FINER FRACTAL GEOMETRY

FOR ANALYTIC FAMILIES

OF

CONFORMAL DYNAMICAL SYSTEMS

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Abstract. We prove several results establishing real analyticity of Hausdorff dimensions of limit sets of analytic families of conformal graph directed Markov systems. With this tool and with iterated functions systems resulting from the existence nice sets in the sense of Rivera-Letelier, we prove that the canonical Hausdorff measure restricted to the radial Julia set of a tame meromorphic function (can be rational) is σ-finite and that the Hausdorff dimension of the radial Julia sets for fairly general families of meromorphic functions (can be rational) is real–analytic.

Contents

1. Introduction

Complex dynamics is a field originated in the works of Pierre Fatou and Gaston Julia. Of course, the problem of linearization for a fixed point was studied before (Böttcher, Koenings and others) and definitely it was an inspiration for the idea of creating this

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separate branch of mathematics, but numerous and extensive works of Fatou and Julia were the place where complex dynamics was born and maturated. The field became widely known and popular when about three decades ago first computer images of Mandelbrot set and Julia sets appeared. Complex dynamics attracted attention of many researches who started to investigate a variety of interesting and exiting topics in this field. One of them is the geometry of Julia sets and one of the ways to describe and analyze the complex nature of this object is its Hausdorff dimension. In this paper we study the behavior of this dimension under analytic perturbations.

Probably the first result indicating how the Hausdorff dimension of Julia sets changes under analytic perturbations is the result of Ruelle in [15]. He studied the family $z \mapsto z^2 + c$ and showed that the Hausdorff dimension of the Julia set is a real-analytic function for a complex parameter c sufficiently close to zero. The main technique Ruelle used was thermodynamic formalism. We refer the reader to the books of M. Zinsmeister [24] and F. Przytycki & M. Urbański [12] for a modern exposition of thermodynamic formalism and contemporary approach to the problem of real analyticity of Hausdorff dimension.

The problem of real analyticity of the Hausdorff dimension was further studied for many families of rational and meromorphic functions (see e.g. [23], [22], [21], [8], [1] and [9]). In the present paper we continue this line of investigation. Our two main results are Theorem 1.1 and Theorem 1.2 stated below. In these theorems we establish real–analyticity of Hausdorff dimension of radial Julia sets under weakest, up to our knowledge, conditions. Having a family $\{f_\lambda\}_{\lambda\in\Lambda}$ of meromorphic functions we commonly abbreviate $\mathcal{J}_\lambda$ for $\mathcal{J}(f_\lambda)$, the Julia set of f_λ. The concept of strong N–regularity appearing below is introduced in Definition 3.6. The class of dynamically regular meromorphic functions was introduced and investigated in [8] and [9]. We deal with them in Section 6 entitled Strong N–Regularity of Dynamically Regular Meromorphic Functions.

Theorem 1.1. *Assume that a tame meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ is strongly N–regular. Let $\Lambda \subset \mathbb{C}^d$ be an open set and let $\{f_\lambda\}_{\lambda\in\Lambda}$ be an analytic family of meromorphic functions such that*

(1) *$f_{\lambda_0} = f$ for some $\lambda_0 \in \Lambda$,*
(2) *there exists a holomorphic motion $H : \Lambda \times \overline{\mathcal{J}_{\lambda_0}} \to \mathbb{C}$ such that each map H_λ is a topological conjugacy between f_{λ_0} and f_λ on $\mathcal{J}_{\lambda_0}$.*

Then the map

$$\Lambda \ni \lambda \mapsto \mathrm{HD}(\mathcal{J}_r(f_\lambda))$$

is real–analytic on some neighborhood of λ_0.

Theorem 1.2. *Suppose that $f\colon \mathbb{C} \to \hat{\mathbb{C}}$ is a dynamically regular meromorphic function of divergence type which belongs to class $\mathcal{S}$. If $\Lambda \subseteq \mathbb{C}$ is an open set, $\{f_\lambda\}_{\lambda\in\Lambda}$ is an analytic family (in the sense of Section 5) of meromorphic functions, and if $f_{\lambda_0} = f$ for some $\lambda_0 \in \Lambda$, then the function $\Lambda \ni \lambda \mapsto \mathrm{HD}(\mathcal{J}_r(f_\lambda))$ is real–analytic in some open neighborhood of λ_0 in Λ.*

Both of these two theorems include rational functions. The former of them is also new for such functions; all hyperbolic, parabolic, semi-hyperbolic, and non-recurrent rational functions are tame. The latter one in the context of rational functions reduces to hyperbolic ones and is well-known.

One of our two main techniques employed in the proofs of these two theorems above is the, recently emerging, concept of nice sets. These sets were introduced and extensively studied by Przytycki and Rivera-Letelier ([13], [11]) in the context of Collet-Eckmann rational mappings. Nice sets in transcendental meromorphic dynamics were used to show that there is no absolutely continuous invariant probability for Misiurewicz exponential maps (see [3]). A general construction of nice sets for transcendental functions can be found in [2]. In our present paper we use them to construct appropriate conformal iterated function systems and then to apply the developed machinery of graph directed Markov systems from [6] and [7] that form a natural generalization of graph directed Markov systems. While doing this, as an actually auxiliary step, we obtain new, up to our knowledge, results about real analyticity of the Hausdorff dimension of limits sets of (infinite) conformal graph directed Markov systems. The following theorems 1.3 and 1.4 in particular extend corresponding assertions from [21] and [1]. While dealing with iterated function systems, or more generally, with graph directed Markov systems, we extensively use the results, definitions, and notation from [7] and [6]. For the convenience of the reader we collect in Appendix all those of them that we will need and use.

Theorem 1.3. *If an analytic family $\{S_\lambda\}_{\lambda\in\Lambda}$ consisting of finitely primitive conformal graph directed Markov systems, is strong, then the function $\Lambda \ni \lambda \mapsto b(S_\lambda) \in \mathbb{R}$ is real–analytic on some neighborhood of every strongly regular parameter $\lambda_0 \in \Lambda$.*

In addition, if we know the Bowen's parameter is equal to the Hausdorff dimension of the limit set (which is due to Theorem 8.1 in Appendix guaranteed for example by the Open Set Condition), we thus automatically get real analyticity of the Hausdorff dimension of the limit sets of conformal graph directed Markov systems S_λ, $\lambda \in \Lambda$, on that same neighborhood of λ_0.

Theorem 1.4. *If $\Lambda \subseteq \mathbb{C}^d$ is an open set and $\{S_\lambda\}_{\lambda\in\Lambda}$ is an analytic family of finitely primitive conformal graph directed Markov systems such that S_{λ_0} is strongly regular for some $\lambda_0 \in \Lambda$ and there exists a holomorphic motion $H\colon \Lambda \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that*

$$\varphi_e^\lambda(H(\lambda, z)) = H(\lambda, \varphi_e^{\lambda_0}(z))$$

for all $\lambda \in \Lambda$ and all $z \in \mathcal{J}_{\lambda_0}(:= \mathcal{J}_{S_{\lambda_0}})$, then the Bowen's parameter function $\Lambda \ni \lambda \to b(S_\lambda)$ is real–analytic on some sufficiently small neighborhood of λ_0.

Again, if the Bowen's parameter is equal to the Hausdorff dimension of the limit set then the Hausdorff dimension is real-analytic.

Note that although we assume in the latter theorem seemingly more, namely the existence of an appropriate holomorphic motion, however, on the other hand, we merely assume here analyticity of the family of graph directed Markov systems, which is much weaker than weakly regular analyticity required in Theorem 4.1 from [21]. Staying in the realm of abstract Conformal Graph Directed Markov Systems we are able to provide a very mild sufficient condition, called periodical separation, which entails the existence of a suitable holomorphic motion. We can then prove Theorem 1.1 under very weak hypotheses indeed. This is however not quite the end of the story about directed Markov systems. The point is that those conformal Markov systems constructed in the proof of Theorem 1.1 are not known to satisfy the Open Set Condition. To remedy this we invoke the theory of conformal Walters expanding maps developed in in [4].

Having Conformal Iterated Function Systems produced with the help of nice sets, we were also able to show (see Theorem 3.4), as a straightforward consequence of the theory of Conformal Graph Directed Markov Systems, that the canonical Hausdorff measure restricted to the radial Julia set of a tame meromorphic function is σ-finite. This is also new for rational functions.

We have already indicated this a few times but as the last remark we would like to say again that all our considerations about tame meromorphic functions also do apply to rational functions, and also for them are new. We do not assume that our meromorphic functions are transcendental.

2. Nice Sets and Corresponding Conformal Iterated Function Systems for Meromorphic Functions

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function. The Fatou set of f consists of all points $z \in \mathbb{C}$ that admit an open neighborhood U_z such that all the forward iterates f^n, $n \geq 0$, of f are well-defined on U_z and the family of maps $\{f^n|_{U_z} : U_z \to \mathbb{C}\}_{n=0}^{\infty}$ is normal. The Julia set $\mathcal{J}(f)$ is then defined as the complement of the Fatou set of f in $\mathbb{C}$. By $\mathrm{sing}(f^{-1})$ we denote the set of singularities of f^{-1}, i. e. such points $w \in \overline{\mathbb{C}}$ that for every spherical ball $B(w, r) \subseteq \overline{\mathbb{C}}$ there exists a connected component C of $f^{-1}(B(w, r))$ for which the map $f : C \to B(w, r)$ is not a homeomorphism. We define the *postsingular set* of $f : \mathbb{C} \to \hat{\mathbb{C}}$ as

$$\mathcal{P}(f) = \overline{\bigcup_{n=0}^{\infty} f^n(\mathrm{sing}\, f^{-1})}.$$

Given $z \in \mathbb{C}$ we say that a complex number w is in $\omega(z)$ if all the forward iterates $f^n(z)$, $n \geq 0$, are well-defined and w is a cluster point of the sequence $\{f^n(z)\}_{n=0}^{\infty}$. The set $\omega(z)$ is then refereed to as the ω-limit set of z. Note that $\omega(z) = \emptyset$ if and only if either z is eventually mapped to infinity or $\lim_{n\to\infty} f^n(z) = \infty$. The primary object of our study in this paper, the *radial Julia set* $\mathcal{J}_r(f)$ of f is defined as

$$\mathcal{J}_r(f) := \{z \in \mathcal{J}(f) : \omega(z) \setminus \mathcal{P}(f) \neq \emptyset\}.$$

Given a set $F \subset \hat{\mathbb{C}}$ and $n \geq 0$, by $\mathrm{Comp}(f^{-n}(F))$ we denote the collection of all connected components of $f^{-n}(F)$. A meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ is called *tame* if its postsingular set does not contain its Julia set. Unless otherwise stated all meromorphic functions considered in the sequel will be tame. As noted in the introduction, J. Rivera-Letelier introduced in [13] the concept of nice sets in the realm of the dynamics of rational maps of the

Riemann sphere. In [2] N. Dobbs proved their existence for tame meromorphic functions from $\mathbb{C}$ to $\hat{\mathbb{C}}$. We quote now his theorem.

