# DYNAMICAL RIGIDITY OF TRANSCENDENTAL MEROMORPHIC FUNCTIONS

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Abstract. We prove the form of dynamical rigidity of transcendental meromorphic functions which asserts that if two tame transcendental meromorphic functions restricted to their Julia sets are topologically conjugate via a locally bi–Lipschitz homeomorphism, then they, treated as functions defined on the entire complex plane $\mathbb{C}$, are topologically conjugate via an affine map, i.e. a map from $\mathbb{C}$ to $\mathbb{C}$ of the form $z \mapsto az + b$. As an intermediate step we show that no tame transcendental meromorphic function is essentially affine.

## 1. Introduction

Let $X$ and $Y$ be arbitrary metric spaces. We say that a homeomorphism $H : X \to Y$ is locally bi-Lipschitz if each point $x \in X$ has some open neighborhood $U_x$ such that both the restriction $H|_{U_x} : U_x \to H(U_x)$ and its inverse $\left(H|_{U_x}\right)^{-1} : H(U_x) \to U_x$ are Lipschitz continuous. The main goal of this paper is to show that if two tame transcendental meromorphic functions restricted to their Julia sets are topologically conjugate via a locally bi–Lipschitz homeomorphism, then they, treated as functions defined on the entire complex plane $\mathbb{C}$, are topologically conjugate via an affine map, i.e. a map from $\mathbb{C}$ to $\mathbb{C}$ of the form $z \mapsto az + b$. As an intermediate step we show that the iterated function system corresponding to any nice set of a transcendental meromorphic function is not essentially affine.

Our work stems from D. Sullivan article [12] treating among others the dynamical rigidity of conformal expanding repellers. Its systematical account can be found in [8]. We make also an essential use of the rigidity result for conformal iterated function systems from [6]. The case of tame rational functions is actually done in [7].

The structure of our argument is this. First we make use of the existence of nice sets for tame transcendental meromorphic functions as proved in [2].

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This means we canonically associate, as in [11], to each nice set $U$ a conformal iterated function system $S_U$ in the sense of [4] and [5]. Then we show (Proposition 3.3) that no such systems $S_U$ are essentially affine. Having this we strengthen the dynamical rigidity result from [6] to conclude that any locally bi–Lipschitz conjugacy between two tame transcendental meromorphic functions yields conformal conjugacy on some neighborhoods of the closures of the limit sets of the associated (via nice sets) iterated function systems. As the last step we prove that such conjugacy extends holomorphically to a holomorphic automorphism of the complex plane $\mathbb{C}$. It thus must be an affine map $z \mapsto az+b$.

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## 2. Preliminaries

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function. The Fatou set of $f$ consists of all points $z \in \mathbb{C}$ that admit an open neighborhood $U_z$ such that all the forward iterates $f^n$, $n \geq 0$, of $f$ are well-defined on $U_z$ and the family of maps $\{f^n|_{U_z} : U_z \to \mathbb{C}\}_{n=0}^{\infty}$ is normal. The Julia set of $f$, denoted by $J_f$, is then defined as the complement of the Fatou set of $f$ in $\mathbb{C}$. By $\mathrm{Sing}(f^{-1})$ we denote the set of singularities of $f^{-1}$. We define the *postsingular* set of $f : \mathbb{C} \to \hat{\mathbb{C}}$ as

$$\mathrm{PS}(f) = \overline{\bigcup_{n=0}^{\infty} f^n(\mathrm{Sing}(f^{-1}))}.$$

Given a set $F \subset \hat{\mathbb{C}}$ and $n \geq 0$, by $\mathrm{Comp}(f^{-n}(F))$ we denote the collection of all connected components of the inverse image $f^{-n}(F)$. A meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ is called *tame* if its postsingular set does not contain its Julia set. This is the primary object of our interest in this paper.

We make a heavy use of the concept of *a nice set* which J. Rivera–Letelier introduced in [10] in the realm of the dynamics of rational maps of the Riemann sphere. In [2] N. Dobbs proved their existence for some meromorphic functions from $\mathbb{C}$ to $\hat{\mathbb{C}}$. The following theorem follows directly from Lemma 11 from [2].

**Theorem 2.1.** *Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a tame meromorphic function. Fix $z \in J_f \setminus \mathrm{PS}(f)$, $K > 1$ and $\kappa > 1$. Then there exists $L > 1$ such that for all $r > 0$ sufficiently small, there exists an open connected set $U = U(z, r) \subset \mathbb{C} \setminus \mathrm{PS}(f)$ such that, for all $n > 0$,*

(a) *if $V \in Comp(f^{-n}(U))$ and $V \cap U \neq \emptyset$, then $\overline{V} \subset U$,*

(b) *if* $V \in Comp(f^{-n}(U))$ *and* $V \cap U \neq \emptyset$, *then, for all* $w, w' \in V$,

$$|(f^n)'(w)| \geq L \quad and \quad \frac{|(f^n)'(w)|}{|(f^n)'(w')|} \leq K,$$

(c) $\overline{B(z,r)} \subset U \subset B(z, \kappa r) \subset \mathbb{C} \setminus \mathrm{PS}(f)$.

Let $\mathcal{U}$ be the collection of all nice sets of $f : \mathbb{C} \to \hat{\mathbb{C}}$, i.e. all the sets $U$ satisfying the above proposition with some $z \in J_f \setminus \mathrm{PS}(f)$ and some $r > 0$. Note that if $U = U(z,r) \in \mathcal{U}$ and $V \in \mathrm{Comp}(f^{-n}(U)$ satisfies the requirements (a), (b) and (c) from Theorem 2.1 then there exists a unique holomorphic inverse branch $f_V^{-n} : B(z, \kappa r) \to \mathbb{C}$ such that $f_V^{-n}(U) = V$. As noted in [11] the collection $S_U$ of all such inverse branches forms obviously an iterated function system in the sense of [4] and [5]. In particular, it clearly satisfies the Open Set Condition. We denote its limit set by $J_U$. We have just mentioned [4] and [5]. In what concerns iterated function systems we try our concepts and notation to be compatible with that of [5].

