FINE INDUCING

AND

EQUILIBRIUM MEASURES

FOR

RATIONAL FUNCTIONS OF THE RIEMANN SPHERE

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Abstract. Let $f:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ be an arbitrary rational map of degree larger than 1. Denote by $J(f)$ its Julia set. Let $\phi: J(f)\to\mathbb{R}$ be a Hölder continuous function such that $P(\phi)>\sup(\phi)$. It is known that there exists a unique equilibrium measure μ_ϕ for this potential. We introduce a special inducing scheme with fine recurrence properties. This construction allows us to prove three results. Dimension rigidity, i.e. we characterize all maps and potentials for which $\mathrm{HD}(\mu_\phi)=\mathrm{HD}(J(f)$. As its fairly straightforward consequence we obtain that $\mathrm{HD}(\mu_\phi)=2$ if and only if both the function $\phi: J(f)\to\mathbb{R}$ is cohomologous to a constant in the class of continuous functions on $J(f)$, and the rational function $f:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ is a critically finite rational map with a parabolic orbifold. Real analyticity of topological pressure $P(t\phi)$ as the function of t. Exponential decay of correlations, and, as its consequence, the Central Limit Theorem and the Law of Iterated Logarithm for Hölder continuous observables. Finally, the Law of Iterated Logarithm for all linear combinations of Hölder continuous observables and the function $\log|f'|$, and its geometric consequences that allow us to compare equilibrium states with the appropriate generalized Hausdorff measures in the spirit of [PUZ].

1. Introduction. Statement of results.

Let $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ be a rational map of degree $\deg(f)\geq 2$. The Julia set of the map $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ is denoted by $J(f)$. Let $\varphi: J(f)\to\mathbb{R}$ be a continuous function, in the sequel frequently referred to as a potential. By $P(\phi)$ we denote the (classical) topological pressure of the potential ϕ with respect to the dynamical system $f: J(f)\to J(f)$. Its definition and a systematic account of properties can be found for example in [PU]. If μ is a Borel probability f-invariant measure on $J(f)$, we denote by $\mathrm{h}_\mu(f)$ its Kolmogorov–Sinai metric entropy. The relation between pressure and entropy is given by the following celebrated

The research of M.Szostakiewicz and A. Zdunik partially supported by the Polish MNiSW grant NN 201 607940. The research of M. Urbański supported in part by the NSF Grant DMS 1001874.

Variational Principle.

$$\mathrm{P}(\varphi) = \sup\left\{\mathrm{h}_\mu(f) + \int \varphi d\mu\right\}, \tag{1}$$

where the supremum is taken over all Borel probability f-invariant measures μ, or equivalently, over all Borel probability f-invariant ergodic measures μ. The measures μ for which

$$\mathrm{h}_\mu(f) + \int \phi\, d\mu = \mathrm{P}(\phi)$$

are called equilibrium states for the potential ϕ. In [Ly] M. Lyubich proved that each continuous potential $\phi : J(f) \to \mathbb{R}$ admits an equilibrium state. The potential $\phi : J(f) \to \mathbb{R}$ is said to have a pressure gap if

$$P(\varphi) > \sup(\varphi).$$

Obviously, the function $\phi = 0$ has a pressure gap, as in this case $P(\phi) = \mathrm{h}_{\mathrm{top}}(f) = \log(\deg(f)) > 0$. Likewise, each constant potential ϕ has a pressure gap. Furthermore, it immediately follows from the Variational Principle (1) that every Hölder continuous function ϕ satisfying $\sup(\phi) - \inf(\phi) < \log d$ has a pressure gap. It was proved in [DU1] (comp. also [Pr1]) that each Hölder continuous potential with a pressure gap admits exactly one equilibrium state; denote it by μ_φ. Some of its ergodic properties have been investigated therein.

The goal of this article is to answer the geometric and stochastic questions that have been attracting the attention of the experts in the field ever since the papers [PUZ], [DU1], and [Z1] have been written. As our main tool, we introduce in this article the method of fine inducing which results in a construction of a conformal iterated function system fitting into the setting of [MU1] and [MU2]. Armed with this inducing scheme and the theory of conformal iterated function systems, we further investigate finer geometric and stochastic properties of the equilibrium measure μ_φ as well as regularity properties of the pressure function $\mathbb{R} \ni t \mapsto P(t\varphi)$. We now describe them in this order.

We recall that two functions $g, k : J(f) \to \mathbb{R}$ cohomologous in a subclass C of real-valued functions defined on $J(f)$ if there is $u \in C$ such that

$$k - g = u - u \circ f.$$

We put

$$P_f = \bigcup_{n=1}^{\infty} f^n(\mathrm{Crit}(f)$$

and call P_f the postcritical set of f.

A point $z \in J(f)$ is called conical for f if there exist $\theta > 0$ and an infinite increasing sequence $n_k \geq 1$ of positive integers such that for each k there exists $f_z^{-n_k}$, a holomorphic inverse branch of f^{n_k}, which is defined on the disk $B(f^{n_k}(z), \theta)$ and sends the point $f^{n_k}(z)$ to z. The set of all conical points of f will be denoted by $J_c(f)$. The hyperbolic dimension

$\mathrm{hD}(J(f))$ is defined to be the supremum of Hausdorff dimensions of all hyperbolic subsets of $J(f)$. Given a Borel probability measure μ on a compact metric space X by $\mathrm{HD}(\mu)$, the Hausdorff dimension of measure μ, we understand the number

$$\inf\{\mathrm{HD}(Y) : \mu(Y) = 1\}.$$

The dynamical dimension $\mathrm{DD}(J(f))$ is defined as

$$\mathrm{DD}(J(f)) = \sup\{\mathrm{HD}(\mu)\},$$

where the supremum is taken over all ergodic invariant measures of positive entropy. Fix $t \geq 0$. A Borel probability measure m on $J(f)$ is said to be t–conformal for the rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, if $m(J(f)) = 1$ and

$$m(f(A)) = \int_A |f'|^t \, dm$$

for every Borel set $A \subset J(f)$ such that $f|_A$ is injective. The number $\delta(f)$ is defined to be the minimal exponent for which a conformal measure exists. The following theorem (see [DU2] and [U1]) gives a full answer to the question of how all these four numbers defined above are related one to each other.

Theorem 1. *For all rational functions $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ we have that*

$$\mathrm{HD}(J_c(f)) = \mathrm{DD}(J(f)) = \mathrm{hD}(J(f)) = \delta(f).$$

Proof. The equality of the last three numbers above originated with the paper [DU2]. Its extensive discussion can be found in [U1]. The inequality $\mathrm{hD}(J(f)) \leq \mathrm{HD}(J_c(f))$ holds since every point of a hyperbolic subset of $J(f)$ is a conical point. Finally the inequality $\mathrm{HD}(J_c(f)) \leq \delta(f)$ follows from the left-hand side of formula (1.2) in [U1]. $\square$

The first main results of the present paper are these.

Theorem (Refined Dimension Rigidity)**.** *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map, let $\phi : J(f) \to \mathbb{R}$ be a Hölder continuous potential such that $\sup(\phi) < P(\phi)$. Let μ_ϕ be a unique equilibrium state corresponding to this potential. Then the following are equivalent.*

(1) $\mathrm{HD}(\mu_\phi) = \mathrm{DD}(J(f))$,
(2) $\mathrm{HD}(\mu_\phi) = \mathrm{HD}(J(f))$,
(3) *The intersection $P_f \cap J(f)$ consists of at most four points, there are no other points in $\overline{P_f} \cap J(f)$ and also the potential $\phi : J(f) \to \mathbb{R}$ is modulo constant cohomologous to $-\mathrm{DD}(J(f)) \log |f'|$ in the class of continuous functions on $J(f) \setminus P_f$. The cohomology constant is equal to $P(\phi)$.*
(4) *The intersection $P_f \cap J(f)$ consists of at most four points, it is equal to $\overline{P_f} \cap J(f)$, and also the potential $\phi : J(f) \to \mathbb{R}$ is cohomologous modulo constant to $-\mathrm{HD}(J(f)) \log |f'|$ in the class of continuous functions on $J(f) \setminus P_f$. The cohomology constant is equal to $P(\phi)$.*

In addition, if the closure of the postcritical set P_f is disjoint from $J(f)$, which equivalently means that the restriction $f_{|J(f)} : J(f) \to J(f)$ is then expanding, and the potential $\phi : J(f) \to \mathbb{R}$ is cohomologous modulo constant to $-\mathrm{HD}(J(f))\log|f'|$ in the class of Hölder continuous functions on $J(f)$.

and

Corollary. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map and let $\phi : J(f) \to \mathbb{R}$ be a Hölder continuous potential such that $\sup(\phi) < P(\phi)$. If μ_ϕ is a unique equilibrium state corresponding to this potential, then $\mathrm{HD}(\mu_\phi) = 2$ if and only if both the function $\phi : J(f) \to \mathbb{R}$ is cohomologous to a constant in the class of continuous functions on $J(f)$, and the rational function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a critically finite rational map with a parabolic orbifold.*

The proof of this theorem is provided in Section 5. As mentioned above, our main method of its proof is a special, called fine, inducing scheme. This theorem has been proved by Anna Zdunik in [Z1] for the case of $\phi = 0$, where even more exceptional set of rational functions has emerged. The idea of her proof was entirely different than ours, based on somewhat complicated regularization process, canonical conditional measures, Rokhlin's natural extension and a heavy use of the Central Limit Theorem. Developing Anna's Zdunik approach Feliks Przytycki has proved in [Pr2] this theorem for all Hölder continuous potential with a pressure gap. He has however not investigated in detail the exceptional set of maps and potentials; he has just required that $\phi : J(f) \to \mathbb{R}$ is not, modulo constant, cohomologous to $-\mathrm{DD}(J(f))\log|f'|$ in $L^2(\mu_\phi)$. Our present proof of this theorem avoids painful regularization, stochastic laws and conditional measures, and, at least to our taste, appears to be substantially simpler.

In Section 8, also endowed with the fine inducing scheme and the corresponding iterated function system, by using a tower method introduced in [LSY], we deduce stochastic properties of the measure–preserving dynamical system (f, μ_φ) from the stochastic properties of the corresponding iterated function system constructed out of the fine inducing scheme. A crucial estimate here is an exponentially small, with respect to n, bound on the measure of points for which the order of the iteration of the original map, appearing in the induced iterated function system, is larger than n. We prove the following.

Theorem. *For the equilibrium measure μ_ϕ the following hold.*

(1) *For every $0 < \alpha \le 1$, every Hölder continuous function $\varphi : J(f) \to \mathbb{R}$ with exponent α, and every bounded measurable function $\psi : J(f) \to \mathbb{R}$, we have*
$$\left| \int \psi \circ f^n \cdot \varphi d\nu - \int \varphi d\nu \int \psi d\nu \right| = O(\kappa^n)$$
with some $0 < \kappa < 1$, depending on α.

(2) *If $\varphi : J(f) \to \mathbb{R}$ is a Hölder continuous function not cohomologous to a constant in $L^2(\mu_\phi)$, i.e. if there is no square integrable function η for which $\varphi = \mathrm{const} + \eta \circ f - \eta$, then the Central Limit Theorem holds. This means that there exists $\sigma > 0$ such that*

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \varphi \circ f^j \to \mathcal{N}(0, \sigma)$$

in distribution. $\mathcal{N}(0, \sigma)$ is here the normal distribution with 0 mean and variance σ.

(3) *The Law of Iterated Logarithm holds for every Hölder continuous function $g : J(f) \to \mathbb{R}$ that is not cohomologous to a constant in $L^2(\mu_\phi)$. This means that there exists a real positive number A_g such that μ_ϕ almost everywhere*

$$\limsup_{n \to \infty} \frac{S_n g - n \int g d\mu}{\sqrt{n \log \log n}} = A_g.$$

Item (1) of this theorem is formulated in [Ha] and item (2) was proved in [DPU]; both using entirely different methods than ours. Item (3), a long standing open problem, is completely new.

The above theorem holds not only for Hölder continuous observable. In fact, we prove the following two results involving the logarithm of the modulus of derivative, a function usually with singularities.

Theorem. *Let $\psi = a\phi + b \log |f'| : J(f) \to \mathbb{R}$, $a, b \in \mathbb{R}$. Then the function $\psi : J(f)) \to \mathbb{R}$ satisfies the Law of Iterated Logarithm with respect to the dynamical system (f, μ_ϕ), provided that ψ_* is not cohomologous to a constant in $L^2(\mu_{\phi_*})$.*

Theorem. *If the pair (f, ϕ) fails to satisfy condition (1) of Theorem 43, then the function $\psi := \phi + \mathrm{HD}(\mu_\phi) \log |f'| : J(f) \to \mathbb{R}$ satisfies the Law of Iterated Logarithm with respect to the dynamical system (f, μ_ϕ). This means that there exists a real positive constant A_ψ such that μ_ϕ almost everywhere*

$$\limsup_{n \to \infty} \frac{S_n \psi - n \int \psi d\mu}{\sqrt{n \log \log n}} = A_\psi.$$

As an important geometric consequence of this latter theorem, we get the following.

Theorem. *Suppose that the pair (f, ϕ) fails to satisfy condition (1) of Theorem 43. Let $c_\phi = A_{\phi + \mathrm{HD}(\mu_\phi) \log |f'|} > 0$ and let $\kappa := \mathrm{HD}(\mu_\phi)$. Then*

- *μ_ϕ is absolutely continuous with respect to $\mathrm{H}_{g_{\kappa,c}}$ for all $0 < c < \sqrt{c_\phi / \chi_{\mu_\phi}}$, where the generalized Hausdorff measures $\mathrm{H}_{g_{\kappa,c}}$ were defined at the very end of our paper.*

- *μ_ϕ is singular with respect to $\mathrm{H}_{g_{\kappa,c}}$ for all $c > \sqrt{c_\phi/\chi_{\mu_\phi}}$.*
- *μ_ϕ is singular with respect to the ordinary Hausdorff measure H_{t^κ}.*

The part (3) of this theorem was proved with different methods in [Z1] in the case when the potential ϕ is identically equal to zero. This theorem was proved in [PUZ] for measures that can be represented as projections of equilibrium states of Hölder continuous potentials, via a coding tree, from an associated symbol space. The question of whether it is true for equilibrium states of Hölder continuous potentials on the Julia set itself, as a matter of fact, the collection of most natural measures there, has been known ever since. We positively answer it now.

In order to set up the next theorem let Δ_ϕ be the set of all those parameters $t \in \mathbb{R}$ for which the potential $t\phi$ is admissible. Note that Δ_ϕ is an open subset of $\mathbb{R}$ containing the point 1. In Section 6 we prove the following.

Theorem. *The topological pressure function*

$$\Delta_\phi \ni t \mapsto P(t\phi) \in \mathbb{R}$$

is real-analytic.

This theorem is classical in the case when the rational function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is hyperbolic, the proof goes back to Ruelle (see [Rue]). Juan Rivera-Letelier has explained to us that it follows from [HP] for all Collet-Eckmann rational functions and asked about the general case. We answer his question now. We heavily use our fine inducing scheme and the appropriate parts of the theory of iterated function systems, see [MU2] and [U2].

Finally, the fine inducing scheme leading to the construction of the conformal iterated function system, which constitutes the main tool of our investigations, is presented in Sections 2 and 3.

2. Fine Inducing Scheme – Preliminaries

We collect here the properties of the measure μ_ϕ which will be used in the sequel. For the proofs, we refer to [DU1] and [DPU].

Proposition 2. *Let $\phi : J(f) \to \mathbb{R}$ be a Hölder continuous function. Assume also that $\sup(\phi) < P(\phi)$. Then there exists a unique equilibrium measure μ_ϕ, i.e μ_ϕ satisfies the equality*

$$P(\phi) = h_{\mu_\phi} + \int \phi \, d\mu_\phi.$$

. In addition, if $\mathcal{L} : C(J(f)) \to C(J(f))$ is the corresponding Perron-Frobenius operator

$$\mathcal{L}(\zeta)(x) = \sum_{y \in f^{-1}(x)} \exp \phi(y)\zeta(y),$$

then

(a) *The spectral radius of the Perron-Frobenius operator $\mathcal{L} : C(J(f)) \to C(J(f))$ is equal to $\lambda = e^{P(\phi)}$.*

(b) *The spectral radius λ is an eigenvalue of both, the Perron-Frobenius operator $\mathcal{L} : C(J(f)) \to C(J(f))$ and its conjugate operator $\mathcal{L}^* : C^*(J(f)) \to C^*(J(f))$.*

(c) *There exists a unique probability eigenmeasure m_ϕ corresponding to the eigenvalue λ of the conjugate operator $\mathcal{L}^* : C^*(J(f)) \to C^*(J(f))$. This measure is in the contemporary literature frequently refereed to as a conformal measure for ϕ.*

(d) *There exists a unique continuous, non-negative eigenfunction $\rho_\phi : J(f) \to \mathbb{R}$ corresponding to the eigenvalue λ of the Perron-Frobenius operator $\mathcal{L}$ which is normalized so that $\int \rho_\phi dm_\phi = 1$*

(e) *The function $\rho_\phi : J(f) \to \mathbb{R}$ is Hölder continuous and everywhere positive.*

(f) $\rho_\phi = \frac{d\mu_\phi}{dm_\phi}$*, i.e. ρ_ϕ is the Radon-Nikodym derivative of μ_ϕ with respect to m_ϕ,*

(g) *The normalized Perron-Frobenius operator $\lambda^{-1}\mathcal{L} : C(J(f)) \to C(J(f))$, is almost periodic.*

Notation. Abbreviate the Julia set $J(f)$ of the map f to J. Let $\phi : J \to \mathbb{R}$ be a Hölder-continuous function, with some Hölder exponent α, such that $\sup(\phi) < P(\phi)$. Since adding to ϕ any constant does not change the equilibrium measure μ_ϕ, subtracting $P(\phi)$ from ϕ, we may in the sequel assume without loss of generality that $P(\phi) = 0$. We now define the function

$$\tilde{\phi} = \log \rho_\phi - \log \rho_\phi \circ f + \phi - P(\phi).$$

Since adding a coboundary does not alter pressure and the set of equilibrium measures, $P(\tilde{\phi}) = 0$, and the potential $\tilde{\phi}$ has also a unique equilibrium measure which is equal to μ_ϕ. What we achieved is that the Jacobian of the map $f : J \to J$ with respect to the invariant measure μ_ϕ is equal to $\exp(-\tilde{\phi})$.

For the sake of simplicity, in what follows we shall write shortly μ for μ_ϕ.

Notation. Given $q \in \mathbb{N}$, we denote

$$\phi_q = \phi + \phi \circ f + \phi \circ f^2 + \cdots + \phi \circ f^{q-1},$$

$$\tilde{\phi}_q = \tilde{\phi} + \tilde{\phi} \circ f + \tilde{\phi} \circ f^2 + \cdots + \tilde{\phi} \circ f^{q-1}$$

and by $S_n\tilde{\phi}_q$ we denote the sum

$$S_n\tilde{\phi}_q = \tilde{\phi}_q + \tilde{\phi}_q \circ f^q + \cdots + \tilde{\phi}_q \circ f^{(n-1)q}.$$

From now on, the iterate f^q will be denoted by g. In this notation

$$S_n\tilde{\phi}_q = \tilde{\phi}_q + \tilde{\phi}_q \circ g + \cdots + \tilde{\phi}_q \circ g^{n-1}$$

We also put

$$\phi_{qr} = S_r\phi_q$$

and

$$\tilde{\phi}_{qr} = S_r\tilde{\phi}_q.$$

Let

$$\tilde{\theta} = P(\phi) - \sup(\phi) \quad \text{and} \quad \theta = \frac{\tilde{\theta}}{4}.$$

Notation. From now on, we shall always use the normalized Perron-Frobenius operator

$$\mathcal{L}_g\eta(x) = \sum_{y\in g^{-1}(x)} \eta(y)\exp\tilde{\phi}_q(y).$$

Note that $\mathcal{L}_g(1) = 1$. Similarly, if $h = g^r$ is an iterate of g, then

$$\mathcal{L}_h\eta(x) = \mathcal{L}_g^r\eta(x) = \sum_{z\in h^{-1}(x)} \eta(z)\exp S_r\tilde{\phi}_q(z)$$

Note that, with $M = 2||\log\rho||_\infty$, we have

$$\tilde{\phi}_q = \log\rho - \log\rho\circ f^{q-1} + (\phi + \phi\circ f + \cdots + \phi\circ f^{q-1}) - qP(\phi) < -q\tilde{\theta} + M < \frac{\tilde{\theta}}{2}q \tag{2}$$

for all $q \geq 1$ large enough, and consequently,

$$S_n\tilde{\phi}_q < -nq\frac{\tilde{\theta}}{2} = -2nq\theta \tag{3}$$

for all such $q \geq 1$. From now on, we assume that $q \geq 1$ is large enough to satisfy (2), and in particular (3) holds.