Theorem 2.1. *Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a tame meromorphic function. Fix $z \in \mathcal{J}(f) \setminus \mathcal{P}(f)$, $\kappa > 1$, and $K > 1$. Then there exists $L > 1$ and for all $r > 0$ sufficiently small there exists an open connected set $U = U(z, r) \subset \mathbb{C} \setminus \mathcal{P}(f)$ such that*

(a) *If $V \in \mathrm{Comp}(f^{-n}(U))$ and $V \cap U \neq \emptyset$, then $V \subset U$.*
(b) *If $V \in \mathrm{Comp}(f^{-n}(U))$ and $V \cap U \neq \emptyset$, then, for all $w, w' \in V$,*

$$|(f^n)'(w)| \geq L \quad \text{and} \quad \frac{|(f^n)'(w)|}{|(f^n)'(w')|} \leq K.$$

(c) *$\overline{B(z, r)} \subset U \subset B(z, \kappa r) \subset \mathbb{C} \setminus \mathcal{P}(f)$.*

Let $\mathcal{U}$ be the collection of all nice sets of $f : \mathbb{C} \to \hat{\mathbb{C}}$, i.e. all the sets U satisfying the above proposition with some $z \in \mathcal{J}(f) \setminus \mathcal{P}(f)$ and some $r > 0$. Note that if $U = U(z, r) \in \mathcal{U}$ and $V \in \mathrm{Comp}(f^{-n}(U))$ satisfies the requirements (a), (b) and (c) from Proposition 2.1 then there exists a unique holomorphic inverse branch $f_V^{-n} : B(z, \kappa r) \to \mathbb{C}$ such that $f_V^{-n}(U) = V$. The collection S_U of all such inverse branches forms obviously an iterated function system in the sense of [6] and [7], comp. Appendix. In particular, S_U clearly satisfies the Open Set Condition. We denote its limit set by $\mathcal{J}_U$.

3. Hausdorff Dimension; Hausdorff, Conformal and Invariant Measures

Recall that $\mathcal{U}$ is the collection of all nice sets of a tame meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$. Since, by Theorem 2.1, $\mathcal{U}$ forms a basis of topology for $\mathcal{J}(f) \setminus \mathcal{P}(f)$ and since this metric space is separable, it follows from Lindelöf's Theorem that $\mathcal{U}$ contains a countable cover of $\mathcal{J}(f) \setminus \mathcal{P}(f)$. We start with the following.

Lemma 3.1. *If $\mathcal{W}$ is a subcover of $\mathcal{U}$, than*

$$\mathcal{J}_r(f) = \bigcup_{U \in \mathcal{W}} \bigcup_{k=0}^{\infty} f^{-k}(\mathcal{J}_U).$$

Proof. Since $\mathcal{J}_U \subset \mathcal{J}_r$ for all U,

$$\bigcup_{U \in \mathcal{W}} \bigcup_{k=0}^{\infty} f^{-k}(\mathcal{J}_U) \subset \mathcal{J}_r.$$

On the other hand, if $x \in \mathcal{J}_r$, then there exists $y \in \omega(x) \setminus \mathcal{P}(f)$ and therefore $U \in \mathcal{W}$ with $y \in U$ such that the set $\{n \geq 0 : f^n(x) \in U\}$ is infinite. So, $x \in f^{-k}(\mathcal{J}_u)$ for some $k \geq 0$. This finishes the proof. $\square$

Now, we aim to prove that for a tame meromorphic the Hausdorff dimension of limit sets of all nice sets is the same and is equal to the Hausdorff dimension of the radial Julia set. To do this we need the following proposition, concerning general Conformal Iterated Function Systems, which is also interesting on its own.

Proposition 3.2. *Let $S = \{\varphi_e\}_{e\in E}$ be a Conformal Iterated Function System. For every $\tau \in E^*$, let*

$$\mathcal{J}_\tau^\infty = \pi(\{\omega \in E^\infty \colon \omega \textit{ contains infinitely many copies of the block } \tau\}).$$

Then $\mathrm{HD}(\mathcal{J}_\tau^\infty) = \mathrm{HD}(\mathcal{J}_s)$.

Proof. Let $F \subseteq E$ be an arbitrary finite subset of E (containing all letters of τ). Let $\tilde{m}_F$ and $\tilde{\mu}_F$ be respectively the corresponding symbolic geometric and invariant h_s-conformal measures; see Appendix for their definitions. Let

$$F_\tau^\infty = \{\omega \in F^\infty \colon \omega \text{ contains infinitely many copies of the block } \tau\}$$

since $\mathrm{supp}(\tilde{\mu_F}) = F^\infty$, it follows from Birkhoff Ergodic Theorem that $\tilde{\mu}_F(F_\tau^\infty) = 1$. Since the measures $\tilde{m_F}$ and $\tilde{\mu}_F$ are equivalent, we conclude that $m_f(F_\tau^\infty) = 1$. Thus

$$m_F(\pi(F_\tau^\infty)) = m_F \circ \pi^{-1}(\pi(F_\tau^\infty)) \geq m_F(F_\tau^\infty) = 1.$$

Since F is finite, the measure m_F coincides on $\mathcal{J}_F$ up to a multiplicative constant with the Hausdorff measure H_{h_F}. So, $\mathrm{H}_{h_F}(\pi(F_\tau^\infty)) > 0$, whence $\mathrm{HD}(\pi(F_\tau^\infty)) = h_F$. Thus,

$$\mathrm{HD}(\mathcal{J}_\tau^\infty) \geq \sup \mathrm{HD}(\pi(F_\tau^\infty)) = \sup h_F = h_E.$$

where both suprema are taken over all F being a finite subsets of E containing all elements of the finite word τ. This finishes the proof. □

Coming back to meromorphic functions, we prove the following.

Lemma 3.3. *If U and W are two arbitrary nice sets of a tame meromorphic function $f \colon \mathbb{C} \to \bar{\mathbb{C}}$, then* $\mathrm{HD}(\mathcal{J}_W) = \mathrm{HD}(\mathcal{J}_U)$.

Proof. Let $S_U = \{\phi_i^U : i \in I_U\}$ be the iterated function system induced by the nice set U. Since $U \cap \mathcal{J}(f) \neq \emptyset$, there exists $q \geq 0$ so large that

$$f^q(U) \cap \mathcal{J}(f) \supseteq \mathcal{J}(f) \cap W.$$

Since

$$\lim_{n\to\infty} \sup\{\mathrm{diam}(\varphi_\omega^U(U) \colon |\omega| = n\} = 0$$

(in fact the rate of convergence is exponential), and since W is an open set, there thus exists $\tau \in E_U^*$ such that

$$f^q(\varphi_\tau^U(U \cap \mathcal{J}(f)) \subseteq \mathcal{J}(f) \cap W.$$

Hence

$$f^q(\mathcal{J}_{U,\tau}^\infty) \subseteq \mathcal{J}_W. \tag{3.1}$$

Therefore, applying Proposition 3.2, we get that

$$\mathrm{HD}(\mathcal{J}_U) = \mathrm{HD}(\mathcal{J}_{U,\tau}^\infty) = \mathrm{HD}(f^q(\mathcal{J}_{U,\tau}^\infty)) \leq \mathrm{HD}(\mathcal{J}_W).$$

Exchanging the roles of the U and W we also get that $\mathrm{HD}(\mathcal{J}_W) \leq \mathrm{HD}(\mathcal{J}_U)$ and the proof is complete. □

Now, as the main result of this section, we show that the number

$$h := \mathrm{HD}(\mathcal{J}_r(f))$$

is equal to the common value of Hausdorff dimensions of the limits sets of all the iterated function systems induced by all the nice sets and that the the corresponding Hausdorff measure on the radial limit set $\mathcal{J}(f)$ is σ-finite.

Theorem 3.4. *Let $f\colon \mathbb{C} \to \bar{\mathbb{C}}$ be a tame meromorphic function. Then the following is true.*

(a) $h = \mathrm{HD}(\mathcal{J}_r(f))) = \mathrm{HD}(\mathcal{J}_U)$ *for every nice set U.*
(b) *The h-dimensional Hausdorff measure H_h restricted to each nice limit set $\mathcal{J}_U$, $U \in \mathcal{U}$, is finite.*
(c) *The h-dimensional Hausdorff measure H_h restricted to $\mathcal{J}_r(f)$ is σ-finite.*

Proof. Fixing $U \in \mathcal{U}$, and choosing a countable subcover of $\mathcal{U}$ containing U, Part (a) follows immediately from Lemma 3.3 and Lemma 3.1. Part (b) follows from Theorem 4.5.1 and 4.5.11 from [7], comp. Fact 8.3 (d) in Appendix, and Part (c) is an immediate consequence of part (b) and Lemma 3.1 applied with an arbitrary countable subcover $\mathcal{W}$ of $\mathcal{U}$. We are done. □

Let us record the following straightforward:

Observation 3.5. If $f\colon \mathbb{C} \to \bar{\mathbb{C}}$ is a meromorphic function, then the following are equivalent.

