THE LAW OF ITERATED LOGARITHM AND EQUILIBRIUM MEASURES VERSUS HAUSDORFF MEASURES FOR DYNAMICALLY SEMI-REGULAR MEROMORPHIC FUNCTIONS

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ABSTRACT. The Law of Iterated Logarithm for dynamically semi-regular meromorphic mappings and loosely tame observables is established. The equilibrium states of tame potentials are compared with an appropriate one-parameter family of generalized Hausdorff measures. The singularity/absolute continuity dichotomy is established. Both results utilize the concept of nice sets and the theory of infinite conformal iterated function systems.

1. INTRODUCTION

It is one of the most urging questions in the ergodic theory of dynamical systems to find out how mixing and how random is a given dynamical system preserving a probability measures. There is an enormous literature on the subject establishing fast, desirably exponential, decay of correlations, the Central Limit Theorem, and the Law of Iterated Logarithm. The classical results concern Bernoulli shifts, Markov chains, and Gibbs states of Hölder continuous potentials for dynamical systems exhibiting some sort of hyperbolic behavior. Strong stochastic laws such as exponential decay of correlations and the Central Limit Theorem were established in [8] for the class of dynamically semi-regular meromorphic functions. As was shown in [7] and [8] this is a large class of functions indeed and its ergodic theory and thermodynamic formalism was well developed and understood. What was

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missing there was the Law of Iterated Logarithm. Some attempt to fill this gap was undertaken in [1]. In the present paper we establish the Law of Iterated Logarithm in full. This means, for all loosely tame observables that in particular include all bounded Hölder continuous observables. Our approach is based on the one hand on our observation that under relatively mild conditions the Law of Iterated Logarithm for an induced (first return) map entails this law for the original system, and on the other hand, on the fact (see [3], [11], and [13]) that each dynamically semi-regular function, as a matter of fact each tame meromorphic function, admits first return maps that form a very well understood class of conformal iterated function systems (see [6]). For this class of system all mentioned above stochastic laws are known ([6]).

Sticking to the realm of dynamically semi-regular meromorphic functions, the second theme of our paper is the issue of comparing the equilibrium states of tame potentials with an appropriate one-parameter family of generalized Hausdorff measures. This circle of investigations goes back to the fundamental work [4] of N. Makarov in potential theory (harmonic measure) and its dynamical counterpart [10]. The dichotomy phenomenon of singularity/absolute continuity observed in [10] has been afterward also detected in the context of parabolic Jordan curves ([2]) and conformal iterated function systems (see [14], comp [6]). In this paper we exhibit it in the realm of meromorphic functions. As for the Law of Iterated Logarithm our approach here utilizes the concept of nice sets that generate infinite conformal iterated function systems in the sense of [6]. For them, as already mentioned, the dichotomy is known (see [14], comp [6]). It is then an easy observation that it also holds for original meromorphic functions. The key technical issues in here are to conclude that the asymptotic variance of an appropriate function related to the induced system (IFS) is positive (this links our first them with the second) and that these functions have finite moments of all orders.

2. Preliminaries

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function. The Fatou set of f consists of all points $z \in \mathbb{C}$ that admit an open neighborhood U_z such that all the forward iterates f^n, $n \geq 0$, of f are well-defined on U_z and the family of maps $\{f^n|_{U_z} : U_z \to \mathbb{C}\}_{n=0}^{\infty}$ is normal. The Julia set of f, denoted by J_f, is then defined as the complement of the Fatou set of f in $\mathbb{C}$. By $\mathrm{Sing}(f^{-1})$ we denote the set of singularities of f^{-1}. We define *the postsingular set* of $f : \mathbb{C} \to \hat{\mathbb{C}}$ as

$$\mathrm{PS}(f) = \overline{\bigcup_{n=0}^{\infty} f^n(\mathrm{Sing}(f^{-1}))}.$$

Given a set $F \subset \hat{\mathbb{C}}$ and $n \geq 0$, by $\mathrm{Comp}(f^{-n}(F))$ we denote the collection of all connected components of the inverse image $f^{-n}(F)$. A meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ is called *tame* if its postsingular set does not contain its Julia set. This is the primary object of our interest in this paper.

We make heavy use of the concept of *a nice set* which J. Rivera–Letelier introduced in [11] in the realm of the dynamics of rational maps of the Riemann sphere. In [3] N. Dobbs proved their existence for tame meromorphic functions from $\mathbb{C}$ to $\hat{\mathbb{C}}$. We quote now his theorem.

Theorem 2.1. *Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a tame meromorphic function. Fix $z \in \mathcal{J}(f) \setminus \mathcal{P}(f)$, $L > 1$ and $K > 1$. Then there exists $\kappa > 1$ such that for all $r > 0$ sufficiently small, there exists an open connected set $U = U(z, r) \subset \mathbb{C} \setminus \mathcal{P}(f)$ such that*

(a) *If $V \in Comp(f^{-n}(U))$ and $V \cap U \neq \emptyset$, then $V \subset U$.*
(b) *If $V \in Comp(f^{-n}(U))$ and $V \cap U \neq \emptyset$, then, for all $w, w' \in V$,*

$$|(f^n)'(w)| > L \text{ and } \frac{|(f^n)'(w)|}{|(f^n)'(w')|} < K.$$

(c) *$\overline{B(z,r)} \subset U \subset B(z, \kappa r) \subset \mathbb{C} \setminus \mathcal{P}(f)$.*

Let $\mathcal{U}$ be the collection of all nice sets of $f : \mathbb{C} \to \hat{\mathbb{C}}$, i.e. all the sets U satisfying the above proposition with some $z \in J_f \setminus \mathrm{PS}(f)$ and some $r > 0$. Note that if $U = U(z, r) \in \mathcal{U}$ and $V \in \mathrm{Comp}(f^{-n}(U)$ satisfies the requirements (a), (b) and (c) from Theorem 2.1 then there exists a unique holomorphic inverse branch $f_V^{-n} : B(z, \kappa r) \to \mathbb{C}$ such that $f_V^{-n}(U) = V$. As noted in [13] the collection $\mathcal{S} = \mathcal{S}_U$ of all such inverse branches forms obviously an iterated function system in the sense of [5] and [6]. In particular, it clearly satisfies the Open Set Condition. We denote its limit set by $J_{\mathcal{S}}$. We have just mentioned [5] and [6]. In what concerns iterated function systems we try our concepts and notation to be compatible with that of [6].

