail address*: urbanski@unt.edu

Web: www.math.unt.edu/~urbanski

Anna Zdunik, Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02-097 Warszawa, Poland

*E-mail address*: A.Zdunik@mimuw.edu.pl