For the construction of our inducing scheme, we need the following.

Lemma 3. *Let G be a union of small discs around critical periodic orbits (if they exist). For every $\gamma \in (0,1)$ there exists an integer $q = q(\gamma) \geq 1$ such that, if $U \subset G^c$ is an open topological disc with piecewise smooth boundary, which contains no critical values of f^q, then for every $n \geq 0$ there exists a family W_n of connected components of $f^{-qn}(U)$ such that $W_0 = \{U\}$ and the following hold.*

- (a_n) *if $V \in W_{n+1}$, then $f^q(V) \in W_n$ and*
- (b_n) *if $V \in W_n$, then the map $f^{qn}_{|V}$ is univalent.*
- (c_n) $\max\{\mathrm{diam}V : V \in W_n\} \leq C_q\gamma^n$

(d_n) *Let Z_n be the family of all connected components of $f^{-q}(V)$, $V \in W_{n-1}$. If we put*

$$B_n(x) = \sum_{V \in Z_n \setminus W_n} \sum_{y \in f^{-qn}(x) \cap V} \exp S_n \tilde{\phi}_q(y),$$

then

$$B_n(x) \le \exp(-nq\theta),$$

where, we recall, $\theta = \frac{\tilde{\theta}}{4}$.

In item (c_n) above the number $C_q > 0$ is a constant depending on q and on the disc U but it is independent of n.

Proof. The proof of this lemma is a simplification of the combined proofs of Lemma 4 in [DU] and Lemma 3.6 in [UZ]. We shall quote a part of this proof, the estimate of $B_n(x)$ from above. The reason is that it is of critical importance for the further steps of our construction of the induced scheme. And indeed, in order to calculate $B_n(x)$, one has to calculate the total sum of the terms $\exp\big(S_n\tilde{\phi}_q(y)\big)$ over all those inverse images $y \in f^{-q}(x)$ that are in "bad" components, namely $y \in V$ and $V \in Z_n \setminus W_n$. It follows from the construction that the maps $f^{qn}|_V$ are univalent, and the elements of $Z_n \setminus W_n$ are exactly those components that either intersect the set of critical values of f^q (its cardinality is bounded above by Nq for some constant N, or their area is large ($\text{area}(V) > \gamma^{2n}$)[1] Obviously, the number of components in $Z_n \setminus W_n$ is thus bounded above by $\gamma^{-2n} + Nq$, and, since each such component contains exactly one preimage of x, we have

$$\begin{aligned} B_n(x) &\le (\gamma^{-2n} + Nq) \exp\left(-nq\frac{\tilde{\theta}}{2}\right) \\ &= \exp\left(-nq\left(\frac{\tilde{\theta}}{2} + \frac{2\log\gamma}{q}\right)\right) + Nq\exp\left(-nq\frac{\tilde{\theta}}{2}\right) \\ &\le \exp(-nq\theta) \end{aligned} \tag{4}$$

if, we recall, $\theta = \tilde{\theta}/4$ and $q \ge 1$ is large enough. ∎

Remark 4. We fix the values q and θ at this step.

[1] As David Simmons has pointed out, there was an inaccuracy in the proof of Lemma 4 in [DU] which can be traced back to Mane's paper [Ma2]. The point is that the closures of inverse images of the set U under f^q need not be closed topological discs. However, one should then consider connected components of inverse images of small open neighborhoods of these closures. The bounded distortion would follow by taking chains of holomorphic inverse branches along these neighborhoods. This is needed to conclude that small area implies small diameter. Since the collections of all these components of the same order will have bounded above multiplicity, the number of them thrown away because of containing critical values, will be still bounded above by Nq with some appropriate constant N.

Corollary 5. *There exists a constant $\tilde{C}_q$ (depending on q and on the disc U) such that for all $n \geq 1$, all $V \in W_n$ and all $x, y \in W$,*

$$\frac{\exp S_n\tilde{\phi}_q(x)}{\exp S_n\tilde{\phi}_q(y)} < \tilde{C}_q \tag{5}$$

Proof. Since $\phi : \hat{\mathbb{C}} \to \mathbb{R}$ is Hölder continuous, the function $\phi_q : \hat{\mathbb{C}} \to \mathbb{R}$ is also Hölder continuous, with the same exponent ω. Denote by H_q the Hölder constant for ϕ_q. If $x, y \in W_n$ then $\mathrm{dist}(g^i(x), g^i(y)) < C_q\gamma^{n-i}$, so $|S_n\phi_q(x) - S_n\phi_q(y)| \leq H_q \sum_{i=0}^n \gamma^{(n-i)\omega}$. Since $S_n\tilde{\phi}_q = S_n\phi_q - \log\rho \circ g^n + \log\rho$ and ρ is bounded both from above and from below, the result follows ☐

Let

$$\tilde{B}_k(x) = \sum_{(k-1)r<n\leq kr} B_n(x)$$

It follows from the previous lemma that

$$\tilde{B}_k \leq e^{-(k-1)r\theta q} + e^{-((k-1)r+1)\theta q} + \cdots + e^{-kr\theta q} \leq re^{-(k-1)r\theta q}.$$

Remark 6. In the calculations below we shall use the important property that $\mathcal{L}_g(1) = 1$, therefore, for every l and for every y

$$\sum_{z\in g^{-l}(y)} \exp S_l\tilde{\phi}_q(z) = 1.$$

Denote

$$\tilde{W}_k = W_{kr}.$$

In particular, $\tilde{W}_1 = W_r$ is the family of "good" components, i.e. "good" preimages of U under f^{qr}. Note that $\tilde{W}_k$ is a subfamily of the family of connected components of $h^{-k}(U)$. The components in $\tilde{W}_k$ will be called good. Thus, denoting by $\tilde{Z}_k$ the family of all connected components of $h^{-1}(\tilde{V})$, over all $\tilde{V} \in \tilde{W}_{k-1}$, we see that

$$\begin{aligned}
\tilde{B}_k(x) &= \sum_{(k-1)r<n\leq kr} B_n(x) = \sum_{(k-1)r<n\leq kr} \left(\sum_{\substack{V\in Z_n\setminus W_n \\ y\in g^{-n}(x)\cap V}} \exp S_n\tilde{\phi}_q(y) \right) \\
&= \sum_{(k-1)r<n\leq kr} \left(\sum_{\substack{V\in Z_n\setminus W_n \\ y\in g^{-n}(x)\cap V}} \exp S_n\tilde{\phi}_q(y) \sum_{z\in g^{-(kr-n)}(y)} \exp S_{kr-n}\tilde{\phi}_q(z) \right) \\
&= \sum_{\tilde{V}\in\tilde{Z}_k\setminus\tilde{W}_k} \sum_{z\in h^{-k}(x)\cap\tilde{V}} \exp S_{kr}\tilde{\phi}_q(z)
\end{aligned} \tag{6}$$

In particular, the only bound, a weak one, we obtain out of this and (4) for $\tilde{B}_1$ is $r\exp(-q\theta)$, but the bounds for $\tilde{B}_2, \tilde{B}_3, \dots$ become much better:

$$\tilde{B}_k(x) = \sum_{(k-1)r<n\leq kr} B_n(x) \leq r\exp(-rq(k-1)\theta). \tag{7}$$

In the sequel, we will need the following simple general observation.

Lemma 7. *Assume that $Q \subset \hat{\mathbb{C}}$ is a set with the following property: there exists $\beta > 0$ such that*

$$\mathcal{L}_h(1_Q)(x) > \beta \tag{8}$$

for almost every $x \in J(f)$. Then there exist $\alpha \in (0,1)$, an integer $n_0 \geq 0$, and $\delta > 0$, all three depending on β only, such that for all $n \geq n_0$

$$\mu(B_\alpha^n) < \exp(-\delta n), \tag{9}$$

where

$$B_\alpha^n = \left\{x : \#\left\{0 \leq i \leq n : h^i(x) \in Q\right\} \leq \alpha n\right\}. \tag{10}$$

Proof. Since

$$\mu(B_\alpha^n) = \int 1_{B_\alpha^n} = \int \mathcal{L}_h^n(1_{B_\alpha^n}),$$

it suffices to estimate $\mathcal{L}^n(1_{B_\alpha^n})$ uniformly from above. For every set $K \subset \{0, 1, 2 \dots n-1\}$ with $\#K > (1-\alpha)n$, let

$$B_{\alpha,K}^n = \{x : h^i(x) \notin Q \;\Leftrightarrow i \in K\}$$

The sets $B_{\alpha,K}^n$ are mutually disjoint and their union is equal to B_α^n. It is now clear that

$$\mathcal{L}_h^n(1_{B_{\alpha,K}^n}) \leq (1-\beta)^k,$$

where $k = \#K \geq (1-\alpha)n$. Thus, summing over all possible choices of the set K, we get

$$\mathcal{L}_h^n(1_{B_\alpha^n}) \leq \sum_{(1-\alpha)n\leq k\leq n-1} \binom{n}{k} (1-\beta)^k. \tag{11}$$

The remaining part of the proof relies on a standard application of Stirling's formula. We shall estimate the first summand in (11):

$$\begin{aligned}\binom{n}{(1-\alpha)n} &= \frac{n!}{((1-\alpha)n)!}(1-\beta)^{(1-\alpha)n} \\ &\asymp \frac{n^{n+\frac{1}{2}}}{((1-\alpha)n)^{((1-\alpha)n+\frac{1}{2}}(\alpha n)^{\alpha n+\frac{1}{2}}}(1-\beta)^{(1-\alpha)n} \\ &= (\alpha(1-\alpha))^{-1/2}\frac{1}{n^{\frac{1}{2}}}\frac{(1-\beta)^{(1-\alpha)n}}{(1-\alpha)^{(1-\alpha)n}\alpha^{\alpha n}} \\ &= (\alpha(1-\alpha))^{-1/2}\frac{1}{n^{\frac{1}{2}}}\left(\frac{(1-\beta)^{(1-\alpha)}}{(1-\alpha)^{(1-\alpha)}\alpha^{\alpha}}\right)^n .\end{aligned} \tag{12}$$

Now, for α sufficiently small (but depending on β only), we have

$$\frac{(1-\beta)^{(1-\alpha)}}{(1-\alpha)^{(1-\alpha)}\alpha^{\alpha}} < \tilde{\delta}_\alpha < 1$$

and $\tilde{\delta}_\alpha \to 1-\beta$ as $\alpha \to 0$. Now, it is straightforward to check that, for all $k \geq (1-\alpha)n$,

$$\binom{n}{k+1}(1-\beta)^{k+1} < \binom{n}{k}(1-\beta)^k$$

if only $\alpha < 1/2$. Therefore, we can estimate (11) from above by $\alpha n\exp(-\tilde{\delta}n) \leq \exp(-\delta n)$ for all $n \geq 1$ large enough, where $\delta = \tilde{\delta}/2$. □

We now want to use Lemma 7 for $h = g^r = f^{qr}$ (where r is not determined yet), and

$$Q = Q(r) = \bigcup_{V\in W_r} V = \bigcup_{V\in \tilde{W}_1} V$$

It is important that in Lemma 7 the constants α and δ depend only on β. The following Proposition says that so defined set Q satisfies the assumptions of Lemma 7.

Proposition 8. *If the disc U in Lemma 3 is chosen so that $\mu_\phi(U) = 1$ then there exists $\beta \in (0,1)$ independent of r such that for*

$$Q = Q(r) = \bigcup_{V\in W_r} V$$

and for every $x \in J \cap U$ (whence for μ_ϕ-almost every $x \in J$)

$$\mathcal{L}_h(1_Q)(x) > \beta, \tag{13}$$

where, as before, $h = g^r$.

Its proof is a direct application of Lemma 4 in [DU1] and Corollary 5. The next lemma will be used for our construction of an appropriate disc U (see Proposition 11 below).

Lemma 9. *For every integer $\lambda \geq 1$ large enough the following holds. Let μ be a probability Borel measure in $\mathbb{R}^2$. Let $x_0 \in \mathbb{R}^2$. For every $\alpha \in [0, 2\pi)$ let l_α be the open ray emanating from x_0 and forming the angle α with the positive x axis. Then for every interval $I \subset [0, 2\pi)$ there exists a subset $I' \subset I$ such that* $\mathrm{Leb}(I') > |I|(1-3(\frac{1}{\lambda}+\frac{1}{\lambda^2}+\dots))$ *and for every $\alpha_0 \in I'$,*

$$\mu\big(L(x_0; (\alpha_0 - |I|\lambda^{-3n}, \alpha_0 + |I|\lambda^{-3n}))\big) < \lambda^{-n},$$

where $L(x_0, A) = \bigcup\{l_\alpha : \alpha \in A\}$.

Proof. Let us partition the interval I into λ^2 subintervals J's of length $|I|\lambda^{-2}$. Let B_1 be the family of intervals J of this partition for which $\mu(L(x_0; J)) < \lambda^{-1}$. Obviously, there are at most λ intervals in B_1^c, thus

$$\#B_1 > \lambda^2 - \lambda = \lambda^2 \left(1 - \frac{\lambda}{\lambda^2}\right)$$

and

$$\mathrm{Leb}\left(\bigcup\{J : J \in B_1\}\right) \geq |I| \left(1 - \frac{\lambda}{\lambda^2}\right) = |I| \left(1 - \frac{1}{\lambda}\right)$$

Next, each interval in B_1 is divided into λ^2 subintervals with disjoint interiors and of length $|I|\frac{1}{(\lambda^2)^2}$, and we remove those subintervals for which $\mu(L(x_0; I)) \geq \lambda^{-2}$. Denoting by B_2 the family of remaining intervals, we see that

$$\#B_2 \geq (\lambda^2)^2 \left(1 - \frac{\lambda}{\lambda^2}\right) - \lambda^2 = (\lambda^2)^2 \left(1 - \frac{1}{\lambda} - \frac{1}{\lambda^2}\right)$$

and

$$\mathrm{Leb}\left(\bigcup\{J : J \in B_2\}\right) \geq |I| \left(1 - \frac{1}{\lambda} - \frac{1}{\lambda^2}\right)$$

Proceeding inductively, we partition the interval I into disjoint intervals of length $|I|\frac{1}{(\lambda^2)^n}$. Next, we define in the same way the family B_n. It is formed by the intervals J of this partition of n'th generation, which are contained in some interval of the family B_{n-1} and for which $\mu(J) < \frac{1}{\lambda^n}$. Then

$$\mathrm{Leb}\left(\bigcup\{J : J \in B_n\}\right) \geq \left(1 - \frac{1}{\lambda} - \frac{1}{\lambda^2} - \dots - \frac{1}{\lambda^n}\right)|I|.$$

For any $\alpha \in I$ let $J_n = J_n(\alpha)$ be the interval of the n'th partition such that $\alpha \in J_n$. Thus, for $\alpha \in \bigcap_{i=1}^\infty \bigcup_{J_n \in B_n} J_n$, we have that $J_n(\alpha) \in B_n$. Consequently, for all $\alpha \in \bigcap_{i=1}^\infty \bigcup_{J_n \in B_n} J_n$, it holds $\mu(L(x_0; J_n(\alpha)) < \lambda^{-n}$ for all n. Let now

$$C_n = \{\alpha \in I : [\alpha - |I|\lambda^{-3n}, \alpha + |I|\lambda^{-3n}] \subset J_n(\alpha)\}.$$

It is easy to see that $\mathrm{Leb}(C_n^c) < 2\frac{|I|}{\lambda^n}$, and, therefore,

$$\mathrm{Leb}\left(\bigcap C_n\right) > |I|\left(1 - 2\left(\frac{1}{\lambda} - \frac{1}{\lambda^2} - \dots\right)\right).$$

Finally, the set $\bigcap_{i=1}^{\infty}\bigcup_{J\in B_n} J_n \cap \bigcap C_n$ is the set which satisfies our requirements. □

The next proposition follows from the special properties of our equilibrium measure measure μ_ϕ.

Lemma 10. *There exist $\tau > 0$ and $r_0 > 0$ such that for every point $p \in \hat{\mathbb{C}}$ and all $0 < r < r_0$, we have that $\mu_\phi(B(p,r)) < r^\tau$.*

Proof. We start with an obvious observation. There exist $\eta > 0$ and $\xi > 0$ such that

(1) if $D \cap B(Crit(g), \eta) \neq \emptyset$ and $\mathrm{diam}(D) < \xi$ then $\mathrm{diam} g(D) \leq \mathrm{diam}(D)$.
(2) if $D \cap B(Crit(g), \eta) = \emptyset$ and $\mathrm{diam}(D) < \xi$ then $g_{|D}$ is one-to-one.

Now, take $r > 0$ and a ball B_r of radius r. If B_r does not intersect the Julia set, we are done as $\mu_\phi(\hat{\mathbb{C}} \setminus J)) = 0$. Otherwise, there exists the least integer $n = n(r) \geq 0$ such that $\mathrm{diam}\big(g^n(B_r)\big) > \xi$. Next, let $0 \leq k \leq n(r)$ be the number of integers $j \in \{0, 1, \dots, n(r)\}$ such that (2) occurs for $D = g^j(B_r)$. Observe that

$$\xi \leq (2r)L^k$$

where L is the Lipschitz constant for the map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Equivalently,

$$k \geq \frac{\log \xi - \log(2r)}{\log L}.$$

At each step when (2) occurs we have

$$\mu_\phi(g(g^j(B_r))) \geq e^{\theta q}\mu_\phi(g^j(B_r))$$

(see (2)) while, if (1) occurs, we have

$$\mu_\phi(g(g^j(B_r))) \geq \mu_\phi(g^j(B_r))$$

as μ_ϕ is an invariant measure. So, $\mu_\phi(B_r) \leq \left(e^{\theta q}\right)^k$. This is suffices to conclude that $\mu(B_r) \leq Cr^{\tau'}$ for some constants C and τ' independent of the ball. Now, taking τ slightly smaller than τ' and $r_0 > 0$ sufficiently small, we can write $\mu(B_r) < r^\tau$ for all $0 < r < r_0$. □

Making use of these two lemmas we now are in a position to prove the following.

Proposition 11. *There exist $\lambda > 1$, $s > 1$, and a topological disc U contained in G^c with piecewise smooth boundary such that $\mu_\phi(U) = 1$, U contains no critical values of f^q and, for all $n \geq 0$,*

$$\mu_\phi(B(\partial U, \tilde{\lambda}^{-n})) < \lambda^{-n}, \tag{14}$$

where $\tilde{\lambda} = \lambda^s$.