Keep $f : \mathbb{C} \to \hat{\mathbb{C}}$ a meromorphic function. The function f is called *topologically hyperbolic* if

$$\mathrm{dist}_{\mathrm{Euclid}}(J_f, \mathrm{PS}(f)) > 0,$$

and it is called expanding if there exist $c > 0$ and $\lambda > 1$ such that

$$|(f^n)'(z)| \geq c\lambda^n$$

for all integers $n \geq 1$ and all points $z \in J_f \setminus f^{-n}(\infty)$. Note that every topologically hyperbolic meromorphic function is tame. A topologically hyperbolic and expanding function is called *hyperbolic*. The meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ is called dynamically *semi-regular* if it is of finite order, denoted in the sequel by ρ, and satisfies the following rapid growth condition for its derivative.

$$|f'(z)| \geq \kappa^{-1}(1+|z|)^{\alpha_1}(1+|f(z)|)^{\alpha_2}, \quad z \in J_f, \tag{2.1}$$

with some constant $\kappa > 0$ and α_1, α_2 such that $\alpha_2 > \max\{-\alpha_1, 0\}$. Let $h : J_f \to \mathbb{R}$ be a weakly Hölder continuous function (frequently referred to as a potential) in the sense of [8]. In particular each bounded, uniformly locally Hölder function $h : J_f \to \mathbb{R}$ is weakly Hölder. Fix $\tau > \alpha_2$ as required in [8]. Let

$$\phi_t = -t \log |f'|_\tau + h$$

where $|f'(z)|_\tau$ is the norm, or, equivalently, the scaling factor, of the derivative of f evaluated at a point $z \in J_f$ with respect to the Riemannian metric $|d\tau(z)| = (1+|z|)^{-\tau}|dz|$. Let $\mathcal{L}_t : C_b(J_f) \to C_b(J_f)$ be the corresponding *Perron-Frobenius operator* given by the formula

$$\mathcal{L}_t g(z) = \sum_{w \in f^{-1}(z)} g(w) e^{\phi_t(w)}.$$

It was shown in [8] that for every $z \in J_f$ the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_t \mathbb{1}(z)$$

exists and takes on the same common value, which we denote by $\mathrm{P}(t)$ and call *the topological pressure* of the potential ϕ_t. The following theorem was proved in [8].

Theorem 2.2. *If $f : \mathbb{C} \to \hat{\mathbb{C}}$ is a dynamically semi-regular meromorphic function and $h : J_f \to \mathbb{R}$ is a weakly Hölder continuous potential, then for every $t > \rho/\alpha$ ($\alpha := \alpha_1 + \alpha_2$) there exist uniquely determined Borel probability measures m_t and μ_t on J_f with the following properties.*

(a) $\mathcal{L}_t^* m_t = m_t$.
(b) $\mathrm{P}(t) = \sup\{h_\mu(f) + \int \phi_t \, d\mu : \mu \circ f^{-1} = \mu$ and $\int \phi_t \, d\mu > -\infty\}$.
(c) $\mu_t \circ f^{-1} = \mu_t$, $\int \phi_t \, d\mu_t > -\infty$, *and*

$$h_{\mu_t}(f) + \int \phi_t \, d\mu_t = \mathrm{P}(t).$$

(d) *The measures μ_t and m_t are equivalent and the Radon–Nikodym derivative $\frac{d\mu_t}{dm_t}$ has a nowhere vanishing Hölder continuous version which is bounded above.*

3. The Law of Iterated Logarithm; Abstract Setting

In this section we deal with issues related to the Law of Iterated Logarithm in the setting of general measure preserving transformations. Let (X,μ) be a probability space and let $T : X \to X$ be a measurable map preserving measure μ. Let $g : Y \to \mathbb{R}$ be a square integrable function. We put

$$\overline{\sigma}_T^2(g) := \limsup_{n\to\infty} \frac{1}{n}\int_X (S_n(g) - n\mu(g))^2\, d\mu$$

and

$$\underline{\sigma}_T^2(g) := \liminf_{n\to\infty} \frac{1}{n}\int_X (S_n(g) - n\mu(g))^2\, d\mu.$$

In the case when these two numbers are equal, we denote by $\sigma_T^2(g)$ their common value and call it the asymptotic variance of g.

Let us now briefly recall *the Rokhlin's natural extension* of the dynamical system (T,μ). The phase space is

$$\tilde{X} = \{(x_n)_{n\le 0} : T(x_n) = x_{n+1}\ \forall\, n \le -1\}$$

The transformation $\tilde{T} : \tilde{X} \to \tilde{X}$ is determined by the the property that

$$\big(\tilde{T}\big(((x_n)_{n\le 0}\big)\big)_k = T(x_k)$$

Let $\pi_0 : \tilde{X} \to X$ be the canonical projection onto the 0th coordinate, i.e,

$$\pi_0\big((x_n)_{n\le 0}\big) = x_0.$$

It is well-known (see [9] for example) that there exists a unique probability $\tilde{T}$-invariant measure $\tilde{\mu}$ on $\tilde{X}$ such that

$$\tilde{\mu}\circ\pi_0^{-1} = \mu.$$

The dynamical system $(\tilde{T},\tilde{\mu})$ is a measure-preserving automorphism and

$$\pi_0\circ\tilde{T} = T\circ\pi_0.$$

This system is referred to as *the Rokhlin's natural extension of* (T,μ).

We say that two functions $g_1 : X \to \mathbb{R}$ and $g_2 : X \to \mathbb{R}$ are *cohomologous* in a class C of function from X to $\mathbb{R}$ if there exists a function $u \in C$ such that

$$g_2 - g_1 = u - u\circ T$$

Any function cohomologous to the zero function is called a *coboundary*.

We shall prove the following generalization and extension of Lemma 53 in [15].

Lemma 3.1. *Let (X, μ) be a probability space and let $T : X \to X$ be a measurable map preserving measure μ. Fix A, a measurable subset of X with positive measure μ. Let $\tau : A \to \mathbb{N}$ be the first return time to A and let $T_A = T^{\tau_A} : A \to A$ be the corresponding first return map. Assume that*

$$\mu(\tau_A^{-1}([n, +\infty))) \leq \text{const}\, n^{-\alpha},$$

for some $\alpha > 8$ and all $n \geq 1$. For every function $g : X \to \mathbb{R}$ let $\hat{g} : A \to \mathbb{R}$ be defined by the formula

$$\hat{g}(x) = \sum_{j=0}^{\tau_A(x)-1} g \circ T^j(x).$$

If $g \in L_4(\mu)$, $\overline{\sigma}_S^2(g) > 0$ and $\mu(g) = 0$, then $\hat{g} : A \to \mathbb{R}$ is not a coboundary in the class of bounded measurable functions on A.

