RELATIVE EQUILIBRIUM STATES AND DIMENSIONS OF FIBERWISE INVARIANT MEASURES FOR DISTANCE EXPANDING RANDOM MAPS

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Abstract. We show that the Gibbs states (known from [9] to be unique) of Hölder continuous potentials and random distance expanding maps coincide with relative equilibrium states of those potentials, proving in particular that the latter exist and are unique. In the realm of conformal expanding random maps we prove that given an ergodic (globally) invariant measure with a given marginal, for almost every fiber the corresponding conditional measure has dimension equal to the ratio of the relative metric entropy and the Lyapunov exponent. Finally we show that there is exactly one invariant measure whose conditional measures are of full dimension. It is the canonical Gibbs state.

1. Introduction

The thermodynamic formalism of random distance expanding maps has been developed in [9]. It comprised and went beyond the previous work of Bogenschütz, Gundlach, Kifer, and others (see [3], [4], [1], [6], [7], and the references therein) on random symbolic dynamical systems and random infinitesimally expanding maps on smooth Riemannian manifolds. The work [9] thoroughly explored the concept of Gibbs states of appropriately defined Hölder continuous potentials. The work [11] substantially developed the theory of relative equilibrium states of holomorphic endomorphisms of the Riemann sphere and Hölder continuous potentials.

In the present paper, after some preliminaries, we firstly deal (in Section 5) with equilibrium states of Hölder continuous potentials and random distance expanding maps as defined in [9]. We prove that the Gibbs states (known from [9] to

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be unique) of such potentials coincide with relative equilibrium states of those potentials, proving in particular that the latter exist and are unique.

Next, in Section 6, we deal with conformal expanding random maps. In [9] the Hausdorff dimension of almost all fibers was identified as the only zero of the expected pressure function, and the corresponding conditional (fiber) measures of the canonical Gibbs state μ_h were shown to have the same dimension. In the present paper we look at an arbitrary ergodic (globally) invariant measure with a given marginal m. As our main result we prove that for m-almost every fiber, the corresponding conditional measure has dimension equal to the ratio of the relative metric entropy and Lyapunov exponent. As a complementary result we show that the canonical Gibbs state μ_h is the only ergodic invariant measure whose conditional measures have dimensions equal to Hausdorff dimensions of their fibers. In short, the Gibbs state μ_h is the only invariant measure of full dimension.

2. Metric Random Dynamical Systems

We first recall the definition of a metric (measurable) random dynamical system.

Definition 2.1. A metric random dynamical system consists of the following objects:

- A Lebesgue probability space $(X, \mathcal{F}, m)$
- An ergodic invertible measure-preserving transformation $\theta : X \to X$
- A Lebesgue measurable space $(\mathcal{J}, \mathcal{B})$ of the form

$$\mathcal{J} = \bigcup_{x \in X} \{x\} \times \mathcal{J}_x$$

  The spaces $\mathcal{J}_x$, $x \in X$ are called the *fibers* of the random dynamical system.
- A measurable transformation $T : \mathcal{J} \to \mathcal{J}$ of the form

$$T(x, y) = (\theta(x), T_x(y)),$$

  where $T_x : \mathcal{J}_x \to \mathcal{J}_{\theta(x)}$.

Notation 2.2.
$\pi_X : \mathcal{J} \to X$ and $\pi_Y : \mathcal{J} \to Y$ are the first and second projections, respectively. For each $x \in X$, $i_x : \mathcal{J}_x \to \mathcal{J}$ is defined by $i_x(y) = (x, y)$.

To do more dynamics, we define for every integer $n \geq 0$ and for every $x \in X$

$$T_x^n := T_{\theta^{n-1}(x)} \circ T_{\theta^{n-2}(x)} \circ \ldots \circ T_x : \mathcal{J}_x \to \mathcal{J}_{\theta^n(x)},$$

so that

$$T^n(x, y) = (\theta^n(x), T_x^n(y))$$

for all $x \in X$ and for all $y \in \mathcal{J}_x$. Throughout the whole paper we are concerned with T-invariant probability measures on $\mathcal{J}$ with marginal m. By $\mathcal{M}^1_m(\mathcal{J})$ we denote the collection of all probability measures μ on $(\mathcal{J}, \mathcal{B})$ such that

$$\mu \circ \pi_X^{-1} = m,$$

and we set

$$\mathcal{M}^1_m(T) = \{\mu \in \mathcal{M}^1_m(\mathcal{J}) : \mu \circ T^{-1} = \mu\}.$$

Denote by ε_X the partition of X into singletons. Since X is a Lebesgue space, the partitions ε_X and $\pi_X^{-1}(\varepsilon_X)$ are measurable, and so each measure $\mu \in \mathcal{M}^1_m(\mathcal{J})$ admits and is uniquely determined by its disintegration $(\mu_x)_{x\in X}$ with respect to the partition $\pi_X^{-1}(\varepsilon_X)$. In other words, there exists a system of measures $(\mu_x)_{x\in X}$ (called the *canonical system of conditional measures*), unique up to a set of m-measure zero, such that for each $x \in X$, μ_x is a measure on the fiber $\mathcal{J}_x$, and such that

$$\mu(g) = \int_X \mu_x(g_x) \mathrm{d}m(x)$$

for every μ-integrable function $g : \mathcal{J} \to \mathbb{R}$, where

$$g_x = g \circ i_x.$$

Moreover, a measure $\mu \in \mathcal{M}^1_m(\mathcal{J})$ is T-invariant, i.e. belongs to $\mathcal{M}^1_m(T)$, if and only if

$$\mu_x \circ T_x^{-1} = \mu_{\theta(x)}$$

for m-a.e. $x \in X$.

The rest of this section is concerned with relative entropies. We follow the standard notation from deterministic entropy theory as used for example in Chapters 1 and 2 of [10]. We recall the following definition:

Definition 2.3. If $\mu \in \mathcal{M}^1_m(T)$, and if α is a measurable partition of $\mathcal{J}$ finer than $\pi_X^{-1}(\varepsilon_X)$, then we let

$$\mathrm{h}_\mu(T|\theta;\alpha) := \lim_{n\to\infty} \frac{1}{n} \mathrm{H}_\sigma\left(\alpha^n \left| \pi_X^{-1}\varepsilon_X\right.\right),$$

where

$$\alpha^n := \bigvee_{j=0}^{n} T^{-j}\alpha.$$

Moreover, let

$$\mathrm{h}_\mu(T|\theta) := \sup\{\mathrm{h}_\mu(T|\theta;\alpha)\},$$

where the supremum is taken over all measurable partitions α of $\mathcal{J}$ finer than $\pi_X^{-1}(\varepsilon_X)$ and having finite entropy relative to $\pi_X^{-1}(\varepsilon_X)$, i.e.

$$\mathrm{H}_\mu(\alpha|\pi_X^{-1}(\varepsilon_X)) := \int_X \mathrm{H}_{\mu_x}(\alpha_x)\mathrm{d}m(x) < +\infty,$$

where $\alpha_x := i_x^{-1}(\alpha)$. The number $\mathrm{h}_\mu(T|\theta)$ is called the *entropy of T relative to θ with respect to the measure μ.*

To shorten notation, given a measure $\mu \in \mathcal{M}_m^1(T)$, we now denote by $\mathcal{A}_\mu(T|\theta)$ the collection of all measurable partitions α of $\mathcal{J}$ finer than $\pi_X^{-1}(\varepsilon_X)$ and having finite entropy relative to $\pi_X^{-1}(\varepsilon_X)$. By $(\mu_{x,y})_{(x,y)\in\mathcal{J}}$ we denote the canonical system of conditional measures of the measure μ with respect to the partition $T^{-1}(\varepsilon_{\mathcal{J}})$, i.e. $\mu_{x,y}$ denotes the conditional measure supported on $T_x^{-1}(T_x(y))$. As in the deterministic case, a good way to calculate relative entropies is by means of generating partitions.

Definition 2.4. A partition α finer than $\pi_X^{-1}(\varepsilon_X)$, i.e. belonging to $\mathcal{A}_\mu(T|\theta)$, is said to be *generating for T relative to θ* if and only if

$$\alpha^\infty := \bigvee_{j=0}^{\infty} T^{-j}\alpha \equiv_\mu \varepsilon_{\mathcal{J}},$$

where for two measurable partitions β_1 and β_2 on $\mathcal{J}$, the relation $\beta_1 \equiv_\mu \beta_2$ means that there exists a measurable set $Z \subseteq \mathcal{J}$ with $\mu(\mathcal{J}\setminus Z) = 0$ such that $\beta_1|_Z = \beta_2|_Z$.

We have the following:

Proposition 2.5. *If $T : \mathcal{J} \to \mathcal{J}$ is a metric random dynamical system, if $\mu \in \mathcal{M}_m^1(T)$, and if $\alpha \in \mathcal{A}_\mu(T|\theta)$ is generating for T relative to θ, then*

$$\begin{aligned} h_\mu(T|\theta) = h_\mu(T|\theta;\alpha) &= \mathrm{H}_\mu(\varepsilon_{\mathcal{J}}|T^{-1}(\varepsilon_{\mathcal{J}})) \\ &= \int_{\mathcal{J}} \mathrm{H}_{\mu_{x,y}}\left(\varepsilon_{\mathcal{J}_x}|_{T_x^{-1}(T_x(y))}\right)\mathrm{d}\mu(x,y) \\ &= \int_X \int_{\mathcal{J}_x} \mathrm{H}_{\mu_{x,y}}\left(\varepsilon_{\mathcal{J}_x}|_{T_x^{-1}(T_x(y))}\right)\mathrm{d}\mu_x(y)\mathrm{d}m(x). \end{aligned}$$

This is an adaptation to the random setting of the well-known Kolmogorov-Sinai generator theorem. Its statement and an outline of its proof can be found in [11].

We end this section with a random version of the Shannon-McMillan-Breiman theorem.