## 3. Essential Affinity

In this section we prove that the iterated function system corresponding to any nice set of a transcendental meromorphic function is not essentially affine. An important step in this proof is provided by Lemma 3.2. Toward this direction let us recall first the following proposition which follows from a theorem of I. N. Baker in [1].

**Proposition 3.1.** *Let* $\psi : \mathbb{C} \to \hat{\mathbb{C}}$ *be a meromorphic function and let* $A_j : \mathbb{C} \to \mathbb{C}$, $j = 1, 2$, *be two affine maps such that for all* $z$

$$\psi(A_1(z)) = \psi(A_2(z)). \tag{3.1}$$

*Then there is a root of unity* $\lambda \in \mathbb{C}$ *such that*

$$A_2^{-1} \circ A_1(z) = \lambda z + \beta.$$

*It follows that there exists* $l \geq 1$ *such that*

$$(A_2^{-1} \circ A_1)^l = T_b, \tag{3.2}$$

*where* $T_b(z) = z + b$.

**Lemma 3.2.** *Let* $f : \mathbb{C} \to \hat{\mathbb{C}}$ *be a transcendental meromorphic function and let* $A_j : \mathbb{C} \to \mathbb{C}$, $j = 1, 2$, *be two affine maps,* $A_1(z) = a_1 z$ *and* $A_2(z) = a_2 z + b_2$ *such that* $0 < |a_j| < 1$ *and* $A_2^k \neq A_1^k$ *for all integers* $k \geq 1$. *Let* $U \subset \mathbb{C}$ *be an open connected set such that* $A_j(U) \subset U$ *for* $j = 1, 2$. *If* $\psi : U \to \hat{\mathbb{C}}$ *is a non-constant meromorphic function, then there is no integer* $q \geq 1$ *such that*

$$\psi(z) = f^q \circ \psi \circ A_j(z)$$

*for $j = 1, 2$ and all $z \in U$.*

As the first step of our proof of this lemma, we shall demonstrate the following claim.

**Claim.** *Suppose that*

$$\psi(z) = f^q \circ \psi \circ A_j(z) \tag{3.3}$$

*for some $q \geq 1$, all $j = 1, 2$ and all $z \in U$ with $U$, $\psi$, $f : \mathbb{C} \to \hat{\mathbb{C}}$ as in Lemma 3.2. Then the function $\psi : U \to \hat{\mathbb{C}}$ has a unique meromorphic extension to all of $\mathbb{C}$ such that (3.3) holds for $j = 1, 2$ and all $z \in U$.*

*Proof of Claim.* First note that in order to prove the claim all what we need to show is that $\psi : U \to \hat{\mathbb{C}}$ has a meromorphic extension onto $\mathbb{C}$. We shall show by induction that for every $n \geq 0$ the map $\psi$ has a unique meromorphic extension to $A_1^{-n}(U)$ such that (3.3) holds on $A_1^{-n}(U)$. Indeed, for $n = 0$ this is our seeking contradiction assumption. For the inductive step suppose it holds for some $n \geq 0$. Define then the function $\psi_* : A_1^{-(n+1)}(U) \to \hat{\mathbb{C}}$ as

$$\psi_*(z) = f^q \circ \psi \circ A_j(z). \tag{3.4}$$

Note that $A_1^{-n}(U) \subset A_1^{-(n+1)}(U)$ and, by our inductive assumption, $\psi_*(z) = \psi(z)$ for all $z \in A_1^{-n}(U)$. Renaming then $\psi_*$ to $\psi$, (3.4) holds for all $z \in A_1^{-(n+1)}(U)$. The uniqueness part follows from the fact that two meromorphic functions defined on the open connected set $A_1^{-n}(U)$ and coinciding on its open subset $U$, are equal. The inductive proof is finished. Since $\bigcup_{n=0}^{\infty} A_1^{-n}(U) = \mathbb{C}$, the claim is proved. $\square$

*Proof of Lemma 3.2.* Seeking contradiction, we suppose that (3.3) holds. It follows from Claim that $\psi = f^q \circ \psi \circ A_1 = f^q \circ \psi \circ A_2$ on $\mathbb{C}$, and therefore $\psi \circ A_1^{-1} = f^q \circ \psi = \psi \circ A_2^{-1}$. This yields

$$\psi = \psi \circ (A_2^{-1} \circ A_1). \tag{3.5}$$

Then by Proposition 3.1 we can write

$$(A_2^{-1} \circ A_1)^l = T_b,$$

where $T_b(z) = z + b$ with some $b \in \mathbb{C} \setminus \{0\}$. Formula (3.5) then yields

$$\psi = \psi \circ (A_2^{-1} \circ A_1)^l = \psi \circ T_b.$$

This just means that the function $\psi$ is periodic. Having this we are ready to show that our equation equation,

$$\psi(mz) = f^q(\psi(z)), \tag{3.6}$$

where $m = a_1^{-1}$, leads to a contradiction. Such equations were studied in [9]. Although, Ritt treated the case of a rational function $R$ instead of a transcendental function $f^q$, his arguments do not depend on a particular form of $R$.