Proof. The idea of the proof is natural. We outline one of its possible materializations. Let $x_1, \dots, x_m$ be the set of all critical values of f^q. One can assume without loss of generality that all the points x_i are in $\mathbb{C}$. Let $W_1 = \mathrm{conv}(x_1, \dots, x_m)$ be their convex hull. Let $x_1, \dots x_s$ be those critical values that lie in ∂W_1. After a possible relabeling. we may assume without loss of generality that they ordered clockwise. Consider the interval $[x_i, x_{i+1}]$ and the unique ray l_α with the initial point x_i, containing x_{i+1}. We call it good if there exists a constant $C > 0$ such that

$$\mu_\phi\big(L(x_i; (\alpha - C\lambda^{-3n}, \alpha + C\lambda^{-3n}))\big) < \lambda^{-n} \tag{15}$$

with the notation introduced in Lemma 9. If the interval is not good, then, using Lemma 9, we replace it by the union of two intervals

$$[x_i, y] \cup [y, x_{i+1}]$$

such that $y \notin W_1$ and for the corresponding rays with initial points x_i and x_{i+1} the statement of Lemma 9 holds. So, we obtain a new domain W_1', bounded by a polygonal line such that every edge is a piece of a ray l_α with an initial point x_i and every ray selected for the construction satisfies (15) for a common constant C_1. In the next step, we consider the set $W_2 = \mathrm{conv}(x_{s+1}, \dots, x_m)$. Obviously, $W_2 \subset \mathrm{int}W_1 \subset \mathrm{int}W_1'$. In the same way, one can modify the convex polygon ∂W_2, which now bounds the region $W_2' \subset \mathrm{int}W_1$ so that each edge of $\partial W_2'$ is a piece of a ray for which the initial point is one of the points $(x_{s+1}, \dots, x_m)$, and for all these selected rays (15) holds with some common constant C_2.

Proceeding inductively, we obtain a finite family of regions W_i', bounded by polygonal lines, such that $W_{i+1}' \subset \mathrm{int}W_i'$ all the points $x_1, \dots x_m$ are in $\bigcup \partial W_i'$, each edge of W_i' is a ray L_α with some x_j as an initial point, and for all the rays selected, the formula (15) holds with some common constant C.

The final domain U is constructed in the following way. First, we remove one of the edges of W_1', thus the remaining part is a curve γ_1 homeomorphic to the interval $[0, 1]$. Let z_1 be one of its endpoints. Consider all rays l_α emanating from the point z_1 that intersect W_2'. The corresponding set of α's has a non-empty interior, so one can, again choose a ray satisfying (15) with some constant C. Let z_2 be the first point of intersection of this ray with W_2'. The interval $[z_1, z_2]$ connects $\partial W_1'$ and $\partial W_2'$. Next, we remove a piece of the edge in the boundary of $\partial W_2'$, an interval joining z_2 with one of the vertices of W_2'. The remaining part is a curve γ_2 homeomorphic to the interval $[0, 1]$. So, $\gamma_1 \cup [z_1, z_2] \cup \gamma_2$ is also homeomorphic to the interval. Proceeding inductively in the same way, we get a curve Γ, homeomorphic to the interval $[0, 1]$, and containing all critical values of f^q. We set

$$U = \hat{\mathbb{C}} \setminus \Gamma.$$

This is the required set if f has no critical periodic points. If there are periodic critical points, and G is their neighborhood one has to modify U again, in an analogous way to get a set U which is contained in $\mathbb{C} \setminus G$. We omit the details. So, ∂U is a union of intervals, each of which satisfies (15) with some common constant C. Invoking also Lemma 10 leads us to

$$\mu_\phi(B(\partial U, C\tilde{\lambda}^{-n})) < \lambda^{-n}$$

with $\tilde{\lambda} = \lambda^s$, $s = \max(3, 1/\tau)$ and all $\lambda > 1$ large enough. Enlarging λ if necessary and taking s slightly larger, we can eventually write

$$\mu_\phi(B(\partial U, \tilde{\lambda}^{-n}) < \lambda^{-n}$$

as required. ☐

Remark 12. At this step we fix several constants and we formulate the first condition on r which produces the map $h = g^r$. Namely, we first fix $\lambda > 1$ for which the statement of Proposition 11 is satisfied. We fix the topological disc U whose existence is guaranteed by this same Proposition 11. Having fixed the domain U, we have the constants C_q and $\tilde{C}_q$, guaranteed by Lemma 3. Since, obviously, one can replace λ in the equation (14) by its power with an exponent larger than 1, we can now require λ to be as large as we wish. We thus demand, for future use in Lemma 16, that

$$\tilde{C}_q \lambda = \lambda' > 1 \tag{16}$$

and

$$\lambda'' = \frac{1}{4}(\lambda')^{\frac{\alpha}{2}} > 1, \tag{17}$$

where α, ascribed to β produced in Proposition 8, comes from Lemma 7. Having U, and thus also C_q, $\tilde{C}_q$, λ and $\tilde{\lambda}$ fixed, we formulate the first condition on the integer r. Namely, r should be so large that

$$C_q \gamma^r \le \tilde{\lambda}^{-2}.$$

Note that because of Lemma 3(c_n), this implies that all the good components V of $h^{-n}(U) = g^{-rn}(U)$ have diameters smaller than $(\tilde{\lambda})^{-2n}$ for all $n \ge 1$.

Definition 13. Given an integer $n \ge 0$, a pullback is a sequence of components of $h^{-k}(U)$, $(V_k)_{0 \le k \le n}$ such that $h(V_{k+1}) = V_k$. A pullback is called good if V_n is a good component, i.e. if $V_n \in \tilde{W}_n$. In particular, $h^n : V_n \to U$ is univalent and, by the construction, all components V_k, $k \le n$ are then in $\tilde{W}_k$).

Definition 14. A good pullback $(V_k)_{k \le n}$ is called very good if in addition

$$\operatorname{dist}(V_k, \partial U) \ge \frac{1}{2}\tilde{\lambda}^{-k}$$

for all $k = 1, 2, \ldots, n$. In particular, $\overline{V_k} \subset U$.

Let

$$Z = \bigcap_{k=0}^{\infty} h^{-k}(U).$$

Given $x \in Z$ let $(V_k)_{k \le n}$ be the only pullback of length n such that $x \in V_n$. This pullback will be denoted by V_n^x. Abusing slightly notation, we will frequently denote by V_n^x also the last element of this pullback, i.e. the set V_n. Let $Q = Q(r)$ be the set defined in Proposition 8.

Lemma 15. *Let Z_n be the set of all points $x \in Z$ such that*

$$\#\left\{i \leq n : h^i(x) \in Q\right\} > \alpha n$$

(we already know that $\mu_\phi(Z \setminus Z_n) < \exp(-\delta n)$ and that the estimate is independent of r, $h = g^r$). Let

$$Z_n \supset Y_n := \left\{x \in Z_n : \#\{0 \leq j \leq n : \text{the pullback } V_j^x \text{ is good}\} < (\alpha/2)n\right\}.$$

Then with $r \geq 1$ sufficiently large, which will be fixed in this proof, we have that

$$\mu_\phi(Y_n) < e^{-\frac{\alpha}{8}nr\theta q}.$$

Proof. For every $x \in Y_n$ consider a piece of trajectory

$$x, h(x), h^2(x), \ldots, h^{n-1}(x).$$

Starting form $j = n-1$, and counting backward, we mark the indices j as follows

(1) j is marked with $*$ if either $h^{j-1}(x) \in Q$ and the pullback V_j^x is good, or if $h^{j-1}(x) \notin Q$.
(2) j is marked with $\Box$ if $h^{j-1}(x) \in Q$ but the pullback V_j^x is not good.

Let $m_1 \leq n-1$ be the largest integer $\leq n-1$ which is marked with $\Box$. Note that, since there are at most $\frac{\alpha}{2}n$ good pullbacks and the frequency of visiting Q is at least αn, the number m_1 cannot be smaller than $\frac{\alpha}{2}n$. Since m_1 is marked with $\Box$, the pullback $V_x^{m_1}$ is not good, which means that there exists $m_1' < m_1$ such that

$$V_{m_1-m_1'}(f^{m_1'}(x)) \in \tilde{Z}_{m_1-m_1'} \setminus \tilde{W}_{m_1-m_1'}.$$

Note that, by the rule of marking with squares $\Box$, $h^{m_1-1}(x) \in Q$. Thus, $m_1' < m_1 - 1$. The index m_1' is now our new starting point; again we mark indices with $*$'s and $\Box$'s. Now, let $m_2 \leq m_1'$ be the largest integer $\leq m_1'$, which is marked with $\Box$ and $m_2' < m_2 - 1$ such that $V_{m_2-m_2'}(f^{m_2'}(x) \in \tilde{Z}_{m_2-m_2'} \setminus \tilde{W}_{m_2-m_2'}$. Proceeding inductively, we divide in this way the whole trajectory $x, h(x), \ldots h^n(x)$ into blocks $(m_i', m_i]$ and "gaps" between them. Since, by the construction, every j in the "gap" is marked with $*$, the total length of "gaps" is less than $(1-\frac{\alpha}{2})n$. Therefore,

$$\sum_i (m_i - m_i') > \frac{\alpha}{2}n. \tag{18}$$

In this way, to every $x \in Y_n$ we attribute a sequence $(m(x)) = m_1, m_1', m_2, m_2', \ldots, m_i, m_i'$. Fix such a sequence (m) and let

$$Y_n^{(m)} = \{x \in Y_n : (m(x)) = (m)\}.$$

Then, using the upper bound on $\tilde{B}_n$ established in (7), we have

$$\tilde{B}_{m_i-m_i'}(x) \leq r\exp(-rq(m_i-m_i'-1)\theta) < r\exp\left(-rq(m_i-m_i')\frac{\theta}{2}\right)$$

as $m_i - m_i' \geq 2$. Having this and using the fact that $\mathcal{L}_h(1) = 1$, the crucial observation is now that

$$\mathcal{L}_h^n(1_{Y_n^{(m)}}(x)) \leq \prod_i re^{-(m_i-m_i')r\frac{\theta}{2}q}.$$

Invoking (18), it gives

$$\mathcal{L}_h^n(1_{Y^{(m)}m_n}) < r^n e^{-\frac{\alpha}{4}nr\theta q}.$$

Summing over all possible choices of sequences (m), we can estimate the above by

$$\mathcal{L}_h^n(1_{Y_n}) < 2^n r^n e^{-\frac{\alpha}{4}nr\theta q} < e^{-\frac{\alpha}{8}nr\theta q}$$

if r has been selected so large that $\frac{r}{\log 2r} > \frac{8}{\alpha\theta q}$. This is our second condition on the integer r. So, finally, we fix r at this step. Since $\mu_\phi(Y_n) = \int_{Y_n} 1 d\mu_\phi = \int 1_{Y_n}\, d\mu_\phi = \int L_h^n(1_{Y_n})\, d\mu$, we conclude that

$$\mu_\phi(Y_n) < e^{-\frac{\alpha}{8}nr\theta q}$$

□

For our construction of the induced system, we will need only the pullbacks which do not intersect the boundary of the disc U along their backward trajectories. Recall that $\lambda'' > 1$ was defined in the formula (17).

Lemma 16. *Let $R_n \subset Z$ be the set of points x satisfying the following.*

(1) *$x \notin Y_n$, i.e. the points in R_n have at least $\frac{\alpha}{2}n$ good pullbacks, but*
(2) *No good pullback V_m^x with $m \leq n$, is very good.*

Then

$$\mu_\phi(R_n) < (\lambda'')^{-n}.$$

Proof. As before, we shall estimate $\mathcal{L}_h^n(1_{R_n})$. Let $x \in R_n$ and let $0 \leq m_1 \leq n$ be the largest integer $\leq n$ such that the pullback from $h^{m_1}(x)$ to x is good. Then $m_1 \geq \frac{\alpha}{2}n$. However, by our assumption, this pullback is not very good, i.e. there exists $m_1' < m_1$, an "obstacle", such that the corresponding preimage of U lands too close to ∂U, meaning that

$$\operatorname{dist}\left(\overline{V_{m_1-m_1'}^{h^{m_1'}(x)}}, \partial U\right) \leq \frac{1}{2}\tilde{\lambda}^{-(m_1-m_1')}.$$

Let now m_2 be the largest integer $\leq m_1'$ such that the pullback from $h^{m_2}(x)$ to x is good, and let $m_2' < m_2$ be the "obstacle", i.e., m_2' is the largest integer for which the corresponding pullback, sending $h^{m_2}(x)$ to $h^{m_2'}(x)$ fails to be very good. Proceeding inductively, define in this way consecutive blocks

$$m_1 > m_1' \geq m_2 > m_2' \geq m_3 > m_3' \ldots .$$

We have thus divided the whole trajectory $x, h(x), \ldots, h^n(x)$ into "blocks" $(m'_j, m_j]$ and "gaps" between them. Since for every k in a gap the pullback from $h^k(x)$ to x is not good, the total length of gaps is smaller than $(1 - \frac{\alpha}{2})n$.

Let now a configuration M of such blocks be fixed and let us consider R_n^M, the subset of R_n consisting of all the points producing the configuration M. We estimate the measure of R_n^M in the same way as in the previous lemma.

$$\mu_\phi(R_n^M) = \int 1_{R_n^M} = \int \mathcal{L}_h^n(1_{R_n^M})(y)\, d\mu_\phi(y).$$

Recall that

$$\mathcal{L}_h^n(1_{R_n^M})(y) = \sum_{w \in h^{-n}(y) \cap R_n^M} \exp S_{nr}\tilde{\phi}_q(w).$$

Again, it is important that here we consider the normalized Perron-Frobenius operator with respect to the invariant measure μ_ϕ, so that $\mathcal{L}_h^n(1)(y) = 1$ for every $y \in J(f)$. Next, let us denote by $GB_n(z)$ the sum

$$\sum_w \exp S_{nr}\tilde{\phi}_q(w),$$

where the summation is taken over over all points $w \in h^{-n}(z)$ that belong to a good component V_n of $h^{-n}(U)$, i.e. to an element of $\tilde{W}_n$, but this component is not very good. Since, by the definition of good and (not) very good components, $\operatorname{diam}(V_n) < (\tilde{\lambda})^{-2n}$ and $\operatorname{dist}(V_n, \partial U) < \frac{1}{2}\tilde{\lambda}^{-n}$, we conclude that

$$V_n \subset B(\partial U, \tilde{\lambda}^{-n}).$$

It then follows from Proposition 11 and Corollary 5 that

$$\begin{aligned} \lambda^{-n} &\geq \mu_\phi(B(\partial U, \tilde{\lambda}^{-n})) = \int 1_{B(\partial U, \tilde{\lambda}^{-n})} d\mu_\phi = \int \mathcal{L}_h^n(1_{B(\partial U, \tilde{\lambda}^{-n})}) d\mu_\phi \\ &\geq \int GB_n(z) d\mu_\phi(z) \geq \tilde{C}_q^{-1} \sup_U GB_n. \end{aligned} \tag{19}$$

So,

$$\sup_U GB_n(z) \leq \tilde{C}_q \lambda^{-n} < (\lambda')^{-n} \tag{20}$$

since λ and λ' have been selected so large that $(\tilde{C}_q)^{-1}\lambda > \lambda' > 1$. Finally, we estimate:

$$\mathcal{L}_h^n(1_{R_n^M})(x) \leq \mathcal{L}_h^{n-m_1}(1)(x) \sup_U GB_{m_1 - m'_1} \cdot \sup_U GB_{m_2 - m'_2} \cdot \ldots \cdot \sup_U GB_{m_i - m'_i} \cdot \ldots$$

Inserting (20) to this inequality, we get

$$\mathcal{L}_h^n(1_{R_n^M})(x) \leq (\lambda')^{-\sum(m_i - m'_i)}.$$

Since $\sum(m_i - m'_i)$ as the sum of lengths of all blocks is $\geq (\alpha/2)n$, we thus get that

$$\mathcal{L}_h^n(1_{R_n^M})(x) \leq (\lambda')^{-\frac{\alpha}{2}n}. \tag{21}$$

Finally, we write

$$R_n = \bigcup_M R_n^M$$

where the summation is over all possible configurations M of blocks $(m_i', m_i]$. Since the total number of such configurations is bounded above by 4^n, inequality (21) gives

$$\mathcal{L}_h^n(1_{R_n})(x) < 4^n \, (\lambda')^{-\frac{\alpha}{2}n} < (\lambda'')^{-n}$$

as λ'' has been chosen in (17) so large that $\lambda'' = \frac{1}{4}(\lambda')^{\frac{\alpha}{2}} > 1$.

□

3. Construction of the induced system

In the previous section, we have constructed a topological disc U with several special properties. Recall that $\mu_\phi(U) = 1$ and that $Z = \bigcap_{i=0}^{\infty} h^{-i}(U)$. So, $\mu_\phi(Z) = 1$. We now describe the construction of an induced map F. It follows from previous results that for almost every point $x \in Z$ there exists a pullback from $h^n(x)$ to x, which is good and very good. We thus fix the smallest integer $n \geq 1$ for which the pullback V_n^x is good and very good, and we define

$$F(x) = h^n(x).$$

Note that, if $y \in V_n^x$ then this procedure, applied to y leads to the same component V_n^x. Indeed, by the definition of the induced map, we use the earliest very good pullback. Thus, if $F(y) \neq h^n(y)$ then $F(y) = h^m(y)$ for some $m < n$. Let V_m^y be the corresponding pullback. Then $V_m^y \cap V_n^x \neq \emptyset$ as y belongs to both of these sets, but $V_n^x \nsubseteq V_m^y$ since $x \in V_n^x \setminus V_m^y$. Let us consider $h^m(V_m^y) = U$ and $h^m(V_n^x)$. The latter is an element of the pullback chosen for x (a component of $h^{-(n-m)}(U)$) and, since V_n^x must intersect ∂V_m^y, also $h^m(V_n^x)$ intersects ∂U. But this is impossible by the definition of very good pullbacks. Let $\mathcal{D}$ be the family of all defined in this way sets V_n^x. We have just shown that the function $n : X \to \mathbb{N}$ is constant on each disc $D \in \mathcal{D}$, and so it can and will be treated as a function from $\mathcal{D} \to \mathbb{N}$. In particular, the map

$$F : \bigcup_{D \in \mathcal{D}} \to U$$

is well-defined and its inverse branches $F_D^{-1} : U \to D$, $D \in \mathcal{D}$, form an infinite conformal iterated function systems, which, with a slight abuse of notation, will be also refereed to as F. Keep in mind that $\bigcup_{D \in \mathcal{D}} D$ is a dense subset of U with full measure μ_ϕ and m_ϕ. Let us record the following essential property of this induced system.

Lemma 17. *If D_1, D_2 are two domains in $\mathcal{D}$, $F_{|D_1} = h^n$, $F_{|D_2} = h^m$ then for $0 \leq s < n$, $0 \leq t < m$ either $h^s(D_1) \cap h^t(D_2) = \emptyset$ or the closure of one of these sets is contained in the other set.*

Proof. This, again, follows from our definition of very good pullbacks. Indeed, if $(m - t) = (n - s)$ then the statement is clear since, in this case $V_1 = h^s(D_1)$ and $V_2 = h^t(D_2)$ are both very good pullbacks of the same order $m-t=n-s$ and are therefore disjoint; it may happen, however, that their closures intersect.

Now, assume without loss of generality that $(m-t)>(n-s)$ and that $V_1 \cap V_2 = h^s(D_1) \cap h^t(D_2) \neq \emptyset$. If $\overline{V}_2 \nsubseteq V_1$ then $\overline{V}_2$ intersects ∂V_1. Considering the sets $h^{n-s}(V_1)=U$ and $h^{n-s}(V_2)=V_2'$, we see that V_2' is in a very good pullback and its closure intersects ∂U. This is impossible. Thus $\overline{V}_2 \subseteq V_1$ and we are done. □

Following the setting of [MU2], we parametrize the family $\mathcal{D}$ by an infinite countable set E so that $\mathcal{D}=\{D_e : e \in E\}$, and given $e \in E$, we let φ_e denote the branch of F^{-1} defined on U and mapping U onto the disc D_e. Recall that the integer $n(e)=n(D_e) \geq 1$ is uniquely determined by the condition that

$$F|_{D_e} = h^{n(e)}|_{D_e}.$$

It immediately follows from the construction of the system F, Lemma 7, Lemma 15, and Lemma 16, that with some $\lambda''' > 1$ small, enough,

$$m_\phi\left(\bigcup\{D_e : n(e)=n\}\right) \leq (\lambda''')^{-n} \tag{22}$$

for all $n \geq 1$. Let E^n be the set of all words of length n, and let $E^* = \bigcup_{n=1}^\infty E^n$. Let E^∞ be the space of all infinite words of the alphabet E. For every $\omega \in E^n$, $\omega=(\omega_1,\dots,\omega_n)$ let

$$\varphi_\omega = \varphi_{\omega_1} \circ \dots \circ \varphi_{\omega_n}.$$

Denote respectively by diam_{hyp} and dist_{hyp} the diameter and distance evaluated with respect to the hyperbolic metric in U, and, as before, by diam, dist respectively the diameter and distance evaluated with respect to the spherical metric.