Proof. Seeking contradiction suppose that $\hat{g} : A \to \mathbb{R}$ is such a coboundary, i.e.

$$\hat{g} = u - u \circ T_A \tag{3.1}$$

with some bounded measurable function $u : A \to \mathbb{R}$. Replace first the dynamical system (T, μ) by its Rokhlin's natural extension $(\tilde{T}, \tilde{\mu})$, the set A by $\pi_0^{-1}(A)$, the function g by $g \circ \pi_0$ and the function u by $u \circ \pi_0$. Then note that after such replacements the equation (3.1) will remain true and the set $\tau_A^{-1}(n)$ (the set of points with first return time n) will be mapped to the sets $\pi_0\big(\tau_A^{-1}(n)\big)$. In particular they will have the same measures, respectively μ and $\tilde{\mu}$. In conclusion, we may assume without loss of generality that the dynamical system (T, μ) is a measure-preserving automorphism. For every $n \geq 0$ let

$$A_n = \{x \in A : \tau_A(x) \geq n\}.$$

Fix an integer $n \geq 1$. For all $x \in A$ let

$$i = i(x) := \min\{0 \leq l \leq n : T^l(x) \in A\}.$$

If no such l exists, set $i = n$. Let

$$j = j(x) := \max\{0 \leq l \leq n : T^l(x) \in A\}.$$

If no such l exists, set $j = 0$. We have,

$$0 \leq i \leq j \leq n$$

and there exists a unique integer $0 \leq k \leq j - i$ such that

$$T^{j-i}(T^i(x)) = T_A^k(T^i(x)).$$

Hence we can write

$$S_n g(x) = S_i g(x) + S_k^{T_A}(\hat{g})(T^i(x)) + S_{n-j} g(T^j(x)) = a(x) + b(x) + c(x).$$

In order to show that $\sigma_T^2(g) = 0$, we shall estimate

$$\left(\int (S_n(g))^2 d\mu\right)^{\frac{1}{2}} = ||(S_n(g))||_2 \leq ||a||_2 + ||b||_2 + ||c||_2. \tag{3.2}$$

We shall deal with each of these three L_2 norms separately. Since $b(x) = S_k^{T_A}(\hat{g})(T^i(x))$ and $|S_k^{T_A}(\hat{g})(T^i(x))| \leq 2||u||_\infty$, we get immediately that

$$||b||_2 \leq 2||u||_\infty. \tag{3.3}$$

Next, we estimate $||a||_2$. We have

$$a(x) = \sum_{l=0}^{n} 1\!\!1_{i^{-1}(l)}(x) S_l g(x).$$

Applying Cauchy-Schwarz inequality, we therefore get

$$\begin{aligned} ||a||_2 &\leq \sum_{l=0}^{n} ||1\!\!1_{i^{-1}(l)} S_l g||_2 = \sum_{l=0}^{n} \left(\int 1\!\!1_{i^{-1}(l)} (S_l g)^2 d\mu\right)^{\frac{1}{2}} \\ &\leq \sum_{l=0}^{n} \Big(\int 1\!\!1_{i^{-1}(l)} d\mu\Big)^{\frac{1}{4}} \Big(\int (S_l g)^4 d\mu\Big)^{\frac{1}{4}} \\ &= \sum_{l=0}^{n} (\mu(i^{-1}(l)))^{\frac{1}{4}} \Big(\int (S_l g)^4 d\mu\Big)^{\frac{1}{4}} \\ &= \sum_{l=0}^{n} (\mu(i^{-1}(l)))^{\frac{1}{4}} ||S_l g||_4 \\ &= \sum_{l=0}^{n} (\mu(i^{-1}(l)))^{\frac{1}{4}} \left\| \sum_{s=0}^{l-1} g \circ T^s \right\|_4 \\ &\leq \sum_{l=0}^{n} (\mu(i^{-1}(l)))^{\frac{1}{4}} \sum_{s=0}^{l-1} ||g \circ T^s||_4 \\ &= \sum_{l=0}^{n} (\mu(i^{-1}(l)))^{\frac{1}{4}} \sum_{s=0}^{l-1} ||g||_4 \\ &= ||g||_4 \sum_{l=1}^{n} l(\mu(i^{-1}(l)))^{\frac{1}{4}}. \end{aligned} \tag{3.4}$$

Now notice that for μ almost every $x \in X$, there exist $x' \in A$ and an integer $k \geq 1$ such that $T^k(x') = x$ and $\tau_A(x') \geq k + i(x)$ (the strict inequality can hold only if

$i(x) = n$ and $T^n(x) \notin A$). Hence,

$$i^{-1}(l) \subset \bigcup_{k=1}^{\infty} T^k(\tau_A^{-1}(k+l)).$$

Thus, as $T : X \to X$ is an automorphism, we get that

$$\mu(i^{-1}(l)) \le \sum_{k=1}^{\infty} \mu\big(T^k(\tau_A^{-1}(k+l))\big) = \sum_{k=1}^{\infty} \mu\big(\tau_A^{-1}(k+l)\big) = \mu(A_{l+1})$$

Inserting this to (3.4), we obtain

$$\begin{aligned} ||a||_2 &\le ||g||_4 \sum_{l=0}^{n} l(\mu(A_{l+1}))^{\frac{1}{4}} \le \mathrm{const}||g||_4 \sum_{l=1}^{n} l(l+1)^{-\alpha/4} \\ &\le \mathrm{const}||g||_4 \sum_{l=1}^{\infty} l^{1-\frac{\alpha}{4}} \\ &< +\infty, \end{aligned} \tag{3.5}$$

where the last inequality was written since $\alpha > 8$. The upper estimate of $||c||_2$ can be done similarly. Indeed, exactly as (3.4), we obtain the following.

$$||c||_2 = ||S_{n-j} g \circ (T^j)||_2 = ||S_{n-j} g||_2 \le ||g||_4 \sum_{l=0}^{n} (n-l)(\mu(j^{-1}(l))^{\frac{1}{4}} \tag{3.6}$$