Let $\mathcal{R}$ be a fundamental domain for $\psi$. Then the map $f^q \circ \psi|_{\mathcal{R}}$ is infinite-to-one, since $f^q$ is transcendental. But if $\psi$ is doubly periodic (so $\mathcal{R}$ is bounded), then obviously the map $\mathcal{R} \ni z \mapsto \psi(mz)$ is finite-to-one map, generating a contradiction. So, assume instead that $\psi$ is simply periodic. We claim that if $h$ is any period of $\psi$, then so is $mh$. Indeed, by (3.6), we get

$$\psi(z + mh) = \psi(m(z/m + h)) = f^q(\psi(z/m + h)) = f^q(\psi(z/m)) = \psi(z).$$

Thus $m$ has to be a real number. Using now Section III and IV of [9] literally, we get that $\psi$ is at most two-to-one on the fundamental stripe $\mathcal{R}$. Since $m$ is real, so is the map $\mathcal{R} \ni z \mapsto \psi(mz)$. This contradiction finishes the proof. $\square$

Let $S = \{\phi_e : X \to X\}_{e\in E}$, $X \subset \mathbb{C}$, be an arbitrary conformal iterated function system whose phase space $X$ is contained in the complex plane $\mathbb{C}$. We require at the moment that $X$ is a bounded set equal to the closure of its interior and it is contained in an open connected set $W \subset \mathbb{C}$ such that each map $\phi_e$ extends to $W$ and maps $W$ into itself. Recalling from [6] we say that the system $S$ is *essentially affine* if the conformal structure on $\overline{J_S}$ admits a Euclidean isometries refinement so that all the maps $\phi_e$, $e \in E$, become affine conformal. More precisely, there exists an atlas $\{\psi_t : U_t \to \mathbb{C}\}_{t\in T}$ with some parameter set $T$ and some open connected simply connected sets $U_t$, $t \in T$, consisting of conformal univalent maps, such that

(a) $\bigcup_{t\in T} U_t \supset \overline{J_S}$ and $\bigcup_{t\in T} U_t \subset W$.
(b) All the sets $U_t \cap U_s$ and $U_t \cap \phi_e(U_s)$, $s, t \in T$, $e \in E$, are connected.
(c) The compositions $\psi_t \circ \psi_s^{-1}$ and $\psi_t \circ \phi_e \circ \psi_s^{-1}$, defined respectively on $\psi_s(U_t \cap U_s)$ and $\psi_s \circ \phi_e^{-1}(U_t \cap \phi_e(U_s))$, $s, t \in T$, $e \in E$, are all affine (of the form $z \mapsto az + b$) with $|(\psi_t \circ \psi_s^{-1})'| = 1$.

Our application of Lemma 3.2 to tame meromorphic functions is this.

**Proposition 3.3.** *If $f : \mathbb{C} \to \hat{\mathbb{C}}$ is a tame transcendental meromorphic function and $U \subset \mathbb{C}$ is a nice set for $f$, then the corresponding iterated function system $S_U = \{\phi_e\}_{e\in E}$ is not essentially affine.*

*Proof.* Suppose on the contrary that the iterated function system $S_U$ is essentially affine and let $\{\psi_t : U_t \to \mathbb{C}\}_{t\in T}$ be the corresponding conformal atlas. Take an element $\psi_s : U_s \to \mathbb{C}$ from this atlas such that $U_S \cap J_S \neq \emptyset$. Let us recall that two words $\omega, \tau \in E^*$, where $E^*$ is the set of all finite words, are *incomparable*, if neither $\omega$ nor $\tau$ is an extension of the other. Take then two incomparable words $\omega, \tau \in E^*$ such that $x_\omega, x_\tau \in U_s$, where $x_\omega$ and $x_\tau$ are the unique fixed points respectively of $\phi_\omega$ and $\phi_\tau$. Taking the words $\omega$ and $\tau$ sufficiently long, we may further assume that

$$\phi_\omega(U_s) \subset U_s \quad \text{and} \quad \phi_\tau(U_s) \subset U_s.$$

Let $k = ||\omega||$ and $l = ||\tau||$, i.e. $f^k \circ \phi_\omega = \mathrm{Id}$ and $f^l \circ \phi_\tau = \mathrm{Id}$. We then have

$$\phi_{\omega^l}(U_s) \subset U_s \quad \text{and} \quad \phi_{\tau^k}(U_s) \subset U_s.$$

Set

$$U' = \psi_s(U_s) \quad \text{and} \quad \psi := \psi_s^{-1} : U' \to \mathbb{C}.$$

It then follows from property (c) of essential affineness that

$$A_1 := \psi^{-1} \circ \phi_{\omega^l} \circ \psi : U' \to U' \quad \text{and} \quad A_2 := \psi^{-1} \circ \phi_{\tau^k} \circ \psi : U' \to U' \tag{3.7}$$

are affine maps. Write $A_j(z) = a_j(z) + b_j$, $j = 1, 2$. Replacing the atlas $\{\psi_t\}_{t\in T}$ by $\{\psi_t - \psi_s(x_\omega)\}_{t\in T}$, we may assume without loss of generality that $\psi_s(x_\omega) = 0$. Then, by (3.7), $b_1 = 0$. Now note that

$$\begin{aligned}
|a_1| &= |A_1'(\psi^{-1}(x_\omega))| \\
&= |(\psi^{-1})'(\phi_\omega(\psi(\psi^{-1}(x_\omega))))| \cdot |\phi_{\omega^l}'(\psi(\psi^{-1}(x_\omega)))| \cdot |\psi'(\psi^{-1}(x_\omega))| \\
&= |(\psi^{-1})'(\phi_\omega(x_\omega))| \cdot |\phi_{\omega^l}'(x_\omega)||\psi'(\psi^{-1}(x_\omega))| \\
&= |(\psi^{-1})'(x_\omega)| \cdot |\phi_{\omega^l}'(x_\omega)| \cdot |\psi'(\psi^{-1}(x_\omega))| \\
&= |\phi_{\omega^l}'(x_\omega)| < 1
\end{aligned}$$