Theorem 2.6 (Theorem 2.2.5(iii) of [1])**.** *If $T : \mathcal{J} \to \mathcal{J}$ is a metric random dynamical system, if $\mu \in \mathcal{M}^1_{m,e}(T)$, and if $\alpha \in \mathcal{A}_\mu(T|\theta)$, then for μ-a.e. $(x, y) \in \mathcal{J}$,*

$$\lim_{n\to\infty} \frac{-1}{n} \log \mu_x(\alpha^n_x(y)) = h_\mu(T|\theta; \alpha).$$

3. Topological Random Dynamical Systems

We say that a metric random dynamical system $T : \mathcal{J} \to \mathcal{J}$ is *topological* if for every $x \in X$, the fiber $\mathcal{J}_x$ is a compact metric space endowed with a metric ρ_x whose Borel σ-algebra is equal to $i_x^{-1}(\mathcal{B})$, and if $\sup_{x\in X} \mathrm{diam}_x(\mathcal{J}_x) < +\infty$.

Definition 3.1. We say that a topological random dynamical system $T : \mathcal{J} \to \mathcal{J}$ is *of global type* if there exists a compact metric space (Y, ρ) such that

- For each $x \in X$, $\mathcal{J}_x \subseteq Y$, and $\rho_x = \rho|_{\mathcal{J}_x}$. (Note that this condition implies that $\mathcal{J} \subseteq X \times Y$.)
- $\mathcal{B} = (\mathcal{F} \otimes \mathcal{B}_Y)|_{\mathcal{J}}$, where $\mathcal{B}_Y$ is the σ-algebra of Borel subsets of Y.

If $\mathcal{J}_x = Y$ for all $x \in X$, then we say that T is *of strongly global type*. We call Y the *vertical space* of the topological random dynamical system $T : \mathcal{J} \to \mathcal{J}$.

Let $T : \mathcal{J} \to \mathcal{J}$ be a topological random dynamical system. We denote by $\mathcal{C}_m(\mathcal{J})$ the space of all measurable functions $g : \mathcal{J} \to \mathbb{R}$ such that g_x is continuous for all $x \in X$. We further define

$$\mathcal{C}^1_m(\mathcal{J}) := \left\{ g \in \mathcal{C}_m(\mathcal{J}) : \int_X \|g_x\|_\infty \, \mathrm{d}m(x) < +\infty \right\}.$$

For each $g \in \mathcal{C}_m(\mathcal{J})$ we let

$$S_n(g) := \sum_{j=0}^{n-1} g \circ T^j.$$

Denote by $\mathcal{M}^1_{m,e}(T)$ the set of all ergodic elements of $\mathcal{M}^1_m(T)$. Let a function $\phi \in \mathcal{C}^1_m(\mathcal{J})$, routinely called in such a context a *potential*, be given. The number

$$\mathrm{P}(\phi, T|\theta) := \sup_{\mu\in\mathcal{M}^1_m(T)} \left(\mathrm{h}_\mu(T|\theta) + \int_{\mathcal{J}} \phi \, \mathrm{d}\mu \right) = \sup_{\mu\in\mathcal{M}^1_{m,e}(T)} \left(\mathrm{h}_\mu(T|\theta) + \int_{\mathcal{J}} \phi \, \mathrm{d}\mu \right)$$

is called the *variationa pressure of ϕ and T relative to θ.*

Given $x \in X$, $\varepsilon > 0$, and an integer $n \geq 0$, we say that a set $F \subseteq \mathcal{J}_x$ is (x, n, ε)-*separated* if for every two distinct points $y, z \in F$ there exists $j \in \{0, 1, \dots, n-1\}$ such that $\rho_{\theta^j(x)}(T^j_x(y), T^j_x(z)) > \varepsilon$. If our random dynamical $T : \mathcal{J} \to \mathcal{J}$ is of strongly global type, then the following quantity, called *topological pressure*, is well-defined (see [2]):

$$\mathrm{P}_t(\phi, T|\theta) := \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\int_X \log \sup_{E\subseteq \mathcal{J}_x}\left(\sum_{y\in E}\exp(S_n\phi(x,y))\right) dm(x), \tag{3.1}$$

where the supremum is taken over all (x, n, ε)-separated subsets E of $\mathcal{J}_x$. By saying that (3.1) is well-defined, we mean that for each $n \in \mathbb{N}$ the integrand is measurable. We have the following variational principle:

Theorem 3.2 ([2]). *If $T : \mathcal{J} \to \mathcal{J}$ is a topological random dynamical system of strongly global type and if $\phi \in \mathcal{C}^1_m(\mathcal{J})$ is a potential, then*

$$\mathrm{P}_t(\phi, T|\theta) = \mathrm{P}(\phi, T|\theta).$$

The hypothesis of strongly global type is strong indeed, and in fact it is too strong for this paper; we brought up the above result only for the sake of completeness. In what follows we will however work with maps of global type which is a much weaker and more frequently satisfied assumption.

We end this section by formulating the following definition of equilibrium states. They will be a primary object of our considerations in Section 5.

Definition 3.3. If $\phi \in \mathcal{C}^1_m(\mathcal{J})$, then a measure $\mu \in \mathcal{M}^1_m(T)$ is called a *relative equilibrium state* for the potential ϕ if

$$\mathrm{h}_\mu(T|\theta) + \int_{\mathcal{J}} \phi\,\mathrm{d}\mu = \mathrm{P}(\phi, T|\theta).$$

In Section 5 we will deal with the issue of the existence and uniqueness of equilibrium states in the context of random distance expanding dynamical systems.

4. Distance Expanding Random Maps; Preliminaries

Modifying a definition from [9] we call a topological random dynamical system $T : \mathcal{J} \to \mathcal{J}$ *distance expanding* if all the mappings $T_x : \mathcal{J}_x \to \mathcal{J}_{\theta(x)}$, $x \in X$, are open and surjective, and if there exist a measurable function $\eta : X \to (0, +\infty)$ and a real number $\xi > 0$ such that following conditions hold:

- (*Uniform Openness*)
$$T_x(B_x(z, \eta_x)) \supseteq B_{\theta(x)}\big(T_x(z), \xi\big)$$
for every $(x, z) \in \mathcal{J}$.

- (*Measurably Expanding*) There exists a measurable function $\gamma : X \to (1, +\infty)$ such that
$$\rho_{\theta(x)}(T_x(y), T_x(z)) \geq \gamma_x \rho_x(y, z)$$
whenever $x \in X$, $y, z \in \mathcal{J}_x$, and $\rho_x(y, z) \leq 2\eta_x$.[1]

- (*Measurability of Degree*) The map
$$X \ni x \mapsto \deg(T_x) := \sup_{y \in \mathcal{J}_{\theta(x)}} \#\big(T_x^{-1}(y)\big)$$
is measurable.

- (*Topological Exactness*) There exists a measurable function $n_\xi : X \to \mathbb{N}$ such that
$$T_x^{n_\xi(x)}(B_x(z, \xi)) = \mathcal{J}_{\theta^{n_\xi(x)}(x)}$$
for all $x \in X$ and for all $z \in \mathcal{J}_x$.

- (*Weak Perfectness*)
$$\widehat{\xi} := \operatorname{ess\,inf}\{\inf\{\operatorname{diam}_x(B_x(z, \xi)) : z \in \mathcal{J}_x\} : x \in X\} > 0.$$

- (*Lipschitz Continuity*) There exists $L \geq 1$ such that
$$\rho_{\theta(x)}(T_x(y), T_x(z)) \leq L\rho_x(y, z)$$
for all $x \in X$ and for all $y, z \in \mathcal{J}_x$.

- (*Log Integrable Compactness*) For every $x \in X$ and for every $r > 0$ let $N_x(r)$ denote the minimal number of open balls with radii r needed to cover $\mathcal{J}_x$. We assume that there exists a measurable function $\widehat{N} : X \to \mathbb{N}$ such that
$$N_x(\widehat{\xi}/4L) \leq \widehat{N}(x) \text{ for } m\text{-a.e. } x \in X$$
and such that
$$\int_X \log \widehat{N}(x)\, \mathrm{d}m(x) < +\infty.$$

Let us make some comments about this definition. Of course, Log Integrable Compactness holds if
$$\operatorname{ess\,sup}\{N_x(\widehat{\xi}/4L) : x \in X\} < +\infty,$$
and in particular if
$$\operatorname{ess\,sup}\{N_x(r) : x \in X\} < +\infty \quad \forall r > 0.$$

[1] In [9] the factor of 2 was incorrectly omitted from this inequality.

Although this last condition looks somewhat restrictive, it is nevertheless always satisfied if the random system $T : \mathcal{J} \to \mathcal{J}$ is of global type.

As a matter of fact the last three conditions of the above definition, i.e. Weak Perfectness, Lipschitz Continuity, and Log Integrable Compactness, were not needed and did not appear in [9]. In the present paper we need them in order to construct generating partitions with finite entropy. Thus the definition of a random distance expanding map is more restrictive in the present paper than in [9].

We call a random distance expanding map $T : \mathcal{J} \to \mathcal{J}$ *accessible* if there exists a partition α of $\mathcal{J}$ finer than $\pi_X^{-1}(\varepsilon_X)$ with the following two properties:

(a) There exists a log-integrable function $\widehat{N} : X \to \mathbb{N}$ such that

$$\#(\alpha_x) \le \widehat{N}(x) \text{ for } m\text{-a.e. } x \in X$$

(b) $\mathrm{diam}_x(\alpha_x) \le \widehat{\xi}/(2L)$ for m-a.e. $x \in X$.

Any such partition α is called *accessible for* T. We note the following.