Now notice that if $T^{i(x)}(x) \notin A$, then $c(x) = 0$ and otherwise $T^{j(x)}(x) \in A$ and $\tau_A(T^{j(x)}(x)) > n - j(x)$. So,

$$j^{-1}(l) \subset T^{-l}(A_{n-l+1}) \subset T^{-l}(A_{n-l})$$

Inserting this to (3.6), we thus get

$$\begin{aligned} ||c||_2 &\le ||g||_4 \sum_{l=0}^{n} (n-l)(\mu(A_{n-l}))^{\frac{1}{4}} v = ||g||_4 \sum_{l=0}^{n-1} (n-l)(\mu(A_{n-l}))^{\frac{1}{4}} \\ &= ||g||_4 \sum_{l=1}^{n} l(\mu(A_l))^{\frac{1}{4}} \le ||g||_4 \sum_{l=1}^{n} l \mathrm{const} l^{-\alpha/4} \\ &\le ||g||_4 \sum_{l=1}^{\infty} \le \mathrm{const}||g||_4 \sum_{l=1}^{\infty} l^{1-\frac{\alpha}{4}} \\ &< +\infty, \end{aligned} \tag{3.7}$$

where the last inequality was written since $\alpha > 8$. Combining this, (3.3), (3.5), and inserting them to (3.2), we see that the integrals $\int (S_n g)^2 d\mu$ remain uniformly bounded as $n \to \infty$. This obviously implies that $\sigma_T^2(g) = 0$. This contradiction finishes the proof. $\square$

We shall now show that under mild conditions if a first return map satisfies the Law of Iterated Logarithm, then so does the original map. Precisely, we say that a μ-integrable function $g : X \to \mathbb{R}$ satisfies the Law of Iterated Logarithm if there exists a positive constant A_g such that

$$\limsup_{n\to\infty} \frac{S_n g - n\int g d\mu}{\sqrt{n \log\log n}} = A_g.$$

From now on we assume without loss of generality that

$$\mu(g) = \int g d\mu = 0.$$

Keep a measurable set $A \subset X$ with $\mu(A) > 0$. Given a point $x \in A$, the sequence $(\tau_n(x))_{n=1}^{\infty}$ is then defined as follows.

$$\tau_1(x) := \tau_A(x) \quad \text{and} \quad \tau_n(x) = \tau_{n-1}(x) + \tau(T^{\tau_{n-1}(x)}(x)).$$

What we were up to is the following theorem. Its proof can be also found in [15].

Theorem 3.2. *Let $T : X \to X$ be a measurable dynamical system preserving a probability measure μ on X. Assume that the dynamical system (T, μ) is ergodic. Fix A, a measurable subset of X having a positive measure μ. Let $g : X \to \mathbb{R}$ be a measurable function such that the function $\hat{g} : A \to \mathbb{R}$ satisfies the Law of Iterated Logarithm with respect to the dynamical system (T_A, μ_A). If in addition,*

$$\int |\hat{g}|^{2+\gamma} d\mu < \infty \tag{3.8}$$

for some $\gamma > 0$, then the function $g : X \to \mathbb{R}$ satisfies the Law of Iterated Logarithm with respect to the original dynamical system (T, μ) and $A_g = A_{\hat{g}}$.

Proof. Since the Law of Iterated Logarithm holds for a point $x \in X$ if and only if it holds for $T(x)$, in virtue of ergodicity of T, it suffices to prove our theorem for almost all points in A. By our assumptions there exists a positive constant $A_{\hat{g}}$ such that

$$\limsup_{n\to\infty} \frac{S_{\tau_n} g(x)}{\sqrt{n \log\log n}} = A_{\hat{g}}.$$

for μ_A-a.e. $x \in A$. Since, by Kac's Lemma,

$$\lim_{n\to\infty} \frac{\tau_n}{n} = \int_X \tau dm = \int_A \tau dm = 1, \tag{3.9}$$

μ_A-a.e. on A, we thus have

$$\limsup_{n\to\infty} \frac{S_{\tau_n} g}{\sqrt{n \log\log n}} = \limsup_{n\to\infty} \frac{S_{\tau_n} g}{\sqrt{\tau_n \log\log \tau_n}} = A_{\hat{g}}. \tag{3.10}$$

μ_A-a.e. on A. Now, for every $n \in \mathbb{N}$ and (almost) every $x \in A$ let $k = k(x, n)$ be the positive integer uniquely determined by the condition that

$$\tau_k(x) \le n < \tau_{k+1}(x).$$

Since

$$S_n g(x) = S_{\tau_k(x)} g(x) + S_{n-\tau_k(x)} g(T^{\tau_k(x)}(x)),$$

we have that

$$\frac{S_n g(x)}{\sqrt{n \log\log n}} = \frac{S_{\tau_k(x)} g(x)}{\sqrt{n \log\log n}} + \frac{S_{n-\tau_k(x)} g(x)}{\sqrt{n \log\log n}} \tag{3.11}$$

Since by (3.9)

$$\lim_{n\to\infty} \frac{\tau_{k+1}(x)}{\tau_k(x)} = 1,$$

we get from (3.10) that,

$$\limsup_{n\to\infty} \frac{S_{\tau_k} g(x)}{\sqrt{n \log\log n}} = \limsup_{n\to\infty} \frac{S_{\tau_k} g(x)}{\sqrt{k \log\log k}} = A_{\hat{g}}.$$

Because of this and because of (3.11), we are only left to show that

$$\lim_{n\to\infty} \frac{S_{n-\tau_{k(n)}} g(x)}{\sqrt{n \log\log n}} = 0. \tag{3.12}$$

μ_A-a.e. on A. To do this, note first that

$$\frac{S_{\tau_{k+1}-\tau_k} |g| (T^{\tau_k}(x))}{\sqrt{k \log\log k}} = \frac{|\hat{g}|(T_A^k(x))}{\sqrt{k \log\log k}}.$$

Take an arbitrary $\varepsilon \in (0, \gamma)$. Since

$$\begin{aligned}
\mu\big(\{x \in A : |\hat{g}|(T_A^k(x)) \ge \varepsilon\sqrt{k \log\log k}\}\big) &= \\
&= \mu\big(\{x \in A : |\hat{g}|(x) \ge \varepsilon\sqrt{k \log\log k}\}\big) \\
&= \mu\big(\{x \in A : |\hat{g}|^{2+\varepsilon}(x) \ge \varepsilon^{2+\varepsilon}(k \log\log k)^{1+\varepsilon/2}\}\big) \\
&\le \frac{\int |\hat{g}|^{2+\varepsilon} d\mu}{\varepsilon^{2+\varepsilon}(k \log\log k)^{1+\varepsilon/2}},
\end{aligned} \tag{3.13}$$

using (3.8) we conclude that

$$\sum_{k=1}^{\infty} \mu\big(\{x \in A : |\hat{g}|(x) \ge \varepsilon\sqrt{k \log\log k}\}\big) < \infty.$$

So, applying Borel-Cantelli lemma, (3.12) follows. We are done. $\square$

4. The Law of Iterated Logarithm; Meromorphic Functions

Let $f:\mathbb{C}\to\hat{\mathbb{C}}$ be a dynamically semi-regular meromorphic function and $t>\rho/\alpha$. Let $\mathcal{S}=\{\phi_e\}_{e\in E}$ be the iterated function system induced by some some nice set U for f. Our first technical result, ultimately aiming at the Law of Iterated Logarithm, is this.