and likewise

$$|a_2| = |\phi_{\tau^k}'(x_\tau)| < 1.$$

Also, for each integer $n \geq 1$, the words $(\omega^l)^n$ and $(\tau^k)^n$ are extensions respectively of $\omega$ and $\tau$, and are therefore different. Hence

$$A_1^n = \psi^{-1} \circ \phi_{\omega^l}^n \circ \psi = \psi^{-1} \circ \phi_{(\omega^l)^n} \circ \psi \neq \psi^{-1} \circ \phi_{\tau^k}^n \circ \psi = \psi^{-1} \circ \phi_{(\tau^k)^n} \circ \psi = A_2^n,$$

and the the assumptions of Lemma 3.2 are verified. But it follows from (3.7) that $\psi \circ A_1 = \phi_{\omega^l} \circ \psi$, and applying $f^{kl}$ to both sides of this equality, we get that $f^{kl} \circ \psi \circ A_1 = f^{kl} \circ \phi_{\omega^l} \circ \psi = \psi$. Likewise $f^{kl} \circ \psi \circ A_2 = \psi$. This however contradicts Lemma 3.2 and ends the proof of our proposition. □

## 4. Conjugacies of Conformal Iterated Function Systems

In this section we deal with bi-Lipschitz conjugacies of conformal iterated function systems. We want to apply them to the systems generated by nice sets of tame meromorphic functions. This however causes two difficulties that have not been addressed in the literature yet. One is that, as noted in the proof of Claim 1 in Theorem 5.1 of [11], the systems thus emerging do not have to satisfy the Open Set Condition. The second difficulty is that they do not have to be regular. Both of these difficulties are taken care of below to the extent which is sufficient for our applications to meromorphic functions.

We call a conformal iterated function system $S = \{\phi_e\}_{e\in E}$ of *W type* if there exists a continuous map

$$F_S : \bigcup_{e\in E} \phi_e(\overline{J}_S) \to \overline{J}_S$$

such that

$$F_S \circ \phi_e = \mathrm{Id}_{\overline{J}_S}$$

for all $e \in E$. The map $F_S$ then induces a conformal Walters expanding map as defined in [3]. For all conformal iterated function systems and all real numbers $t \geq 0$ the topological pressure $\mathrm{P}(t) \in \mathbb{R}$ is well-defined (though can take up the value $+\infty$), however the existence of $e^{\mathrm{P}(t)}$-conformal measures requires either the Open Set Condition (see [4] and [5] or the W property). By an $e^{\mathrm{P}(t)}$-conformal measure we mean a Borel probability measure $m_t$ on the limit set $J_S$ such that

$$m_t(\phi_e(A)) = e^{-\mathrm{P}(t)} \int_A |\phi_e'|^t dm_t \tag{4.1}$$

for all $e \in E$ and all Borel sets $A \subset J_S$, and also that

$$m_t\big(\phi_a(J_S) \cap \phi_b(J_S)\big) = 0 \tag{4.2}$$

whenever $a, b \in E$ and $a \neq b$. Applying (4.1) and (4.2) inductively gives

$$m_t(\phi_\omega(A)) = e^{-\mathrm{P}(t)|\omega|} \int_A |\phi_\omega'|^t dm_t \tag{4.3}$$

for all $\omega \in E^*$ and all Borel sets $A \subset J_S$, and also that

$$m_t\big(\phi_\omega(J_S) \cap \phi_\tau(J_S)\big) = 0 \tag{4.4}$$

whenever $\omega, \tau \in E$ are incomparable. If an $e^{\mathrm{P}(t)}$-conformal measure exists, it is unique and by $\mu_t$ or $\mu_{S,t}$ we denote the unique Borel probability measure on $J_S$ that is absolutely continuous with respect to the measure $m_t$ and such that

$$\sum_{e\in E} \mu_t(\phi_e(A)) = \mu_t(A)$$

for all Borel sets $A \subset J_S$. If the system $S$ is of $W$ type then this condition just means that the measure $\mu_t$ is $F_S$-invariant. We shall prove the following.

**Proposition 4.1.** *If two conformal iterated function systems $S = \{\phi_e\}_{e\in E}$ and $Q = \{\psi_e\}_{e\in E}$ are bi-Lipschitz conjugate via a bi-Lipschitz map $H : J_S \to J_Q$, then*

$$\mathrm{P}_Q(t) = \mathrm{P}_S(t)$$

*for all $t \geq 0$. If in addition both systems either satisfy the Open Set Condition or the W property, then*

$$\mu_{Q,t} = \mu_{S,t} \circ H^{-1}$$

*for all $t \in \mathcal{F}in(S) = \mathcal{F}in(Q)$ (the sets of $t$ for which the pressure is finite).*

*Proof.* Since the systems $S$ and $Q$ are bi-Lipschitz conjugate and $||\phi'_\omega|| \asymp \mathrm{diam}(\phi_\omega(J_S))$, $||\psi'_\omega|| \asymp \mathrm{diam}(\psi_\omega(J_Q))$ for all $\omega \in E^*$, we conclude that

$$C^{-1} \leq \frac{||\psi'_\omega||}{||\phi'_\omega||} \leq C$$

for some constant $C \geq 1$ and all $\omega \in E^*$. Therefore, $\mathcal{F}in(S) = \mathcal{F}in(Q)$ and

$$\mathrm{P}_Q(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{|\omega|=n} ||\psi'_\omega||^t = \lim_{n\to\infty} \frac{1}{n} \sum_{|\omega|=n} ||\phi'_\omega||^t = \mathrm{P}_S(t). \tag{4.5}$$