Proposition 4.1. *Every random distance expanding map of global type is accessible.*

Proof. Let $\beta = \{B_1, B_2, \ldots, B_N\}$ be a finite cover of Y by open balls with radii $\widehat{\xi}/4L$. Define the partition $\gamma = \{C_1, C_2, \ldots, C_N\}$ of Y by induction as follows.

$$C_1 := B_1$$
$$C_{n+1} := B_{n+1} \setminus (C_1 \cup C_2 \cup \ldots \cup C_n), \quad 1 \le n \le N-1.$$

Let

$$\alpha := \pi_Y^{-1}(\gamma) \vee \pi_X^{-1}(\varepsilon_X).$$

By its very definition α is finer than $\pi_X^{-1}(\varepsilon_X)$. Define $\widehat{N}(x) := N$; then (a) clearly holds. Also, for every $1 \le j \le n$ and every $x \in X$, we have that

$$\mathrm{diam}(C_j \cap \mathcal{J}_x) \le \mathrm{diam}(C_j) \le \widehat{\xi}/(2L),$$

and so (b) holds too. The proof is complete. $\square$

The importance of accessible partitions lies primarily in the following fact:

Proposition 4.2. *Every accessible partition for a random distance expanding map $T : \mathcal{J} \to \mathcal{J}$ belongs to $\mathcal{A}_\mu(T|\theta)$ and is generating for every measure $\mu \in \mathcal{M}_m^1(T)$.*

Proof. Let α be an accessible partition for $T : \mathcal{J} \to \mathcal{J}$. Then

$$\mathrm{H}_\mu(\alpha|\pi_X^{-1}(\varepsilon_X)) = \int_X \mathrm{H}_{\mu_x}(\alpha_x)\, \mathrm{d}m(x) \le \int_X \log \widehat{N}(x)\, \mathrm{d}m(x) < +\infty.$$

It immediately follows from the conditions of Uniform Openness, Lipschitz Continuity, and Weak Perfectness that

$$\eta_x \geq \widehat{\xi}/(2L) \tag{4.1}$$

for m-a.e. $x \in X$.

Now fix an $x \in X$ satisfying (4.1), and suppose that $\alpha_x^\infty \neq \varepsilon_{\mathcal{J}_x}$. Then there exists $A \in \alpha_x^\infty$ with $\#(A) \geq 2$. Fix distinct $y, z \in A \in \alpha_x^\infty$; then

$$\rho_{\theta^n(x)}(T_x^n(y), T_x^n(z)) \leq \mathrm{diam}_{\theta^n(x)}(T_x^n(A)) \leq \mathrm{diam}_{\theta^n(x)}(\alpha_{\theta^n(x)}) \leq \widehat{\xi}/(2L) \leq \eta_x.$$

Thus by the condition of Measurably Expanding we have

$$\rho_{\theta^{n+1}(x)}(T_x^{n+1}(y), T_x^{n+1}(z)) \geq \gamma_{\theta^n(x)} \rho_{\theta^n(x)}(T_x^n(y), T_x^n(z)). \tag{4.2}$$

Let us write

$$\gamma_x^n := \prod_{i=0}^{n-1} \gamma_{\theta^i(x)}.$$

Then iterating (4.2) gives

$$\eta_x \geq \rho_{\theta^n(x)}(T_x^n(y), T_x^n(z)) \geq \gamma_x^n \rho_x(y, z)$$

and in particular the sequence $(\gamma_x^n)_1^\infty$ is bounded. On the other hand, by Birkhoff's Ergodic Theorem,

$$\lim_{n\to\infty} \frac{1}{n} \log \gamma_x^n = \int_X \log \gamma \, \mathrm{d}m > 0 \tag{4.3}$$

for m-a.e $x \in X$. Thus, if we let Z be the set of x for which both (4.1) and (4.3) hold, then $X \setminus Z$ has m-measure zero and for all $x \in Z$, $\alpha_x^\infty = \varepsilon_{\mathcal{J}_x}$. Thus the partition α is generating and the proof is complete. $\square$

Combining Proposition 4.2 with Proposition 2.5 and then with Theorem 2.6 yields the following corollary which we will use twice in Section 6:

Corollary 4.3. *If $T : \mathcal{J} \to \mathcal{J}$ is a random distance expanding map, if $\mu \in \mathcal{M}_{m,e}^1(T)$, and if α is a partition which is accessible for T, then for μ-a.e. $(x, y) \in \mathcal{J}$,*

$$\lim_{n\to\infty} \frac{-1}{n} \log \mu_x(\alpha_x^n(y)) = h_\mu(T|\theta).$$

5. Gibbs and Equilibrium Measures for Distance Expanding Random Maps

Throughout this whole section $T : \mathcal{J} \to \mathcal{J}$ is an accessible random distance expanding map. Following [9] fix $\alpha \in (0,1]$. By $\mathcal{H}^\alpha(\mathcal{J}_x)$ we denote the space of Hölder continuous functions on $\mathcal{J}_x$ with exponent α. This means that $\phi_x \in \mathcal{H}^\alpha(\mathcal{J}_x)$ if and only if $\phi_x \in \mathcal{C}(\mathcal{J}_x)$ and $v_\alpha(\phi_x) < \infty$ where

$$v_\alpha(\phi_x) := \sup\left\{ \frac{|\phi_x(y) - \phi_x(z)|}{\rho_x^\alpha(y,z)} : \ y, z \in \mathcal{J}_x \right\}.$$

A function $\phi \in \mathcal{C}_m^1(\mathcal{J})$ is called *Hölder continuous with exponent* α provided that there exists a measurable function $H : X \to [1, +\infty)$ such that

$$\log H \in L^1(m)$$

and

$$v_\alpha(\phi_x) \le H_x \ \text{ for } m\text{-a.e. } x \in X.$$

We denote the space of all Hölder continuous functions with fixed α and H by $\mathcal{H}^\alpha(\mathcal{J}, H)$ and the space of all Hölder continuous functions with exponent α by $\mathcal{H}^\alpha(\mathcal{J})$. Now, for every $x \in X$, we consider the *transfer operator*

$$\mathcal{L}_x = \mathcal{L}_{\phi,x} : \mathcal{C}(\mathcal{J}_x) \to \mathcal{C}(\mathcal{J}_{\theta(x)})$$

given by the formula

$$\mathcal{L}_x g_x(w) = \sum_{z \in T_x^{-1}(w)} g_x(z) e^{\phi_x(z)}, w \in \mathcal{J}_{\theta(x)}.$$

This is obviously a positive linear operator; moreover, it is bounded and its operator norm is bounded above by

$$\|\mathcal{L}_x\|_\infty \le \deg(T_x) \exp(\|\phi\|_\infty).$$

For every $n \ge 0$ and for μ-a.e. $x \in X$, we put

$$\mathcal{L}_x^n := \mathcal{L}_{\theta^{n-1}(x)} \circ \ldots \circ \mathcal{L}_x : \mathcal{C}(\mathcal{J}_x) \to \mathcal{C}(\mathcal{J}_{\theta^n(x)}).$$

Note that

$$\mathcal{L}_x^n g_x(w) = \sum_{z \in T_x^{-n}(w)} g_x(z) e^{S_n \phi(x,z)} \ , \ w \in \mathcal{J}_{\theta^n(x)} \, .$$

Then the dual operator $\mathcal{L}_x^* : \mathcal{C}^*(\mathcal{J}_{\theta(x)}) \to \mathcal{C}^*(\mathcal{J}_x)$ is defined by

$$\mathcal{L}_x^*(\mu_{\theta(x)}) g_x := \mu_{\theta(x)}(\mathcal{L}_x g_x).$$

Proposition 5.1. *Let $T : \mathcal{J} \to \mathcal{J}$ be a distance expanding random map, and let $\xi > 0$ and $\eta : X \to (0, +\infty)$ be as in the definition of distance expanding. For* *every* $(x,y)\in\mathcal{J}$*, the map* T_x *has a unique inverse branch* $T_{x,y}^{-1}: B_{\theta(x)}(T_x(y),\xi)\to B_x(y,\eta_x)$*. Furthermore,*

$$\rho_x(T_{x,y}^{-1}(z),T_{x,y}^{-1}(w))\le\frac{1}{\gamma_x}\rho_{\theta(x)}(z,w)\text{ for all } z,w\in B_{\theta(x)}(T_x(y),\xi). \tag{5.1}$$

Proof. The existence of a unique inverse branch $T_{x,y}^{-1}: B_{\theta(x)}(T_x(y),\xi)\to B_x(y,\eta_x)$ is equivalent to the assertion that $\#(T_x^{-1}(z)\cap B_x(y,\eta_x))=1$ for all $z\in B_{\theta(x)}(T_x(y),\xi)$. Now $\ge$ is due to the Uniform Openness condition, while $\le$ is due to the Measurably Expanding condition (and the fact that $\rho_x(z_1,z_2)\le 2\eta_x$ for all $z_1,z_2\in B_x(y,\eta_x)$). Finally, (5.1) is due to the Measurably Expanding condition. □

For $(x,y)\in\mathcal{J}$ and $n>1$, we let

$$T_{x,y}^{-n}=T_{x,y}^{-1}\circ T_{\theta(x),T_x(y)}^{-1}\circ\ldots\circ T_{\theta^{n-1}(x),T_x^{n-1}(y)}^{-1}.$$

Then iterating (5.1) gives

$$\rho_x(T_{x,y}^{-n}(z),T_{x,y}^{-n}(w))\le\frac{1}{\gamma_x^n}\rho_{\theta^n(x)}(z,w)\text{ for all } z,w\in B_{\theta^n(x)}(T_x^n(y),\xi).$$

Remark 5.2. In [9], the map $T_{x,y}^{-n}$ was improperly called the "unique continuous inverse branch of T_x^n defined on $B_{\theta^n(x)}(T_x^n(y),\xi)$ that sends $T_x^n(y)$ to y." Strictly speaking this is incorrect, since the shift maps give counterexample to uniqueness. However, if $n=1$, then it is the unique inverse branch of T_x defined on $B_{\theta(x)}(T_x(y),\xi)$ and taking values in $B_x(y,\eta_x)$, as the above proposition shows.

Following [9] we call a measure $\mu\in\mathcal{M}_m^1(T)$ a *Gibbs state* for a potential $\phi:\mathcal{J}\to\mathbb{R}$ if there exist measurable functions $D_\phi: X\to[1,+\infty)$ and $R: X\to\mathbb{R}$ such that

$$\frac{1}{D_\phi(x)D_\phi(\theta^n(x))}\le\frac{\mu_x(T_{x,y}^{-n}(B_{\theta^n(x)}(T_x^n(z),\xi)))}{\exp(S_n\phi(x,z)-S_nR(x))}\le D_\phi(x)D_\phi(\theta^n(x))$$

for every $n\ge 0$, for m-a.e. $x\in X$, and for every $z\in\mathcal{J}_x$.