(a) For every point $z \in \mathcal{J}(f)$ there exists a radius $R(z) > 0$ such that $\mathcal{J}(f) \cap B(z, R(z))$ is contained in a real-analytic (so connected) curve.
(b) There exists a point $z \in \mathcal{J}(f)$ and a radius $R(z) > 0$ such that $\mathcal{J}(f) \cap B(z, R(z))$ is contained in a real-analytic (so connected) curve.
(c) For every point $z \in \mathcal{J}(f)$ there exists a radius $R(z) > 0$ such that $\mathcal{J}(f) \cap B(z, R(z))$ is contained in a countable union of real-analytic curves.
(d) There exists a point $z \in \mathcal{J}(f)$ and a radius $R(z) > 0$ such that $\mathcal{J}(f) \cap B(z, R(z))$ is contained in a countable union of real-analytic curves.

If one, or equivalently all, of the above condition fails, we call the meromorphic function $f\colon \mathbb{C} \to \bar{\mathbb{C}}$ of fractal type A; otherwise we call it of fractal type B.

Definition 3.6. We call a tame meromorphic function $f\colon \mathbb{C} \to \bar{\mathbb{C}}$ (*strongly*) *N–regular* if there exists at least one nice set $U \in \mathcal{U}$ giving rise to a (strongly) regular iterated function system (IFS) S_U.

In Section 6 we shall provide same sufficient conditions for a meromorphic function to be strongly N–regular. Strong N–regularity will turn out to be a much harder issue than mere N–regularity.

Let us now recall another fundamental concept. Its origins go back to work of S. Paterson ([10]) in the realm of Fuchsian groups and of D. Sullivan([18], [20], ([17], ([19],) in the contexts of Kleinian groups and rational functions. Namely, a Borel σ-finite measure m_h on $\mathcal{J}(f)$ is called h-conformal for $f\colon \mathbb{C} \to \bar{\mathbb{C}}$ if

$$m_h(f(A)) = \int_A |f'|^h dm_h \tag{3.2}$$

for every Borel set $A \subseteq \mathcal{J}(f)$ (or $\subseteq \mathbb{C}$) such that $f|_A$ is one–to–one. This concept is clearly similar to the one for graph directed Markov systems (see Appendix) although it is a verbatim copy of Sullivan's definition for rational functions, and it therefore should not be regarded as historically later than that of graph directed Markov systems.

The existence of this kind of measures is of enormous help in the investigation of geometric properties of Julia sets. Therefore, we now prove their existence and establish some properties of these measures.

Theorem 3.7. *If a tame meromorphic function $f : \mathbb{C} \to \overline{\mathbb{C}}$ is N–regular, then the following hold.*

(a) *Each nice set $W \in \mathcal{U}$ gives rise to a regular IFS.*
(b) *There exists a σ-finite h-conformal measure m_h for $f\colon \mathbb{C} \to \bar{\mathbb{C}}$. In addition, $m_h(\mathcal{J}(f)\backslash\mathcal{J}_r(f)) = 0$, and for every nice set $U \in \mathcal{U}$, we have that $m_h(\mathcal{J}_U) > 0$ and the measure $m_h|_{\mathcal{J}_U}$ is h-conformal for the IFS S_U.*
(c) *There exist a Borel σ-finite f-invariant measure μ_h on $\mathcal{J}(f)$ such that $\mu_h(\mathcal{J}(f)\backslash\mathcal{J}_r(f)) = 0$, $0 < \mu_h(\mathcal{J}_U) < +\infty$, for every nice set $U \in \mathcal{U}$, and $\mu_h|_{\mathcal{J}_U} = \mu_U$ is equivalent to the h-conformal probability measure m_U on $\mathcal{J}_U$.*
(d) *The Radon-Nikodym derivative $\frac{d\mu_h}{dm_h}$ has a unique real-analytic extension to an open neighborhood of the set $\mathcal{J}(f) \setminus \mathcal{P}(f)$ if the meromorphic function $f\colon \mathbb{C} \to \bar{\mathbb{C}}$ is of fractal type A, and a unique holomorphic extension to such a neighborhood if the function f is of fractal type B.*

Proof. Let $U \in \mathcal{U}$ be a nice set giving rise to a regular IFS $S_U = \{\varphi_e^U\colon e \in E\}$. Denote $\|e\|$ the number $n \in \mathbb{N}$ such that $f^n \circ \varphi_e^U = \mathrm{id}_U$. For every $n \geq 1$ let I_n parametrize all holomorphic branches $\{f_i^{-n}\}_{i\in I_n}$ of f^{-n} that are defined on U, let $I = \bigcup_{n=1}^\infty I_n$ and for every $i \in I$ let $n(i)$ be a unique integer $k \geq 0$ such that $i \in I_k$. Let I_* be the subset of I consisting all elements i such that $f^k(f_i^{-n(i)}(U)) \cap U = \emptyset$ for all $0 \leq k \leq n(i) - 1$. Notice that the family $\{f_i^{-n(i)}(U)\}_{i\in I_*}$ consists of mutually disjoint sets and define the measure m_h on $U \cup \bigcup_{i\in I_*} f_i^{-n(i)}(U)$ by the following formula. If $i \in I_*$, $A \subseteq f_i^{-n(i)}(U)$ is an arbitrary Borel set, then

$$m_h(A) = \int_{f^{n(i)}(A)} |(f_i^{-n(i)})'(z)|^h dm_U(z). \tag{3.3}$$

Otherwise, if $A \subseteq U$ is a Borel set, then

$$m_h(A) = m_U(A). \tag{3.4}$$

It immediately follows (3.3) that (3.2) holds for all Borel set $A \subseteq f_i^{-n(i)}(U)$ where $i \in I_*$, since $n(i) \geq 1$. Now, for any $z \in \mathcal{J}_U$ let $N(z) \geq 1$ be the first return time to U, *i.e.* $N(z) \geq 1$

is the least integer such that $f^{N(z)}(z) \in U$. Note that $N(z) < +\infty$ and $f^{N(z)}(z) \in \mathcal{J}_U$. For every Borel set $A \subseteq \mathcal{J}_U$ and every $n \geq 1$ let

$$A_n = \{z \in A \colon N(z) = n\},$$

then $\{A_n\}_{n=1}^\infty$ is a partition of A into measurable sets. Notice that

$$A_n = \bigcup_{||e||=n} \varphi_e^U(f^n(A_n)).$$

Assume that $f|_A$ is 1-1. Then also $f|_{A_n}$ is also 1-1, and by (3.3), (3.4) and conformality of the measure m_U for the IFS S_U,

$$\begin{aligned}
m_h(f(A)) &= \sum_{n=1}^\infty m_h(f(A_n)) = \sum_{n=1}^\infty \sum_{e:||e||=n} m_h(f \circ \varphi_e^U(f^n(A_n))) \\
&= \sum_{n=1}^\infty \sum_{e:||e||=n} \int_{f^n(A_n)} |(f \circ \varphi_e^U)'|^h dm_U \\
&= \sum_{n=1}^\infty \sum_{e:||e||=n} \int_{f^n(A_n)} |f' \circ \varphi_e^U|^h |(\varphi_e^U)'|^h dm_U \\
&= \sum_{n=1}^\infty \sum_{e:||e||=n} \int_{\varphi_e^U(f^n(A_n))} |f'|^h dm_U \\
&= \sum_{n=1}^\infty \sum_{||e||=n} \int_{\varphi_e^U(f^n(A_n))} |f'|^h dm_h \\
&= \sum_{n=1}^\infty \int_{\bigcup\{\varphi_e^U(f^n(A_n)):||e||=n\}} |f'|^h dm_h = \sum_{n=1}^\infty \int_{A_n} |f'|^h dm_h \\
&= \int_{\bigcup_{n=1}^\infty A_n} |f'|^h dm_h = \int_A |f'|^h dm_h.
\end{aligned}$$

Thus, (3.2) holds for all Borel sets

$$A \subseteq U \cup \bigcup_{i \in I_*} f^{-n(i)}(U)$$

such that $f|_A$ is 1-1. Observe that then all sets $f(A \cap U)$ and $f(A \cap f^{-n(i)}(U))$, $i \in I_*$ are mutually disjoint and $m_h(A \cap U) = m_U(A \cap \mathcal{J}_U)$ as well as $m_h(f(A \cap U)) = m_h(f(A \cap \mathcal{J}_U))$. Since also

$$m_h(\mathbb{C} \backslash (\mathcal{J}_U \cup \bigcup_{i \in I_*} f_i^{-n(i)}(\mathcal{J}_U))) = 0 \tag{3.5}$$

and since

$$f\big(\mathcal{J}_U \cup \bigcup_{i \in I_*} f_i^{-n(i)}(\mathcal{J}_U)\big) = f^{-1}\big(\mathcal{J}_U \cup \bigcup_{i \in I_*} f_i^{-n(i)}(\mathcal{J}_U)\big) = \mathcal{J}_U \cup \bigcup_{i \in I_*} f_i^{-n(i)}(\mathcal{J}_U),$$

we conclude that m_h is a Borel σ-finite h-conformal measure for $f\colon \mathbb{C} \to \bar{\mathbb{C}}$ such that $m_h|_{\mathcal{J}_U} = m_U$. Since

$$\mathcal{J}_U \cup \bigcup_{i\in I_*} f_i^{-n(i)}(\mathcal{J}_U) \subseteq \mathcal{J}_r \tag{3.6}$$

we further conclude from (3.5) that $m_h(\mathcal{J}(f) \setminus \mathcal{J}_r(f) = \emptyset$.

If now W is an arbitrary nice set, then it follow from (3.1), conformality property (3.2), and the fact that $m_h(\mathcal{J}_{U,\tau}^\infty) = m_U(\mathcal{J}_{U,\tau}^\infty) = 1$, that $m_h(\mathcal{J}_\omega) > 0$. Items (a) and (b) of our theorem are then proved.

It is known that one can spread out the measure μ_U and get a unique ergodic and conservative f-invariant measure μ_h on $\mathcal{J}_U \cup \bigcup_{i\in I_*} f_i^{-n(i)}(\mathcal{J}_U)$ such that $\mu_h|_{\mathcal{J}_U} = \mu_U$. Hence, by (3.6), $\mu_h(\mathcal{J}(f)\setminus\mathcal{J}_r(f)) = 0$ and the item (c) is proved.