Proposition 18. *There exist constants $C>0$ and $0<\tau<1$, such that, if $\omega \in E^n$ then*

$$\mathrm{diam}_{hyp}(\varphi_\omega(U)),\ \mathrm{diam}(\varphi_\omega(U)) \leq C\tau^n.$$

If $\omega=(\omega_1,\omega_2,\dots,\omega_n,\dots) \in E^\infty$ then

$$\bigcap_{n=1}^\infty \varphi_{\omega_1} \circ \dots \circ \varphi_{\omega_n}(U)$$

is a singleton.

Proof. For every $e \in E$ the map φ_e is an isometry in hyperbolic metrics in U and $D_i = \varphi_e(U)$, respectively. We know that φ_e is an inverse branch of the iterate $h^{n(e)}$. Since $\mathrm{diam}(D_e) < \tilde{\lambda}^{-2n(e)}$ and $\mathrm{dist}(D_e, \partial U) > \frac{1}{2}\tilde{\lambda}^{-n(e)}$, there exists a constant $C_1>0$ such that

$$\mathrm{diam}_{hyp}(D_e) \leq C_1 \tag{23}$$

for all $e \in E$. The inclusion $D_e \to U$, considered in hyperbolic metrics in D_e and U, respectively, is a contraction by some factor $\tau < 1$ independent of $e \in E$. Therefore,

$\varphi_e : U \to U$ is a contraction by some factor less than or equal equal to τ, with respect to the hyperbolic metric in U. Thus, for every $\omega \in E^{n-1}$, every set $A \subset U$,

$$\mathrm{diam}_{hyp}(\varphi_\omega(A)) \le \tau^{n-1}\mathrm{diam}_{hyp}(A).$$

Now, if $\omega = (\omega_1, \dots, \omega_n) \in E^n$, then $(\omega_1, \dots, \omega_{n-1}) \in E^{n-1}$ and

$$\varphi_\omega(U) = \varphi_{\omega_1,\dots,\omega_{n-1}}(\varphi_{\omega_n}(U)).$$

But $\mathrm{diam}_{hyp}(\varphi(D)) \le C_1$, and, so

$$\mathrm{diam}_{hyp}(\varphi_\omega(D)) \le C_1\tau^{n-1} = C\tau^n, \tag{24}$$

where $C = \frac{C_1}{\tau}$. Since the diameter evaluated with respect to the hyperbolic metric in U is larger than the spherical diameter, multiplied by some positive constant, we get from (24) that

$$\mathrm{diam}(\varphi_\omega(D)) \le C\tau^n \tag{25}$$

with some modified constant $C > 0$. Moreover, since for every very good pullback V_k we have $\overline{V}_k \subset U$, it follows that also that

$$\overline{\varphi_{\omega_1,\dots\omega_n}(U)} \subset \varphi_{\omega_1,\dots\omega_{n-1}}(U)$$

This implies that the intersection

$$\bigcap_{n=1}^{\infty} \varphi_{\omega_1\dots\omega_n}(U) = \bigcap_{n=1}^{\infty} \overline{\varphi_{\omega_1\dots\omega_n}(U)}. \ne \emptyset$$

As the diameters converge to zero, this intersection is a singleton. We are done. $\square$

Let X be the limit set of the iterated function system F, i.e.

$$X = \bigcap_{n=1}^{\infty} \bigcup_{\omega \in E^n} \varphi_\omega(U).$$

AS $\mu_\phi(U) = 1$, guaranteed by Proposition 11, it immediately follows from (22) that

$$\mu_\phi(X) = 1. \tag{26}$$

In virtue of Proposition 18 the map $\pi : E^\infty \to X$,

$$\omega \mapsto \bigcap_{n=1}^{\infty} \varphi_{\omega_1} \circ \varphi_{\omega_1}, \circ \dots \circ \varphi_{\omega_n}(U)$$

is a well defined bijection. The proposition below follows directly from the definition of the map $\pi : E^\infty \to X$ and from Proposition 18.

Proposition 19. *Let $\omega, \omega' \in E^\infty$. Assume that $\omega_1 = \omega'_1$. Let $s = s(\omega, \omega') \geq 1$ be the largest integer such that $(\omega_1, \ldots, \omega_s) = (\omega'_1, \ldots, \omega'_s)$. Then*

$$\mathrm{dist}(\pi(\omega), \pi(\omega')) \leq C\tau^{s(\omega,\omega')}$$

and

$$\mathrm{dist}_{hyp}(\pi(\omega), \pi(\omega)') \leq C\tau^{s(\omega,\omega')}.$$

As we have passed to induced system, we shall modify the potential ϕ accordingly to this inducing process. First, if $D_e \in \mathcal{D}$, $e \in E$ is one of discs on which F is defined, and if $F_{|D_e} = h^{n(e)}$, then we put, for all $x \in D_e$,

$$\overline{\phi}(x) = \sum_{k=0}^{n(e)-1} \phi_{qr}(h^k(x)).$$

Then, for all Borel sets $A \subset D_e$ we have that,

$$m_\phi(F(A)) = m_\phi(h^{n(e)}(A)) = \int_A \exp\left(-\sum_{k=0}^{n(e)-1} \phi_{qr} \circ h^k\right) dm_\phi = \int_A \exp(-\overline{\phi}(x)) dm_\phi(x).$$

Along with (26) this entails the following.

Lemma 20. *The probability measure m_ϕ is $\exp(-\overline{\phi})$-conformal for the map $F : X \to X$.*

For the sake of the next proposition, we need to extend the potential ϕ beyond the Julia sets $J(f)$.

Lemma 21. *The function ϕ can be extended in a Hölder continuous manner, with the same Hölder exponent, to the whole Riemann sphere $\hat{\mathbb{C}}$.*

This lemma is well-known; a proof can be found in [UZ]. From now on, we assume that the potential ϕ is defined and Hölder continuous in the whole Riemann sphere $\hat{\mathbb{C}}$. Having Lemmas 20 and 21, the general theory of infinite iterated function systems, as developed in [MU2] along with [MU3], gives the following.

Proposition 22. *There exists a unique probability F-invariant measure $\mu_{\overline{\phi}}$ which is equivalent to m_ϕ. Moreover the Radon-Nikodym derivative $\overline{\rho} := \frac{d\mu_{\overline{\phi}}}{dm_\phi}$ is bounded above and separated below from zero. This Radon-Nikodym derivative $\overline{\rho}$ has a continuous extension $\overline{\rho} : U \to (0; +\infty)$ to the whole disc U and this extension is a fixed point of the following transfer operator.*

$$\mathcal{L}_{\overline{\phi}}(v)(x) = \sum_{y \in F^{-1}(x)} \exp \overline{\phi}(y) v(y).$$

This is a bounded linear operator acting on the Banach space $C_b(U)$ of all bounded real-valued continuous functions defined on U, and it is easy to see that this operator is almost periodic.

4. A modified Iterated Function System

Let us prove the following.

Proposition 23. *The function $\log|F'|$ is integrable with respect to the measure m_ϕ.*

Proof. We check first that

$$\int (\log|F'|)_- \, dm_\phi < +\infty.$$

Note that $|F'|$ may be arbitrarily close to 0 near the boundary of some components $D \in \mathcal{D}$, as the construction of good inverse branches starts from considering all components of $f^{-q}(U)$ and some of them may have critical points of f^q in their boundaries. However, $\log|(f^q)'|$ is integrable, by Lemma 10, and, by the same reason, $\log|h'|$ is integrable over every component D of $h^{-1}(U)$ ($D \in \mathcal{D}$), used for our construction, for which $n(e) = 1$, i.e. F is defined as h. If $e \in E$, put $V_e = h^{n(e)-1}(D_e)$. Then it is easy to see from our construction that

$$\log|F'|(z) \geq -C + \log|h'(h^{n-1})|(z)$$

for some constant $C > 0$ independent of $e \in E$, and for all $z \in D_e$. So, we can estimate as follows. Then

$$\begin{aligned} \int_{D_e} (\log|F'|)_-(z)\, dm_\phi(z) &\leq \int_{D_e} \big(C + (\log|h'(h^{n(e)-1})|)_-(z)\big)\, dm_\phi(z) \\ &= Cm_\phi(D_e) + \int_{V_e} (\log|h'|)_-(z) Jac_{m_\phi}\big((h^{n(e)-1}|_{D_e})\big)^{-1}(z)\, dm_\phi(z) \end{aligned} \tag{27}$$

Now

$$Jac_{m_\phi}\big((h^{n(e)-1}|_{D_e})\big)^{-1}(z) = \exp\big(S_{n(e)-1})\phi_{qr} \circ h^{n(e)-1}|_{D_e})\big)^{-1}(z)\big)$$

has a bounded distortion on V_e independent of e. So, picking an arbitrary point $z_e \in V_e$, we can write

$$\begin{aligned}\int_{V_e} (\log |h'|)_-(z) Jac_{m_\phi}\big((h^{n(e)-1}|_{D_e})\big)^{-1}(z)\, dm_\phi(z) \le \\ &\le C Jac_{m_\phi}\big((h^{n(e)-1}|_{D_e})\big)^{-1}(z_e) \int_{V_e} (\log |h'|)_-(z)\, dm_\phi(z) \\ &\le C^2 \int_{V_e} (\log |h'|)_-(z)\, dm_\phi(z) \frac{m_\phi(D_e)}{m_\phi(V_e)} \\ &\le M_1^{-1} C^2 \int_{V_e} (\log |h'|)_-(z)\, dm_\phi(z) m_\phi(D_e) \\ &\le M_1^{-1} C^2 \int_{J(f)} (\log |h'|)_-(z)\, dm_\phi(z) m_\phi(D_e) \\ &= M_1^{-1} M_2 C^2 m_\phi(D_e)\end{aligned}$$

where $M_1 = \min\{m_\phi(V_e) : n(e) = 1\} > 0$ since the set $\{e \in E : n(e) = 1\}$ is finite, and $M_2 = \int_{J(f)} (\log |h'|)_-\, dm_\phi < +\infty$. It remains to check that $\int \log |F'|_+ d\mu_\phi < \infty$. Indeed, since $||h'||_\infty < +\infty$, we we have

$$\log |F'|_+(z) \le \log |(h^n)'(z)|_+ \le n \log ||h'||_\infty$$

for all $e \in E_n := \{e \in E : n(e) = n\}$ and all $z \in H_n := \bigcup_{e \in E_n} D_e$. Therefore, using (22), we get

$$\int_X \log |F'|_+ dm_\phi = \sum_{n=1}^{\infty} \int_{H_n} \log |F'|_+ dm_\phi \le \sum_{n=1}^{\infty} n \log ||h'||_\infty (\lambda'')^n < +\infty$$

We are done. $\square$

In this section, rather for technical reasons, we modify our induced map in the following way. Let $D = D_e$ with some $e \in E$ be such that $n(e) = 1$. We define $F_* : D \to D$ as the first return map under $F(!)$ to the set D. In fact F_* is then defined $\mu_{\overline{\phi}}$ almost everywhere, thus also m_ϕ-almost everywhere, on D. It is easy to see that, since F was an iterated function system with non-overlapping domains, and D is one of these domains, the system F_* has also this property. More precisely, this new induced map F_* is defined on a union of pairwise disjoint (although their closures may touch each other) topological discs $\hat{D}$ whose closures are contained in D and such that $m_\phi(\bigcup \hat{D}) = m_\phi(D)$. Let X_* be the limit set of this new iterated function system F_*. Note that X_* is not necessarily closed. Parametrize the set of all discs $\hat{D}$ by some countable set I and for every $i \in I$ denote by $\hat{\varphi}_i : D \to \hat{D}_i$ the corresponding branch of F_*^{-1}, mapping D onto $\hat{D}_i$. Let the integer $N(i) \ge 1$ be determined by the property that $F_* = f^{N(i)}$ on $\hat{D}_i$.

Proposition 24. *The induced system F_* is expanding. Precisely, there exist constants $C > 0$ and $0 < \eta < 1$ such that, for every $n \geq 1$, we have $|(F_*^n)'| > C\eta^{-n}$ everywhere in the domain of F_*^n.*

Proof. By the construction, every map $\hat{\varphi}_i : D \to \hat{D}_i$, $i \in I$, is the restriction to D of a composition of several very good pullbacks, all defined on U. Since the closure of D is contained in U, the inclusion $D \to U$ is a strict contraction in the hyperbolic metric in U. Since the map $\hat{\varphi}_i : U \to \hat{D}_i$ is an isometry with respect to the hyperbolic metrics in U and its image $\hat{D}_i$, respectively, it follows that $\hat{\varphi}_i : U \to U$ contracts the hyperbolic metric in U by a constant factor less than 1. Finally, since in D both spherical metric and hyperbolic metric (with respect to U) are comparable, there thus exists an integer $N \geq 1$ such that the spherical metric is also contracted by every composition of of maps $(\hat{\varphi}_i)|_D$, of length N. □

Remark 25. Since all the maps $\hat{\varphi}_i$, $i \in I$, are defined on U and $\overline{D} \subset U$, the distortion of all maps $\hat{\varphi}_i$ and all their compositions is bounded above by a common positive constant. We denote by K.

As a straightforward consequence of Proposition 23, and Kac's lemma applied for the function $\log|(F_*)'|$ and the invariant measure $\mu_{\overline{\phi}}$, we get the following.

Proposition 26. *The function $\log|(F_*)'|$ is m_ϕ-integrable.*

We now modify now the potential $\overline{\phi}$ according to the second inducing scheme, putting, for every $i \in I$ and every point $x \in \hat{D}_i$,

$$\phi_*(x) = \sum_{k=0}^{N(i)-1} \overline{\phi}(F^k(x)),$$

Again, it is straightforward to verify that the measure m_ϕ is conformal for the system F_*. Similarly as for F, the following proposition holds for the new system F_*.

Proposition 27. *There exists a unique probability F_*-invariant measure μ_{ϕ_*} on X_* which is equivalent to $m_\phi|_{X_*}$. Moreover the Radon-Nikodym derivative $\rho_* := \frac{d\mu_{\phi_*}}{dm_\phi}$ is bounded above and separated below from zero. This Radon-Nikodym derivative ρ_* has a continuous extension $\rho_* : D \to (0; +\infty)$ to the whole disc D and this extension is a fixed point of the following transfer operator*

$$\mathcal{L}_{\phi_*}(v)(x) = \sum_{y \in F^{-1}(x)} \exp \phi_*(y) v(y).$$

This is a bounded linear operator acting on the Banach space $C_b(D)$ of all bounded real-valued continuous functions defined on D, and it is easy to see that this operator is almost periodic. In fact μ_{ϕ_*} is the measure $\mu_{\overline{\phi}}$, conditioned on D, and the Radon-Nikodym derivative ρ_* extends continuously even to U.

As an immediate consequence of Proposition 26 we get the following.

Corollary 28. *The Lyapunov exponent*

$$\int \log |F_*'| d\mu_{\phi_*}$$

is finite.

Having this corollary, in virtue of Theorem 4.4.2 in [MU2], which is a version of Volume Lemma, we can express the Hausdorff dimension of the measure μ_{ϕ_*} as follows.

Corollary 29.

$$\mathrm{HD}(\mu_{\phi_*}) = \frac{h_{\mu_{\phi_*}}}{\int \log |F_*'| d\mu_{\phi_*}} = \frac{h_{\mu_{\phi_*}}}{\chi_{\mu_{\phi_*}}}.$$

Remark 30. Let α be the partition into sets $\hat{D}_i = \hat{\varphi}_i(D)$, $i \in I$. Then α is a generator for F_*. It follows from the proof of Theorem 4.4.2 in [MU2] that, if the Lyapunov exponent of F_* is finite then also $H_{\mu_{\phi_*}}(\alpha)$, the entropy of the partition α, is finite. In consequence, α is a countable generator with finite entropy.

5. Dimension of the equilibrium measure–rigidity

In this section we provide the proof of Theorem 43 which characterizes all rational maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ for which

$$\beta := \mathrm{HD}(\mu_\phi) = \mathrm{DD}(J(f)).$$

or, as will turn out to be equivalent, for which

$$\beta := \mathrm{HD}(\mu_\phi) = \mathrm{HD}(J(f)).$$

Therefore, throughout the present section we keep the following

Assumption. $\mathrm{HD}(\mu_\phi) = \beta = \mathrm{DD}(J(f))$.

5.1. **Homology equation for F_*.** With the above assumption we get

$$\mathrm{HD}(\mu_{\phi_*}) = \mathrm{HD}(m_\phi) = \mathrm{HD}(\mu_\phi) = \beta$$

and, therefore, in virtue of Corollary 29,

$$h_{\mu_{\phi_*}} - \beta\chi_{\mu_{\phi_*}} = 0.$$

Invoking the Variational Principle (see Theorem 2.1.7 in [MU2]), this implies that

$$P_*(-\beta \log |F_*'|) \geq 0,$$

where P_* refers to topological pressure defined with respect to the iterated function system F_*. But, as X_* consists of radial points, we get in virtue of Theorem 1, that $\beta = \mathrm{DD}(J(f)) \geq \mathrm{HD}(X_*)$. Hence, Bowen's formula (see Theorem 4.2.13 in [MU2]) yields this.

$$P_*(-\beta \log |F_*'|) \leq 0.$$

These two inequalities, taken together, give the following.

Corollary 31. *If $\beta = \mathrm{HD}(J(f)) = \mathrm{HD}(\mu_\phi)$, then*

$$P_*(-\beta \log |F_*'|) = 0.$$

With this corollary, the results from [MU2] and [MU3], give the following.

Proposition 32. *There exists a unique conformal measure m_β on X_*, i.e. a Borel probability measure m_β on X_*, such that*

$$m_\beta(F_*(A)) = \int_A |F_*'|^\beta dm_\beta$$

for every Borel subset of X_* on which F_* is one-to one. Moreover there exists a unique Borel probability F_*-invariant measure (a Gibbs state) $\mu_\beta \asymp m_\beta$. In fact, $\mu_\beta = \rho \cdot m_\beta$, where the Radon-Nikodym derivative ρ has a continuous extension to D, and this extension $\rho : D \to (0, +\infty)$ is a fixed point of the Perron-Frobenius operator

$$\mathcal{L}(g)(x) = \sum_{y \in F_*^{-1}(x)} g(y) \frac{1}{|F_*'((y))|^\beta}$$

acting, as a bounded linear operator, on the Banach space $C_b(D)$.

Remark 33. We do not know whether, $\int \log |F_*'| d\mu_\beta$, the Lyapunov exponent of the measure μ_β, is finite. Fortunately, we do not need this for the next steps of the proof.

We now prove the following.

Proposition 34. *If* $\mathrm{HD}(\mu_{\phi_*}) = \beta = \mathrm{HD}(J(f))$, *then*

$$\frac{\rho}{\rho \circ F_*} \cdot \frac{1}{|F_*'|^\beta} \cdot \frac{1}{J_{\phi_*}^{-1}} = 1 \tag{28}$$

everywhere in X_*, *where* $J_{\phi_*} = \exp(-\phi_*) \cdot \frac{\rho_* \circ \hat{F}}{\rho_*}$ *is the Jacobian of the map* F_* *evaluated with respect to the invariant measure* μ_{ϕ_*}.