Let us recall from [9] (p. 18) that a random distance expanding map $T:\mathcal{J}\to\mathcal{J}$ is said to satisfy the *measurability of cardinality of covers* condition if there exists a measurable function $a: X\to\mathbb{N}$ such that for m-a.e. $x\in X$, there exists a finite sequence $w_x^1,\ldots,w_x^{a_x}\in\mathcal{J}_x$ such that $\bigcup_{j=1}^{a_x}B_x(w_x^j,\xi)=\mathcal{J}_x$, where ξ is as in the definition of distance expanding. Every accessible random distance expanding map satisfies the measurability of cardinality of covers condition since $\widehat{\xi}/(2L)\le\xi$. This means that for accessible random distance expanding maps, we can use Theorem 3.3 and Proposition 4.8 of [9], which we repeat (slightly modified) below.[2]

[2]The measurability of cardinality of covers condition is not mentioned in the statement of Proposition 4.8 of [9], but it is mentioned in Lemma 3.29 from which Proposition 4.8 follows.

Theorem 5.3 (Theorems 3.1 and 3.3 of [9]). *Let $T : \mathcal{J} \to \mathcal{J}$ be an accessible random distance expanding map, and let $\phi : \mathcal{J} \to \mathbb{R}$ be a Hölder continuous potential. Then*

(a) *There exists a unique measurable function $\lambda : X \to (0, +\infty)$ and a unique measure $\nu_\phi \in \mathcal{M}_m^1(\mathcal{J})$ such that*
$$\mathcal{L}_x^* \nu_{\phi,\theta(x)} = \lambda_x \nu_{\phi,x} \quad \text{for } m\text{-a.e. } x \in X$$
(b) *There exists a unique function $q \in \mathcal{C}_m(\mathcal{J})$ such that*
$$\mathcal{L}_x q_x = \lambda_x q_{\theta(x)} \text{ and } \nu_{\phi,x}(q_x) = 1 \text{ for } m\text{-a.e. } x \in X.$$
Moreover $q_x \in \mathcal{H}^\alpha(\mathcal{J}_x)$ for m-a.e. $x \in X$.

(c) *If we let*
$$\mu_{\phi,x} = q_x \nu_{\phi,x},$$
and if we let μ_ϕ be the unique measure in $\mathcal{M}_m^1(\mathcal{J})$ whose system of conditional measures is $(\mu_{\phi,x})_{x\in X}$, then μ_ϕ is a T-invariant Gibbs state.

Theorem 5.4 (Proposition 4.8 of [9]). *If $T : \mathcal{J} \to \mathcal{J}$ is an accessible random distance expanding map, and if $\phi : \mathcal{J} \to \mathbb{R}$ is a Hölder continuous potential, then the measure $\mu_\phi \in \mathcal{M}_m^1(T)$ defined in the previous theorem is the unique T-invariant Gibbs state for ϕ.*

Following [9] we put
$$\mathcal{E}\mathrm{P}(\phi) = \int_X \log \lambda \, \mathrm{d}m.$$
The goal of this section is to prove that μ_ϕ is the unique equilibrium state for the potential $\phi : \mathcal{J} \to \mathbb{R}$. Let $\widehat{\mathcal{L}}_x := \mathcal{L}_{\widehat{\phi},x}$ be the transfer operator with potential
$$\widehat{\phi}_x := \phi_x + \log q_x - \log q_{\theta(x)} \circ T_x - \log \lambda_x.$$
Then
$$\widehat{\mathcal{L}}_x g_x = \frac{1}{q_{\theta(x)}} \lambda_x^{-1} \mathcal{L}_x(g_x q_x) \quad \text{for every } g_x \in L^1(\mu_x).$$
Consequently
$$\widehat{\mathcal{L}}_x \mathbb{1}_x = \mathbb{1}_{\theta(x)} \tag{5.2}$$
and
$$\widehat{\mathcal{L}}_x^* \mu_{\phi,\theta(x)} = \mu_{\phi,x} \tag{5.3}$$
for m-a.e. $x \in X$.

Our first step towards showing that μ_ϕ is a relative equilibrium state for ϕ is the following lemma:

Lemma 5.5. *If we let*

$$\widehat{\mu}_{x,y} = \widehat{\mathcal{L}}_x^*(\delta_{T_x(y)}), \tag{5.4}$$

then the collection of measures $(\widehat{\mu}_{x,y})_{(x,y)\in\mathcal{J}}$ *is a disintegration of* μ_ϕ *with respect to the partition* $T^{-1}(\varepsilon_{\mathcal{J}})$.

Proof. First of all, note that the expression (5.4) depends only on $T(x,y) = (\theta(x), T_x(y))$, and thus

$$\widehat{\mu}_{x,y} = \widehat{\mu}_{x,y'} \tag{5.5}$$

for every $y' \in T_x^{-1}(T_x(y)) = T_x^{-1}(\varepsilon_{\mathcal{J}_{\theta(x)}})(y)$. Thus we may write $\widehat{\mu}_{x,T_x^{-1}(w)}$ to denote the measure $\widehat{\mu}_{x,y}$ for any $y \in T_x^{-1}(w)$. Also,

$$\begin{aligned} \widehat{\mu}_{x,y}(T_x^{-1}(T_x(y))) &= \widehat{\mathcal{L}}_x^*(\delta_{T_x(y)})(\mathbb{1}_{T_x^{-1}(T_x(y))}) \\ &= \delta_{T_x(y)}(\widehat{\mathcal{L}}_x \mathbb{1}_{T_x^{-1}(T_x(y))}) \\ &= \left(\widehat{\mathcal{L}}_x \mathbb{1}_{T_x^{-1}(T_x(y))}\right)(T_x(y)) \\ &= 1. \end{aligned} \tag{5.6}$$

In general, by (5.3), we have

$$\begin{aligned} \mu_{\phi,x}(g_x) = \big(\widehat{\mathcal{L}}_x^* \mu_{\phi,\theta(x)}\big)(g_x) = \mu_{\phi,\theta(x)}\big(\widehat{\mathcal{L}}_x g_x\big) &= \int_{\mathcal{J}_{\theta(x)}} \big(\widehat{\mathcal{L}}_x g_x\big)(y)\, \mathrm{d}\mu_{\phi,\theta(x)}(y) \\ &= \int_{\mathcal{J}_{\theta(x)}} \delta_y(\widehat{\mathcal{L}}_x g_x)\, \mathrm{d}\mu_{\phi,\theta(x)}(y) \\ &= \int_{\mathcal{J}_{\theta(x)}} \big(\widehat{\mathcal{L}}_x^* \delta_y\big)(g_x)\, \mathrm{d}\mu_{\phi,\theta(x)}(y) \\ &= \int_{\mathcal{J}_{\theta(x)}} \widehat{\mu}_{x,T_x^{-1}(y)}(g_x)\, \mathrm{d}\mu_{\phi,\theta(x)}(y). \end{aligned}$$

Hence,

$$\begin{aligned}
\mu_\phi(g) &= \int_X \mu_{\phi,x}(g_x)\,\mathrm{d}m(x) = \int_X\int_{\mathcal{J}_{\theta(x)}} \widehat{\mu}_{x,T_x^{-1}(y)}(g_x)\,\mathrm{d}\mu_{\phi,\theta(x)}(y)\,\mathrm{d}m(x)\\
&= \int_X\int_{\mathcal{J}_{\theta(x)}} \widehat{\mu}_{x,T_x^{-1}(y)}(g_x)\,\mathrm{d}\mu_{\phi,x}\circ T_x^{-1}(y)\,\mathrm{d}m(x)\\
&= \int_X\int_{\mathcal{J}_{\theta(x)}} \widehat{\mu}_{x,T_x^{-1}(T_x(w))}(g_x)\,\mathrm{d}\mu_{\phi,x}(w)\,\mathrm{d}m(x)\\
&= \int_X\int_{\mathcal{J}_{\theta(x)}} \widehat{\mu}_{x,w}(g_x)\,\mathrm{d}\mu_{\phi,x}(w)\,\mathrm{d}m(x)\\
&= \int_{\mathcal{J}} \widehat{\mu}_{x,y}(g_x)\,\mathrm{d}\mu(x,y).
\end{aligned}$$

This equality along with (5.5) and (5.6) completes the proof. $\square$

Now we are in position to prove the main result of this section.

Theorem 5.6. *If $T:\mathcal{J}\to\mathcal{J}$ is an accessible distance expanding RDS and if $\phi:\mathcal{J}\to\mathbb{R}$ is a Hölder continuous potential, then $\mathrm{P}(T,\phi)=\mathcal{E}\mathrm{P}(\phi)$, and μ_ϕ is the only relative equilibrium state for ϕ.*

Proof. Consider an arbitrary measure $\mu\in\mathcal{M}_m^1(T)$. Recall that $(\mu_{x,y})_{(x,y)\in\mathcal{J}}$ denotes its canonical system of conditional measures with respect to the partition $T^{-1}(\varepsilon_{\mathcal{J}})$. By the standard variational principle for finite sets we have for all $(x,y)\in\mathcal{J}$ that

$$\begin{aligned}
&\int_{T_x^{-1}(T_x(y))}\left[\mathrm{H}_{\mu_{x,y}}\left(\varepsilon_{\mathcal{J}_x}|_{T_x^{-1}(T_x(y))}\right)+\widehat{\phi}_x\right]\,\mathrm{d}\mu_{x,y} =\\
&\qquad = \mathrm{H}_{\mu_{x,y}}\left(\varepsilon_{\mathcal{J}_x}|_{T_x^{-1}(T_x(y))}\right)+\int_{T_x^{-1}(T_x(y))}\widehat{\phi}_x\,\mathrm{d}\mu_{x,y}\\
&\qquad \le \log\sum_{w\in T_x^{-1}(T_x(y))} e^{\widehat{\phi}_x(x,w)}\\
&\qquad = \log\left(\widehat{\mathcal{L}}_x\mathbb{1}_x\right)(T_x(y)) = \log 1 = 0.
\end{aligned} \tag{5.7}$$