Lemma 4.1. *If $f:\mathbb{C}\to\hat{\mathbb{C}}$ is a dynamically semi-regular meromorphic function and $t>\rho/\alpha$, then*

$$\limsup_{n\to\infty}\frac{1}{n}\log\mu_t\Big(\bigcup_{|\omega|\ge n}\phi_\omega(U)\Big)<0.$$

Proof. Noting that $U\cap\bigcup_{k=1}^{n-1}f^k(\phi_\omega(U))=\emptyset$ and repeating the proof of Proposition 6.3 from [13], we show that

$$\mathrm{P}_c(t):=\limsup_{n\to\infty}\frac{1}{n}\log\sum_{|\omega|=n}\exp\big(\sup\big(S_{n-1}(\phi_t\circ f\circ\phi_\omega)\big)\big)<\mathrm{P}(t).$$

Hence,

$$\begin{aligned}
\limsup_{n\to\infty}&\frac{1}{n}\log m_t\Big(\bigcup_{|\omega|=n}f(\phi_\omega(U)\Big)\le\\
&\le\limsup_{n\to\infty}\frac{1}{n}\log\sum_{|\omega|=n}\exp\big(\sup\big(S_{n-1}(\phi_t\circ f\circ\phi_\omega)\big)-\mathrm{P}(t)(n-1)\big)\\
&=\mathrm{P}_c(t)-\mathrm{P}(t)<0.
\end{aligned}$$

Since, see Theorem 2.2(d), the Radon–Nikodym derivative $\frac{d\mu_t}{dm_t}$ is uniformly bounded above, we thus get that

$$\limsup_{n\to\infty}\frac{1}{n}\log\mu_t\bigg(f\Big(\bigcup_{|\omega|=n}\phi_\omega(U)\Big)\bigg)\le\mathrm{P}_c(t)-\mathrm{P}(t).$$

Since the probability measure μ_t is f-invariant, we have

$$\mu_t\bigg(f\Big(\bigcup_{|\omega|=n}\phi_\omega(U)\Big)\bigg)\ge\mu_t\Big(\bigcup_{|\omega|=n}\phi_\omega(U)\Big),$$

and therefore,

$$\limsup_{n\to\infty}\frac{1}{n}\log\mu_t\Big(\bigcup_{|\omega|=n}\phi_\omega(U)\Big)\le\mathrm{P}_c(t)-\mathrm{P}(t).$$

So, finally,

$$\limsup_{n\to\infty}\frac{1}{n}\log\mu_t\Big(\bigcup_{|\omega|\geq n}\phi_\omega(U)\Big)\leq \mathrm{P}_c(t)-\mathrm{P}(t)<0.$$

The proof is complete. □

Because of Lemma 4.1 we obviously have some constant $C>0$ such that

$$\mu_t\Big(\bigcup_{|\omega|\geq n}\phi_\omega(U)\Big)\leq Cn^{-9} \tag{4.1}$$

for all $n\geq 1$. For every $e\in E$ let $N_e\geq 1$ be the unique integer determined by the property that $f^{N_e}\circ\phi_e=\mathrm{Id}$. Let $\hat{f}:J_{\mathcal{S}}\to\ J_{\mathcal{S}}$ be the first return map on $J_{\mathcal{S}}$, i.e. $\hat{f}$ is defined by the formula

$$\hat{f}(\phi_e(z))=f^{N_e}(\phi_e(z))=z$$

for all $e\in E$ and all $z\in J_{\mathcal{S}}$. N_e is then the first return time to $J_{\mathcal{S}}$. Recall from the previous section that given $g:J_f\to\mathbb{R}$, the function $\hat{g}:J_{\mathcal{S}}:\mathbb{R}$ is given by the following formula.

$$\hat{g}(\phi_e(z))=\sum_{j=0}^{N_e-1}g\circ f^j(\phi_e(z))$$

for all $e\in E$ and all $z\in J_{\mathcal{S}}$. Let $\hat{m}_t$ and $\hat{\mu}_t$ be the probability conditional measures on $J_{\mathcal{S}}$ respectively of m_t and μ_t. The measure $\hat{\mu}_t$ is then $\hat{f}$-invariant. Moreover, $\hat{m}_t$ is the F-conformal measure for F, the summable Hölder family consisting of functions $\{\phi_t^{(e)}\}_{e\in E}$, defined by the following formula.

$$\phi_t^{(e)}(z)=\hat{\phi}_t(\phi_e(z))-\mathrm{P}(t)N_e.$$

In consequence, all the results proved in [6] for summable Hölder families apply, in particular to measures $\hat{m}_t$ and $\hat{\mu}_t$. As an immediate consequence of Theorem 3.2, we get the following.

Theorem 4.2. *Let $f:\mathbb{C}\to\hat{\mathbb{C}}$ be a dynamically semi-regular meromorphic function and fix $t>\rho/\alpha$. Let $g:J_f\to\mathbb{R}$ be a measurable function such that the function $\hat{g}:J_{\mathcal{S}}\to\mathbb{R}$ satisfies the Law of Iterated Logarithm with respect to the dynamical system $(\hat{f},\hat{\mu}_t)$. If in addition,*

$$\int_{J_f}|\hat{g}|^{2+\gamma}d\mu_t<\infty \tag{4.2}$$

for some $\gamma>0$, then the function $g:J_f\to\mathbb{R}$ satisfies the Law of Iterated Logarithm with respect to the dynamical system (f,μ_t) and $A_g=A_{\hat{g}}$.

In order to be able to apply this theorem, we need a technical result establishing (4.2) for a large class of functions from J_f to $\mathbb{R}$. This is the content of the following lemma.