Proof. This proof follows an argument of habilitation thesis of Gerhard Keller. We consider two Perron-Frobenius operators

$$\hat{\mathcal{L}}_{\phi_*}(g)(x) = \sum_{y \in F_*^{-1}(x)} g(y)(J_{\hat{\phi}}(y))^{-1}$$

and

$$\mathcal{L}(g)(x) = \sum_{y \in F_*^{-1}(x)} g(y) \frac{1}{|F_*'(y)|^\beta}, \tag{29}$$

and write

$$\begin{aligned}
1 = \int 1 d\mu_{\phi_*} = \int \frac{\mathcal{L}\rho}{\rho} d\mu_{\phi_*} &= \int \mathcal{L}_{\hat{\phi}} \left(\frac{\rho \cdot \frac{1}{|\hat{F}'|^\beta}}{J_{\phi_*}^{-1} \cdot \rho \circ F_*} \right) d\mu_{\phi_*} \\
&= \int \frac{\rho \cdot \frac{1}{|F_*'|^\beta}}{J_{\phi_*}^{-1} \cdot \rho \circ F} d\mu_{\phi_*} \\
&\geq 1 + \int \log \left(\frac{\rho}{\rho \circ F_*} \cdot \frac{1}{|F_*'|^\beta} \cdot \frac{1}{J_{\phi_*}^{-1}} \right) d\mu_{\phi_*} \\
&= 1 + \int \log \rho \, d\mu_{\phi_*} - \int \log \rho \circ F_* d\mu_{\phi_*} + \int \log J_{\phi_*} d\mu_{\phi_*} - \int \log |F_*'|^\beta d\mu_{\phi_*}.
\end{aligned} \tag{30}$$

Since the partition α, introduced in Remark 30, is a countable generator with finite entropy for F_*, we have that $\int \log J_{\phi_*} d\mu_{\phi_*} = h_{\mu_{\phi_*}}$. Therefore, the last sum in (30) is equal to 1. This means that

$$\frac{\rho}{\rho \circ F_*} \cdot \frac{1}{|F_*'|^\beta} \cdot \frac{1}{J_{\phi_*}^{-1}} = 1 \tag{31}$$

μ_{ϕ_*} almost everywhere. Since all the functions appearing in this equation are continuous in each disc $\hat{D}_i$, $i \in I$, the equality (31) holds everywhere in each set $\hat{D}_i \cap X_*$ $i \in I$. □

Therefore, we can write

$$|F_*'|^\beta = \exp(-\phi_*) \frac{\rho_* \circ F_*}{\rho_*} \cdot \frac{\rho}{\rho \circ F_*} = \exp(-\phi_*) \frac{r \circ F_*}{r}$$

everywhere in each set $\hat{D}_i \cap X_*$, where $r = \rho_*/\rho$. Put $u_0 = \log r$. We rewrite the above displayed formula in the following.

Corollary 35. *If* $\dim_H(\mu_{\phi_*}) = \beta = \dim_H(J)$*, then everywhere in* X_**, we have that*

$$\beta \log |F_*'| = -\phi_* + u_0 \circ F_* - u_0, \tag{32}$$

and u_0 *has a continuous extension to* D *given by the formula* $u_0 = \log(\hat{\rho}/\rho)$*.*

Recalling Theorem Proposition 32 and Proposition 27, we obtain from this corollary and Theorem 2.2.7 in [MU2], the following.

Corollary 36. *If* $\dim_H(\mu_{\phi_*}) = \beta = \dim_H(J)$*, then* $\mu_{\phi_*} = \mu_\beta$*.*

5.2. **Cohomology equation for** h**.** We start this section with proving the following.

Lemma 37. *There exists a continuous function* $u : U \to \mathbb{R}$ *such that*

(a) $u|_{X_*} = u_0$.
(b) $\beta \log |F'(x)| = u \circ F(x) - u(x) - \overline{\phi}(x)$ *for all* $x \in F(X_*) \cap F^{-1}(F(X_*))$.

Proof. Since $F(D) = U$ and since $F|_D$ is one-to-one, we may define $u : U \to \mathbb{R}$ by declaring that

$$u(F(x)) = \beta \log |F'(x)| + \overline{\phi}(x) + u(x) \tag{33}$$

for all $x \in D$. In order to prove (a) fix a point $x \in X_*$. Since $U = F(D)$ and $F|_D$ is one-to-one, there exists a unique $y \in D$ such that $x = F(y)$. But now, since both x and y belong to D and since F_* is the first return map to D, we have $F_*(y) = F(y)$. In consequence $y \in X_*$, and applying Corollary 35, we get

$$\begin{aligned} u_0(x) &= u_0 \circ F(y) = u_0 \circ F_*(y) = \phi_*(y) + u_0(y) + \beta \log |(F_*)'(y)| \\ &= \overline{\phi}(y) + u_0(y)) + \beta \log |F'(y)| \\ &= u(F(y)) = u(x). \end{aligned}$$

In order to prove (b) suppose that $x \in F(X_*) \cap F^{-1}(F(X_*))$. Then $x = F(y)$ and $F(x) = F(z)$ with $y, z \in X_*$. Write $F_*(y) = F^m(y)$, $m \geq 1$, and $F_*(z) = F^n(z)$, $n \geq 1$. Corollary 35 then gives

$$\begin{aligned} u_0(F_*(y)) &= u_0(y) + \phi_*(y) + \beta \log |(F_*)'(y)| \\ &= u_0(y) + \overline{\phi}(y)) + \sum_{j=0}^{m-2} \overline{\phi}(F^j(x) + \beta \log |(F^{m-1})'(x)| + \beta \log |F'(y)| \\ &= u(x)) + \sum_{j=0}^{m-2} \overline{\phi}(F^j(x)) + \beta \log |(F^{m-1})'(x)|, \end{aligned} \tag{34}$$

and likewise

$$u_0(F_*(z)) = u(F(x)) + \sum_{i=0}^{n-2} \overline{\phi}(F^i(F(x))) + \beta \log |(F^{n-1})'(F(x))|. \tag{35}$$

Now consider two cases. Suppose first that $x \notin D$. Then $F_*(y) = F_*(z)$ and $m = n+1$. Equating (34) and (35), we then get

$$\begin{aligned} &u(F(x)) + \sum_{i=0}^{n-2} \overline{\phi}(F^i(F(x))) + \beta \log |(F^{n-1})'(F(x))| \\ &= \\ &u(x) + \sum_{j=0}^{n-1} \overline{\phi}(F^j(x)) + \beta \log |(F^n)'(x)| \end{aligned}$$

or equivalently,

$$u(F(x)) - u(x) = \log |F'(x)| + \overline{\phi}(x),$$

which is just the equation (b). If, on the other hand, $x \in D$, then $x \in X_*$ and, by (a), $u(x) = u_0(x)$, and hence, (33) implies that $u(F(x)) = u(x) + \beta \log |F'(x)| + \overline{\phi}(x)$. We are done. $\square$

Proposition 38. *The function $u : U \to \mathbb{R}$ satisfies the homology equation for the map $h = g^r = f^{rq}$ throughout $J \cap U$. Precisely,*

$$u \circ h - u = \beta \log |h'| + \phi_{qr} \tag{36}$$

everywhere in $J \cap U$.

Proof. Since $m_\phi(X_*) = m_\phi(J \cap D)$, we get that

$$m_\phi(F(X_*)) = m_\phi(F(J \cap D)) = m_\phi(J) = 1. \tag{37}$$

Since also $m_\phi(F^{-1}(U) = 1$ and $F(X_*) \subset U$, we thus conclude that $m_\phi(F^{-1}(F(X_*)) = 1$. Along with (37) this implies that

$$m_\phi\big(F(X_*) \cap F^{-1}(F(X_*))\big) = 1.$$

Hence, $F(X_*) \cap F^{-1}(F(X_*))$ is a dense subset of the Julia set J. Therefore, $D_e \cap F(X_*) \cap F^{-1}(F(X_*))$ is a dense subset of $D_e \cap J$ for every $e \in E$. Since $F|_{D_e}$ and $u : U \to \mathbb{R}$ are continuous, we thus conclude from Lemma 37 (b) that

$$\beta \log |F'(x)| = -\overline{\phi}(x) + u \circ F(x) - u(x) \tag{38}$$

for all $e \in E$ and all $x \in D_e \cap J$. Now keep $e \in E$ and $x \in D_e \cap J$. Because of the way the point $F(x)$ is defined, i.e. the earliest very good pullback from $h^n(x)$ to x with some

$n \geq 1$, we have that $F^m(h(x)) = F(x)$ with some $0 \leq m \leq n-1$. Formula (38) applied to x then gives,

$$\begin{aligned} \beta \log |(h^n)'(h(x))| &= \beta \log |F'(x)| = u \circ F(x) - u(x) - \phi_*(x) \\ &= u(h^n(x)) - u(x) - \sum_{j=0}^{n-1} \phi_{qr}(h^j(x)) \end{aligned} \tag{39}$$

and started from $h(x)$ and iterated m times, it gives

$$\begin{aligned} \beta \log |(h^{n-1})'(x)| &= \beta \log |(F^m)'(h(x))| \\ &= u \circ F^m(h(x)) - u(h(x)) - \overline{\phi}(h(x)) \\ &= u(h^n(x)) - u(h(x)) - \sum_{j=1}^{n-1} \phi_{qr}(h^j(x)). \end{aligned}$$

Subtracting this from (39), we get that

$$\beta \log |h'(x)| = u(h(x)) - u(x) - \phi_{qr}(x).$$

Hence (36) is established throughout the set $J \cap \bigcup_{e \in E} D_e$. Since this set is dense in $J \cap U$ and all the functions appearing in (36) are continuous, formula (36) is thus proved throughout $J \cap U$. The proof is complete. □

The following proposition is now easy to prove.

Proposition 39. *If $p \in J$ is not an element of the postcritcal set for the iterate $h = f^{qr} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, then there exists a neighborhood V of p such that the restriction $u_{|V \cap J \cap U}$ is uniformly continuous and bounded.*

Proof. Since u is continuous in U, it is uniformly continuous, and thus bounded, in every open set W such that $\overline{W} \subset U$. This proves the statement for all points $p \in U$. If $p \in J \setminus U$, then $p \in J \cap \partial U$ and $U \cap h^{-k}(p) \neq \emptyset$ for some integer $k \geq 0$ sufficiently large. Pick $\tilde{p} \in h^{-k}(p) \cap U$. Since $p \in J$ is not an element of the postcritcal set for h, the logarithm $\log |(h^k)'|(x)$ is uniformly continuous and bounded in some neighborhood V of $\tilde{p}$. So, the statement of our theorem follows from the equation (36) and the already proved part of the theorem. □

Corollary 40. *If $p \in J$ is in the postcritical set of $h = f^{qr}$, i.e. if $p = h^k(c)$ for some integer $k \geq 1$, then each point $\tilde{p} \in h^{-1}(p)$ is either a critical point of h or belongs to the postcritical set of h. In particular, the trajectory of every critical point $c \in J$ is finite, i.e. every critical point in J is eventually periodic.*

Proof. Since the rational function $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ has only finitely many critical points, the second assertion follows from the first one. So, we will prove the first assertion of our corollary. Since J contains no critical periodic cycles of h, we may assume without loss of generality that c does not belong to the postcritical set of h. Thus, by the previous corollary, the function u is bounded in $U \cap V \cap J$, for some neighborhood V of c. Note that this intersection is nonempty, even infinite, and c is its accumulation point, even if $c \in \partial U$. Thus, keeping in mind that $h^k(V)$ is a neighborhood of $h^k(c)$ and invoking (36), we see that the function u is unbounded in the set $h^k(U \cap V) \cap J$. Even more, $u(z) \to \infty$ as $z \to h^k(c)$, $z \in h^k(U \cap V) \cap J$. If now $p = h^k(c)$ had a preimage $\tilde{p} \in h^{-1}(p)$ which is neither a critical point of h nor an element of a postcritical set h, then, by Proposition 39, there would exists a neighborhood W of $\tilde{p}$ such that u is bounded in $J \cap W \cap U$. But then, by (36) again, the function u would bounded in some neighborhood of p in J. This contradiction finishes the proof of the first assertion, and the whole corollary is established. $\square$

For the final conclusion we will need the following observation whose proof can be found for example in [DH]).

Proposition 41. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. Recall that P_R is the postcritical set of R. If some set $Z \subset P_R$ satisfies $R^{-1}(Z) \subset P_R \cup \mathrm{Crit}(R)$, then $\#(Z) \le 4$. If $\#(Z) = 4$ then all critical points are ordinary, Z contains the set of all critical values of R, and $Z \cap \mathrm{Crit}(R) = \emptyset$.*

Remark 42. Actually, the authors in [DH] assume that the map R itself is critically finite, in particular, that the trajectories of all critical points outside $J(R)$ are also finite, but this assumption is not used in the proof.

We can now classify all rational maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ for which $\dim_H(\mu_\phi) = \dim_H(J(f))$. Notice that passing to the iterate $h = f^{qr}$ does not alter the postcritical set, i.e. $P_f = P_h$. We prove the following.

Theorem 43. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map, let $\phi : J(f) \to \mathbb{R}$ be a Hölder continuous potential such that $\sup(\phi) < P(\phi)$, and let μ_ϕ be a unique equilibrium state corresponding to this potential. Then the following are equivalent.*

(1) $\mathrm{HD}(\mu_\phi) = \mathrm{DD}(J(f))$,
(2) $\mathrm{HD}(\mu_\phi) = \mathrm{HD}(J(f))$,
(3) *The intersection $P_f \cap J(f)$ consists of at most four points, there are no other points in $\overline{P_f} \cap J(f)$ and also the potential $\phi : J(f) \to \mathbb{R}$ is cohomologous modulo constant to $-\mathrm{DD}(J(f)) \log |f'|$ in the class of continuous functions on $J(f) \setminus P_f$. The cohomology constant is equal to $P(\phi)$.*

(4) *The intersection $P_f \cap J(f)$ consists of at most four points, it is equal to $\overline{P_f} \cap J(f)$, and also the potential $\phi : J(f) \to \mathbb{R}$ is cohomologous modulo constant to $-\mathrm{HD}(J(f))\log|f'|$ in the class of continuous functions on $J(f) \setminus P_f$. The cohomology constant is equal to $P(\phi)$.*

In addition, if the closure of the postcritical set P_f is disjoint from $J(f)$, which equivalently means that the restriction $f_{|J(f)} : J(f) \to J(f)$ is then expanding, and the potential $\phi : J(f) \to \mathbb{R}$ is cohomologous modulo constant to $-\mathrm{HD}(J(f))\log|f'|$ in the class of Hölder continuous functions on $J(f)$.

Proof. Assume without loss of generality that $P(\phi) = 0$. Apart from the fact that $\overline{P}_f \cap J(f) = P_f \cap J(f)$, the implication (1)⇒(3) follows immediately from Proposition 38, Proposition 39, Corollary 40, and Proposition 41. In order to prove this implication in full, suppose for a contrary that there exists a point $w \in \overline{P}_f \cap J(f) \setminus P_f \cap J(f)$. In virtue of the Fatou-Sullivan classification of connected components of the Fatou set, and since the boundaries of a Siegel disc and a Herman ring are contained in $\overline{P}_f$, we conclude that w must be a rationally indifferent periodic point of f. Passing to a sufficiently high iterate, we may assume without loss of generality that w is a fixed point of f. But since $w \in J(f) \setminus P_f$ it follows from Proposition 39 that the function $u : U \to \mathbb{R}$ can be extended in a continuous manner to a neighborhood of w in $J(f)$ and the cohomology equation holds there. Equating this equation at w, we conclude that $\phi(w) = 0$. Thus

$$\sup(\phi) \geq 0. \tag{40}$$

Since $P(\phi) = 0$ this however contradicts the existence of a pressure gap for the potential ϕ. The implication (1)⇒(3) is established.

Now let show that (3)⇒(1). Applying in turn the dimension formula due to Mané (see [Ma2], comp. [PU]), the fact that μ_ϕ is an equilibrium state for ϕ, vanishing of $P(\phi)$, and the cohomology equation of (3), we get that

$$\mathrm{HD}(\mu_\phi) = \frac{\mathrm{h}_{\mu_\phi}(f)}{\chi_{\mu_\phi}(f)} = \frac{P(\phi) - \int \phi\, d\mu_\phi}{\chi_{\mu_\phi}(f)} = \frac{-\int \phi\, d\mu_\phi}{\chi_{\mu_\phi}(f)} = \frac{\mathrm{DD}(J(f))\chi_{\mu_\phi}(f)}{\chi_{\mu_\phi}(f)} = \mathrm{DD}(J(f)).$$

The implication (3)⇒(1) is established.

The conditions (3) and (4) are equivalent since if either of them holds then all but countably many points of the Julia set $J(f)$ are radial and therefore, by Theorem 1, $\mathrm{DD}(J(f)) = \mathrm{HD}(J(f))$.

The implication (2)⇒(1) holds since $\mathrm{HD}(\mu_\phi) \leq \mathrm{DD}(J(f)) \leq \mathrm{HD}(J(f))$.

Finally if (1), (3), and (4) hold, then, as we noted in the proof of the equivalence of (3) and (4), $\mathrm{DD}(J(f)) = \mathrm{HD}(J(f))$. Therefore, because of (1) also holds (2). We are done. □

As a fairly straightforward consequence of this theorem we get the following remarkable corollary.

Corollary 44. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map and let $\phi : J(f) \to \mathbb{R}$ be a Hölder continuous potential such that $\sup(\phi) < P(\phi)$. If μ_ϕ is a unique equilibrium state corresponding to this potential, then $\mathrm{HD}(\mu_\phi) = 2$ if and only if both the function $\phi : J(f) \to \mathbb{R}$ is cohomologous to a constant in the class of continuous functions on $J(f)$, and the rational function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a critically finite rational map with a parabolic orbifold.*

Proof. If $\mathrm{HD}(\mu_\phi) = 2$, then using the Koebe's Distortion Theorem it is straightforward to see that the measures m_2 and μ_2 produced in Proposition 32 are equivalent to 2-dimensional Lebesgue measure l_2 restricted to X_*. So, invoking Corollary 36, we get that the measures μ_{ϕ_*} and l_2 restricted to X_* are equivalent. Consequently

$$l_2(X_*) > 0. \tag{41}$$

We shall now show that

$$J(f) = \hat{\mathbb{C}} \tag{42}$$

Indeed, seeking contradiction, suppose that $J(f) \neq \hat{\mathbb{C}}$. Then $J(f)$ is a nowhere dense subset of $\hat{\mathbb{C}}$, and therefore $l_2\big(\mathrm{Int}(D) \setminus \bigcup_{i\in I} \varphi_i(D)\big) > 0$. Invoking Proposition 4.5.9 from [MU2], we get that $l_2(X_*) = 0$, contrary to (41). Formula (42) is proved. Having this formula, Theorem 43 tells us that the entire postcritical set P_f consists of at most four points. Furthermore, Corollary 40, with h replaced by f yields $f^{-1}(P_f) \subset P_f \cup \mathrm{Crit}(f)$, and, because of the cohomology equation in Theorem 43, $\deg_x(f) = \deg_y(f)$ for every $z \in \hat{\mathbb{C}}$ and all $x, y \in f^{-1}(z)$. This enables us to conclude in exactly the same way as in [Z1] that the map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ has a parabolic orbifold. But then this map is semi-conjugate to map $\tilde{f} : \mathbb{C}/\mathbb{Z}^2 \to \mathbb{C}/\mathbb{Z}^2$, which is the multiplication by some integer ≥ 2. The semi-conjugacy in question is the canonical projection from $\mathbb{C}$ to $\mathbb{C}/\mathbb{Z}^2$. So, $\log|f'|$ is cohomologous to a constant in the class of continuous functions, and so, the same is true for $-2\log|f'|$. In consequence the Hölder continuous map $\phi : \hat{\mathbb{C}} \to \mathbb{R}$ is cohomologous to a constant in the class of continuous functions on $\hat{\mathbb{C}} \setminus P_f$. But since ϕ and this constant are both continuous on $\hat{\mathbb{C}}$, the cohomology equation yields the the function giving cohomology to extend continuously to all points of P_f. Of course then the cohomology equation holds in the whole space $\hat{\mathbb{C}}$.