Now let $t \in \mathcal{F}in(S) = \mathcal{F}in(Q)$. Assuming either the Open Set Condition or the W property (so in particular conformal measures $m_{Q,t}$ and $m_{S,t}$, as well as their invariant versions $\mu_{Q,t}$ are well-defined), it follows from (4.5) and (4.3) that

$$C^{-t}K^{-t} \leq \frac{m_{Q,t}(\psi_\omega(J_Q))}{m_{S,t}(\phi_\omega(J_S))} \leq C^t K^t$$

for all $\omega \in E^*$. Thus the measures $m_{Q,t}$ and $m_{S,t} \circ H^{-1}$ are equivalent (even with Radon Nikodym derivatives bounded by $(CK)^t$ and $(CK)^{-t}$ respectively from above and from below). Since also $\mu_{Q,t} \asymp m_{Q,t}$ and $\mu_{S,t} \asymp m_{S,t}$, we therefore conclude that the measures $\mu_{Q,t}$ and $\mu_{S,t} \circ H^{-1}$ are equivalent. Since they are ergodic (Theorem 2.2.9 and formula (3.10) in [5] if the Open Set Condition is satisfied and Theorem 2.5(b) in [3] if the W property holds), they must coincide. We are done. □

Now the proof of Theorem 3.1 in [6] goes through with Jacobians $\tilde{D}\phi_i$ replaced by $\tilde{D}^t\phi_i$ with respect to any measure $\mu_{S,t}$ with $t \in (0, +\infty) \cap \mathcal{F}in(S)$. This in turn permits to prove, with an analogous proof, the following refinement of Theorem 4.2 in [6]. In the present paper we only need the implication (c) $\Longrightarrow$ (a).

**Theorem 4.2.** *Let $S = \{\phi_e\}_{e\in E}$ and $Q = \{\psi_e\}_{e\in E}$ be two complex plane conformal iterated function systems either of W type or satisfying the Open Set Condition. If at least one of these two systems is not essentially affine and they are topologically conjugate by a homeomorphism $H : J_S \to J_Q$, then the following conditions are equivalent.*

(a) *The conjugacy* $H : J_S \to J_Q$ *extends in a conformal manner to an open neighborhood of* $\overline{J_S}$.
(b) *The conjugacy* $H : J_S \to J_Q$ *extends in a real–analytic manner to an open neighborhood of* $\overline{J_S}$.
(c) *The conjugacy* $H : J_S \to J_Q$ *is bi-Lipschitz continuous.*
(d) $|\psi'_\omega(y_\omega)| = |\phi'_\omega(x_\omega)|$ *for all* $\omega \in E^*$, *where* $x_\omega$ *and* $y_\omega$ *are the only fixed points of* $\phi_\omega$ *and* $\psi_\omega$ *respectively.*
(e) $\exists C \geq 1\ \forall \omega \in E^*$

$$C^{-1} \leq \frac{\operatorname{diam}(\psi_\omega(J_Q))}{\operatorname{diam}(\phi_\omega(J_S))} \leq C.$$

(f) $\exists D \geq 1\ \forall \omega \in E^*$

$$D^{-1} \leq \frac{||\psi'_\omega||}{||\phi'_\omega||} \leq D.$$

(g) *For every* $t \in (0, +\infty) \cap \mathcal{F}in(S)$ *the measures* $m_{Q,t}$ *and* $m_{S,t} \circ H^{-1}$ *are equivalent.*
(h) *For every* $t \in (0, +\infty) \cap \mathcal{F}in(S)$, $\mu_{Q,t} = \mu_{S,t} \circ H^{-1}$.
(i) *There exists* $t \in (0, +\infty) \cap \mathcal{F}in(S)$ *such that the measures* $m_{Q,t}$ *and* $m_{S,t} \circ H^{-1}$ *are equivalent.*
(j) *There exists* $t \in (0, +\infty) \cap \mathcal{F}in(S)$ *such that* $\mu_{Q,t} = \mu_{S,t} \circ H^{-1}$.

## 5. Conjugacies of tame Meromorphic Functions

Making use of the main results of the two previous sections we now shall prove the following main result of our paper. We want to emphasize that in this theorem we do not require the conjugacy to be defined on the whole complex plane but merely on the Julia sets.

**Theorem 5.1.** *If the restrictions to their Julia sets of two tame transcendental meromorphic functions* $f : \mathbb{C} \to \hat{\mathbb{C}}$ *and* $g : \mathbb{C} \to \hat{\mathbb{C}}$ *are topologically conjugate by a locally bi–Lipschitz homeomorphism* $H : J_f \to J_g$, *then this conjugacy extends to an affine linear* ($z \mapsto az+b$) *conjugacy from* $\mathbb{C}$ *to* $\mathbb{C}$ *between the meromorphic maps* $f, g : \mathbb{C} \to \hat{\mathbb{C}}$.

*Proof.* Since both sets $\mathrm{PS}(f) \cap J_f$ and $\mathrm{PS}(g) \cap J_g$ are nowhere dense respectively in $J_f$ and $J_g$, and since $H : J_f \to J_g$ is a homeomorphism, there thus exists a point $b \in J_f \setminus \mathrm{PS}(f)$ such that $H(b) \notin \mathrm{PS}(g)$ and $b$ is not a periodic. Hence, there in turn exists $\eta > 0$ so small that

$$B(H(b), 4\eta) \cap \mathrm{PS}(g) = \emptyset \quad \text{and} \quad |(g^k)'(z)| > 4K \tag{5.1}$$

whenever $k \geq 1$, $z \in B(H(b), \eta)$ and $g^k(z) = H(b)$. Here the constant $K$ is Koebe's distortion constant on $B(H(b), \eta)$. Now take $\eta' > 0$ so small that the following conditions are satisfied:

(i) the map $H|_{J_f \cap B(b,2\eta')} : J_f \cap B(b, 2\eta') \to H(J_f \cap B(b, 2\eta')$ is bi–Lipschitz,
(ii) $H(J_f \cap B(b, 2\eta')) \subset B(H(b), \eta)$.