Lemma 4.3. *Let $\psi : J_f \to \mathbb{R}$ be a function, following* [8] *called loosely tame, of the following form*
$$\psi(z) = -s \log |f'(z)|_\tau + k(z),$$
where $s \in \mathbb{R}$ and $h : J_f \to \mathbb{R}$ is a weakly Hölder continuous function. Then for every $\gamma > 0$,
$$\int_{J_{\mathcal{S}}} |\hat{\psi}|^\gamma d\hat{m}_t < +\infty.$$

Proof. Since the measure $\hat{m}_t$ is proportional to m_t on $J_{\mathcal{S}}$, our equivalent task is to show that
$$\int_{J_{\mathcal{S}}} |\hat{\psi}|^\gamma dm_t < +\infty.$$
Fix $\varepsilon > 0$. Because of expanding properties of the function $f : \mathbb{C} \to \hat{\mathbb{C}}$ there exists a constant $C > 0$ such that
$$|\psi(z)| \le C|f'(z)_\tau^\varepsilon$$
for all $z \in J(f)$. Therefor, for every $e \in E$ and all $z \in J_{\mathcal{S}}$ we get,
$$\begin{aligned}
|\hat{\psi}(\phi_e(z))| &= \left| \sum_{j=0}^{N_e-1} \psi(f^j(\phi_e(z))) \right| \le \sum_{j=0}^{N_e-1} \left|\psi(f^j(\phi_e(z)))\right| \\
&\le C \sum_{j=0}^{N_e-1} |f'(f^j(\phi_e(z))|_\tau^\varepsilon \le C \prod_{j=0}^{N_e-1} |f'(f^j(\phi_e(z))|_\tau^\varepsilon \\
&= C|(f^{N_e})'(\phi_e(z))|_\tau^\varepsilon.
\end{aligned}$$
Thus
(4.3)
$$\begin{aligned}
&|\hat{\psi}\phi_e(z))|^\gamma m_t(\phi_e(J_{\mathcal{S}})) \asymp \\
&\quad \asymp |\hat{\psi}(\phi_e(z))|^\gamma \exp\big(S_{N_e}\phi_t(\phi_e(z)) - \mathrm{P}(\phi_t)N_e\big) \\
&\quad \le C^\gamma \exp\big(\gamma\varepsilon \log |(f^{N_e})'(\phi_e(z))|_\tau - t \log |(f^{N_e})'(\phi_e(z))|_\tau + S_{N_e}k(z)) - \mathrm{P}(\phi_t)N_e\big) \\
&\quad = C^\gamma \exp\big(S_{N_e}\phi_{t-\gamma\varepsilon}(\phi_e(z)) - \mathrm{P}(\phi_t)N_e\big) \\
&\quad = C^\gamma \exp\big(S_{N_e}\phi_{t-\gamma\varepsilon}(\phi_e(z)) - \mathrm{P}(\phi_{t-\gamma\varepsilon})N_e\big) \exp\big((\mathrm{P}(\phi_{t-\gamma\varepsilon} - \mathrm{P}(\phi_t))N_e\big) \\
&\quad \asymp C^\gamma \exp\big((\mathrm{P}(\phi_{t-\gamma\varepsilon}) - \mathrm{P}(\phi_t))N_e\big) m_{t-\gamma\varepsilon}(\phi_e(J_{\mathcal{S}}).
\end{aligned}$$

Since, by Lemma 7.5 in [8], the function $(t-\delta_1, t+\delta_1) \ni u \mapsto \mathrm{P}(\phi_u)$ ($\delta_1 > 0$ sufficiently small) is real-analytic, we get a constant $M > 0$ such that for all $\varepsilon > 0$ sufficiently small, we have that

$$|\mathrm{P}(\phi_{t-\gamma\varepsilon}) - \mathrm{P}(\phi_t)| \le M\varepsilon.$$

Formula (4.3) then yields

$$|\hat{\psi}(\phi_e(z))|^\gamma m_t(\phi_e(J_{\mathcal{S}}) \le C^\gamma e^{M\varepsilon N_e} m_{t-\gamma\varepsilon}(\phi_e(J_{\mathcal{S}})). \tag{4.4}$$

Now, for every $k \ge 1$ let

$$U_k^c = \bigcap_{j=0}^{k} f^{-j}(\mathbb{C} \setminus U).$$

Fixing $u > \rho/\alpha_2$, we have for every $n \ge 1$ that

$$\begin{aligned} m_u\Bigg(\bigcup_{e\in E: N_e = n} \phi_e(J_{\mathcal{S}})\Bigg) &= m_u\big(U \cap f^{-1}(U_{n-1}^c) \cap f^{-n}(U)\big) \le m_u(f^{-1}(U_{n-1}^c)) \\ &= m_u(f^{-1}(\mathbb{1}_{U_{n-1}^c})) = m_u(\mathbb{1}_{U_{n-1}^c} \circ f) \\ &= m_u\big(e^{-\mathrm{P}(\phi_u)n}\mathcal{L}_u\big(\mathbb{1}_{U_{n-1}^c} \circ f\big)\big) \\ &= m_u\big(e^{-\mathrm{P}(\phi_u)(n-1)}\mathcal{L}_u^{n-1}\big(e^{-\mathrm{P}(\phi_u)}\mathcal{L}_u\big(\mathbb{1}_{U_{n-1}^c} \circ f\big)\big)\big) \\ &= m_u\big(e^{-\mathrm{P}(\phi_u)(n-1)}\mathcal{L}_u^{n-1}\big(\mathbb{1}_{U_{n-1}^c}(e^{-\mathrm{P}(\phi_u)}\mathcal{L}_u\mathbb{1})\big) \\ &\le C_1 m_u\big(e^{-\mathrm{P}(\phi_u)(n-1)}\mathcal{L}_u^{n-1}\big(\mathbb{1}_{U_{n-1}^c}\big)\big) \end{aligned} \tag{4.5}$$

with some constant $C_1 > 0$. Looking at this moment at the proof of Proposition 6.3 in [12] and taking into account continuity properties of the Perron-Frobenius operator $\mathcal{L}_u$, we conclude that there exist $\kappa > 0$ and $c_2 > 0$ such that

$$m_u\big(\mathcal{L}_u^{n-1}\big(\mathbb{1}_{U_{n-1}^c}\big)\big) \le C_2 e^{-\kappa n} e^{\mathrm{P}(\phi_u)(n-1)}$$

for all $u \in (t-\delta, t+\delta)$ with some $0 < \delta \le \delta_1$ small enough and all integers $n \ge 1$. Substituting this to (4.5) we get that

$$m_u\Bigg(\bigcup_{e\in E: N_e = n} \phi_e(J_{\mathcal{S}})\Bigg) \le C_1 C_2 e^{-\kappa n} \tag{4.6}$$

for all $\in (t-\delta, t+\delta)$. Fix $0 < \varepsilon < \min\{\delta/\gamma, \kappa/(2M)\}4$. Inserting then (4.6) into (4.4), we obtain