In the other direction the proof is immediate once one invokes the semi–conjugacy fact used in the first part of the proof. □

6. Real Analyticity of the Pressure Function

We start with some abstract symbolic preparation. Let I be an arbitrary countable set and let $\sigma : I^\infty \to I^\infty$ be the shift map. We describe now in detail the Hölder continuity concepts appropriate in this setting. Recall that for $\omega, \tau \in I^\infty$, we define $\omega \wedge \tau \in I^\infty \cup I^*$ to be the longest initial block common to both ω and τ. We say that a function $f : I^\infty \to \mathbb{C}$ is Hölder continuous with an exponent $\alpha > 0$ if

$$v_\alpha(f) := \sup_{n\ge 1}\{v_{\alpha,n}(f)\} < \infty,$$

where

$$v_{\alpha,n}(f) = \sup\{|f(\omega) - f(\tau)|e^{\alpha(n-1)} : \omega, \tau \in I^\infty \text{ and } |\omega \wedge \tau| \ge n\}.$$

For every $\alpha > 0$ let $\mathcal{K}_\alpha$ be the set of all complex-valued Hölder continuous (not necessarily bounded and allowing $-\infty$ with the convention that $e^{-\infty} = 0$ and $-\infty - (-\infty) = 0$) functions on I^∞. Set

$$\mathcal{K}_\alpha^s := \left\{\rho \in \mathcal{K}_\alpha : \sum_{e\in E} \exp\big(\sup\big(\mathrm{Re}\rho|_{[e]}\big)\big) < +\infty\right\}.$$

An element of $\mathcal{K}_\alpha^s$ is called an α-Hölder summable potential. Moreover, H_α is defined to be the set of all bounded Hölder continuous functions. By endowing H_α with the norm

$$||f||_\alpha := ||f||_\infty + v_\alpha(f),$$

the set H_α becomes a complex Banach space. Also, the set H_α forms a vector subspace of the Banach space $C_b := C_b(I^\infty)$ of bounded continuous complex-valued functions defined on I^∞. This Banach space is equipped with the supremum norm $||\cdot||_\infty$. Now, fix $\rho \in \mathcal{K}_\alpha^s$ and notice that for every $g \in C_b$, the function $\mathcal{L}_\rho(g)$ given by the formula

$$\mathcal{L}_\rho(g)(\omega) = \sum_{i\in I} e^{\rho(i\omega}g(i\omega), \quad \omega \in I^\infty, \tag{43}$$

is well-defined, it belongs to C_b and $||\mathcal{L}_\rho(g)||_\infty \le \sum_{e\in E}\exp\big(\sup\big(\mathrm{Re}\rho|_{[e]}\big)\big)||g||_\infty$. The operator $\mathcal{L}_\rho$, called the transfer, or Perron-Frobenius, operator, acts continuously on C_b with

$$||\mathcal{L}_\rho||_\infty \le \sum_{i\in I}\exp\big(\sup\big(\mathrm{Re}\rho|_{[e]}\big)\big) < \infty.$$

The transfer operator $\mathcal{L}_\rho$ preserves the Banach space H_α and acts continuously on this space. Now notice that the function v_α alone is a pseudo-norm on the vector space $\mathcal{K}_\alpha$. So, it induces a pseudo-metric on $\mathcal{K}_\alpha$ ($(f, g) \mapsto v_\alpha(f - g)$), and this pseudo-metric restricted to $\mathcal{K}_\alpha^s$ induces a topology on $\mathcal{K}_\alpha^s$, which in the sequel will be called the α–Hölder topology. We quote now Theorem 3.8 from [U2].

Theorem 45. *Suppose that Λ is an open subset of $\mathbb{C}^d$, $d \ge 1$, and that the function $\Lambda \ni \lambda \mapsto \rho_\lambda \in \mathcal{K}_\alpha^s$ ($\mathcal{K}_\alpha^s$ endowed with the α–Hölder topology) is continuous. If the function $\Lambda \ni \lambda \mapsto \rho_\lambda(\omega) \in \mathbb{C}$ is holomorphic for every $\omega \in I^\infty$, then the function $\Lambda \ni \lambda \mapsto \mathcal{L}_{\rho_\lambda} \in L(\mathrm{H}_\alpha)$ is also holomorphic.*

This theorem was in fact proved in [U2] for $d = 1$ only, but the general case follows immediately from Hartogs Theorem. We naturally identify $\mathbb{R}^d$ with a subset of $\mathbb{C}^d$. As a fairly straightforward consequence of Theorem 45 and Kato-Rellich Perturbation Theorem we shall prove the following.

Theorem 46. *Suppose that Λ is an open subset of $\mathbb{C}^d$, $d \geq 1$, and that the function $\Lambda \ni \lambda \mapsto \rho_\lambda \in \mathcal{K}_\alpha^s$ ($\mathcal{K}_\alpha^s$ endowed with the α–Hölder topology) is continuous. Suppose also that the function $\Lambda \ni \lambda \mapsto \rho_\lambda(\omega) \in \mathbb{C}$ is holomorphic for every $\omega \in I^\infty$. If $\rho_\lambda(I^\infty) \subset \mathbb{R}$ for all $\lambda \in \Lambda \cap \mathbb{R}^d$, then the topological pressure function $\Lambda \cap \mathbb{R}^d \ni \lambda \mapsto P(\rho_\lambda)$ is real-analytic.*

Proof. Fix $\lambda_0 \in \Lambda \cap \mathbb{R}^d$. In view of Theorem 2.4.6 from [MU2], $e^{P(\rho_{\lambda_0})}$ is a simple isolated eigenvalue of the operator $\mathcal{L}_{\lambda_0} := \mathcal{L}_{\rho_{\lambda_0}} : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$. Hence, in view of Theorem 45, Kato-Rellich Perturbation Theorem ([K], Theorem XII.8) is applicable to yield a number $r_1 > 0$ and a holomorphic function $\gamma : D_q(\lambda_0; r_1) \to \mathbb{C}$ such that

(a) $D_q(\lambda_0; r_1) \subset \Lambda$, where $D_q(\lambda; r) \subset \mathbb{C}^d$ is the polydisc centered at λ and with all radii equal to r.
(b) $\gamma(\lambda_0) = e^{P(\rho_{\lambda_0})}$.
(c) For all $\lambda \in D_q(\lambda_0; r_1)$, $\gamma(\lambda)$ is a simple isolated eigenvalue of the operator $\mathcal{L}_\lambda : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$, with the remainder of the spectrum uniformly separated from $\gamma(\lambda)$.

In particular there exist $r_2 \in (0, r_1]$ and $\eta > 0$ such that

$$\sigma(\mathcal{L}_\lambda) \cap B_{\mathbb{C}}(e^{P(\rho_{\lambda_0})}, \eta) = \{\gamma(\lambda)\} \tag{44}$$

for all $\lambda \in D_q(\lambda_0; r_2)$. Now, $e^{P(\rho_\lambda}$ is the spectral radius $r(\mathcal{L}_\lambda)$ of the operator $\mathcal{L}_\lambda$ for all $\lambda \in D_q(\lambda_0; r_2) \cap \mathbb{R}^d$. In view of semi-continuity of the spectral set function (see Theorem 10.20 on p.256 in [Rud]), taking $r_2 > 0$ appropriately smaller, we will also have that $r(\mathcal{L}_\lambda) \in [0, e^{P(\rho_{\lambda_0})} + \eta)$. Along with (44), these facts imply that $e^{P(\rho_\lambda)} = \gamma(\lambda)$ for all $\lambda \in D_q(\lambda_0; r_2) \cap \mathbb{R}^d$. Consequently, the function $D_q(\lambda_0; r_2) \cap \mathbb{R}^d \ni \lambda \mapsto P(\rho_\lambda) \in \mathbb{R}$ is real-analytic. □

Recall that for every Hölder continuous potential $\phi : J(f) \to \mathbb{R}$,

$$\Delta_\phi = \left\{ t \in \mathbb{R} : \sup_{n \geq 1} \left(P(t\phi) - \frac{1}{n} \sup(S_n(t\phi)) \right) > 0 \right\}.$$

Obviously, Δ_ϕ is an open subset of $\mathbb{R}$. Our goal in this section is to prove the following.

Theorem 47. *The topological pressure function*

$$\Delta_\phi \ni t \mapsto P(t\phi) \in \mathbb{R}$$

is real-analytic.

Proof. Since analyticity is a local property, we may assume without loss of generality that $1 \in \Delta_\phi$ and to prove real analyticity in some sufficiently small neighborhood of 1. So, fix η so small that $(1-\eta, 1+\eta) \subset \Delta_\phi$ and, moreover, $P(t\phi) - \sup(t\phi) > \kappa$ for all $t \in (1-\eta, 1+\eta)$

and some $\kappa > 0$. Denote by $\mathbb{Q}$ the set of all rational numbers and arrange $Q \cap (1-\eta, 1+\eta)$ in a sequence $\{t_k\}_{k=1}^{\infty}$. Define

$$\mu_0 := \sum_{k=1}^{\infty} 2^{-k} \mu_{t_k \phi}.$$

Notice (see the proof) that by the choice of η, Lemma 10 is satisfied with the same constants τ and r_0 for all $t \in (1-\eta, 1+\eta)$, and the measures $m_{t\phi}$. So the same estimate is valid for the measure μ_0. Now, applying Lemma 9 and Lemma 10 for this measure μ_0, we arrive at the following version of Lemma 11.

Lemma 48. *There exists a topological disc U contained in E^c, with piecewise smooth boundary, such that U does not contain any critical value of f^q, $\mu_{t_k}(U) = 1$ for all $k \geq 1$, and there exist $\lambda > 1$, $\tilde{\lambda} > 1$ such that*

$$\mu_{t_k}(B(\partial U, \tilde{\lambda}^{-n})) \leq \lambda^{-n}.$$

Remark 49. Having the disc U as appearing in Lemma 48, all the objects, i.e. inducing schemes and iterated function systems, produced in Section 2, depend only on U, and not on the measure involved. In particular, all statements about the measure μ_ϕ are now true for all measures $\mu_{t_k\phi}$, $k \geq 1$, with the constants, particularly λ and $\tilde{\lambda}$, as coming from Lemma 48.

Recall that $F = \{\varphi_e : U \to D_e\}_{e \in E}$ is the corresponding iterated function system and X is its limit set. Let $\pi : E^\infty \to X$ be the corresponding canonical projection. Recall that $\overline{\phi}(x) = \sum_{k=0}^{n(e)-1} \phi_{qr}(h^k(x))$ whenever $x \in D_e$. Put $\overline{\psi} = \overline{\phi} \circ \pi$ and $\psi_{qr} = \phi_{qr} \circ \pi$. Consider the following 2–parameter family of potentials.

$$\overline{\psi}_{s,t}(\omega) = t\overline{\psi}(\omega) - sqrn(\omega_1),\ \omega \in E^\infty.$$

Notice that, equivalently, one can write

$$\overline{\psi}_{s,t}(\omega) = tS_{n(\omega_1)}\psi_{qr}(\omega) - sqrn(\omega_1).$$

Remember that $\overline{\phi} : E^\infty \to \mathbb{R}$ is Hölder continuous. It then follows from (22) that there exist $\gamma > 0 > 0$ and $C_1, C_2 > 0$ such that

$$\begin{aligned} \sum_{n(e)=k} \exp(\sup(\overline{\psi}_{|[i]}) - P(\phi)qrk) &= \sum_{n(e)=k} \exp(\sup(S_n\psi_{qr\,|[i]}) - P(\phi)qrk) \\ &\leq C_1 \sum_{n(e)=k} \mu_\phi(\varphi_i(U)) \\ &\leq C_2 e^{-\gamma k}. \end{aligned} \tag{45}$$

In view of Remark 49, for every $k \geq 1$ there exists a Borel probability measure m_k on E^∞ such that, for all $j \geq 1$, for all $\tau \in E^j$, and for all Borel sets $A \subset E^\infty$

$$m_k(\tau A) = \int_A \exp\big(t_k S_n \overline{\psi}(\tau\omega) - qr(n(\tau_1) + n(\tau_2) + \cdots + n(\tau_j))P(t_k\phi)\big)\, dm_k(\omega).$$

Indeed, Remark 49 assures us that for all integers $k \geq 1$ the limit set X of our induced system F is of full $\mu_{t_k\phi}$ measure. Since the measure $\mu_{t_k\phi}$ is equivalent to the $\exp(-t_k\phi + P(t_k\phi))$–conformal measure $m_{t_k\phi}$, it follows that $m_{t_k\phi}$ itself, serves as a conformal measure for the induced system. Thus, m_k is just equal to $m_{t_k\phi} \circ \pi$. We thus conclude that

$$P(\overline{\psi}_{P(t_k\phi),t_k}) = 0 \tag{46}$$

for all k, where P denotes topological pressure with respect to the shift map $\sigma : E^\infty \to E^\infty$. Since $||\overline{\psi}_{|[e]}||_\infty \leq n(e)||\psi_{qr}||_\infty$, we conclude from (45) that

$$\sum_{n(e)=j} \exp\Big(\sup(\overline{\psi}_{s,t_{|[e]}})\Big) \leq e^{-\frac{\gamma}{2}j} \tag{47}$$

for all $j \geq 1$, all $t \in T := \Big(1 - \frac{\gamma}{4||\psi_{qr}||_\infty}, 1 + \frac{\gamma}{4||\psi_{qr}||_\infty}\Big)$ and all $s \in S := (P(\phi) - \frac{\gamma}{4qr}, P(\phi) + \frac{\gamma}{4qr})$. Therefore, for all $(s,t) \in \Lambda := \{(s,t) \in \mathbb{C} \times \mathbb{C} : \mathrm{Re}s \in S, \mathrm{Re}t \in T\}$ the operator $\mathcal{L}_{s,t} : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ is well-defined if given by the following formula

$$\mathcal{L}_{s,t}(g)(\omega) = \sum_{e \in E} g(e\omega) \exp\big(\overline{\psi}_{s,t}(e\omega)\big).$$

Obviously, the function $\Lambda \ni (s,t) \mapsto \overline{\psi}_{s,t}(\omega)$ is holomorphic for all $\omega \in \Lambda$ and $\overline{\psi}_{s,t}(E^\infty) \subset \mathbb{R}$ for all $(s,t) \in \Lambda \cap \mathbb{R} = S \times T$. Also, since the function $\overline{\psi}_{qr} : E^\infty \to \mathbb{R}$ is Hölder continuous, the standard distortion argument shows that

$$v_\alpha\big(\overline{\psi}_{s,t} - \overline{\psi}_{a,b}\big) = |t-b| v_\alpha\big(S_{n(\omega_1)}\overline{\psi}_{qr}(\omega)\big) \leq Q|t-b|$$

with some constant $Q > 0$ depending only on $\overline{\psi}$. Hence, the map $\Lambda \ni (s,t) \mapsto \overline{\psi}_{s,t} \in \mathcal{K}_\alpha^s$ is Lipschitz continuous, thus continuous. So, we have verified all the hypotheses of Theorem 46. It gives the following.

Lemma 50. *The function*

$$S \times T \ni (s,t) \mapsto P(\overline{\psi}_{s,t})$$

is real-analytic.

Since the map $T \ni t \mapsto P(t\phi)$ is continuous, even Lipschitz continuous with a Lipschitz constant equal to 1, there exists an open interval $\Gamma \subset \Delta_\phi$ containing the number 1 such that $P(t\phi) \in S$ for all $t \in \Gamma$. Using this, continuity of the map $T \ni t \mapsto (P(t\phi), t)$, and Lemma 50, we thus conclude that the function $\Gamma \ni t \mapsto P(\overline{\psi}_{P(t\phi),t})$ is continuous as a composition of two continuous functions. We thus derive from (46) that

$$P(\overline{\psi}_{P(t\phi),t}) = 0 \tag{48}$$

for all $t \in \Gamma$. Now, in virtue of Proposition 2.6.13 in [MU2] we have that

$$\frac{\partial}{\partial s}\bigg|_{(P(\phi),1)} P(\psi_{s,t}) = -\int_{E^\infty} n(\omega_1)\, d\mu_{\overline{\psi}_{P(\phi),1}} \in (-\infty, 0).$$

It thus follows from the Implicit Function Theorem, Lemma 50 and (48) applied with $t = 1$ that there exists $\eta > 0$ and a unique continuous function $Q : (1-\eta, 1+\eta) \to S$ such that $(1-\eta, 1+\eta) \subset \Gamma$,

$$Q(1) = P(\phi)$$

and

$$P(\overline{\psi}_{Q(t),t}) = 0$$

for all $t \in (1-\eta, 1+\eta)$. Moreover, the function $Q : (1-\eta, 1+\eta) \to S$ is real–analytic. Invoking (48) we thus conclude that $Q(t) = P(t\phi)$ for all $t \in (1-\eta, 1+\eta)$. As a consequence, the function $(1-\eta, 1+\eta) \ni t \mapsto P(t\phi)$ is real–analytic. We are done.

□

7. The Law of Iterated Logarithm; Abstract Setting

This section is of general abstract character. We consider a probability space $(Y, \mathcal{F}, \mu)$ and an $\mathcal{F}$-measurable mapping $T : Y \to Y$ preserving the probability measure μ. Our goal is to show that if a "sufficiently good" induced map satisfy the Law of Iterated Logarithm, then so does the original map. Precisely, we say that a μ-integrable function $g : Y \to \mathbb{R}$ satisfies the Law of Iterated Logarithm if there exists a positive constant A_g such that

$$\limsup_{n\to\infty} \frac{S_n g - n\int g d\mu}{\sqrt{n \log\log n}} = A_g.$$

From now on we always assume without loss of generality that

$$\mu(g) = \int g d\mu = 0.$$

Fix a set $F \in \mathcal{F}$ with $\mu(F) > 0$. Let $\tau : F \to \mathbb{N} \cup \infty$ be the first return time to F, i.e.

$$\tau(x) = \min\{n \geq 1 : T^n(x) \in F\}.$$

Poincare's Return Theorem assures us that the function τ is μ-almost everywhere finite in F. The first return (induced) map is then defined by the following formula.

$$T_F(x) := T^{\tau(x)}(x).$$

It is well-known that the conditional measure μ_F on F is T_F-invariant. Given $x \in F$, the sequence $(\tau_n(x))_{n=1}^\infty$ is then defined as follows.

$$\tau_1(x) := \tau(x) \quad \text{ and } \quad \tau_n(x) = \tau_{n-1}(x) + \tau(T^{\tau_{n-1}(x)}(x)).$$

Define the function $\hat{g} : F \to \mathbb{R}$ as follows.

$$\hat{g} = \sum_{j=0}^{\tau-1} g \circ T^j.$$

The main result of this section is the following.

Theorem 51. *Let $T : Y \to Y$ be a measurable dynamical system preserving a probability measure μ on Y. Assume that the dynamical system (T, μ) is ergodic. Fix F, a measurable subset of Y having a positive measure μ. Let $g : Y \to \mathbb{R}$ be a measurable function such that the function $\hat{g} : F \to \mathbb{R}$ satisfies the Law of Iterated Logarithm with respect to the dynamical system (T_F, μ_F). If in addition,*

$$\int |\hat{g}|^{2+\gamma} d\mu < \infty \tag{49}$$

for some $\gamma > 0$, then the function $g : Y \to \mathbb{R}$ satisfies the Law of Iterated Logarithm with respect to the original dynamical system (T, μ) and $A_g = A_{\hat{g}}$.