By Theorem 2.1 there exist $0 < \beta \leq \eta'$ and a nice set $U_1$, for the map $f$, centered at the point $b$ and such that

$$B(b, \beta) \subset U_1 \subset B(b, \eta').$$

Let $S_{U_1} = \{\phi_e\}_{e \in E}$ be the iterated function system induced by the nice set $U_1$. Then $S_{U_1}$ satisfies the Open Set condition and is of W type. For every $e \in E$ let $||e|| \geq 1$ be uniquely determined by the requirement that $\phi_e : U_1 \to U_1$ is a holomorphic inverse branch of $f^{||e||}$. Furthermore, let $\phi_e^* : B(H(b), 2\eta) \to \mathbb{C}$ be the unique holomorphic inverse branch of $g^{||e||}$ defined on $B(H(b), 2\eta)$ and sending $H(b)$ to $H(\phi_e(b))$. Note that in fact $\phi_e^*$ is well defined and univalent on the ball $B(H(b), 4\eta)$.

**Claim 1:** $S_{U_1}^* = \{\phi_e^*\}_{e \in E}$ is a conformal iterated function system of W type on $B(H(b), \eta)$ and it is bi–Lipschitz conjugate to $S_{U_1}$.

*Proof.* By virtue of (ii) and (5.1) we get for every $e \in E$ that

$$\begin{aligned} \phi_e^*\big(B(H(b), 4\eta)\big) &\subset B(H(b), \eta + \operatorname{diam}\big(\phi_e^*\big(B(H(b), 4\eta)\big)\big) \\ &\subset B(H(b), \eta + K(8K)^{-1}8\eta) \\ &= B(H(b), 2\eta). \end{aligned}$$

So, $S_{U_1}^*$ is a conformal iterated function system on $B(H(b), \eta)$, and by its very definition and conjugation of $f$ and $g$ by $H$, $S_{U_1}^*$ is bi–Lipschitz conjugate to $S_{U_1}$. Thus in particular, since $S_{U_1}$ is of W type, so is $S_{U_1}^*$. Claim 1 is proved.

By Proposition 3.3 the system $S_{U_1}$ is not essentially affine, and therefore, by virtue of Theorem 4.2, there exist an open set $\overline{J_{U_1}} \subset U_1^* \subset U_1$ and a conformal map $H_1 : U_1^* \to \mathbb{C}$ such that

$$H_1|_{\overline{J_{U_1}}} = H. \tag{5.2}$$

Now fix $0 < \eta'' \leq \beta$ so small that $B(b, \eta'') \subset U_1^*$ and

(iii) The map $H_1|_{B(b,\eta'')}$ is 1-to-1.

Take finally $0 < \eta''' \leq \eta''$ so small that

$$H(J_f \cap B(b, \eta'')) \supset J_g \cap H_1(B(b, \eta''')). \tag{5.3}$$

At this moment apply Theorem 2.1 again to produce a nice set $U_2$ for $f$, centered at $b$, and such that

$$U_2 \subset B(b, \eta'''). \tag{5.4}$$

As above, let $S_{U_2} = \{\phi_e\}_{e \in E}$ be the iterated function system induced by the nice set $U_2$. Let $S^*_{U_2}$ has the corresponding meaning as $S^*_{U_1}$. Then Claim 1 holds for $S^*_{U_2}$ too. As for $U_1$, there exist an open set $\overline{J_{U_2}} \subset U_2^* \subset U_2$ and a conformal map $H_2 : U_2^* \to \mathbb{C}$ such that

$$H_2|_{\overline{J_{U_2}}} = H. \tag{5.5}$$

Since $J_{U_2} \subset J_{U_1}$, we may assume without loss of generality that

$$U := U_2^* \subset U_1^*. \tag{5.6}$$

Since $J_{U_2}$ is an uncountable subset of $U_2$, and since $H_1$ and $H_2$ are holomorphic functions, we conclude from (5.2) and (5.5) that $H_1|_U = H_2$. Along with (5.6), (5.4), (5.3), and (iii), this implies that

$$H_2(U \setminus J_f) \subset \mathbb{C} \setminus J_g. \tag{5.7}$$

For an ease of notation set from now on

$$\tilde{H} := H_2.$$

Fix $e$, an arbitrary element in $E$, and put $n = ||e||$. Then $g^n \circ \tilde{H} \circ \phi_e|_{J_U} = \tilde{H}|_{J_U}$, and as both maps $g^n \circ \tilde{H} \circ \phi_e : U \to \hat{\mathbb{C}}$ and $\tilde{H} : U \to \hat{\mathbb{C}}$ are holomorphic (the former omits infinity on $U \setminus J_f$ because of (5.7) and since the Fatou set of $g$ contains no inverse images of $\infty$ under any iterate of $g$, and on the Julia sets we have topological conjugacy between $f$ and $g$ which respects inverse images of $\infty$), we conclude that

$$g^n \circ \tilde{H} \circ \phi_e = \tilde{H}, \tag{5.8}$$

both maps defined on $U$. Let

$$E_2(f) = \hat{\mathbb{C}} \setminus \bigcup_{k=0}^{\infty} f^k(U).$$

Montel's Theorem tells us that the set $E_2(f)$ consists of at most two points. Let $\mathrm{Sing}(f^{-1})$ and $\mathrm{PS}(f)$ be the set defined in Preliminaries. In order to apply Kuratowski–Zorn Lemma, consider the family $\mathcal{F}$ of all open connected subsets $W$ of $\mathbb{C} \setminus E_2(f)$ containing $U$ for which there exists a holomorphic function $\tilde{H}_W : W \to \hat{\mathbb{C}}$ with the following two properties.