$$
\begin{aligned}
\int |\hat{\psi}|^\gamma dm_t &= \sum_{n=1}^{\infty} \int_{\bigcup_{e\in E: N_e=n} \phi_e(J_{\mathcal{S}})} |\hat{\psi}|^\gamma dm_t = \sum_{n=1}^{\infty} \sum_{N_e=n} \int_{\phi_e(J_{\mathcal{S}})} |\hat{\psi}|^\gamma dm_t \\
&\le \sum_{n=1}^{\infty} \sum_{N_e=n} ||\hat{\psi}|_{\phi_e(J_{\mathcal{S}})}||_\infty^\gamma m_t(\phi_e(J_{\mathcal{S}})) \\
&\le C^\gamma \sum_{n=1}^{\infty} e^{M\varepsilon n} \sum_{N_e=n} m_{t-\gamma\varepsilon}(\phi_e(J_{\mathcal{S}})) \\
&= C^\gamma \sum_{n=1}^{\infty} e^{M\varepsilon n} m_{t-\gamma\varepsilon}\Bigg(\bigcup_{N_e=n} \phi_e(J_{\mathcal{S}}) \Bigg) \\
&\le C^\gamma C_1 C_2 \sum_{n=1}^{\infty} e^{M\varepsilon n} e^{-\kappa n} \\
&\le C^\gamma C_1 C_2 \sum_{n=1}^{\infty} e^{-\frac{1}{2}\kappa n}.
\end{aligned} \tag{4.7}
$$

The proof is complete. □

Now we are in position to provide a short proof of the following theorem and its corollary, both forming the main results of this sections.

Theorem 4.4. *Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a dynamically semi-regular meromorphic function and fix $t > \rho/\alpha$. Let $\psi : J_f \to \mathbb{R}$ be a loosely tame function. Then the asymptotic variance $\sigma_f^2(\psi)$ exists and, if $\sigma_f^2(\psi) > 0$, equivalently if $\psi : J_f \to \mathbb{R}$ is not cohomologous to a constant in the class of Hölder continuous functions on J_f, then the function $\psi : J_f \to \mathbb{R}$ satisfies the Law of Iterated Logarithm with respect to the dynamical system (f, μ_t) with $A_\psi = \sqrt{2}\sigma_{\hat{f}}(\hat{\psi}) > 0$.*

Proof. Adding a constant to ψ we may assume without loss of generality that $\int \psi d\mu_t = 0$. The existence of the asymptotic variance $\sigma_f^2(\psi)$ was established in Theorem 6.17 of [8]. The fact that $\sigma_f^2(\psi) > 0$ if and only if $\psi : J_f \to \mathbb{R}$ is not cohomologous to a constant in the class of Hölder continuous functions on J_f is the content of Proposition 6.21 in [8]. In view of Lemma 5.2 in [8] the function $\hat{\psi}$ is Hölder continuous, precisely its composition with the canonical projection from $E^{\mathbb{N}}$ onto $J_{\mathcal{S}}$ is Hölder continuous. Along with Lemma 4.3 this implies (see Lemma 2.5.6 in [6] and the beginning of the page 41 in [6]) that the asymptotic variance $\sigma^2_{\hat f}(\hat\psi)$ exists and, in addition with Theorem 2.5.5 and Lemma 2.5.6 in [6], both in [6], that the function $\hat\psi$ satisfies the Law of Iterated Logarithm with respect to the dynamical system $(\hat f, \hat\mu_t)$ (with $A_{\hat\psi} = \sigma^2_{\hat f}(\hat\psi))$ provided that $sg^2_{\hat f}(\hat\psi) > 0$. But since, by Lemma 7.11 in [8], the function ψ has all moments with respect to the measure μ_t, we in particular have that $\psi \in L_4(\mu_t)$. Then, using (4.1), Lemma 3.1 implies that $\hat\psi$ is not a coboundary in the class of bounded measurable functions on $J_{\mathcal S}$. It then directly follows from Lemma 4.8.8 that $\sigma^2_{\hat f}(\hat\psi) > 0$. Now, with the help of Lemma 4.3, the application of Theorem 3.2 finishes the proof. $\square$

As an immediate consequence of this theorem, with the help of Theorem 6.20 in [8], we get the following.

Corollary 4.5. *Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a dynamically semi-regular meromorphic function and fix $t > \rho/\alpha$. If $\psi : J_f \to \mathbb{R}$ is a loosely tame function ($\psi(z) = -s\log|f'(z)|_\tau + k(z)$) with $s \neq 0$, then the function $\psi : J_f \to \mathbb{R}$ satisfies the Law of Iterated Logarithm with respect to the dynamical system (f, μ_t) with $A_\psi = \sqrt{2}\sigma_{\hat f}(\hat\psi) > 0$.*

5. Equilibrium States versus Hausdorff Measures

Keep $f : \mathbb{C} \to \hat{\mathbb{C}}$ a dynamically semi-regular meromorphic function and $t > \rho/\alpha$. Let

$$D_t := \mathrm{HD}(\mu_t),$$

let the function $\zeta : J_{\mathcal S} \to (0, +\infty)$ be defined by the formula

$$\zeta(\phi_e(z)) = -\log|\phi_e'(z)|,$$

and let

$$\chi_t := \int \zeta d\mu_t.$$

The number χ_t is called the Lyapunov exponent of the measure μ_t. Let $h : (a, +\infty) \to (0, +\infty)$ ($a > 0$ small enough) be a non-decreasing function. This function h is said to belong to to the lower class if

$$\int_a^\infty \frac{h(r)}{r} \exp\big(-\frac{1}{2}(h(r)^2\big)\, dr < +\infty$$

and to the upper class if

$$\int_a^\infty \frac{h(r)}{r} \exp\big(-\frac{1}{2}(h(r)^2\big)\, dr = +\infty.$$

Associated to the function h is the function $\tilde{h}$ defined for all sufficiently small $t > 0$ by the following formula.

$$\tilde{h}(r) = r^{D_t} \exp\left(\frac{\sigma(\hat{\psi})}{\sqrt{\chi_t}} h(-\log r)\sqrt{-\log r}\right).$$

Let finally $\mathrm{H}_{\tilde{h}}$ be the Hausdorff measure on $\mathbb{C}$ induced by the gauge function $\tilde{h}$ and let

$$\psi_t = \phi_t + D_t\zeta - \mathrm{P}(\phi_t) = -(t - D_t)\log|f'|_\tau + (h - \mathrm{P}(\phi_t)).$$

Since the function $\hat{\psi}_t : J_{\mathcal{S}} \to \mathbb{R}$ is Hölder continuous and since all the integrals $\in |\hat{\psi}_t|^\gamma d\mu_t$ $(\gamma > 2)$ are finite (see Lemma 4.3 where this is proved for all $\gamma > 0$), Theorem 4.8.3 in [6]) applies to give the following.