Proof. Since the Law of Iterated Logarithm holds for a point $x \in X$ if and only if it holds for $T(x)$, in virtue of ergodicity of T, it suffices to prove our theorem for almost all points in F. By our assumptions there exists a positive constant $A_{\hat{g}}$ such that

$$\limsup_{n\to\infty} \frac{S_{\tau_n} g(x)}{\sqrt{n \log\log n}} = A_{\hat{g}}.$$

for μ_F-a.e. $x \in F$. Since, by Kac's Lemma,

$$\lim_{n\to\infty} \frac{\tau_n}{n} = \int_Y \tau dm = \int_F \tau dm = 1, \tag{50}$$

μ_F-a.e. on F, we thus have

$$\limsup_{n\to\infty} \frac{S_{\tau_n} g}{\sqrt{n \log\log n}} \limsup_{n\to\infty} \frac{S_{\tau_n} g}{\sqrt{\tau_n \log\log \tau_n}} = A_{\hat{g}}. \tag{51}$$

μ_F-a.e. on F. Now, for every $n \in \mathbb{N}$ and (almost) every $x \in F$ let $k = k(x, n)$ be the positive integer uniquely determined by the condition that

$$\tau_k(x) \le n < \tau_{k+1}(x).$$

Since

$$S_n g(x) = S_{\tau_k(x)} g(x) + S_{n-\tau_k(x)} g(T^{\tau_k(x)}(x)),$$

we have that

$$\frac{S_n g}{\sqrt{n \log\log n}} = \frac{S_{\tau_k(x)} g}{\sqrt{n \log\log n}} + \frac{S_{n-\tau_k(x)} g}{\sqrt{n \log\log n}} \tag{52}$$

Since by (50)

$$\lim_{n\to\infty} \frac{\tau_{k+1}(x)}{\tau_k(x)} = 1,$$

we get from (51) that,

$$\limsup_{n\to\infty} \frac{S_{\tau_k} g(x)}{\sqrt{n \log\log n}} = \limsup_{n\to\infty} \frac{S_{\tau_k} g(x)}{\sqrt{k \log\log k}} = A_{\hat{g}}.$$

Because of this and because of (52), we are only left to show that

$$\lim_{n\to\infty} \frac{S_{n-\tau_{k(n)}} g(x)}{\sqrt{n \log\log n}} = 0. \tag{53}$$

μ_F-a.e. on F. To do this, note first that

$$\frac{S_{\tau_{k+1}-\tau_k}|g|(T^{\tau_k}(x))}{\sqrt{k\log\log k}} = \frac{|\hat{g}|(T_F^k(x))}{\sqrt{k\log\log k}}.$$

Take ean arbitrary $\varepsilon \in (0,\gamma)$. Since

(54)

$$\begin{aligned}\mu\big(\{x\in F : |\hat{g}|(T_F^k(x)) \geq \varepsilon\sqrt{k\log\log k}\}\big) &= \mu\big(\{x \in F : |\hat{g}|(x) \geq \varepsilon\sqrt{k\log\log k}\}\big) \\ &= \mu\big(\{x\in F : |\hat{g}|^{2+\varepsilon}(x) \geq \varepsilon^{2+\varepsilon}(k\log\log k)^{1+\varepsilon/2}\}\big) \\ &\leq \frac{\int |\hat{g}|^{2+\varepsilon} d\mu}{\varepsilon^{2+\varepsilon}(k\log\log k)^{1+\varepsilon/2}},\end{aligned}$$

using (49) we conclude that

$$\sum_{k=1}^{\infty} \mu\big(\{x\in F : |\hat{g}|(x) \geq \varepsilon\sqrt{k\log\log k}\}\big) < \infty.$$

So, applying Borel-Cantelli lemma, (53) follows. We are done. □

8. Stochastic properties of the equilibrium measure μ_ϕ

In this section we obtain strong transparent stochastic properties of the dynamical system (f,μ_ϕ). We deduce them from the corresponding properties of the induced system (F,μ_{ϕ_D}), where μ_{ϕ_D} is the conditional measure μ_ϕ on D. We follow the scheme worked out in [LSY]. We recall it briefly now. We do this in an abstract context. Let $(\Delta_0,\mathcal{B}_0,m_0)$ be a measure space with a finite measure m_0, let $\mathcal{P}_0$ be a countable measurable partition of Δ_0 and let $T_0 : \Delta_0 \to \Delta_0$ be a measurable map such that, for every $\Delta' \in \mathcal{P}_0$ the map $T_0 : \Delta' \to \Delta_0$ is a bijection onto Δ_0. Moreover, we assume that the partition $\mathcal{P}_0$ is generating, i.e. for every $x,y\in\Delta_0$ there exists $s\geq 0$ such that $T_0^s(x), T_0^s(y)$ are in different elements of the partition $\mathcal{P}_0$. We denote by $s = s(x,y)$ the smallest integer with this property and we call it a separation time for the pair x,y. We assume also that for each $\Delta'\in\mathcal{P}_0$ the map $(T_{0|\Delta'})^{-1}$ is measurable and that the Jacobian $Jac_{m_0}(T_0)$ with respect to the measure m_0 is well–defined and positive a.e. in Δ'. The following distortion property is assumed to be satisfied.

$$\left|\frac{Jac_{m_0}T_0(x)}{Jac_{m_0}T_0(y)} - 1\right| \leq C\beta^{s(T_0(x),T_0(y))}. \tag{55}$$

We have also a function $R : \Delta_0 \to \mathbb{N}$ ("return time") which is constant on each element of the partition $\mathcal{P}_0$. We assume that the greatest common divisor of the values of R is equal to 1. Finally, let

$$\Delta = \{(z,n)\in\Delta_0\times\mathbb{N}\cup\{0\} : 0\leq n < R(z)\}$$

and each point $z \in \Delta_0$ is identified with $(z, 0)$. The map T acts on Δ as

$$T(z,n) = \begin{cases} (z, n+1) \text{ if } n+1 < R(z) \\ (T_0(z), 0) \text{ if } n+1 = R(z) \end{cases}$$

The measure m_0 is spread over the whole space Δ by putting

$$\tilde{m}_{|\Delta_0} = m_0 \quad \text{and} \quad \tilde{m}_{|\Delta' \times \{j\}} = m_{0|\Delta'} \circ \pi_j^{-1}, \ \Delta' \in \mathcal{P},$$

where $\pi_j(z,0) = (z,j)$. Thus, the measure m is finite if and only if $\int_{\Delta_0} R dm_0 < \infty$. The separation time $s((x,n),(y,m))$ is defined to be equal to $s(x,y)$ if $n = m$ and x, y are in the same set of the partition $\mathcal{P}$. Otherwise we set $s(x,y) = 0$. Given $\beta > 0$ we define the space

$$C_\beta(\Delta) = \{\varphi : \Delta \to \mathbb{R} : \exists\, C_\varphi \text{ such that } |\varphi(x) - \varphi(y)| < C_\varphi \beta^{s(x,y)} \ \forall x, y \in \Delta\}.$$

We refer to the quadrupole $\mathcal{Y} = (\Delta_0, T_0, \mathcal{P}_0, R)$ as a Young tower. The following basic result has been proved in [LSY].

Theorem 52. *If $\mathcal{Y} = (\Delta_0, T_0, \mathcal{P}_0, R))$ is a Young tower and $\int R dm_0 < \infty$ then there exists a unique probability T–invariant measure ν, absolutely continuous with respect to $\tilde{m}$. The Radon-Niokodym derivative $d\nu/d\tilde{m}$ is bounded from below by a positive constant. The dynamical system (T, ν) is exact, thus ergodic.*

Let us make now an abstract digression. Let (Y, μ) be a probability space and let $S : Y \to Y$ be a measurable map preserving measure μ. Let $g : Y \to \mathbb{R}$ be a square integrable function. We put

$$\overline{\sigma}_S^2(g) := \limsup_{n\to\infty} \frac{1}{n} \int_Y (S_n(g) - n\mu(g))^2 \, d\mu$$

and

$$\underline{\sigma}_S^2(g) := \liminf_{n\to\infty} \frac{1}{n} \int_Y (S_n(g) - n\mu(g))^2 \, d\mu.$$

In the case when these two numbers are equal, we denote by $\sigma_S^2(g)$ their common value and call it the asymptotic variance of g. Coming back to our dynamical system $T : \Delta \to \Delta$, we shall prove the following technical fact interesting itself and needed for the Law of Iterated Logarithm.

Lemma 53. *For every $s \geq 1$ let A_s be the union of all those elements Δ' of the partition $\mathcal{P}$ for which $R_{|\Delta'} \geq s$. Assume that there exist $\alpha > 3$ and $C > 0$ such that $m_0(A_s) \leq Cs^{-\alpha}$ for all $s \geq 1$. If a bounded measurable function $g : \Delta \to \mathbb{R}$ is such that $\nu(g) = 0$ and $\overline{\sigma}_T^2(g) > 0$, then the measurable function $\hat{g} : \Delta_0 \to \mathbb{R}$ is not a coboundary in the class of real-valued bounded measurable functions defined on Δ_0.*

Proof. Seeking contradiction suppose that $\hat{g} : \Delta_0 \to \mathbb{R}$ is such a coboundary, i.e.

$$\hat{g} = u - u \circ T$$

with some bounded measurable function $g : \Delta_0 \to \mathbb{R}$. Fix an integer $n \geq 1$. For all $x \in \Delta$ let

$$i = i(x) := \min\{0 \leq s \leq n : T^s(x) \in \Delta_0\}.$$

If no such s exists, set $i = n$. Let

$$j = j(x) := \max\{0 \leq s \leq n : T^s(x) \in \Delta_0\}.$$

Likewise, if no such s exists, set $j = n$. We have,

$$0 \leq i \leq j \leq n,$$

and there exists a unique integer $0 \leq k \leq j - i$ such that

$$T^{j-i}(x) = T_0^k(x).$$

Hence we can write

$$S_n g(x) = S_i g(x) + S_k^{T_0}(\hat{g})(T^i(x)) + S_{n-j} g(T^j(x)) = a(x) + b(x) + c(x).$$

In order to show that $\sigma_T^2(g) = 0$, we shall estimate

$$\int (S_n(g))^2 d\nu = ||(S_n(g))||_2^2 \leq (||a(x)||_2 + ||b(x)||_2 + ||c(x)||_2)^2.$$

We shall deal with each of these three norms separately. Since $b(x) = S_k^{T_0}(\hat{g})(T^i(x))$ and $|S_k^{T_0}(\hat{g})(T^i(x))| \leq 2||u||_\infty$, we get immediately that

$$||b||_2 \leq 2||u||_\infty.$$

Next, we estimate $||a(x)||_2$. Since g is bounded, we have

$$\int a(x)^2 d\nu(x) \leq ||g||_\infty \int (i(x))^2 d\nu(x) \leq M||g||_\infty \int (i(x))^2 d\tilde{m}(x).$$

Note that, for all $0 \leq s \leq n$ we have,

$$s = i(x) \iff x \in T^{(R_{|\Delta'} - s)}(\Delta') \tag{56}$$

for some $\Delta' \in A_s$ and, putting also the value given by the formula (56) on the remaining part of the set X, gives us an estimate of the function $i(x)$ for above. Thus,

$$\begin{aligned}
\int i^2(x)d\tilde{m}(x) &= \sum_{s=1}^{n} \tilde{m}\{x : i(x) = s\} \cdot s^2 \\
&\leq \sum_{s=1}^{\infty} \sum_{\Delta' \in A_s} \tilde{m}(T^{(R_{|\Delta'} - s)}(\Delta')) \cdot s^2 \\
&= \sum_{s=1}^{\infty} \sum_{\Delta' \in A_s} m_0(\Delta') \cdot s^2 = \sum_{s=1}^{\infty} s^2 m_0(A_s) \\
&\leq \sum_{s=1}^{\infty} s^{2-\alpha} < +\infty.
\end{aligned} \tag{57}$$

The estimate of $||c(x)||_2$ can be treated similarly. Indeed, note first that for $0 \leq s \leq n0$, we have

$$j(x) = n - s \iff T^{n-s}(x) \in A_{s+1}.$$

Thus,

$$\begin{aligned}
\int (b(x))^2 d\nu(x) &\leq ||g||_\infty \int (n - j(x))^2 d\nu(x) \\
&= \sum_{s=1}^{n} s^2 \nu(\{x : T^{n-s}(x) \in A_{s+1}\}) = \sum_{s=1}^{n} s^2 \nu(A_{s+1}) \\
&\leq C \sum_{s=1}^{n} s^{2-\alpha} \\
&\leq C \sum_{s=1}^{\infty} s^{2-\alpha} < +\infty.
\end{aligned} \tag{58}$$

Therefore, the integrals $\int (S_n g)^2 d\nu$ remain bounded as $n \to \infty$. This obviously implies that $\sigma_T^2(g) = 0$. □

The first two items of the following theorem concerning stochastic properties of the dynamical system (T, ν) form an extract from the results proved in [LSY] while the item (3) has been basically proved in Section7.

Theorem 54. *Let $\mathcal{Y} = (\Delta_0, T_0, \mathcal{P}_0, R))$ be a Young tower. Then the following hold.*

(1) *If $m_0(R > n) = O(\theta^n)$ for some $0 < \theta < 1$, then there exists $0 < \tilde{\theta} < 1$ such that for all functions $\psi \in L^\infty$ and we have $g \in C_\beta$,*

$$Cov(\psi \circ T^n, g) = \left| \int (\psi \circ T^n) g d\nu - \int \psi d\nu \int g d\nu \right| = O(\tilde{\theta}^n) \tag{59}$$

(2) *If* $m_0(R > n) = O(n^{-\alpha})$ *with some* $\alpha > 1$ *(in particular, if* $m_0(R > n) = O(\theta^n)$*), then the Central Limit Theorem is satisfied for all functions* $g \in C_\beta$*, that are not cohomologous to a constant in* $L^2(\nu)$*.*

(3) *If* $m_0(R > n) = O(n^{-\alpha})$ *with some* $\alpha > 4$ *(in particular, if* $m_0(R > n) = O(\theta^n)$*), then the Law of Iterated Logarithm holds for all functions* $g \in C_\beta$*, that are not cohomologous to a constant in* $L^2(\nu)$*.*

Proof. As we have already said, the first two items are extracted from [LSY]. We will show that (3) holds. Assume without loss of generality that $\nu(g) = 0$. First note that the boundedness of the function g and the condition $m_0(R > n) = O(n^{-\alpha})$ with some $\alpha > 4$ entail condition (49) in Theorem 51. Since the item (2) of our theorem holds, we must have that $\sigma_T^2(g) > 0$. It the follows from Lemma 53 that the measurable function $\hat{g} : \Delta_0 \to \mathbb{R}$ is not a coboundary in the class of real-valued bounded measurable functions defined on Δ_0. It is also easy to see that the function $\hat{g} : \Delta_0 \to \mathbb{R}$ is Hölder continuous with respect to the "symbolic" metric $d_\beta(x, g) = \beta^{s(x,y)}$. These two facts inserted to Theorem 5.5 in [MU2] imply that the Law of Iterated Logarithm holds for the function $\hat{g} : \Delta_0 \to \mathbb{R}$. So, it holds for the map g itself because of Theorem 51. We are done. □

Passing to our rational function dynamical system (f, μ_ϕ) we shall check that the assumptions of this theorem are satisfied for our induced system $(F, m_\phi|_D)$. The space Δ_0 is now X, the limit set of the iterated function system F. The partition $\mathcal{P}_0$ consists of the sets $\Delta_e := D_e \cap X$, $e \in E$. The measure m_0 is the conformal measure m_ϕ, restricted to X. The map T_0 is, in our setting, the map F. The function R, the return time, is, naturally, defined as $R_{|D_e} = n(e)$, where $h^{n(e)}(D_e) = U$. Fix $e \in E$, write $F_{|D_e} = h^{n(e)} = g^{rn(e)}$ and fix $x, x' \in \Delta_e$. Recall that $\phi_{qr}(x) = \sum_{j=0}^{r-1} \phi_q(x)$. Since ϕ is Hölder continuous, ϕ_{qr} is also Hölder continuous; we denote the Hölder constant of ϕ_{qr} by H_{qr}. Then

$$\log Jac_{m_\phi} F(x) = -\sum_{k=0}^{n(e)-1} \phi_{qr}(h^k(x)) = -\sum_{j=0}^{rn(e)-1} \phi_q(g^j(x)),$$

and similarly for x'. So,

$$\begin{aligned}
\left|\log \frac{Jac_{m_\phi} F(x)}{Jac_{m_\phi} F(x')}\right| &\leq \sum_{k=0}^{n(e)-1} |\phi_{qr}(h^k(x)) - \phi_{qr}(h^k(x'))| \\
&\leq H_{qr} \sum_{k=0}^{n(e)-1} \operatorname{dist}^\alpha(h^k x, h^k y) \qquad (60)\\
&\leq \tilde{H}_{qr} \sum_{k=0}^{n(e)-1} \operatorname{dist}^\alpha_{hyp}(h^k x, h^k y)
\end{aligned}$$

where the last inequality follows from the fact that the spherical distance in U can be estimated from above by the hyperbolic distance in U multiplied by some constant $C > 0$;

so $\tilde{H}_{qr} = C^\alpha H_{qr}$). Now, as the map $F : \bigcup_{e\in E} D_e \to U$ expands the hyperbolic metric in U by some factor $\frac{1}{\tau} > 1$ along every good pullback (see the proof of Proposition 18), we get the following.

$$\begin{aligned}\sum_{k=0}^{n(e)-1} \mathrm{dist}_{hyp}^\alpha(h^k x, h^k y) &\le \sum_{k=0}^{n(e)-1} \tau^{\alpha(n(e)-1-k)} \mathrm{dist}_{hyp}^\alpha(h^{n(e)-1}x, h^{n(e)-1}x') \\ &\le \mathrm{Const}\, \mathrm{dist}_{hyp}^\alpha(h^{n(e)}x, h^{n(e)}x').\end{aligned} \tag{61}$$

First assume that $s_0((F(x), F(y)) \ge 1$. This corresponds to the case when $F(x)$ and $F(y)$ are both in the same disc D_e, $e \in E$, from the domain of F. Then, applying Proposition 19 to the points $h^n(x) = F(x)$ and $h^n(x') = F(x')$, we see that

$$\left|\log \frac{Jac_{m_\phi} F(x)}{Jac_{m_\phi} F(x')}\right| \le \mathrm{Const}\, \beta^{s_0(F(x),F(x'))}$$

with $\beta := \tau^\alpha$. If, on the other hand, $s_0(F(x), F(y)) = 0$ then, using (23), we see that $\mathrm{dist}_{hyp}^\alpha(h^{n(e)-1}x, h^{n(e)-1}x') \le C_1^\alpha$. Therefore, the second term of (61) is readily bounded by a constant. So, (55) is established in our context. The fact that the partition $\mathcal{P}_0$ is generating follows directly from formula (25).

The last assumption in Theorem 54 is that the greatest common divisor of all the values of $n(e)$, $e \in E$, is equal to 1. If for our induced system this value is equal to some integer $s > 1$, then we replace the map h by its iterate h^s. The return times are now equal to $\frac{n(e)}{s}$, $e \in E$, and their greatest common divisor is equal to 1.

Therefore, using (22) we conclude from Theorems 52 and 54 that

$$\tilde{m}_\phi(\Delta) < +\infty, \tag{62}$$

the map $T : \Delta \to \Delta$ admits a probability T-invariant measure ν which is absolutely continuous with respect to $\tilde{m}_\phi$, that for each function $g \in C_\beta$ and every $\psi \in C_\infty(\nu)$, (59) holds, and that both, the Central Limit Theorem and the Law of Iterated Logarithm are true for all functions $g \in C_\beta(\Delta)$ that are not cohomologous to a constant in $L^2(\nu)$.