(a) $\tilde{H}_W|_U = \tilde{H}$.
(b) If $z \in U$ and $f^n(z) \in W$, then $g^n \circ \tilde{H}(z) = \tilde{H}_W \circ f^n(z)$.

The family $\mathcal{F}$ is partially ordered by inclusion and, by (5.8), it contains $U$, so $\mathcal{F}$ is not empty. If $\mathcal{C}$ is a linearly ordered subset of $\mathcal{F}$, then $\tilde{H}_{W_2}|_{W_1} = \tilde{H}_{W_1}$ whenever $W_1, W_2 \in \mathcal{C}$ and $W_1 \subset W_2$. This is so since $W_1 \supset U$ and (a) holds. Thus putting $W = \bigcup\{G : G \in \mathcal{C}\}$ and defining $\tilde{H}_W(z) = \tilde{H}_G(z)$ if $z \in G \in \mathcal{C}$, we see that $\tilde{H}_W : W \to \hat{\mathbb{C}}$ is a well-defined holomorphic function satisfying the requirements (a) and (b). So, $W$ is an upper bound of $\mathcal{C}$. We therefore conclude from Kuratowski–Zorn Lemma that $\mathcal{F}$ contains a maximal element, and we denote it by $G$. We claim that

$$G = \mathbb{C} \setminus E_2(f). \tag{5.9}$$

Indeed, seeking contradiction suppose that there exists a point $w \in \partial G \setminus E_2(f)$. Then there exist $k \geq 0$ and $\xi \in U$ such that $f^k(\xi) = w$. Take $R > 0$ so small that if $C_\xi(w, R)$ is the connected component of $f^{-k}(B(w, R))$ containing $\xi$, then

$$C_\xi(w, R) \cap f^{-k}(w) = \{\xi\}, \tag{5.10}$$

$$C_\xi(w, R) \subset U, \tag{5.11}$$

$$E_2(f) \cap B(w, R) = \emptyset, \tag{5.12}$$

and the map $f^k|_{C_\xi(w,R)} : C_\xi(w, R) \to B(w, R)$ has no other critical points except possibly $\xi$. Let $l$ be an arbitrary closed line segment joining $w$ and $\partial B(w, R)$. There then exists $f_l^{-k} : B(w, R) \setminus l \to \mathbb{C}$, a holomorphic branch of $f^{-k}$ such that

$$f_l^{-k}\big(B(w, R) \setminus l\big) \subset C_\xi(w, R).$$

Define the holomorphic map $\tilde{H}_l : B(w, R) \setminus l \to \mathbb{C}$ as

$$\tilde{H}_l = g^k \circ \tilde{H} \circ f_l^{-k}.$$

Because of (5.11) and (b) applied to $G$, we have that

$$\tilde{H}_l|_{G \cap (B(w,R) \setminus l)} = \tilde{H}_G|_{G \cap (B(w,R) \setminus l)},$$

and therefore, if $q$ is another closed line segment joining $w$ and $\partial B(w, R)$, then $\tilde{H}_l$ and $\tilde{H}_q$ coincide on the uncountable set $G \cap (B(w, R) \setminus (l \cup q))$. Hence, they glue together to a single holomorphic map $\tilde{H}_w : B(w, R) \setminus \{w\} \to \mathbb{C}$. In virtue of (5.10), $\lim_{z \to w} f_l^{-k}(z) = \xi$ and $\lim_{z \to w} f_q^{-k}(z) = \xi$. Therefore,

$$\lim_{z \to w} \tilde{H}_w(z) = g^k(\tilde{H}(\xi)).$$

Consequently $\tilde{H}_w$ extends holomorphically to a function from $B(w, R)$ to $\mathbb{C}$. Since $\tilde{H}_w$ and $\tilde{H}_G$ coincide on $G \cap B(w, R)$, they glue together to a single holomorphic function $\hat{H} : G_w \to \mathbb{C}$, such that

$$\hat{H}|_G = \tilde{H}_G,$$

where $G_w = G \cup B(w,R)$. We shall prove that

$$G_w \in \mathcal{F}.$$

Indeed, $G_w$ is an open connected subset of $\mathbb{C} \setminus E_2(f)$ containing $U$. Moreover, property (a) holds since $G_w \supset G \supset U$ and $\hat{H}|_U = \tilde{H}_G|_U = \tilde{H}$. We shall show that (b) holds too. In order to prove it consider an integer $n \geq 0$ and $C$, a connected component of $U \cap f^{-n}(G_w)$. If $f^n(C) \cap G \neq \emptyset$, then (b) holds for $G_w$ because it holds for $G$. So, we may assume without loss of generality that

$$f^n(C) \cap G = \emptyset,$$

in particular

$$f^n(C) \subset B(w,R). \tag{5.13}$$

Let $\gamma$ be a compact topological arc in $B(w,R)$ joining $f^n(C)$ and $G$, and disjoint from $\bigcup_{j=0}^{\infty} f^j(\mathrm{Sing}(f^{-1}))$. Let $V_* \subset B(w,R)$ be an open connected simply connected neighborhood of $\gamma$ disjoint from $\bigcup_{j=0}^{n} f^j(\mathrm{Sing}(f^{-1}))$. Let $f_*^{-n} : V_* \to \mathbb{C}$ be a unique holomorphic inverse branch of $f^n$ defined on $V_*$ and determined by the condition that