Theorem 5.1. *Suppose that $\sigma^2(\hat{\psi}_t) > 0$ and that $h : (a, +\infty) \to (0, +\infty)$ is a slowly growing function. Then*

(a) *If h belongs to the upper class, then the measures $\hat{\mu}_t$ and $\mathrm{H}_{\tilde{h}}|_{J_{\mathcal{S}}}$ are mutually singular.*
(b) *If h belongs to the lower class, then $\hat{\mu}_t$ is absolutely continuous with respect to $\mathrm{H}_{\tilde{h}}$.*

We shall now prove a sufficient condition for $\sigma^2(\hat{\psi}_t)$ to be positive. It is trivially verifiable. Let $J_{r,f}$ be the set of points in J_f that do not escape to infinity under the action of the map $: \mathbb{C} \to \hat{\mathbb{C}}$. It is called in the literature the radial (or conical) Julia set of f.

Lemma 5.2. *If $t \neq \mathrm{HD}(J_{r,f})$, then the function $\psi_t = -(t - D_t)\log|f'|_\tau + (h - \mathrm{P}(\phi_t))$ is not cohomologous to a constant in the class of Hölder continuous functions on J_f and $\sigma^2(\hat{\psi}_t) > 0$. In particular this is true for all $t \geq 2$.*

Proof. First observe that because of Theorem 8.1 (Volume Lemma) and Theorem 6.25 (Variational Principle), both in [8], we have

$$\begin{aligned}\int \psi_t d\mu_t &= -t\chi_{\mu_t} + \frac{\mathrm{h}_{\mu_t}(f)}{\chi_{\mu_t}}\chi_{\mu_t} + \frac{\mathrm{h}_{\mu_t}}{t\chi_{\mu_t}} + \int h d\mu_t - \mathrm{P}(\phi_t) \\ &= \mathrm{h}_{\mu_t}(f) - t\chi_{\mu_t} + \int h d\mu_t - \mathrm{P}(\phi_t) \\ &= 0.\end{aligned} \tag{5.1}$$

We already know that ψ_t is cohomologous to a constant in the class of Hölder continuous functions on J_f if and only if $\sigma^2(\hat{\psi}_t) = 0$. So, assume that ψ_t is cohomologous to a constant. By (5.1) ψ_t is then a coboundary. By Theorem 6.20 in [8], we get that

$$t = D_t. \tag{5.2}$$

The fact that ψ_t is a coboundary equivalently means that the function $-D_t \log|f'|_\tau$ is cohomologous to $\phi_t - \mathrm{P}(\phi_t)$. But the topological pressure of the latter function vanishes, whence $\mathrm{P}(-D_t \log|f'|_\tau) = 0$. Theorem 8.3 in [8] (Bowen's Formula) then implies that

$$D_t = \mathrm{HD}(J_{r,f}). \tag{5.3}$$

This theorem is in fact in [8] formulated for dynamically regular functions only but apart from dynamical semiregularity all what was needed there was the existence of a zero of the pressure function of potentials $-t\log|f'|_\tau$. Combining (5.2) and (5.3) yields $t = \mathrm{HD}(J_{r,f})$. To complete the proof we are thus only left to notice that $\mathrm{HD}(J_{r,f}) < 2$. We are done. □

As an immediate consequence of Theorem 5.1 and Lemma 5.2 we get the following main result of this paper.

Theorem 5.3. *Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a dynamically semi-regular meromorphic function and for every $t > \rho/\alpha$ let $\phi_t = -t\log|f'|_\tau + h$. Suppose that $\sigma^2(\hat{\psi}_t) > 0$ (this is in particular true if $t \neq \mathrm{HD}(J_{r,f})$, more particularly if $t \geq 2$) and that $h : (a, +\infty) \to (0, +\infty)$ is a slowly growing function. Then*

(a) *If h belongs to the upper class, then the measures μ_t and $\mathrm{H}_{\tilde{h}}|_{J_f}$ are mutually singular.*
(b) *If h belongs to the lower class, then μ_t is absolutely continuous with respect to $\mathrm{H}_{\tilde{h}}$.*

Towards the end of the paper note that the function $h_c(t) = c\sqrt{\log\log t}$, $c \geq 0$, belongs to the upper class if and only if $c \leq \sqrt{2}$. With the consistent notation

$$\begin{aligned} \tilde{h}_c(r) &= r^{D_t} \exp\left(\frac{\sigma(\hat{\psi})}{\sqrt{\chi_t}} h_c(-\log r)\sqrt{-\log r}\right) \\ &= r^{D_t} \exp\left(c\frac{\sigma(\hat{\psi})}{\sqrt{\chi_t}} \sqrt{\log(1/r)\log_3(1/r)}\right), \end{aligned}$$

we therefore immediately obtain the following consequence of Theorem 5.3.

Theorem 5.4. *Let $f:\mathbb{C}\to\hat{\mathbb{C}}$ be a dynamically semi-regular meromorphic function and for every $t>\rho/\alpha$ let $\phi_t=-t\log|f'|_\tau+h$. Suppose that $\sigma^2(\hat{\psi}_t)>0$; this is in particular true if $t\neq \mathrm{HD}(J_{r,f})$, more particularly if $t\geq 2$. Then*

(a) *The measures μ_t and $\mathrm{H}_{\tilde{h}}|_{J_f}$ are mutually singular for all $0\leq c\leq\sqrt{2}$.*
(b) *The measure μ_t is absolutely continuous with respect to $\mathrm{H}_{\tilde{h}}$ for all $c>\sqrt{2}$.*

Given $\kappa>0$ let H_κ be the standard Hausdorff measure corresponding to the parameter κ, i.e. $\mathrm{H}_\kappa=\mathrm{H}_{r\mapsto r^\kappa}$ with the notation introduced above. Taking in Theorem 5.4 $c=0$, we obtain the following.

Corollary 5.5. *Let $f:\mathbb{C}\to\hat{\mathbb{C}}$ be a dynamically semi-regular meromorphic function and for every $t>\rho/\alpha$ let $\phi_t=-t\log|f'|_\tau+h$. Suppose that $\sigma^2(\hat{\psi}_t)>0$; this is in particular true if $t\neq \mathrm{HD}(J_{r,f})$, more particularly if $t\geq 2$. Then the measures μ_t and $\mathrm{H}_{\mathrm{HD}(\mu_t)}|_{J_f}$ are mutually singular.*

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