Let us now prove item (d). For any two open balls $B_1, B_2 \subset \mathbb{C}$ let $B_1 \wedge B_2 = B_1$ and $B_2 \wedge B_1 = B_2$ if either $B_1 \cap B_2 = \emptyset$ or $\mathcal{J}(f) \cap B_1 \cap B_2 \neq \emptyset$. Otherwise, let $\{a, b\} = \partial B_1 \cap \partial B_2$ and let l be the unique straight line passing through the points a and b. Let then H_i, $i = 1, 2$, be the connected component (open half-plane) of $\mathbb{C} \setminus l$ containing the center of the ball B_i. Set then

$$B_1 \wedge B_2 = B_1 \cap H_1 \quad \text{and} \quad B_2 \wedge B_1 = B_2 \cap H_2.$$

Let s be the spherical metric on $\overline{\mathbb{C}}$ normalized so that $\operatorname{diam}_s(\overline{\mathbb{C}}) = 1$. For every integer $n \geq 0$ consider the set

$$\Gamma_n = \{z \in \mathcal{J}(f) \setminus \mathcal{P}(f) : 2^{-(n+1)} \leq s(z, \mathcal{P}(f) \cup \{\infty\}) \leq 2^{-n}\}.$$

Obviously, each set Γ_n, $n \geq 0$, is compact and

$$\bigcup_{n=0}^{\infty} \Gamma_n = \mathcal{J}(f) \setminus \mathcal{P}(f).$$

For every integer $n \geq 0$ and every point $z \in \mathcal{J}(f) \setminus \mathcal{P}(f)$ let U_z be a nice set containing z. Fix then an open ball $B_n(z) = B(z, r_n(z))$ with $0 < r_n(z) < 2^{-(n+3)}$ so small that

$$B_n(z) \subset U_z \cap \mathbb{C} \setminus \mathcal{P}(f). \tag{3.7}$$

Since each set Γ_n, $n \geq 0$, is compact, we can find for every $n \geq 0$ a finite set $E_n \subset \Gamma_n$ such that

$$\bigcup_{w\in E_n} B_n(w) \supset \Gamma_n.$$

An immediate observation is that

$$\bigcup_{w\in E_n} B_n(w) \cap \bigcup_{k=n+2}^{\infty} \bigcup_{z\in E_k} B_k(z) = \emptyset \tag{3.8}$$

for all $n \geq 0$. Hence, for every $n \geq 0$ and every $w \in E_n$, the set

$$A(n, w) := \{(k, z) \in \mathbb{N} \times \mathbb{C} : z \in E_k \quad \text{and} \quad B_n(w) \cap B_k(z) \neq \emptyset\}$$

is finite. Set then

$$\hat{B}_n(w) = \bigcap_{(k,z)\in A(n,w)} B_n(w) \wedge B_k(z).$$

It immediately follows from our construction that

(a) $\mathcal{J}(f) \setminus \mathcal{P}(f) \subset G := \bigcup_{n=0}^{\infty} \bigcup_{w \in E_n} \hat{B}_n(w) \subset \mathbb{C} \setminus \mathcal{P}(f)$.

(b) All the sets $\hat{B}_n(w)$, $n \geq 0$, $w \in E_n$, are open and convex; so simply connected.

(c) If $\hat{B}_n(w) \cap \hat{B}_k(z) \neq \emptyset$, then $\mathcal{J}(f) \cap \hat{B}_n(w) \cap \hat{B}_k(z) \neq \emptyset$, in fact
$$(\mathcal{J}(f) \setminus \mathcal{P}(f)) \cap \hat{B}_n(w) \cap \hat{B}_k(z) \neq \emptyset$$
as $\mathcal{P}(f)$ is a nowhere dense subset of $\mathcal{J}(f)$.

In virtue of Theorem 6.1.3 from [7] (comp. Fact 8.3 (c) in Appendix), for every $w \in \mathcal{J}(f) \setminus \mathcal{P}(f)$ the Radon-Nikodym derivative $\frac{d\mu_{U_w}}{dm_{U_w}}$ defined on J_{U_w}, has a real-analytic extension $\hat{\rho}_w : U_w \to \mathbb{R}$. Since $\frac{d\mu_h}{dm_h}\big|_{J_{U_w}}$ is a constant multiple of $\frac{d\mu_{U_w}}{dm_{U_w}}$, we thus infer that $\frac{d\mu_h}{dm_h}\big|_{J_{U_w}}$ has a real-analytic extension, being a constant multiple of $\hat{\rho}_w$, $\rho_w : U_w \to \mathbb{R}$. Assume now that our meromorphic function $f : \mathbb{C} \to \overline{\mathbb{C}}$ is of fractal type A. Define a function $\rho : G \to \mathbb{R}$ by setting
$$\rho(z) := \rho_w(z) \;\text{ if }\; z \in \hat{B}_n(w) \;\text{ with some }\; n \geq 0 \;\text{ and some } w \in E_n.$$
Since G is an open set containing $\mathcal{J}(f) \setminus \mathcal{P}(f)$, all what what we are to check is that the function ρ is well-defined. But from our construction (item (c) above), if $n, k \in \mathbb{N}$, $w \in E_n$, $z \in E_k$, and $\hat{B}_n(w) \cap \hat{B}_k(z) \neq \emptyset$, then $\mathcal{J}(f) \cap \hat{B}_n(w) \cap \hat{B}_k(z) \neq \emptyset$. But as
$$\mathcal{J}(f) \cap \hat{B}_n(w) \cap \hat{B}_k(z) \subset \{\xi \in \hat{B}_n(w) \cap \hat{B}_k(z) : (\rho_w - \rho_z)(\xi) = 0\},$$
if ρ_z and ρ_w are not identically equal on $\hat{B}_n(w) \cap \hat{B}_k(z)$, then the right hand side of the above formula is a countable union of real-analytic curves, contrary to item (d) of Observation 3.5. Thus $\rho_w = \rho_z$ on $\hat{B}_n(w) \cap \hat{B}_k(z)$ and we are done in the case meromorphic functions of fractal type A.

In the case of fractal type B, in the construction of the balls $B_n(z)$, $n \in \mathbb{N}$, $z \in E_n$, in addition to (3.7) we require the radii $r_n(z) > 0$ to be so small that the real-analytic function ρ_z restricted to $\Delta_{z,n}$ (where $\Delta_{z,n}$ denotes the real-analytic curve contained in $B_n(z)$ and containing $\mathcal{J}(f) \cap B_n(z)$) extends uniquely to a holomorphic function $\tilde{\rho}_z : B_n(z) \to \mathbb{C}$. Now the argument analogous to that employed in the case of fractal type A, shows that all the functions $\tilde{\rho}_z|_{\hat{B}_n(z)}$, $n \in \mathbb{N}$, $z \in E_n$, glue together to a holomorphic function $\tilde{\rho} : G \to \mathbb{C}$ which extends the function $\rho : \mathcal{J}(f) \setminus \mathcal{P}(f)$. The proof is complete. $\square$

4. Real Analyticity of Hausdorff Dimension for Conformal Graph Directed Markov Systems

The results of this section form a far going strengthening of existing theorems about real analyticity (see [21] and references therein) or even continuity (see [14]) of the Hausdorff dimension of limit sets of Conformal Graph Directed Markov Systems. All notions needed are explained in Appendix. We would also like to emphasize that our results in this section will primarily concern Conformal Graph Directed Markov Systems (conformal GDMSs) without assuming the Open Set Condition to hold and will be primarily formulated as real analyticity of Bowen's parameter defined below. Real analyticity of the Hausdorff dimension of limit sets will be then obtained as an immediate corollary in the case when the Open set Condition holds. However in Section 5 devoted to proving real analyticity of the Hausdorff dimension of radial Julia sets of tame meromorphic functions that are strongly regular, we will construct conformal Graph Directed Markov Systems which will not be known to satisfy the Open Set Condition. Real analyticity of Bowen's parameter will result from the present section whereas its equality to Hausdorff dimension will come from the theory of conformal Walters expanding maps laid down in [4].

Let $\Lambda \subseteq \mathbb{C}^d$ be a complex manifold. Let $\Gamma = (E, V, t, i, A)$ be a finitely primitive multigraph with edges E, vertexes V, initial and terminal function t and i, and a incidence matrix $A\colon E \times E \to \{0, 1\}$ (see Appendix). For every vertex $v \in V$ let bounded open sets $W_v, W'_r \subseteq \mathbb{C}$ be given satisfying that $\overline{W_r} \subset W'_r$. Further more, for every $\lambda \in \Lambda$, let

$$S_\lambda = \{\varphi_e^\lambda \colon W_{t(e)} \to W_{i(e)}\}$$

be a conformal GDMS generated over the multigraph Γ with the properties that W_r is connected, $\overline{W_r} \subset W'_r$, $\varphi_e : W'_{t(e)} \to W'_{i(e)}$ and $\varphi_e : W_{t(e)} \to W_{i(e)}$. Although for our applications to meromorphic dynamics considered in this paper all the sets X_r^λ will be independent of λ, here we do not assume that the corresponding compact seed sets $X_v^\lambda \subset W_v$ are independent of λ.

Fix $\lambda_o \in \Lambda$ and for every $\omega \in E_A^\infty$, let $\Psi_\omega\colon \Lambda \to \mathbb{C}$ be given by the following formula

$$\Psi_\omega(\lambda) = \frac{(\varphi_{\omega_1}^\lambda)'(\pi_\lambda(\sigma_\omega))}{(\varphi_{\omega_1}^{\lambda_0})'(\pi_{\lambda_0}(\sigma_\omega))}$$

where $\pi_\lambda\colon E_A^\infty \to \mathcal{J}_{S_\lambda}$ is the canonical projection induced by the GDMS S_λ. The family $\{S_\lambda\}_{\lambda\in\Lambda}$ is called analytic if

(ra-a) For any $e \in E$ and every $z \in W_{t(e)}$, the function $\Lambda \ni \lambda \mapsto \varphi_e^\lambda(z) \in \mathbb{C}$, $\lambda \in \Lambda$ is holomorphic.

The analytic family $\{S_\lambda\}_{\lambda\in\Lambda}$ is called *strong* if the following conditions are satisfied.

(ra-b) The GDMS S_{λ_0} is strongly regular, and then we simply say the parameter λ_0 is strongly regular.