Thus, in view of Propositions 2.5 and 4.2, we get

$$\begin{aligned}
\mathrm{h}_\mu(T|\theta)+&\int_{\mathcal{J}} \widehat{\phi}\,\mathrm{d}\mu = \\
&= \int_{\mathcal{J}} \mathrm{H}_{\mu_{x,y}}\left(\varepsilon_{\mathcal{J}_x}|_{T_x^{-1}(T_x(y))}\right)\,\mathrm{d}\mu(x,y) + \int_{\mathcal{J}}\int_{T_x^{-1}(T_x(y))} \widehat{\phi}_x\,\mathrm{d}\mu_{x,y}\,\mathrm{d}\mu(x,y) \\
&= \int_{\mathcal{J}}\int_{T_x^{-1}(T_x(y))} \left[\mathrm{H}_{\mu_{x,y}}\left(\varepsilon_{\mathcal{J}_x}|_{T_x^{-1}(T_x(y))}\right) + \widehat{\phi}_x\right]\,\mathrm{d}\mu_{x,y}\,\mathrm{d}\mu(x,y) \\
&\le 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{h}_\mu(T|\theta) + \int_{\mathcal{J}} \phi\,\mathrm{d}\mu &= \mathrm{h}_\mu(T|\theta) + \int_{\mathcal{J}} \widehat{\phi}\,\mathrm{d}\mu + \int_{\mathcal{J}} \log\lambda(x)\,\mathrm{d}\mu(x,y) \\
&\le \int_X \log\lambda\,\mathrm{d}m = \mathcal{E}\mathrm{P}(\phi).
\end{aligned} \tag{5.8}$$

The variational principle for finite sets also asserts that the equality in (5.7) holds if and only if

$$\begin{aligned}
\mu_{x,y} = \frac{\sum_{w\in T_x^{-1}(T_x(y))} e^{\widehat{\phi}_x(w)}\delta_w}{\sum_{w\in T_x^{-1}(T_x(y))} e^{\widehat{\phi}_x(w)}} &= \sum_{w\in T_x^{-1}(T_x(y))} e^{\widehat{\phi}_x(w)}\delta_w \\
&= \widehat{\mathcal{L}}_x^*(\delta_{T_x(y)}) = \widehat{\mu}_{x,y}.
\end{aligned} \tag{5.9}$$

Along with (5.8) and Lemma 5.5, this already shows that μ_ϕ is an equilibrium state for ϕ and that $\mathrm{P}(T,\phi) = \mathcal{E}\mathrm{P}(\phi)$. We are thus left only to show that (5.9) implies $\mu = \mu_\phi$. And indeed, it follows from (5.9) that for every $g_x \in \mathcal{C}(\mathcal{J}_x)$ we have

$$\begin{aligned}
\big(\widehat{\mathcal{L}}_x^*\mu_{\theta(x)}\big)(g_x) = \mu_{\theta(x)}(\widehat{\mathcal{L}}_x g_x) = \big(\mu_x\circ T_x^{-1}\big)(\widehat{\mathcal{L}}_x g_x) &= \mu_x\left((\widehat{\mathcal{L}}_x g_x)\circ T_x\right) \\
&= \int_{\mathcal{J}_x} \big(\widehat{\mathcal{L}}_x^*\delta_{T_x(y)}\big)(g_x)\,\mathrm{d}\mu_x(y) \\
&= \int_{\mathcal{J}_x} \mu_{x,y}(g_x)\mathrm{d}\mu_x(y) \\
&= \mu_x(g_x).
\end{aligned}$$

Thus, $\widehat{\mathcal{L}}_x^*\mu_{\theta(x)} = \mu_x$, and so by induction,

$$\widehat{\mathcal{L}}_x^{*n}\mu_{\theta^n(x)} = \mu_x$$

for all $n \ge 0$. Therefore, applying Proposition 3.19 of [9] (uniform convergence of the Perron-Frobenius operator), we get for m-a.e. $x \in X$ and for every $g_x \in \mathcal{C}(\mathcal{J}_x)$

that

$$\begin{aligned}\mu_x(g_x) &= \lim_{n\to\infty}\big(\widehat{\mathcal{L}}_x^{*n}\mu_{\theta^n(x)}\big)(g_x) = \lim_{n\to\infty}\mu_{\theta^n(x)}(\widehat{\mathcal{L}}_x^n g_x)\\ &= \lim_{n\to\infty}\mu_{\theta^n(x)}(\mu_{\phi,x}(g_x)\mathbb{1}_{\theta^n(x)})\\ &= \lim_{n\to\infty}\mu_{\phi,x}(g_x)\\ &= \mu_{\phi,x}(g_x).\end{aligned}$$

So, $\mu_x = \mu_{\phi,x}$ for m-a.e. $x \in X$, meaning that $\mu = \mu_\phi$. The proof is complete. $\square$

6. Fiberwise Hausdorff Dimension of Invariant Measures

In this section we deal with random conformal expanding maps.Their definition is as follows. Firstly, we are given a topological random map $f : \mathcal{J} \to \mathcal{J}$ which is of global type with a vertical space M. We further assume that M is a smooth compact Riemannian manifold (possibly with boundary) and that there exists a number $\kappa > 0$ such that

(a) $\overline{B(\mathcal{J}_x, \kappa)} \cap \partial M = \emptyset$ for m-a.e $x \in X$.[3]

(b) There exists $\alpha > 0$ such that all the fiber maps $f_x : \mathcal{J}_x \to \mathcal{J}_{\theta(x)}$ can be extended to $C^{1+\alpha}$–conformal maps from $B(\mathcal{J}_x, \kappa)$ to M, denoted by the same symbols f_x, which satisfy:

(b1) There exists a number $L' > 1$ and a measurable function $\gamma : X \to (1, L')$ such that

$$1 < \gamma_x \le |f_x'(z)| \le L' < +\infty$$

for m-a.e. $x \in X$ and for all $z \in B(\mathcal{J}_x, \kappa)$, where $|f_x'(z)|$ is the norm of the derivative $f_x'(z)$ of the map f_x and the point z. By the conformality of f_x, the number $|f_x'(z)|$ is simultaneously the similarity factor of $f_x'(z)$.

(b2) $\mathcal{J}$ is fully invariant i.e.

$$\mathcal{J}_x = f_x^{-1}(\mathcal{J}_{\theta(x)}) \text{ for } m\text{-a.e. } x \in X.$$

(b3) The function

$$\mathcal{J}_\kappa \ni (x, y) \mapsto \log|f_x'(y)| \in \mathbb{R}$$

belongs to $\mathcal{H}^\alpha(\mathcal{J}_\kappa)$, where

$$\mathcal{J}_\kappa = \bigcup_{x\in X}\{x\} \times B(\mathcal{J}_x, \kappa).$$

[3]The notation $B(\cdot, \cdot)$ with no subscript indicates that the ball is taken with respect to the ambient space M. In particular $B_x(y, r) = B(y, r) \cap \mathcal{J}_x$.

(c) The conditions of Measurability of the Degree, Topological Exactness, and Weak Perfectness from the definition of distance expanding hold for every $\xi > 0$.

If (a)-(c) are satisfied, then the map f is called a *random conformal expanding map*.

The proofs of the following two statements are quite standard; see for instance [5] p. 72 and [8] p. 73 for the corresponding proofs in their appropriate contexts. We presnt them here for the convenience of the reader.

Proposition 6.1. *Every random conformal expanding map is a random distance expanding map.*

Proof. Let f be a random conformal expanding map. Let $\beta' > 0$ be the minimum of the injectivity radius over the set $M \setminus B(\partial M, \kappa)$, and let $\beta = \min(\beta', \kappa)$. Let $\xi = \beta/(2L')$ and $L = \max(L', \mathrm{diam}(M)/\kappa)$. Let $\eta_x = \xi$ for every $x \in X$. We now claim that the conditions of Uniform Openness, Measurably Expanding, Lipschitz Continuity, and Log Integrable Compactness from the definition of distance expanding are satisfied. Since the others were assumed to be satisfied, this will show that f is a random distance expanding map. We will show the condition of Measurably Expanding while the proof of the other three will be left to the reader. Log Integrable Compactness follows just since f is of global type.

Fix $x \in X$ and $y, z \in \mathcal{J}_x$ with $\rho(y, z) \leq 2\eta_x = 2\xi$. Let $\gamma_1 : [0, 1] \to M$ be a length-minimizing geodesic from $f_x(y)$ to $f_x(z)$. Let t be the largest number such that there exists a path $\delta_1 : [0, t] \to M$ such that $f_x \circ \delta_1 = \gamma_1$. Then

$$\ell(\delta_1) \leq \frac{1}{\gamma_x}\ell(\gamma_1|_{[0,t]}) \leq \frac{1}{\gamma_x}\ell(\gamma) = \frac{1}{\gamma_x}\rho(f_x(y), f_x(z)) \leq \frac{1}{\gamma_x}2L'\xi < 2L'\xi = \beta \leq \kappa,$$

which implies that $\delta_1(t) \in B(\mathcal{J}_x, \kappa)$. If $t < 1$, then this implies that δ_1 can be extended, so $t = 1$. Let $w = \delta_1(1)$, and note that

$$\rho(y, w) \leq \ell(\delta_1) \leq \frac{1}{\gamma_x}\rho(f_x(y), f_x(z)).$$

On the other hand, $f_x(w) = \gamma_1(1) = f_x(z)$. To complete the proof we need to show that $w = z$. Let $\delta_2 : [0, 1] \to M$ be a length-minimizing geodesic from y to z. Then $\rho(y, \delta_2(t)) \leq 2\xi \leq \kappa$ for all $t \in [0, 1]$, so $\gamma_2 := f_x \circ \delta_2 : [0, 1] \to M$ is well-defined. Furthermore

$$\ell(\gamma_2) \leq L'\ell(\delta_2) \leq 2L'\xi = \beta;$$

on the other hand

$$\ell(\gamma_1) = \rho(f_x(y), f_x(z)) \leq 2L'\xi = \beta.$$

Thus, γ_1 and γ_2 are both paths from $f_x(y)$ to $f_x(z)$ of lengths less than or equal to β. On the other hand, $f_x(y) \in M \setminus B(\partial M, \kappa)$ by (a); thus the exponential map $\exp_y$ is injective on $B(0,\beta)$. Thus γ_1 and γ_2 can be pulled back to $B(0,\beta)$, a simply connected space. Thus γ_1 and γ_2 are homotopic in $B(f_x(y),\beta)$. The homotopy can be pulled back via f_x and thus δ_1 and δ_2 are homotopic. In particular their endpoints must be equal, so $w = z$. $\square$

For each $(x,y) \in \mathcal{J}$ and $n \in \mathbb{N}$, let $f_{x,y}^{-n} : B(f_x^n(y),\xi) \to B(y,\xi)$ be the unique continuous inverse branch of f_x^n such that $f_{x,y}^{-n}(f_x^n(y)) = y$. In the sequel we will need the following lemma:

Lemma 6.2. *There exists a measurable function $Q : X \to [0,+\infty]$ which is finite m-a.e. such that*

(6.1)
$$B\left(y, e^{-Q_{\theta^n(x)}\xi^\alpha}|(f_x^n)'(y)|^{-1}r\right) \subseteq f_{x,y}^{-n}\left(B(f_x^n(y),r)\right) \subseteq B\left(y, e^{Q_{\theta^n(x)}\xi^\alpha}|(f_x^n)'(y)|^{-1}r\right).$$

for every $n \in \mathbb{N}$, for m-a.e. $x \in X$, for every $y \in \mathcal{J}_x$, and for every $r \in (0,\xi]$.