Now consider $\pi : \Delta \to \hat{\mathbb{C}}$, the natural projection from the abstract tower Δ to the Riemann sphere $\mathbb{C}$ given by the formula

$$\pi(z, n) = h^n(z). \tag{63}$$

Then

$$\pi \circ T = h \circ \pi, \tag{64}$$

$$\tilde{m}_{\phi|\Delta_0} \circ \pi^{-1} = m_0 = m_\phi,$$

and

$$\tilde{m}_{\phi_{D_e\times\{n\}}} \circ \pi^{-1} = m_{\phi_{D_e\times\{0\}}} \circ h^{-n} = m_{0|D_e} \circ h^{-n}$$

for all $e \in E$ and all $0 \le n \le n(e)$. Now, $\tilde{m}_{\phi_{D_e \times \{n\}}} \circ \pi^{-1}$ is absolutely continuous with respect to m_ϕ, with the Radon-Nikodym derivative equal to $J_{D_e,n} := Jac_{m_\phi}(h^{-n})$ in $h^n(D_e)$ and zero elsewhere. Therefore, using (62), we get that

$$\int \sum_{e\in E} \sum_{0\le n<n(e)} J_{D_e,n} dm_\phi = \sum_{e\in E} \sum_{0\le n<n(e)} \int J_{D_e,n} dm_\phi = \tilde{m}_\phi \circ \pi^{-1}(J(f)) = \tilde{m}_\phi(\Delta) < +\infty.$$

Thus, the function $\sum_{e\in E}\sum_{0\le n<n(i)} J_{D_e,n}$ is integrable with respect to the measure m_ϕ. This implies immediately that the measure $\tilde{m}_\phi \circ \pi^{-1}$ is absolutely continuous with respect to the measure m_ϕ with the Radon–Nikodym derivative equal to $\sum_{e\in E}\sum_{0\le n<n(e)} J_{D_i,n}$. Hence, the measure $\nu \circ \pi^{-1}$ is also absolutely continuous with respect to m_ϕ. Since ν is F–invariant and $\pi \circ T = h \circ \pi$, the measure $\nu \circ \pi^{-1}$ is h–invariant. But the measure μ_ϕ is h–invariant ergodic and equivalent with the conformal measure m_ϕ. Hence, $\nu \circ \pi^{-1}$ is absolutely continuous with respect to the ergodic measure μ_ϕ. Invariance and ergodicity of $\nu \circ \pi^{-1}$ yield this.

Lemma 55.

$$\nu \circ \pi^{-1} = \mu_\phi.$$

We are now in position to prove the following.

Theorem 56. *For the dynamical system (f, μ_ϕ) the following hold.*

(1) *For every $\alpha \le 1$, every α–Hölder continuous function $g : J(f) \to \mathbb{R}$ and every bounded measurable function $\psi : J(f) \to \mathbb{R}$, we have that*

$$\left| \int \psi \circ f^n \cdot g d\mu_\phi - \int g d\mu_\phi \int \psi d\mu_\phi \right| = O(\theta^n)$$

for some $0 < \theta < 1$ depending on α.

(2) *The Central Limit Theorem holds for every Hölder continuous function $g : J(f) \to \mathbb{R}$ that is not cohomologous to a constant in $L^2(\mu_\phi)$, i.e. for which there is no square integrable function η for which $g = \mathrm{const} + \eta \circ f - \eta$. Precisely this means that there exists $\sigma > 0$ such that*

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} g \circ f^j \to \mathcal{N}(0, \sigma)$$

in distribution.

(3) *The Law of Iterated Logarithm holds for every Hölder continuous function $g : J(f) \to \mathbb{R}$ that is not cohomologous to a constant in $L^2(\mu_\phi)$. This means that there exists a real positive constant A_g such that such that μ_ϕ almost everywhere*

$$\limsup_{n\to\infty} \frac{S_n g - n \int g d\mu}{\sqrt{n \log\log n}} = A_g.$$

Proof. Let $g : J(f) \to \mathbb{R}$ be a Hölder continuous function with a Hölder exponent α and let $\psi \in L_\infty(\mu_\phi)$. Consider the functions $\tilde{g} = g \circ \pi, \tilde{\psi} = \psi \circ \pi : \Delta \to \mathbb{R}$. We shall prove the following.

Claim 1: The function $\tilde{g}$ belongs to the space C_β for an appropriate exponent $\beta \in (0,1)$.

Indeed (see the definition of the space C_β), it is enough to check that $|\tilde{g}(x,n) - \tilde{g}(x',n)| \le C\beta^{s(T_0(x),T_0(x'))}$. Equivalently, we are to check that

$$|g(h^n(x)) - g(h^n(x'))| \le C\beta^{s(T_0(x),T_0(x'))}.$$

The left-hand side of this inequality can be estimated from above by $\text{dist}^\alpha(h^n(x), h^n(x')$. Since both points $h^n(x), h^n(x')$ are images of $T_0(x), T_0(x')$ under a very good pullback, the same reasoning as in the proof of Proposition 19 shows that

$$\text{dist}(h^n(x), h^n(x')) \le C\text{dist}_{hyp}(h^n(x), h^n(x')) \le \text{dist}_{hyp}(T_0(x), T_0(x')) \le C\tau^{s(T_0(x),T_0(x'))}.$$

Therefore, $\tilde{g} \in C_\beta$ with $\beta = \tau < 1$.

Claim 2: The function $\tilde{g}$ is not cohomologous to a constant in $L^2(\nu)$.

Indeed, assume without loss of generality that $\mu_\phi(g) = 0$. Let $\mathcal{L}_{\mu_\phi} : L^2(\mu_\phi) \to L^2(\mu_\phi)$ be the Perron–Frobenius operator corresponding to the function μ_ϕ. The fact that $g : J(f) \to \mathbb{R}$ that is not a coboundary in $L^2(\mu_\phi)$ equivalently means that the sequence $\big(S_n(g)\big)_{n=0}^\infty$ is not uniformly bounded in $L^2(\mu_\phi)$. But because of Lemma 55, $||S_n(\tilde{g})||_{L^2(\nu)} = ||S_n(g)||_{L^2(\mu_\phi)}$. So, the sequence $\big(S_n(\tilde{g})\big)_{n=0}^\infty$ is not uniformly bounded in $L^2(\nu)$. Thus, it is not a coboundary in $L^2(\nu)$.

Having these two claims, all items, (1), (2), and (3), now follow immediately from Theorem 54 with the use of Lemma 55 and formula (64). The proof is finished. ☐

9. The Law of Iterated Logarithm for $a\phi + b\log|f'|$ and Refined Geometry of the Equilibrium State μ_ϕ.

In this section we prove the Law of Iterated Logarithm for the functions of the form $a\phi + b\log|f'|$ whenever an appropriate cohomology equation fails and derive from it finer geometrical properties of the equilibrium states μ_ϕ. In particular we show that if condition (1) of Theorem 43 fails then the measure μ_ϕ is singular with respect to the Hausdorff measure $\text{H}_{\text{HD}(\mu_\phi)}$. Keep D the disc used to define the system F_*, and X_* the limit set of this system. For every $\psi : J(f) \to \mathbb{R}$ let

$$\psi_* = \hat{\hat{\psi}} : X_* \to \mathbb{R},$$

where the first "ˆ" refers to the inducing scheme with respect to h on $X = \Delta_0$, while the second ‘ˆ" refers to the inducing scheme with respect to F on X_*. We start with the following.

Lemma 57. *If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational function with degree* $\deg(f) \geq 2$ *and a having pressure gap Hölder continuous potential $\phi : J(f) \to \mathbb{R}$ fails to satisfy condition (1) of Theorem 43, then the function $\psi_* : X_* \to \mathbb{R}$ is not cohomologous to a constant in $L^2(\mu_{\phi_*})$, where $\psi := \phi + \mathrm{HD}(\mu_\phi) \log |f'| : J(f) \to \mathbb{R}$.*

Proof. Suppose for a contrary that ψ_* is cohomologous to a constant in $L^2(\mu_{\phi_*})$. Let

$$\kappa = \mathrm{HD}(\mu_\phi).$$

We know from Corollary 28 that $\int \log |F_*'| d\mu_{\phi_*} < +\infty$. Therefore, using Theorem 2.2.7 in [MU2] we conclude that $\mu_{\phi_*} = \mu_\kappa$, where $\mu_\kappa := \mu_{-\kappa \log |F_*'|}$ is the equilibrium (or Gibbs) state corresponding to the potential $-\kappa \log |F_*'| : \Delta_0 \to \mathbb{R}$. Using this and Theorem 4.4.2 (Volume Lemma) in [MU2], as well as the definition of a Gibbs state, we can write as follows.

$$\begin{aligned}\mathrm{HD}(\mu_\phi) = \mathrm{HD}(\mu_{\phi_*}) = \mathrm{HD}(\mu_\kappa) &= \frac{\mathrm{h}_{\mu_\kappa}(F_*)}{\chi_{\mu_\kappa}} = \frac{P(-\kappa \log |F_*'|) + \kappa \chi_{\mu_\kappa}}{\chi_{\mu_\kappa}} \\ &= \frac{P(-\kappa \log |F_*'|)}{\chi_{\mu_\kappa}} + \kappa \\ &= \mathrm{HD}(\mu_\phi) + \frac{P(-\kappa \log |F_*'|)}{\chi_{\mu_\kappa}}.\end{aligned}$$

Hence

$$P(-\mathrm{HD}(\mu_\phi) \log |F_*'|) = 0.$$

Starting from this formula and proceeding in exactly the same was as in our current paper with β replaced by $\kappa = \mathrm{HD}(\mu_\phi)$, we end up with Theorem 43 with β replaced by κ. As, in addition, ϕ was normalized so that $\mathrm{P}(\phi) = 0$, the function $\phi + \kappa \log |f'|$ is therefore a coboundary. Le μ be an arbitrary ergodic probability f-invariant measure on $J(f)$ having positive entropy. We then get

$$\mathrm{HD}(\mu) = \frac{\mathrm{h}_\mu}{\chi_\mu} \leq \frac{P(\phi) - \int \phi \, d\mu}{\chi_\mu} = -\int \phi \, d\mu \chi_\mu = \frac{\kappa \chi_\mu}{\chi_\mu} = \kappa = \mathrm{HD}(\mu_\phi).$$

Thus, we get that $\mathrm{DD}(J(f)) = \mathrm{HD}(\mu)$, and so, $\phi + \mathrm{DD}(J(f)) \log |f'| : J(f) \to \mathbb{R}$ is a coboundary, contrary to our assumption. We are done. □

Notice that Koebe's Distortion Theorem readily entails the following.

Proposition 58. *Let $\psi = -\log |f'| : J(f) \to \mathbb{R}$. Then $\psi_* : X_* \to \mathbb{R}$ is a Hölder continuous function in the sense of Section 6.*

We shall now prove the following lemma in an abstract context of subshifts of finite type with a countable alphabet. Apart from being needed for the proof of finiteness of all momenta of $(\log |f'|)_*$, this lemma is of intrinsic interest itself.

Lemma 59. *Being in the context of Section 6 let I be a countable set, called alphabet, consisting of at least two distinct points. Let $\rho \in \mathcal{K}_\alpha^s$ and let μ_ρ be the corresponding Gibbs (equilibrium) state; see* [MU2] *for their definition, existence and an account of properties. Recall that for every $i \in I$, $\tau_i : [i] \to \mathbb{N}$ is the first return map to $[i]$, i.e. $\tau_i(\omega) = \min\{n \geq 1 : \sigma^n(\omega) \in [i]\}$. We claim that for every $a \in I$ there exists $\kappa > 0$ such that*

$$\mu_\rho(\tau_a^{-1}(n)) \leq e^{-\kappa n} \tag{65}$$

for all $n \geq 1$.

Proof. Fix $a \in I$. Since I contains at least two distinct elements and since measure μ_ρ has a full topological support, it suffices to prove (65) for all $n \geq$ large enough. In fact it suffices to show that

$$\mu_\rho(\tau_a > n) \leq e^{-\kappa n} \tag{66}$$

for some $\kappa > 0$ and all $n \geq 1$. Let $\mathcal{L}_\rho : C_b(I^\infty) \to C_b(I^\infty)$ be the Perron-Frobenius operator defined in (43). Let m_ρ be the probability eigenmeasure corresponding to the eigenvalue $e^{P(\rho)}$ of the dual operator $\mathcal{L}_\rho^* : C_b^*(I^\infty) \to C_b^*(I^\infty)$ and let $g = \frac{d\mu_\rho}{dm_\rho}$. Set $\rho_0 := \exp\big(\rho - P(\rho)\big)\frac{g}{g\circ\sigma}$. Since

$$\sum_{i\in I} \exp\big(\rho_0(i\omega)\big) = 1,$$

we can write as follows:

$$\begin{aligned}
\mu_\rho(\tau_a > n) &= \sum_{i\neq a} \mu_\rho\big([e(\tau_a > n-1)]\big) = \sum_{i\neq a} \int_{\{\tau_a > n-1\}} \exp\big(\rho_0(i\omega)\big)\, d\mu_\rho(\omega) \\
&= \int_{\{\tau_a > n-1\}} \Big(\sum_{i\neq a} \exp\big(\rho_0(i\omega)\big)\Big)\, d\mu_\rho(\omega) = \int_{\{\tau_a > n-1\}} \big(\mathbb{1} - \exp\big(\rho_0(a\omega)\big)\big)\, d\mu_\rho(\omega) \\
&\leq \int_{\{\tau_a > n-\}\}} \big(\mathbb{1} - \exp\big(\inf\big(\rho_0|_{[i]}\big)\big)\big)\, d\mu_\rho(\omega) \\
&= \big(\mathbb{1} - \exp\big(\inf\big(\rho_0|_{[i]}\big)\big)\big)\mu_\rho(\tau_a > n-1).
\end{aligned} \tag{67}$$

Since $\mu_\rho(\tau_a > 1) < 1$ and since $\big(\mathbb{1} - \exp\big(\inf\big(\rho_0|_{[i]}\big)\big)\big) < 1$, formula (66) follows from (67) by induction. We are done. $\square$

Now we are in position to prove the following.

Proposition 60. *Let $\psi = \log|f'| : J(f) \to \mathbb{R}$. Then all moments of the function $\psi_* : X_* \to \mathbb{R}$ are finite, i.e. $\int |\psi_*|^\gamma \, d\mu_{\phi_*} < +\infty$ for all $\gamma > 0$.*

Proof. Let $n \geq 1$ be an arbitrary integer. For every $i \in I$ such that $N(i) = n$ and for all $z \in \hat{D}_i$, in virtue of Koebe's Distortion Theorem we get

$$K^{-1} \leq K^{-1}\frac{\operatorname{diam}(D)}{\operatorname{diam}(\hat{D}_i)} \leq |(h^n)'(z)| \leq ||h'||_\infty^n$$

with some $K \geq 1$ independent of i, n, and $z \in \hat{D}_i$. Therefore,

$$\big|\log|(h^n)'(z)|\big| \leq n\max\{\log K, \log||h'||_\infty\}.$$

Putting $L := \max\{\log K, \log||h'||_\infty\}$, and applying Lemma 59 we thus get for all $\gamma > 0$ that

$$\begin{aligned}
\int |\psi_*|^\gamma\, d\mu_{\phi_*} &= \sum_{n=1}^\infty \sum_{N(i)=n} \int_{\hat{D}_i} |\psi_*|^\gamma\, d\mu_{\phi_*} \\
&= \sum_{n=1}^\infty \int_{\tau_D=n} |\psi_*|^\gamma\, d\mu_{\phi_*} \\
&\leq \sum_{n=1}^\infty (nL)^\gamma \mu_{\phi_*}\big(\tau_D = n\big) \\
&\leq L^g \sum_{n=1}^\infty n^\gamma e^{-\kappa n} < +\infty.
\end{aligned}$$

The proof is complete. $\square$

Inserting these last two propositions to Theorem 2.5.5 and Lemma 2.5.6 in [MU2], we get the following.

Lemma 61. *Let $\psi = a\phi + b\log|f'| : J(f) \to \mathbb{R}$, $a, b \in \mathbb{R}$. Then the function $\psi_* : X_* \to \mathbb{R}$ satisfies the Law of Iterated Logarithm with respect to the dynamical system (F_*, μ_{ϕ_*}), provided that ψ_* is not cohomologous to a constant in $L^2(\mu_{\phi_*})$.*

Notice that the function $\psi_* : X_* \to \mathbb{R}$ can be also represented in the form $\psi_* = \widehat{\widehat{(\psi \circ \pi)}}$, where $\pi : \Delta \to J(f)$ is the projection from the tower Δ to the Julia set $J(f)$, defined by (63), the first "ˆ" refers to the inducing scheme on Δ_0 with respect to the map $T : \Delta \to \Delta$, while the second 'ˆ" refers to the inducing scheme with respect to $T_0 = F$ on X_*. So, as an immediate consequence of lemma 61 and Lemma 53, by passing from $\psi_* : X_* \to \mathbb{R}$ to $\widehat{(\psi \circ \pi)}$, from $\widehat{(\psi \circ \pi)}$ to $\psi \circ \pi$, and from $\psi \circ \pi$ to ψ, in the same way as in the proof of Theorem 56, we get the following.

Theorem 62. *Let $\psi = a\phi + b\log|f'| : J(f) \to \mathbb{R}$, $a, b \in \mathbb{R}$. Then the function $\psi : J(f)) \to \mathbb{R}$ satisfies the Law of Iterated Logarithm with respect to the dynamical system (f, μ_ϕ), provided that ψ_* is not cohomologous to a constant in $L^2(\mu_{\phi_*})$.*

Now, as an immediate consequence of this theorem and Lemma 57, we obtain the following.

Theorem 63. *If the pair (f,ϕ) fails to satisfy condition (1) of Theorem 43, then the function $\psi := \phi + \mathrm{HD}(\mu_\phi)\log|f'| : J(f) \to \mathbb{R}$ satisfies the Law of Iterated Logarithm with respect to the dynamical system (f,μ_ϕ). This means that there exists a real positive constant A_g such that μ_ϕ almost everywhere*

$$\limsup_{n\to\infty} \frac{S_n\psi - n\int \psi d\mu}{\sqrt{n\log\log n}} = A_g.$$

We now derive geometric consequences of this theorem. We recall first the definition of generalized Hausdorff measures. Let $g : [0,\infty) \to [0,\infty)$ be a non-decreasing function continuous at 0, positive on $(0,\infty)$ and such that $g(0) = 0$. Such functions are frequently referred to as gauge functions. Let (X,ρ) be a metric space. For every $\delta > 0$ define

$$\mathrm{H}_g^\delta(A) = \inf\big\{\sum_{i=1}^\infty g(\mathrm{diam}(U_i))\big\}$$

where the infimum is taken over all countable covers $\{U_i : i = 1,2,\ldots\}$ of A of with the diameter of each U_i not exceeding δ. The following limit

$$\mathrm{H}_g(A) = \lim_{\delta\to 0} \mathrm{H}_\delta^\delta(A) = \sup_{\delta>0} \mathrm{H}_\delta(A)$$

exists, but may be infinite, since $\mathrm{H}_g^\delta(A)$ increases as δ decreases. Since all the functions H_g^δ are outer measures, H_g is an outer measure too. In addition, H_g turns out to be a metric outer measure and therefore all Borel subsets of X are H_g–measurable. At the moment we are particularly interested in gauge functions of the form

$$g_{\kappa,c}(t) = t^\kappa \exp\big(c\sqrt{\log(1/t)\log_3(1/t)}\big) \quad \kappa, c > 0.$$

Having Theorem 64 and proceeding then in the same way as in [PUZ] (comp. [PU] for an easier, expanding, case), we can prove the following.

Theorem 64. *Suppose that the pair (f,ϕ) fails to satisfy condition (1) of Theorem 43. Let $c_\phi = A_{\phi+\mathrm{HD}(\mu_\phi)\log|f'|} > 0$ and let $\kappa := \mathrm{HD}(\mu_\phi)$. Then*

μ_ϕ is absolutely continuous with respect to $\mathrm{H}_{g_{\kappa,c}}$ for all $0 < c < \sqrt{c_\phi/\chi_{\mu_\phi}}$.

μ_ϕ is singular with respect to $\mathrm{H}_{g_{\kappa,c}}$ for all $c > \sqrt{c_\phi/\chi_{\mu_\phi}}$.

μ_ϕ is singular with respect to the ordinary Hausdorff measure H_{t^κ}.

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