(ra-c) There exists a function $\kappa\colon E \to (0, +\infty)$ such that

$$\sup\{\|(\varphi_e^\lambda)'\| \exp(\kappa(e))\colon e \in E, \lambda \in \Lambda\} < +\infty.$$

(ra-d) The family of real-valued function $\Lambda \ni \lambda \mapsto \kappa(\omega_1)^{-1} \log|\Psi_\omega(\lambda)|$, $\omega \in E_A^\infty$, $\lambda \in \Lambda$, is bounded.

There are two differences of the above setting as related to Section 4 of [21]. The first one, as a matter of fact, inessential, is that we do not require in the present paper the sets X_V^λ to be independent of λ, and the second one, more important, is that condition (ra-d) involves $\log|\Psi_\omega(\lambda)|$ rather than $\log \Psi_\omega(\lambda)$. Somewhat awkwardly, a family of such maps was called in [21] *weakly regularly analytic*. Having all this said, Theorem 4.2 from [21] can be reformulated as follows.

Theorem 4.1. *If $\{S_\lambda\}_{\lambda\in\Lambda}$, a family of finitely primitive conformal GDMS, is weakly regularly analytic, then the function $\Lambda \ni \lambda \mapsto b(S_\lambda) \in \mathbb{R}$ is real–analytic on some neighborhood of every strongly regular parameter $\lambda_0 \in \Lambda$.*

In fact the assertion in here is different than in the original theorem in [21]. The point is that since we now did not assume the Open Set Condition to hold, we have replaced Hausdorff dimension by Bowen's parameter; the Open Set Condition was used in Section 4 of [21] exclusively to guarantee that Bowen's parameter is equal to the Hausdorff dimension of the limit set. With condition (ra-d) weaker then its analog in [21] as explained above, our strengthened version of Theorem 4.1 is the following.

Theorem 1.3. *If an analytic family $\{S_\lambda\}_{\lambda\in\Lambda}$ consisting of finitely primitive conformal GDMS, is strong, then the function $\Lambda \ni \lambda \mapsto b(S_\lambda) \in \mathbb{R}$ is real–analytic on some neighborhood of every strongly regular parameter $\lambda_0 \in \Lambda$.*

Proof. Fix a strongly regular parameter $\lambda_0 \in \Lambda$. We shall show that on same sufficiently small open neighborhood of λ_0, the family of functions

$$\{\lambda \mapsto (\kappa(\omega_1))^{-1} \log \Psi_\omega(\lambda)\}_{\omega\in E_A^\infty},$$

is uniformly bounded. Then the theorem follows from Theorem 4.1. So assume without loss of generality that Λ is simply connected. First, for every $\omega \in E_A^\infty$, choose an analytic branch of logarithm $\log_\omega(\Psi_\omega)\colon \Lambda \to \mathbb{C}$ such that

$$\log_\omega \Psi_\omega(\lambda_0) = 0. \tag{4.1}$$

Then set

$$\Psi_\omega^{1/\kappa(\omega_1)}(\lambda) := \exp\left(\frac{1}{\kappa(\omega_1)} \log_\omega \Psi_\omega(\lambda)\right). \tag{4.2}$$

Let $B > 0$ be the bound coming from (rad). We then have that, for all $\omega \in E_A^\infty$ and all $\lambda \in \Lambda$,

$$\begin{aligned} \left|\Psi_\omega^{1/\kappa(\omega_1)}(\lambda)\right| &= \exp\left(\mathrm{Re}\left(\frac{1}{\kappa(\omega_1)} \log_\omega \Psi_\omega(\lambda)\right)\right) \\ &= \exp\left(\frac{1}{\kappa(\omega_1)} \mathrm{Re}\left(\log_\omega \Psi_\omega(\lambda)\right)\right) \\ &= \exp\left(\frac{1}{\kappa(\omega_1)} \log|\Psi_\omega(\lambda)|\right) \le e^B \end{aligned} \tag{4.3}$$

Put

$$g_\omega := \Psi_\omega^{1/\kappa(\omega_1)}.$$

Fix an arbitrary $r > 0$ so small that $B(\lambda_0, 2r) \subseteq \Lambda$ and let $\lambda \in B(\lambda_0, r)$. Set

$$\Gamma_r = \{\gamma \in \Lambda\colon |\gamma_j - \lambda_j| = r \text{ for all } j = 1, \dots, d\}.$$

In virtue of Cauchy's formula, and of (4.3), we have

$$\left|\frac{dg_\omega}{d\lambda}(\lambda)\right| = \left|\frac{1}{(2\pi i)^d} \int_{\Gamma_r} \frac{g_\omega(\gamma)}{(\gamma_1 - \lambda_1)^2 \dots (\gamma_d - \lambda_d)^2} d\gamma_1 \dots d\gamma_d\right|$$

$$
\begin{aligned}
&\le \frac{1}{(2\pi)^d}\int_{\Gamma_r}\frac{|g_\omega(\gamma)|}{|\gamma_1-\lambda_1|^2\dots|\gamma_d-\lambda_d|^2}|d\gamma_1|\dots|d\gamma_d| \\
&= \frac{1}{(2\pi r^2)^d}\int_{\Gamma_r}|g_\omega(\gamma)||d\gamma_1|\dots|d\gamma_d| \\
&\le \frac{e^B r^d}{(2\pi r^2)^d} = \frac{e^B}{(2\pi r)^d}.
\end{aligned}
$$

Since $g_\omega(\lambda_0)=1$, we therefore get, for all $\lambda\in B(\lambda_0,r)$, that

$$|g_\omega(\lambda)-1| = |g_\omega(\lambda)-g_\omega(\lambda_0)| \le \frac{e^B}{(2\pi r)^d}|\lambda-\lambda_0|.$$

Fix now $\delta\in(0,r)$ so small that $e^B(2\pi r)^{-d}\delta<\frac{1}{4}$. Let $\log_0\colon B(1,\frac{1}{2})\to\mathbb{C}$ be an analytic branch of logarithm such that $\log_0(1)=0$. Then $\log_0\circ g_\omega\colon B(\lambda_0,\delta)\to\mathbb{C}$ is an analytic branch of logarithm of g_ω, and, by (4.2). It follows from (4.1) and the fact that $\log_0\circ g_\omega(\lambda_0)=\log_0(1)=0$, that

$$\frac{1}{\kappa(\omega_1)}\log_\omega\Psi_\omega(\lambda) = \log_0\circ g_\omega(\lambda)$$

for all $\omega\in E_A^\infty$ and all $\lambda\in B(\lambda_0,\delta)$. Then

$$\left|\frac{1}{\kappa(\omega_1)}\log_\omega\Psi_\omega(\lambda)\right| \le \sup\{|\log\circ g_0(z)|\colon z\in B(1,1/4)\}<+\infty.$$

We are done. □

Remark 4.2. With the hypotheses of Theorem 1.3, if we knew in addition that Bowen's parameters are equal to the Hausdorff dimensions of the corresponding limit sets (which due to Theorem 8.1 is guaranteed for example by the Open Set Condition), then we would automatically have a corresponding real analyticity statement for Hausdorff dimension.

We now provide a useful sufficient condition for an analytic family of GDMS to be strong.

Definition 4.3. An analytic family $\{S_\lambda\}_{\lambda\in\Lambda}$, consisting of finitely primitive conformal GDMS, is called *Hölderly stable* if

(a) there exists $\lambda_0\in\Lambda$, called the center of Λ, such that S_{λ_0} is strongly regular,
(b) there are two constant $c>0$ and $\alpha\in(0,1)$ and for every $\lambda\in\Lambda$ there exists a homeomorphism $H_\lambda\colon\mathcal{J}_{\lambda_0}\to\mathcal{J}_\lambda$ such that

$$C^{-1}|z-\omega|^{\frac{1}{\alpha}} \le |H_\lambda(z)-H_\lambda(\omega)| \le C|z-\omega|^\alpha \text{ for all } z,w\in\mathcal{J}_{\lambda_0}$$

and

$$\varphi_e^\lambda\circ H_\lambda = H_\lambda\circ\varphi_e^{\lambda_0} \text{ for all } e\in E.$$

Proposition 4.4. *If S, an analytic family consisting of finitely primitive conformal GDMS, is Hölderly stable, then S is strong and, in consequence, the associated Bowen's parameter function is real–analytic in some sufficiently small neighborhood of the center of S.*

Proof. In order to show the first part we need to prove that there exists a function $\kappa\colon E \to (o,+\infty)$ such that (ra-c) and (ra-d) holds. Indeed, it follows from condition (b) of Definition 4.3 that

$$C^{-1}\ \mathrm{diam}^{\frac{1}{\alpha}}(\varphi_e^{\lambda_0}(\mathcal{J}_{\lambda_0})) \leq\ \mathrm{diam}(\varphi_e^{\lambda}(\mathcal{J}_\lambda)) \leq C\ \mathrm{diam}^{\alpha}(\varphi_e^{\lambda_0}(\mathcal{J}_{\lambda_0}))$$

for all $e \in E$ and all $\lambda \in \Lambda$. Therefore, by uniform distortion, there exists $\hat{C} > 0$ such that

$$\hat{C}^{-1}|(\varphi_e^{\lambda_0})'(z)|^{\frac{1}{\alpha}} \leq |(\varphi_e^{\lambda})'(\omega)| \leq \hat{C}|(\varphi_e^{\lambda_0})'(z)|^{\alpha} \tag{4.4}$$

for all $e \in E$, all $z \in \mathcal{J}_{\lambda_0}$, all $\lambda \in \Lambda$ and all $\omega \in \mathcal{J}_\lambda$. It follows

$$\begin{aligned}-\log\hat{C} + (1/\alpha - 1)\log|(\varphi_e^{\lambda_0})'(z)| &\leq \log|(\varphi_e^{\lambda})'(\omega)| - \log|(\varphi_e^{\lambda_0})'(z)| \\ &\leq \log\hat{C} + (\alpha-1)\log|(\varphi_e^{\lambda_0})'(z)|,\end{aligned}$$

and then for the distortion constant $K > 1$ we have

$$\begin{aligned}&-\log\hat{C} + (1-1/\alpha)\log K + (1/\alpha - 1)\log\|(\varphi_e^{\lambda_0})'\| \leq \\ &\qquad \leq \log|(\varphi_e^{\lambda})'(\omega)| - \log|(\varphi_e^{\lambda_0})'(z)| \\ &\qquad \leq \log\hat{C} + (\alpha-1)\log K + (\alpha-1)\log\|(\varphi_e^{\lambda_0})'\|.\end{aligned}$$