Proof. Define $\phi : \mathcal{J} \to \mathbb{R}$ by

$$\phi(x,y) := -\log|f_x'(y)|.$$

Since $\phi \in \mathcal{H}^\alpha(\mathcal{J}_\kappa)$ by assumption, there exists a log-integrable function $H : X \to \mathbb{R}$ such that $\phi \in \mathcal{H}^\alpha(\mathcal{J}_\kappa, H)$. Following [9] we define the function $Q : X \to [0,+\infty]$ by

$$Q_x := \sum_{j=1}^{\infty} \left(H_{\theta^{-j}(x)}\right)\left(\gamma^j_{\theta^{-j}(x)}\right)^{-\alpha}. \tag{6.2}$$

By Lemma 2.3 of [9], Q is finite m-a.e. and

$$|S_n\phi(x, f_{x,y}^{-n}(w_1)) - S_n\phi(x, f_{x,y}^{-n}(w_2))| \le Q_{\theta^n(x)}\left(\rho(w_1,w_2)\right)^\alpha$$

for all $n \in \mathbb{N}$, for m-a.e. $x \in X$, for every $y \in \mathcal{J}_x$, and for every $w_1, w_2 \in B(f_x^n(y),\xi)$. In particular, for every $w \in B(f_x^n(y),\xi)$ we have

$$|S_n\phi(x, f_{x,y}^{-n}(w)) - S_n\phi(x,y)| \le Q_{\theta^n(x)}\xi^\alpha$$

i.e.

$$e^{-Q_{\theta^n(x)}\xi^\alpha}\frac{1}{|(f_x^n)'(y)|} \le |(f_{x,y}^{-n})'(w)| \le e^{Q_{\theta^n(x)}\xi^\alpha}\frac{1}{|(f_x^n)'(y)|}. \tag{6.3}$$

Now fix $x \in X$ satisfying (6.3) for every $n \in \mathbb{N}$, $y \in \mathcal{J}_x$, and $w \in B(f_x^n(y),\xi)$. Fix $n \in \mathbb{N}$, $y \in \mathcal{J}_x$, and $r \in (0,\xi]$. Now the second inclusion of (6.1) follows directly from the second inequality of (6.3) and the mean value inequality. The first inclusion of (6.1) is more subtle.

Suppose that $z \in B\left(y, e^{-Q_{\theta^n(x)}\xi^\alpha}|(f_x^n)'(y)|^{-1}r\right)$. Let $\delta : [0,1] \to M$ be a length-minimizing geodesic from y to z.

Claim 6.3. *$\delta(t) \in f_{x,y}^{-n}(B(f_x^n(y), \xi))$ for all $t \in (0,1)$.*

Proof. Suppose not; let $t_0 \in (0,1)$ be the smallest value where the claim fails. Then

$$f_x^n(\delta(t_0)) \in \partial B(f_x^n(y), \xi). \tag{6.4}$$

On the other hand, for all $t \in [0, t_0)$, we have $\delta(t) = f_{x,y}^{-n}(f_x^n(\delta(t)))$; letting $w = f_x^n(\delta(t))$, the first inequality of (6.3) can be rearranged to show that

$$|(f_x^n)'(\delta(t))| \leq e^{Q_{\theta^n(x)}\xi^\alpha}|(f_x^n)'(y)|. \tag{6.5}$$

Now by the mean value inequality we have

$$\begin{aligned} \rho(f_x^n(y), f_x^n(\delta(t_0))) &\leq e^{Q_{\theta^n(x)}\xi^\alpha}|(f_x^n)'(y)|\rho(y, \delta(t_0)) \\ &< e^{Q_{\theta^n(x)}\xi^\alpha}|(f_x^n)'(y)|\rho(y, z) \leq r \leq \xi, \end{aligned}$$

which contradicts (6.4). ◁

Thus, (6.5) holds for all $t \in (0,1)$. Applying the mean value inequality yields that $z \in f_{x,y}^{-n}\left(B(f_x^n(y), r)\right)$, proving the first inclusion of (6.1). □

If $\mu \in \mathcal{M}_m^1(f)$, then we call the number

$$\chi_\mu(f) := \int_{\mathcal{J}} \log|f_x'(z)| \, \mathrm{d}\mu(x,z) \geq \int_X \log\gamma \, \mathrm{d}m > 0$$

the *Lyapunov exponent of the invariant measure μ with respect to the conformal map f*. For each $x \in X$, the map $f_x : \mathcal{J}_x \to \mathcal{J}_{\theta(x)}$ is Lipschitz and expanding, and so since $\mu_{\theta(x)} = \mu_x \circ f_x^{-1}$, we have that $\mathrm{HD}(\mu_{\theta(x)}) = \mathrm{HD}(\mu_x)$. So, by the ergodicity of m, the function $X \ni x \mapsto \mathrm{HD}(\mu_x)$ is constant m-a.e. We denote this common number $\mathrm{FD}(\mu)$ and call it the *fiberwise Hausdorff dimension* of the measure μ. The first main result of this section is the following:

Theorem 6.4. *If $f : \mathcal{J} \to \mathcal{J}$ is a random conformal expanding map and if $\mu \in \mathcal{M}_m^1(f)$ is ergodic then*

$$\mathrm{FD}(\mu) = \liminf_{r \to 0} \frac{\log \mu_x(B_x(y,r))}{\log r} = \frac{h_\mu(f|\theta)}{\chi_\mu(f)}$$

for m-a.e. $x \in X$ and for μ_x-a.e. $y \in \mathcal{J}_x$.

We will prove the second equality first, as two directions, which will be separated as lemmas.

Lemma 6.5 ($\leq$ direction)**.** *With the hypotheses of Theorem* 6.4 *above,*

$$\liminf_{r\to 0} \frac{\log \mu_x(B_x(y,r))}{\log r} \leq \frac{h_\mu(f|\theta)}{\chi_\mu(f)}$$

for m-a.e. $x \in X$ and for μ_x–a.e. $y \in \mathcal{J}_x$.

Proof. By Proposition 4.1 there exists a partition α accessible for the map $f : \mathcal{J} \to \mathcal{J}$.

Claim 6.6.

$$\alpha_x^n(y) \subseteq f_{x,y}^{-n}(B_{\theta^n(x)}(f_x^n(y),\xi)) \tag{6.6}$$

for all $(x,y) \in \mathcal{J}$ and for all integers $n \geq 0$.

Proof. Fix $z \in \alpha_x^n(y)$. Then for all $j = 0,\ldots,n$, we have $f_x^j(z) \in \alpha_{\theta^j(x)}(f_x^j(y))$; since $\operatorname{diam}(\alpha_{\theta_j(x)}) \leq \xi$, we have

$$f_x^j(z) \in B_{\theta^j(x)}(f_x^j(y),\xi).$$

Let $w = f_{x,y}^{-n}(f_x^n(z))$; then

$$f_x^j(w) \in B_{\theta^j(x)}(f_x^j(y),\xi) \ \ \forall j = 0,\ldots,n,$$

and thus $\rho(f_x^j(z), f_x^j(w)) \leq 2\xi$ for all $f = 0,\ldots,n$. Thus by the Measurably Expanding condition

$$f_x^{j+1}(z) = f_x^{j+1}(w) \Rightarrow f_x^j(z) = f_x^j(w)$$

for all $j = 0,\ldots,n-1$. Iterating gives $z = w$, so $z \in f_{x,y}^{-n}(B_{\theta^n(x)}(f_x^n(y),\xi))$. $\triangleleft$