$$f_*^{-n}(V_* \cap f^n(C)) \subset C.$$

Now, fix a point $y \in f_*^{-n}(\gamma \cap G) \setminus E_2(f)$. There then exist a point $x \in U$ and an integer $k \geq 0$ such that $f^k(x) = y$. Let $V \subset V_*$ be an open connected simply connected neighborhood of $\gamma$ disjoint from $\bigcup_{j=0}^{n+k} f^j(\mathrm{Sing}(f^{-1}))$. Let $f_x^{-(n+k)} : V \to \mathbb{C}$ be a unique holomorphic inverse branch of $f^{n+k}$ defined on $V$ and sending $f^n(y)$ to $x$. Decreasing $V$ if necessary, we may assume without loss of generality that

$$f_x^{-n}(V) \subset U.$$

The immediate observations are that

(1) $f^k \circ f_x^{-(n+k)} = f_*^{-n}|_V$,
(2) $f_x^{-k} := f_x^{-(n+k)} \circ f^n : f^{-n}(V) \to V$
is the unique holomorphic inverse branch of $f^k$ defined on $f_*^{-n}(V)$ and sending $y$ to $x$.

Furthermore,

(3) $U \cap f_x^{-(n+k)}(G \cap V) \neq \emptyset \quad$ as $\quad x \in U \cap f_x^{-(n+k)}(G \cap V)$

and

(4) $U \cap f_x^{-k}(U \cap f_*^{-n}(V)) \neq \emptyset$
as

$$U \cap f_*^{-n}(V) \supset f_*^{-n}(V \cap f^n(C)),\ V \cap f^n(C) \neq \emptyset,$$

and
$$f_x^{-k}(U\cap f_*^{-n}(V))\subset f_x^{-k}\circ f_*^{-n}(V)=f_x^{-(n+k)}(V)\subset U.$$
It follows from (3) that
$$\tilde H|_{G\cap V}=g^{n+k}\circ \tilde H\circ f_x^{-(n+k)}|_{G\cap V},$$
whence
$$\tilde H|_V=g^{n+k}\circ \tilde H\circ f_x^{-(n+k)}.$$
And it follows from (4) that
$$\tilde H|_{U\cap f_*^{-n}(V)}=g^k\circ \tilde H\circ f_x^{-k}|_{U\cap f_*^{-n}(V)}.$$
Therefore
$$\begin{aligned}\hat H|_{f^n(U\cap f_*^{-n}(V))}&=g^{n+k}\circ \tilde H\circ f_x^{-(n+k)}|_{f^n(U\cap f_*^{-n}(V))}\\&=g^n(g^k\circ \tilde H\circ f_x^{-k}|_{U\cap f_*^{-n}(V)}\circ f_x^{-n}|_{f^n(U\cap f_*^{-n}(V))}\\&=g^n\circ \tilde H\circ f_x^{-n}|_{f^n(U\cap f_*^{-n}(V))}.\end{aligned}$$
Thus
$$g^n\circ \tilde H|_{U\cap f_*^{-n}(V)}=\hat H\circ f^n|_{U\cap f_*^{-n}(V)}.$$
Since $C\subset U$, $f_*^{-n}(V)\supset f_*^{-n}(V\cap f^n(C))\neq\emptyset$, and $f_*^{-n}(V\cap f^n(C))\subset C$, we thus obtain that
$$g^n\circ \tilde H|_C=\hat H\circ f^n|_C.$$
So, $G_w\in\mathcal F$, contrary to maximality of $G$. Formula (5.9) is thus proved. Put
$$\hat H_f:=\tilde H_{\mathbb C\setminus E_2(f)}:\mathbb C\setminus E_2(f)\to\mathbb C.$$
By the symmetry of the situation we also have now a holomorphic function $H_g^{-1}:\mathbb C\setminus E_2(f)\to\mathbb C$ which extends $H^{-1}:J_g\to J_f$ from some neighborhood of a point in $J_g$. But $H_g^{-1}\circ H_f$ is a holomorphic function well-defined on the set $\mathbb C\setminus\big(E_2(f)\cup H_f^{-1}(E_2(g))\big)$. We might have constructed $H_g^{-1}$ starting with a nice set $U_g$ such that $H^{-1}(J_g\cap U_g)\subset J_f\cap U$. Then
$$H_g^{-1}\circ H_f|_{H^{-1}\left(J_{U_g}\right)}=H_g^{-1}|_{J_{U_g}}\circ H_f|_{H^{-1}\left(J_{U_g}\right)}=\mathrm{Id}_{H^{-1}\left(J_{U_g}\right)}.$$
Thus,
$$H_g^{-1}\circ H_f=\mathrm{Id}_{\mathbb C\setminus\left(E_2(f)\cup H_f^{-1}(E_2(g))\right)}.\tag{5.14}$$
In particular, $H_f|_{\mathbb C\setminus\left(E_2(f)\cup H_f^{-1}(E_2(g))\right)}$ is one–to–one. Hence, $H_f:\mathbb C\setminus E_2(f)\to\mathbb C$ is one–to–one. Therefore $E_2(f)$ consists only of removable singularities of the function $H_f:\mathbb C\setminus E_2(f)\to\mathbb C$. Consequently $H_f$ extends holomorphically to $\mathbb C$. The same holds for $H_g^{-1}$, and thus, because of (5.14), $H_g^{-1}\circ H_f=\mathrm{Id}_{\mathbb C}$. So, $H_f:\mathbb C\to\mathbb C$ is a holomorphic isomorphism and $H_f\circ f=g\circ H_f$ on $\mathbb C$. But then

the map $H_f : \mathbb{C} \to \mathbb{C}$ must be affine linear, i.e. of the form $\mathbb{C} \ni z \mapsto az + b \in \mathbb{C}$. The proof is complete. $\square$

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