Hence

$$\begin{aligned}-M &\leq \frac{(1-\frac{1}{\alpha})}{\alpha} + \frac{\log\hat{C} + (1-\frac{1}{\alpha})\log K}{-\alpha\log||(\varphi_e^{\lambda_0})'||} \\ &\leq \frac{1}{-\alpha\log||(\varphi_e^{\lambda_0})'||}\log\frac{|(\varphi_e^{\lambda})'(\omega)|}{|(\varphi_e^{\lambda_0})'(z)|} \\ &\leq \frac{(1-\alpha)}{\alpha} + \frac{(1-\alpha)\log K + \log\hat{C}}{-\alpha\log||(\varphi_e^{\lambda_0})'||} \\ &\leq M\end{aligned}$$

for some $M > 0$ large enough since, for all $e \in E$, $||(\varphi_e^{\lambda_0})'|| \leq S < 1$ and $\lim_{e\to\infty}||(\varphi_e^{\lambda_0})'|| = 0$. Since in addition, by (4.4),

$$\begin{aligned}||(\varphi_e^{\lambda})'|| &\leq \hat{C}||(\varphi_e^{\lambda_0})'||^{\alpha} \\ &= \hat{C}\exp\left(\alpha\log||(\varphi_e^{\lambda_0})'||\right) \\ &= \hat{C}\left(\exp(-(-\alpha\log||(\varphi_e^{\lambda_0})'||))\right),\end{aligned}$$

we conclude that conditions (rac) and (rad) are satisfied if we set

$$\kappa(e) = -\alpha\log||(\varphi_e^{\lambda_0})'||.$$

The first assertion of our proposition is thus proved. The second one follows now immediately from Theorem 1.3 $\square$

Remark 4.5. Similarly as in Remark 4.2, if we knew in addition to the hypotheses of Proposition 4.4, that Bowen's parameters are equal to the Hausdorff dimensions of the corresponding limit sets, which due to Theorem 8.1 is guaranteed for example by the Open

Set Condition, then we would automatically have a corresponding real analyticity statement for Hausdorff dimension.

Now we may prove the following main general result of this section already stated in the introduction.

Theorem 1.4. *Let $\Lambda \subseteq \mathbb{C}^d$ be an open set. Assume that $\{S_\lambda\}_{\lambda\in\Lambda}$ is an analytic family consisting of finitely primitive GDMS such that for some $\lambda_0 \in \Lambda$ the system S_{λ_0} is strongly regular. If there exists a holomorphic motion $H\colon \Lambda \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that*

$$\varphi_e^\lambda(H(\lambda, z)) = H(\lambda, \varphi_e^{\lambda_0}(z))$$

for all $\lambda \in \Lambda$ and all $z \in \mathcal{J}_{\lambda_0}$, then the Bowen's parameter function $\Lambda \ni \lambda \to b(S_\lambda)$ is real–analytic on some sufficiently small neighborhood of λ_0.

Proof. Fix a radius $r > 0$ such that $B(\lambda_0, r) \subseteq \Lambda$. Then, decreasing $r > 0$ if necessary, the λ-lemma (see [5], [16]) asserts that for all $\lambda \in B(\lambda_0, r)$, the maps $\hat{\mathbb{C}} \ni z \to H(\lambda, z) \in \hat{\mathbb{C}}$ are Hölder continuous with both, a common Hölder exponent and a common Hölder constant. Hence, the analytic family $\{S_\lambda\}_{\lambda\in B(\lambda_0,r)}$ is Hölderly stable. Applying Proposition 4.4 finishes then the proof. □

Remark 4.6. Similarly as in Remarks 4.2 and 4.5, if we knew in addition to the hypotheses of Theorem 1.4, that Bowen's parameters are equal to the Hausdorff dimensions of the limit sets, which due to Theorem 8.1 in Appendix is guaranteed for example by the Open Set Condition, then we would automatically have a corresponding real analyticity result for Hausdorff dimension.

For our applications to meromorphic functions, we will need Theorem 1.4 in the form as stated above, explicitly involving holomorphic motion. However, we can already now provide some mild quite general sufficient conditions for an analytic family of conformal GDMSs to admit a holomorphic motion. These conditions are frequently fairly easy to verify, though not in the context of meromorphic functions.

Definition 4.7. Let S be a finitely primitive GDMS and let

$$E_p^* = \{\omega \in E^* : \omega_1 = \omega_{|\omega|} \text{ and } \omega \neq \tau^k \text{ for any } \tau \in E \text{ and } k \geq 2\}.$$

For every $\omega \in E_p^*$ let $x_\omega \in \overline{W_{t(\omega)}}$ be the only fixed point of the map $\varphi_\omega : \overline{W}_{t(\omega)} \to \overline{W}_{t(\omega)}$. We say that the system S is *periodically separated*, if $x_\omega \neq x_\tau$ wherever $\omega, \tau \in E_p^*$ and the words are *incomparable* (that is none of them is an extension of the other).

Let us now record two obvious sufficient conditions for a GDMS to be periodically separated. The following proposition will not be used later.

Proposition 4.8. *If either*

(a) *for every* $\omega \in E_p^*$, $x_\omega \in W_{t(\omega)}$ *and* $\varphi_a(W_{t(a)}) \cap \varphi_b(W_{t(b)}) = \emptyset$ *whenever* $a, b \in E$ *with* $a \neq b$

or

(b) $\varphi_a(\overline{W_{t(a)}}) \cap \varphi_b(\overline{W_{t(b)}}) = \emptyset$ *whenever* $a, b \in E$ *with* $a \neq b$,

then S *is a periodically separated.*

Lemma 4.9. *If* $\Lambda \subseteq \mathbb{C}^d$ *is an open simply connected set and* $\{S_\lambda\}_{\lambda \in \Lambda}$ *is an analytic family consisting of finitely primitive conformal GDMS that are periodically separated, then for every* $\lambda_0 \in \Lambda$ *there exists a holomorphic motion* $H\colon \Lambda \times \overline{\mathcal{J}_{\lambda_0}} \to \hat{\mathbb{C}}$ *such that*

$$\varphi_e^\lambda(H(\lambda, z)) = H(\lambda, \varphi_e^{\lambda_0}(z)) \text{ for all } \lambda \in \Lambda \text{ and all } z \in \overline{\mathcal{J}_{\lambda_0}}.$$

In addition

$$H(\{\lambda\} \times \overline{\mathcal{J}_{\lambda_0}}) = \overline{\mathcal{J}_\lambda} \text{ and } H(\{\lambda\} \times \mathcal{J}_{\lambda_0}) = \mathcal{J}_\lambda \text{ for all } \lambda \in \Lambda.$$

Proof. Fix $\omega \in E_p^*$. Since the map

$$\Lambda \times W_{t(\omega)} \ni (\lambda, z) \mapsto \varphi_\omega^\lambda(z) \in \mathbb{C}$$

is holomorphic and since $(\varphi_\omega^\lambda)'(\xi) \neq 1$ for all $\xi \in W_{t(\omega)}$, it follows from the Implicit Function Theorem that for $r_{\lambda_0,\omega} > 0$ small enough there exists a unique holomorphic function

$$B(\lambda_0, r_{\lambda_0,\omega}) \ni \lambda \mapsto x_{\lambda_0,\omega}^\lambda \in \; W_{t(\omega)}$$

such that $\varphi_\omega^\lambda(x_{\lambda_0,\omega}^\lambda) = x_{\lambda_0,\omega}^\lambda$ and $x_{\lambda_0,\omega}^\lambda$ is (of course) the unique fixed point of the map $\varphi_\omega^\lambda\colon W_{t(\omega)} \to W_{t(\omega)}$. By this uniqueness, all the maps $x_{\lambda_0,\omega}^{(\cdot)}$ glue together to a unique holomorphic function $\Lambda \ni \lambda \mapsto x_\omega^\lambda \in W_{t(\omega)}$ such that

$$\varphi_\omega^\lambda(x_\omega^\lambda) = x_\omega^\lambda. \tag{4.5}$$

Let

$$Y_\lambda := \{x_\omega^\lambda\colon \omega \in E_p^*\}.$$

Since all the systems $\{S_\lambda\}_{\lambda\in\Lambda}$ are periodically separated, there exists a bijection $P_\lambda\colon Y_\lambda \to E_p^*$ sending each point $x \in Y_\lambda$ to the unique $\omega \in E_p^*$ such that $x_\omega^\lambda = P_\lambda(x)$ and for every fixed $\lambda \in \Lambda$, the map $Y_{\lambda_0} \ni z \mapsto W_{P_{\lambda_0}(z)}^\lambda \in Y_\lambda$ is bijective. Thus the map $H\colon \Lambda \times Y_{\lambda_0} \to \mathbb{C}$ given by the formula

$$H(\lambda, z) = x_{P_{\lambda_0}(z)}^\lambda \in \mathbb{C}$$

is a holomorphic motion, and by the λ-lemma it uniquely extends to a holomorphic motion $H\colon \Lambda \times \overline{Y_{\lambda_0}} \to \mathbb{C}$. But, by finite primitivity, $\overline{Y_\lambda} = \overline{\mathcal{J}_\lambda}$. By (4.5) on the other hand only by continuity of the map H, we have that

$$\varphi_\omega^\lambda(H(\lambda, z)) = H(\lambda, \varphi_\omega^{\lambda_0}(z)) \tag{4.6}$$

for all $\lambda \in \Lambda$ and all $z \in \overline{\mathcal{J}_{\lambda_0}}$. Also, for all $\lambda \in \Lambda$, we have

$$H(\{\lambda\} \times \overline{\mathcal{J}_{\lambda_0}}) = \overline{H(\{\lambda\} \times Y_{\lambda_0})} = \overline{Y_\lambda} = \overline{\mathcal{J}_\lambda}.$$

Finally, it follows from (4.6) that $\pi_\lambda(\omega) = H(\lambda, \pi_{\lambda_0}(\omega))$, and therefore

$$\mathcal{J}_\lambda = \pi_\lambda(E_A^\infty) = H(\{\lambda\} \times \pi_{\lambda_0}(E_A^\infty)) = H(\{\lambda\} \times \mathcal{J}_{\lambda_0}).$$

We are done. ∎

Theorem 4.10. *Let $\Lambda \subseteq \mathbb{C}$ be an open simply connected set whose complement $\mathbb{C}\backslash\Lambda$ contains at least two points. If $\{S_\lambda\}_{\lambda\in\Lambda}$ is an analytic family consisting of finitely primitive conformal GDMS that are periodically separated, then for every $\lambda_0 \in \Lambda$ there exists a holomorphic motion $H\colon \Lambda \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that*

$$\varphi_e^\lambda(H(\lambda, z)) = H(\lambda, \varphi_e^{\lambda_0}(z)) \text{ for all } \lambda \in \Lambda \text{ and all } z \in \overline{\mathcal{J}_{\lambda_0}}.$$

In addition,

$$H(\{\lambda\} \times \mathcal{J}_{\lambda_0}) = \mathcal{J}_\lambda \text{ for all } \lambda \in \Lambda,$$

and if the system S_{λ_0} is strongly regular, then the Bowen's parameter function $\lambda \mapsto b(S_\lambda)$ is real–analytic on some sufficiently small neighborhood of λ_0.