On the other hand, by Lemma 6.2, we have

$$f_{x,y}^{-n}\left(B_{\theta^n(x)}(f_x^n(y),\xi)\right) \subseteq B_x\left(y, e^{Q_{\theta^n(x)}\xi^\alpha}|(f_x^n)'(y)|^{-1}\xi\right), \tag{6.7}$$

where $Q : X \to [0,+\infty]$ is the (m-a.e. finite) function defined by (6.2). Now, by Birkhoff's Ergodic Theorem and by the ergodicity of the dynamical system $(\theta : X \to X, m)$, there exists a constant $A \in (0,+\infty)$ such that for m-a.e. $x \in X$ there exists an unbounded increasing sequence $(n_k(x))_1^\infty$ such that

$$Q_{\theta^{n_k(x)}}(x) \leq A \tag{6.8}$$

for all integers $k \geq 1$. Also, by Birkhoff's Ergodic Theorem and by the ergodicity of the dynamical system $(f : \mathcal{J} \to \mathcal{J}, \mu)$, there exists a measurable set $\widetilde{X} \subseteq X$ such that $m(\widetilde{X}) = 1$, and such that for all $x \in \widetilde{X}$ there exists a Borel set $\widetilde{\mathcal{J}}_x \subseteq \mathcal{J}_x$ with $\mu_x(\widetilde{\mathcal{J}}_x) = 1$ such that for every $\varepsilon > 0$, every $x \in \widetilde{X}$, and every $y \in \widetilde{\mathcal{J}}_x$ there exists $N(x,y) \geq 1$ such that

$$|(f_x^n)'(y)| \geq e^{A\xi^\alpha}\xi \exp\left((\chi_\mu(f) - \varepsilon)n\right)$$

for all $n \geq N(x,y)$. Inserting this into (6.7) and using (6.8), formula (6.6) yields

$$\alpha_x^{n_k(x)}(y) \subseteq B_x\left(y, \exp\left(-(\chi_\mu(f)-\varepsilon)n_k(x)\right)\right)$$

for all $x \in \widetilde{X}$, for all $y \in \widetilde{\mathcal{J}}_x$, and for all $k \geq 1$ large enough so that $n_k(x) \geq N(x,y)$. Thus, putting

$$r_k(x) = \exp\left(-(\chi_\mu(f)-\varepsilon)n_k(x)\right),$$

we get

$$\frac{\log \mu_x(B_x(y, r_k(x)))}{\log r_k(x)} \leq \frac{\log \mu_x(\alpha_x^{n_k(x)}(y))}{\log r_k(x)} = -\frac{\log \mu_x(\alpha_x^{n_k(x)}(y))}{(\chi_\mu(f)-\varepsilon)n_k(x)}.$$

Applying Corollary 4.3, we have the following:

$$\begin{aligned}\liminf_{r\to 0}\frac{\log \mu_x(B_x(y,r))}{\log r} &\leq \liminf_{k\to\infty}\frac{\log \mu_x(B_x(y,r_k(x)))}{\log r_k(x)} \\ &\leq \frac{-1}{\chi_\mu(f)-\varepsilon}\lim_{k\to\infty}\frac{1}{n_k(x)}\log \mu_x(\alpha_x^{n_k(x)}(y)) \\ &= \frac{\mathrm{h}_\mu(f|\theta)}{\chi_\mu(f)-\varepsilon}.\end{aligned}$$

So, letting $\varepsilon \searrow 0$, we get that

$$\liminf_{r\to 0}\frac{\log \mu_x(B_x(y,r))}{\log r} \leq \frac{\mathrm{h}_\mu(f|\theta)}{\chi_\mu(f)}$$

for all $x \in \widetilde{X}$ and for all $y \in \widetilde{\mathcal{J}}_x$. The proof is complete. $\square$

Now, we shall prove the other inequality, i.e.

Lemma 6.7 ($\geq$ direction)**.** *With the hypotheses of Theorem* 6.4 *above,*

$$\liminf_{r\to 0}\frac{\log \mu_x(B_x(y,r))}{\log r} \geq \frac{h_\mu(f|\theta)}{\chi_\mu(f)}$$

for m-a.e. $x \in X$ and for μ_x-a.e. $y \in \mathcal{J}_x$.

Proof. Let $Q : X \to [0,+\infty]$ be the (m-a.e. finite) function defined by (6.2). By Birkhoff's Ergodic Theorem and by the ergodicity of m, there exists a constant $A \in (0,+\infty)$ and a measurable set $X_0 \subseteq X$ such that $m(X_0)=1$, and such that for all $x \in X_0$, if $(n_k(x))_1^\infty$ is the increasing sequence consisting of all $n \geq 0$ such that $Q_{\theta^n(x)} \leq A$, then

$$\lim_{k\to\infty}\frac{n_{k+1}(x)-n_k(x)}{n_k(x)} = 0.$$

Consequently, for all $x \in X_0$ and for all $\varepsilon > 0$ there exists $k_\varepsilon(x) \geq 1$ such that if $k \geq k_\varepsilon(x)$, then

$$\frac{n_{k+1}(x) - n_k(x)}{n_k(x)} < \varepsilon. \tag{6.9}$$

Applying Corollary 9.1.9 of [10] (in the printed version) with $\nu = \mu \circ \pi_Y^{-1}$ and $r = \widehat{\xi}/(2L)$ yields the existence of a finite partition $\mathcal{P} = \{P_1, P_2, \ldots, P_p\}$ of Y into Borel sets such that

$$\operatorname{diam}(\mathcal{P}) < \frac{\widehat{\xi}}{2L}, \quad \nu(\partial \mathcal{P}) = 0,$$

and such that for some constant $C > 0$, we have

$$\nu(\partial_a \mathcal{P}) \leq Ca \ \ \forall a > 0,$$

where

$$\partial_a \mathcal{P} := \bigcap_{j=1}^{p} \bigcup_{i \neq j} B(P_i, a).$$

Then

$$\alpha := \pi_Y^{-1}(\mathcal{P}) \vee \pi_X^{-1}(\varepsilon_X)|_{\mathcal{J}}$$

is an accessible partition.

Claim 6.8. *For every $\varepsilon > 0$ there exists a measurable set $X_\varepsilon^1 \subseteq X_0$ with $m(X_\varepsilon^1) = 1$, such that for every $x \in X_\varepsilon^1$ there exists a measurable set $\mathcal{J}_{x,\varepsilon}^1 \subseteq \mathcal{J}_x$ with $\mu_x\left(\mathcal{J}_{x,\varepsilon}^1\right) = 1$ such that for every $y \in \mathcal{J}_{x,\varepsilon}^1$, there exists an integer $n_\varepsilon^{(1)}(x,y) \geq n_{k_\varepsilon(x)}(x)$ such that for every $n \geq n_\varepsilon^{(1)}(x,y)$,*

$$B_{\theta^n(x)}(f_x^n(y), e^{-\varepsilon n}) \subseteq \alpha_{\theta^n(x)}(f_x^n(y)). \tag{6.10}$$

Proof. By the fiberwise invariance of μ, we have for all $(x,y) \in \mathcal{J}$

$$\sum_{n=0}^{\infty} \mu_x(f_x^{-n}(\partial_{e^{-\varepsilon n}} \alpha_{\theta^n(x)})) = \sum_{n=0}^{\infty} \mu_{\theta^n(x)}(\partial_{e^{-\varepsilon n}} \alpha_{\theta^n(x)}),$$

and hence

$$\begin{aligned}\int_X \left(\sum_{n=0}^{\infty} \mu_x(f_x^{-n}(\partial_{e^{-\varepsilon n}}\alpha_{\theta^n(x)}))\right) \mathrm{d}m(x) &= \sum_{n=0}^{\infty} \int_X \mu_{\theta^n(x)}(\partial_{e^{-\varepsilon n}}\alpha_{\theta^n(x)})\,\mathrm{d}m(x) \\ &= \sum_{n=0}^{\infty} \int_X \mu_x(\partial_{e^{-\varepsilon n}}\alpha_x)\,\mathrm{d}m(x) \\ &\le \sum_{n=0}^{\infty} \nu(\partial_{e^{-\varepsilon n}}\mathcal{P}) \\ &\le C\sum_{n=0}^{\infty} e^{-\varepsilon n} < +\infty.\end{aligned}$$

Thus, there exists a measurable set $X_\varepsilon^1 \subseteq X_0$ with $m(X_\varepsilon^1) = 1$ such that

$$\sum_{n=0}^{\infty} \mu_x(f_x^{-n}(\partial_{e^{-\varepsilon n}}\alpha_{\theta^n(x)})) < +\infty \ \ \forall x \in X_\varepsilon^1.$$

Thus by the Borel–Cantelli lemma, for every $x \in X_\varepsilon^1$ there exists a measurable set $\mathcal{J}_{x,\varepsilon}^1 \subseteq \mathcal{J}_x$ with $\mu_x\left(\mathcal{J}_{x,\varepsilon}^1\right) = 1$, such that for every $y \in \mathcal{J}_{x,\varepsilon}^1$ there exists $n_\varepsilon^{(1)}(x,y) \ge n_{k_\varepsilon(x)}(x)$ such that $y \notin f_x^{-n}(\partial_{e^{-\varepsilon n}}\alpha_{\theta^n(x)})$ for all $n \ge n_\varepsilon^{(1)}(x,y)$. Equivalently,

$$f_x^n(y) \notin \partial_{e^{-\varepsilon n}}\alpha_{\theta^n(x)}.$$

This means that there exists $j \in \{1,2,\dots,p\}$ such that $f_x^n(y) \notin \bigcup_{i\neq j} B(P_i, e^{-\varepsilon n})$. On the other hand, we have $f_x^n(y) \in \alpha_{\theta^n(x)}(f_x^n(y)) = P_s$ for some $s \in \{1,2,\dots,p\}$. We must have $j = s$, and so $f_x^n(y) \in P_s \setminus \bigcup_{i\neq s} B(P_i, e^{-\varepsilon n})$. Therefore,

$$B_{\theta^n(x)}(f_x^n(y), e^{-\varepsilon n}) \subseteq P_s = \alpha_{\theta^n(x)}(f_x^n(y)),$$

implying that (6.10) holds. ◁

Let

$$\widehat{X}_1 := \bigcap_{k=1}^{\infty} X_{1/k}^1 \cap \bigcap_{j=0}^{\infty} \theta^{-j}(\{x \in X : \mu_x(\partial\alpha_x) = 0\}),$$

and for each $x \in \widehat{X}_1$, let

$$\widehat{\mathcal{J}}_{x,1} = \bigcap_{k=1}^{\infty} \mathcal{J}_{x,1/k}^1 \setminus \bigcup_{j=0}^{\infty} f_x^{-j}\partial\alpha_{\theta^j(x)}$$

Then $m(\widehat{X}_1) = 1$, and $\mu_x(\widehat{\mathcal{J}}_{x,1}) = 1$ for all $x \in \widehat{X}_1$.