Proof. In virtue of Lemma 4.9 there exists a holomorphic motion on $\lambda \times \overline{\mathcal{J}_{\lambda_0}}$ satisfying the required properties. By Slodkowski's Theorem [16] it can be extended to a holomorphic motion on $\Lambda \times \hat{\mathbb{C}}$ with uniformly bounded dilatation. The last assertion of the theorem follows then immediately from Proposition 1.4. □

Remark 4.11. Similarly as in the three preceding remarks, if we knew in addition to the hypotheses of Theorem 4.10 that Bowen's parameters are equal to the Hausdorff dimension of the corresponding limit sets, which due to Theorem 8.1 is guaranteed for example by the Open Set Condition, then we would automatically have a corresponding real analyticity result for Hausdorff dimension.

5. Real Analyticity of Hausdorff Dimension for Meromorphic Functions

Recall that a meromorphic function $f\colon \mathbb{C} \to \hat{\mathbb{C}}$ belongs to Speiser class $\mathcal{S}$ if the set $\text{sing}(f^{-1})$ of all singularities of f^{-1} is finite. Let $\Lambda \subseteq \mathbb{C}^d$ be an open set. We say that a family $\{f_\lambda\}_{\lambda\in\Lambda}$ of meromorphic functions from $\mathbb{C}$ to $\hat{\mathbb{C}}$ is analytic if

(a) The function $f_{\lambda_0} : \mathbb{C} \to \hat{\mathbb{C}}$ belongs to Speiser class $\mathcal{S}$.
(b) The function $\Lambda \ni \lambda \mapsto \text{sing}(f_\lambda^{-1})$ is continuous.
(c) Each point of $\text{sing}(f_{\lambda_0}^{-1}) \setminus \mathcal{J}(f_{\lambda_0})$ belongs to the attraction basin of some attracting periodic orbit of f_{λ_0}.
(d) The function $\Lambda \ni \lambda \mapsto f_\lambda(z) \in \hat{\mathbb{C}}$ is meromorphic for all $z \in \mathbb{C}$.

The main result of this section is the following theorem already stated in the introduction.

Theorem 1.1. *Assume that a tame meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ is strongly N–regular. Let $\Lambda \subset \mathbb{C}^d$ be an open set and let $\{f_\lambda\}_{\lambda\in\Lambda}$ be an analytic family of meromorphic functions with the following properties:*

1. *$f_{\lambda_0} = f$ for some $\lambda_0 \in \Lambda$,*
2. *there exists a holomorphic motion $H : \Lambda \times \overline{\mathcal{J}_{\lambda_0}} \to \mathbb{C}$ such that each map H_λ is a topological conjugacy between f_{λ_0} and f_λ on $\mathcal{J}_{\lambda_0}$.*

Then the map

$$\Lambda \ni \lambda \mapsto \mathrm{HD}(\mathcal{J}_\lambda)$$

is real–analytic on some neighborhood of λ_0*.*

Proof. The idea of the proof of this theorem is to associate, by means of nice sets, to the analytic family $f_\lambda : \mathbb{C} \to \hat{\mathbb{C}}$ of meromorphic maps a strong analytic family of conformal iterated function systems (the simplest subclass of conformal Graph Directed Markov Systems) that have the same Hausdorff dimensions of their limit sets as the Hausdorff dimensions of the radial Julia sets of the corresponding maps f_λ. Having this, we can use the real analyticity results of Section 4 to conclude the proof.

Since the function f_{λ_0} is tame, it has at least one nice set U. Let $S_{\lambda_0} = \{\varphi_e^{\lambda_0}\}_{e\in E}$ be iterated function system generated by the nice set U. We can require that

$$B(\xi, R) \subseteq U \subseteq B(\xi, 2R) \tag{5.1}$$

with some non-periodic point $\xi \in \mathcal{J}_{\lambda_0}$. Because of analyticity of our family $f_\lambda : \mathbb{C} \to \hat{\mathbb{C}}$, $\lambda \in \Lambda$, (this takes care of singular points of f_λ^{-1} lying in the Fatou set of f_λ) and because of topological conjugacy guaranteed by (2) (this takes care of singular points of f_λ^{-1} lying in the Julia set of f_λ), we may further require that

$$B(\xi, 12R) \cap \bigcup_{n=0}^{\infty} f_\lambda^n\left(\mathrm{sing}(f_\lambda^{-1})\right) = \emptyset$$

and

$$|(f_\lambda^k)'(z)| \geq 6K \text{ whenever } z \in B(\xi, 6R) \text{ and } f_\lambda^k(z) = \xi \tag{5.2}$$

for all $\lambda \in \Gamma^2_{\lambda_0}$, where $\Gamma^2_{\lambda_0} \subseteq \Lambda$ is a sufficiently small open neighborhood of $\lambda_0 \in \Lambda$. The number $K \geq 1$ is here the Koebe's constant corresponding to the scale $1/2$. Now for every $\lambda \in \Gamma^2_{\lambda_0}$ form an iterated function system S_λ acting on $\overline{B}(\xi, 6R)$ as follows. If $e \in E$, let φ_e^λ be the unique holomorphic inverse branch of $f^{\|e\|}$ defined on $B(\xi, 6R)$ and sending ξ to $H_\lambda(\varphi_e^{\lambda_0}(\xi))$. We shall prove the following

Claim 1. *For any* $\lambda \in \Gamma^2_{\lambda_0}$ *sufficiently close to* λ_0*,* $S_\lambda = \{\varphi_e^\lambda\}_{e\in E}$ *is a strongly regular conformal iterated function system on* $B(\xi, 6R)$*.*

Proof. Conformality of the maps φ_e^λ, $e \in E$, is immediate from their definitions. The distortion properties follows immediately from Koebe's Distortion Theorems and the fact that all maps φ_e^λ have unique univalent holomorphic extensions to $B(\xi, 12R)$. In order to complete the proof that S_λ is a conformal IFS it thus suffices to show that

$$\varphi_e^\lambda(B(\xi, 5R)) \subseteq B(\xi, 5R).$$

Indeed, since for any $e \in E$, $\varphi_e^{\lambda_0}(U) \subseteq U$, we get that $\varphi_e^{\lambda_0}(\mathcal{J}(S_{\lambda_0})) \cap U \neq \emptyset$, and therefore (see also (5.1)) $\varphi_e^\lambda(H_\lambda(\mathcal{J}(S_{\lambda_0})) \cap B(\xi, 3R) \neq \emptyset$ for all λ sufficiently close to λ_0 (independently of e), say $\lambda \in B(\lambda_0, \delta_1) \subseteq \Gamma^2_{\lambda_0}$. Since $\|(\varphi_e^\lambda)'\| \leq \frac{1}{6}$ (see (5.2)), using the triangle inequality, we conclude that

$$\begin{aligned}
\varphi_e^\lambda\left(B(\xi, 6R)\right) &\subseteq B\left(\xi, 3R + 12R\|(\varphi_e^\lambda)'\|\right) \\
&\subseteq B(\xi, 3R + 2R) \\
&= B(\xi, 5R).
\end{aligned}$$

We are left to show that all the systems S_λ are strongly regular for all λ sufficiently close to λ_0. Indeed, since the system S_λ and S_{λ_0} are quasi-conformally conjugated on $\overline{\mathcal{J}(S_{\lambda_0})}$ and $\overline{\mathcal{J}(S_\lambda)}$ respectively, we get that

$$C^{-1}||(\varphi_\omega^{\lambda_0})'||^{\frac{1}{\alpha_\lambda}} \le ||(\varphi_\omega^{\lambda})'|| \le C||(\varphi_\omega^{\lambda_0})'||^{\alpha_\lambda} \tag{5.3}$$

for all $\omega \in E^*$ and some C and numbers $\alpha_\lambda \in (0,1)$ such that

$$\lim_{\lambda\to\lambda_0} \alpha_\lambda = 1. \tag{5.4}$$

For every $\lambda \in B(\lambda_0, \delta_1)$ put

$$b_\lambda = b(S_\lambda).$$

Since the function $f_{\lambda_0} : \mathbb{C} \to \hat{\mathbb{C}}$ is strongly N–regular, the system S_{λ_0} is strongly regular, and so, there exist $t_0 < b_{\lambda_0}$ and $0 < \kappa < 1$ such that

$$0 < P_{\lambda_0}(t) < +\infty \tag{5.5}$$

for all $t \in (t_0 - \kappa(b_{\lambda_0} - t_0), t_0 + \kappa(b_{\lambda_0} - t_0))$. In view of (5.4) there exists $\delta_2 \in (0, \delta_1)$ such that

$$\alpha_\delta t_0, \alpha_\delta^{-1} t_0 \in (t_0 - \kappa(b_{\lambda_0} - t_0), t_0 + \kappa(b_{\lambda_0} - t_0))$$

for all $\lambda \in B(\lambda_0, \delta_2)$. Formulas (5.3) and (5.5) imply then that

$$P_\lambda(t) \le P_{\lambda_0}(\alpha_\lambda t_0) < +\infty$$

and

$$P_\lambda(t) \ge P_{\lambda_0}(\alpha_\lambda^{-1} t) > 0$$

for all $\lambda \in B(\lambda_0, \delta_2)$. Thus, all the systems S_λ, $\lambda \in B(\lambda_0, \delta_2)$, are strongly regular, and the proof of Claim 1 is complete. $\square$

Claim 2. $\mathrm{HD}(\mathcal{J}_\lambda) = b_\lambda$ *for all* $\lambda \in B(\lambda_0, \delta_2)$.