As in the previous lemma, by Birkhoff's Ergodic Theorem and by the ergodicity of the dynamical system $(f : \mathcal{J} \to \mathcal{J}, \mu)$, there exists a measurable set $\widehat{X}_2 \subseteq \widehat{X}_1$ with $m(\widehat{X}_2) = 1$ such that for all $x \in \widehat{X}_2$ there exists a measurable set $\widehat{\mathcal{J}}_{x,2} \subseteq \widehat{\mathcal{J}}_{x,1}$

such that $\mu_x\left(\widehat{\mathcal{J}}_{x,2}\right) = 1$ and such that for every $y \in \widehat{\mathcal{J}}_{x,2}$ and for every $\varepsilon > 0$, there exists $n_\varepsilon^{(2)}(x,y) \geq n_\varepsilon^{(1)}(x,y)$ such that

$$|(f_x^n)'(y)| \leq e^{-A\xi^\alpha} \exp\left((\chi_\mu(f) + \varepsilon)n\right) \tag{6.11}$$

for all $n \geq n_\varepsilon^{(2)}(x,y)$. Now fix $x \in \widehat{X}_2$ and $y \in \widehat{\mathcal{J}}_{x,2}$. Since $y \notin f_x^{-j}(\partial\alpha_{\theta^j(x)})$ for all $j \in \mathbb{N}$, there exists $K \geq 1$ (depending on x, y, and ε) large enough so that that $K^{-1}\xi \leq 1$ and so that

$$f_x^q\left(B_x\left(y, K^{-1}\xi \exp\left(-(\chi_\mu(f) + (2 + \log L)\varepsilon)n_\varepsilon^{(2)}(x,y)\right)\right)\right) \subseteq \alpha_{\theta^q(x)}(f_x^q(y)) \tag{6.12}$$

for all $0 \leq q \leq n_\varepsilon^{(2)}(x,y)$. Now for each $n \in \mathbb{N}$ let

$$s_n(\varepsilon) := K^{-1}\xi \exp\left(-(\chi_\mu(f) + (2 + \log L)\varepsilon)n\right).$$

Fix an arbitrary $n \geq n_\varepsilon^{(2)}(x,y)$. By (6.12), we have that

$$f_x^q\left(B_x\left(y, s_n(\varepsilon)\right)\right) \subseteq \alpha_{\theta^q(x)}(f_x^q(y)) \tag{6.13}$$

for all $0 \leq q \leq n_\varepsilon^{(2)}(x,y)$.

Claim 6.9. (6.13) *also holds for q such that $n_\varepsilon^{(2)}(x,y) \leq q \leq n$.*

Proof. Fix such a q. Since $q \geq n_{k_\varepsilon(x)}(x)$, there exists a unique $k \geq k_\varepsilon(x)$ such that

$$n_k(x) \leq q < n_{k+1}(x). \tag{6.14}$$

Put

$$r_{k,n}(x) := K^{-1}\xi \exp\left(-\chi_\mu(f)(n - n_k(x))\right) \exp\left(\varepsilon(n_k(x) - (2 + \log L)n)\right).$$

Then by (6.11) we get

$$e^{-A\xi^\alpha} r_{k,n}(x) |(f_x^{n_k(x)})'(y)|^{-1} \geq s_n(\varepsilon)$$

and combining with Lemma 6.2 gives

$$\begin{aligned} f_{x,y}^{-n_k(x)}\left(B_{\theta^{n_k(x)}(x)}\left(f_x^{n_k(x)}(y), r_{k,n}(x)\right)\right) &\supseteq B_x\left(y, e^{-A\xi^\alpha} r_{k,n}(x) |(f_x^{n_k(x)})'(y)|^{-1}\right) \\ &\supseteq B_x\left(y, s_n(\varepsilon)\right). \end{aligned}$$

Rearranging, we see that

$$f_x^{n_k(x)}\left(B_x\left(y, s_n(\varepsilon)\right)\right) \subseteq B_{\theta^{n_k(x)}(x)}\left(f_x^{n_k(x)}(y), r_{k,n}(x)\right).$$

Now, by (6.9) and (6.14) we have $q - n_k(x) < \varepsilon n_k(x)$. We therefore obtain

$$
\begin{aligned}
f_x^q\big(B_x\big(y,s_n(\varepsilon)\big)\big) = & \\
&= f_{\theta^{n_k(x)}(x)}^{q-n_k(x)}\left(f_x^{n_k(x)}\left(B_x\left(y, s_n(\varepsilon)\right)\right)\right) \\
&\subseteq f_{\theta^{n_k(x)}(x)}^{q-n_k(x)}\left(B_{\theta^{n_k(x)}(x)}\left(f_x^{n_k(x)}(y), r_{k,n}(x)\right)\right) \\
&\subseteq B_{\theta^q(x)}\big(f_x^q(y), r_{k,n}(x)\exp\big((q-n_k(x))\log L\big)\big) \\
&\subseteq B_{\theta^q(x)}\big(f_x^q(y), r_{k,n}(x)\exp\big(\varepsilon n_k(x)\log L\big)\big) \\
&\subseteq B_{\theta^q(x)}\left(f_x^q(y), K^{-1}\xi\exp\left(\varepsilon\left(n_k(x)-(2+\log L)n+n_k(x)\log L\right)\right)\right) \\
&= B_{\theta^q(x)}\left(f_x^q(y), K^{-1}\xi\exp\left(\varepsilon(n_k(x)-n)-\varepsilon n+\varepsilon(n_k(x)-n)\log L\right)\right) \\
&\subseteq B_{\theta^q(x)}\left(f_x^q(y), K^{-1}\xi e^{-\varepsilon n}\right) \\
&\subseteq B_{\theta^q(x)}\left(f_x^q(y), e^{-\varepsilon n}\right) \\
&\subseteq \alpha_{\theta^q(x)}(f_x^q(y)).
\end{aligned}
$$

The last inclusion was written due to (6.10). $\triangleleft$

Thus, (6.13) holds for all $0 \le q \le n$. It follows that

$$B_x\left(y, s_n(\varepsilon)\right) \subseteq \alpha_x^n(y) \tag{6.15}$$

for all $x \in \widehat{X}_2$, for all $y \in \widehat{\mathcal{J}}_{x,2}$, and for all $n \ge n_\varepsilon^{(2)}(x,y)$.

Now fix $x \in \widehat{X}_2$ and $y \in \widehat{\mathcal{J}}_{x,2}$. Since the sequence $(s_n(\varepsilon))_1^\infty$ is geometric, we have

$$\liminf_{r\to 0}\frac{\log\mu_x(B_x(y,r))}{\log r} = \liminf_{n\to\infty}\frac{\log\mu_x(B_x(y,s_n(\varepsilon)))}{\log s_n(\varepsilon)}.$$

On the other hand, by (6.15) we have

$$
\begin{aligned}
\liminf_{n\to\infty}\frac{\log\mu_x(B_x(y,s_n(\varepsilon)))}{\log s_n(\varepsilon)} &\ge \liminf_{n\to\infty}\frac{\log\mu_x(\alpha_x^n(y))}{\log s_n(\varepsilon)} \\
&= \liminf_{n\to\infty}\frac{\log\mu_x(\alpha_x^n(y))}{-(\chi_\mu(f)+(3+\log L)\varepsilon)n}.
\end{aligned}
$$

Finally, Corollary 4.3 yields

$$\lim_{n\to\infty}\frac{\log\mu_x(\alpha_x^n(y))}{-n} = \mathrm{h}_\mu(f|\theta) \text{ for } \mu\text{-a.e. } (x,y)\in\mathcal{J}$$

and combining everything gives

$$\liminf_{r\to 0}\frac{\log\mu_x(B_x(y,r))}{\log r} \ge \frac{\mathrm{h}_\mu(f|\theta)}{\chi_\mu(f)+(3+\log L)\varepsilon} \text{ for } \mu\text{-a.e. } (x,y)\in\mathcal{J}.$$

Now letting $\varepsilon \searrow 0$ gives

$$\liminf_{r\to 0} \frac{\log \mu_x(B_x(y,r))}{\log r} \geq \frac{\mathrm{h}_\mu(f|\theta)}{\chi_\mu(f)}.$$

The proof of Lemma 6.7 is complete. $\square$

Combining Lemmas 6.5 and 6.7 together with Theorem 8.6.5 of [10] (printed version) proves Theorem 6.4.

We now want to address and fully answer the question of which invariant measures are of full Hausdorff dimension. Specifically, for each $t \in \mathbb{R}$ consider the Hölder continuous potential

$$\mathcal{J} \ni (x,y) \mapsto \phi_t(x,y) := -t \log |f_x'(y)|.$$

Let μ_t be its unique relative Gibbs/equilibrium state. It was proven in [9] that there exists a unique $h \in \mathbb{R}$ such that $\mathcal{E}\mathrm{P}(h) = 0$. The proof of Theorem 5.2 (Bowen's Formula) in [9] gives that

$$\mathrm{FD}(\mu_h) = h = \mathrm{HD}(\mathcal{J}_x) \geq 0 \text{ for } m\text{-a.e. } x \in X. \tag{6.16}$$

Our second main theorem in this section is the following:

Theorem 6.10. *If $f : \mathcal{J} \to \mathcal{J}$ is a random conformal expanding map, then μ_h, the unique relative Gibbs/equilibrium state for the potential $\phi_h(x,y) := -h \log |f_x'(y)|$, is the unique ergodic measure $\mu \in \mathcal{M}^1_{m,e}(\mathcal{J})$ such that*

$$\mathrm{FD}(\mu) = h (= \mathrm{HD}(\mathcal{J}_x)) \text{ for } m\text{-a.e. } x \in X. \tag{6.17}$$

Proof. Obviously, formula (6.16) shows that μ_h satisfies (6.17). For the uniqueness, suppose that μ satisfies (6.17), i.e. $\mathrm{FD}(\mu) = h$. Then by Theorem 6.4 we have that $\mathrm{h}_\mu(f|\theta) = h\chi_\mu(f)$, or equivalently, that $\mathrm{h}_\mu(f|\theta) + \int \phi_h \, \mathrm{d}\mu = 0$. Since $\mathcal{E}\mathrm{P}(h) = 0$, by Theorem 5.6 this means that μ is an equilibrium state for the potential ϕ_h. But since μ_h is such a state, a different assertion of Theorem 5.6 yields $\mu = \mu_h$. The proof is complete. $\square$

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