Proof. By the very definition of the nice sets all the sets $\varphi_e^{\lambda_0}(\overline{U})$, $e \in E$, are mutually disjoint, and therefore, so are the sets $\{\varphi_e^{\lambda_0}(\overline{\mathcal{J}(S_{\lambda_0})}\}_{e\in E}$. So, because of topological conjugacy, the sets $\{\varphi_e^{\lambda}(\overline{\mathcal{J}(S_{\lambda})}\}_{e\in E}$ are mutually disjoint for every fixed $\lambda \in B(\lambda_0, \delta_2)$. This means that the global map $F_\lambda\colon \bigcup_{e\in E} \varphi_e^\lambda(\overline{\mathcal{J}(S_\lambda)}) \to \overline{\mathcal{J}(S_\lambda)}$ is well-defined if given by the formula

$$F_\lambda(\varphi_e^\lambda(z)) = z \text{ where } z \in \overline{\mathcal{J}_\lambda}.$$

It is straightforward to see that all transformations F_λ, $\lambda \in B(\lambda_0, \delta_2)$, are Walter expanding conformal maps in the sense of [4]. Therefore, Theorem 2.7 in [4] yields $\mathrm{HD}(\mathcal{J}(S_\lambda)) = b_\lambda$ for all $\lambda \in B(\lambda_0, \delta_2)$. The proof of Claim 2 is complete. $\square$

Claim 3. *There exits* $\delta_3 \in (0, \delta_2]$ *such that, for every* $\lambda \in B(\lambda_0, \delta_3) \subset \Lambda$, *we have that*

$$\mathrm{HD}(\mathcal{J}(S_\lambda)) = \mathrm{HD}(\mathcal{J}_r(f_\lambda)).$$

Proof. Clearly $\mathcal{J}(S_\lambda) \subseteq \mathcal{J}_r(f_\lambda)$. Hence,

$$\mathrm{HD}(\mathcal{J}(S_\lambda)) \le \mathrm{HD}\left(\mathcal{J}_r(f_\lambda)\right). \tag{5.6}$$

In order to prove the opposite inequality take $\lambda \in B(\lambda_0, \delta_2)$ and consider a nice set $U_\lambda \subseteq B(\xi, R)$ for the tame meromorphic map f_λ. If ψ_e^λ is a member of the iterated functions system S'_λ induced by the nice set U_λ, then $\psi_e^\lambda(U_\lambda) \subseteq U_\lambda$. So, if λ is taken sufficiently close to λ_0 (independently of e), say $\lambda \in B(\lambda_0, \delta_3)$, with $0 < \delta_3 \le \delta_2$, then

$$H_\lambda^{-1} \circ \psi_e^\lambda (\mathcal{J}_r(f_\lambda) \cap U_\lambda) \subseteq B(\xi, 2R).$$

Thus the map $\psi_e^{\lambda_0} \colon U \to \mathbb{C}$, the unique holomorphic inverse branch of $f_{\lambda_0}^{||e||}$, determined by the condition that $\psi_e^{\lambda_0}(\xi) = H_\lambda^{-1} \circ \psi_e^\lambda(\xi)$ is a member of S_{λ_0}. But then $\psi_e^\lambda = \varphi_e^\lambda$, where $\varphi_e^\lambda \in S_\lambda$. Consequently, the limit set $\mathcal{J}(S'_\lambda)$ of S'_λ, $\lambda \in B(\lambda_0, \delta_3)$, is contained in $\mathcal{J}(S_\lambda)$. Hence $\mathrm{HD}(\mathcal{J}(S'_\lambda)) \le \mathrm{HD}(\mathcal{J}(S_\lambda))$, and, in virtue of Theorem 3.4, $\mathrm{HD}(\mathcal{J}_r(f_\lambda)) \le \mathrm{HD}(\mathcal{J}(S_\lambda))$. Along with (5.6), this finishes the proof of Claim 3. □

Conclusion of the proof of Theorem 1.1 This proof is now straightforward. Since the family $(f_\lambda)_{\lambda \in \Lambda}$ is analytic, so is the family $\{S_\lambda\}_{\lambda \in B(\lambda_0, \delta_3)}$. By the very definition of the systems S_λ, the map $H|_{B(\lambda_0,\delta_3) \times \mathcal{J}(S_{\lambda_0})}$ forms a holomorphic motion such that

$$H_\lambda(\mathcal{J}(S_{\lambda_0})) = \mathcal{J}(S_\lambda)$$

and

$$\varphi_e^\lambda(H(\lambda, z)) = H(\lambda, \varphi_e^{\lambda_0}(z))$$

for all $\lambda \in B(\lambda_0, \delta)$ and all $z \in \mathcal{J}(S_{\lambda_0})$. By Slodkowski's Theorem ([16]) this holomorphic motion extends to a holomorphic motion of the entire extended complex plane $\hat{\mathbb{C}}$. Thus Proposition 1.4 with the help of the Claims 1, 2 and 3, complete the proof of our theorem. □

6. Strong N–Regularity of Dynamically Regular Meromorphic Functions

In this section we deal with *dynamically regular functions* as defined in [9]. Our goal is to show first that they are all strongly N–regular and then, in the next section, to prove real analyticity of Hausdorff dimension of radial Julia sets of analytic families consisting of dynamically regular meromorphic functions. We refer the reader to [9] for the definition and specific facts about *dynamically regular functions*. In what follows we use the notation of that article.

Let $f \colon \mathbb{C} \to \overline{C}$ be a dynamically regular meromorphic function. Let $|d\tau(z)|$ be the Riemannian metric defined in section 5.1 of [9]. Remember that metric $|d\tau|$ is conformally equivalent to the standard Euclidean metric $|dz|$. It was proved in [9] that the limit

$$P(t) = \lim_{n \to \infty} \frac{1}{n} \log \mathcal{L}_t^n \mathbf{1}(z)$$

exists for any $z \in \mathcal{J}(f)$, where $\mathcal{L}_t \colon C_b(\mathcal{J}(f)) \to C_b(\mathcal{J}(f))$ is the bounded linear operator defined by the formula

$$\mathcal{L}_t g(z) = \sum_{w \in f^{-1}(z)} g(w) |f'(w)|_\tau^{-t}.$$

It is referred to as *the Perron-Frobenius operator associated to the parameter t*. The number $P(t)$ is called *the topological pressure at t*. It was proved in [9] that there is a certain number $c > 0$, that if $t > c$, then

$$\|\mathcal{L}_t \mathbf{1}\|_\infty < \infty \text{ and therefore } P(t) < \infty.$$

For every open set $U \subseteq \mathbb{C}$, we define

$$K(U) = \bigcap_{n=0}^{\infty} f^{-n}(U^c \cap \mathcal{J}(f)) = \{z \in \mathcal{J}(f)\colon f^n(z) \notin U \text{ for all } n \geq 0\},$$

and

$$K_r(U)\colon = \bigcap_{n=0}^{\infty} f^{-n}(U^c \cap \mathcal{J}_r(f)) = \{z \in \mathcal{J}_r(f)\colon f^n(z) \notin U \text{ for all } n \geq 0\}.$$

Of course $K(U)$ is a closed subset of $\mathcal{J}(f)$ and $K_r(U)$ is a closed subset of $\mathcal{J}_r(f)$. Both $K(U)$ and $K_r(U)$ are forward invariant in the sense that

$$f(K(U)) \subseteq K(U) \text{ and } f(K_r(U)) \subseteq K_r(U).$$

Put

$$U_n^c = \bigcap_{j=0}^{n} f^{-j}(U^c),$$

and define

$$\overline{P}_U^c(t) = \sup\{\overline{P}_U^c(z,t)\colon z \in \mathcal{J}(f) \cap U^c\},$$

where

$$\overline{P}_U^c(z,t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in U_n^c \cap f^{-n}(z)} |(f^n)'(\omega)|_\tau^{-t} = \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{L}_t^n \mathbf{1}_{U_n^c}(z).$$

We shall prove the following.

Lemma 6.1. *If U is an open subset of $\mathcal{J}(f)$ and $\overline{P}_U^c(t) < 0$, then* $\mathrm{HD}(K_r(U)) \leq t$.

Proof. For every $k \geq 0$ set

$$K_r^k(U) = \{z \in K_r(U)\colon \limsup_{n\to\infty} |f^n(z)| < k\}.$$

From topological hyperbolicity of f, guaranteed by its dynamical regularity, there exists $\delta > 0$ such that each open ball $B(z, 2\delta)$, $z \in \mathcal{J}(f)$, is disjoint from the forward orbit of the singular set of f^{-1}. Cover the ball $\mathcal{J} \cap \overline{B}(0,k)$ with finitely many balls $\{B(x_j, \delta)\}_{x\in E}$, where $E \subset \mathcal{J}(f) \cap U^c$. Fix arbitrary $\eta > 0$. By hyperbolicity of f and the definition of $\overline{P}_U^c(t)$, there exists an integer $l(\eta) \geq 0$ such that, for all $n \geq l(\eta)$, all $x \in E$, and all $w \in f^{-n}(x)$, we have that

$$|(f^n)'(w)|_\tau \geq 2K\eta^{-1}\delta$$

and

$$\sum_{y \in f^{-n}(x)} |(f^n)'(y)|_\tau^{-t} \leq \exp\big(P(t)/2\big).$$

But the family

$$\Big\{f_w^{-n}\big(B(x,\delta)\big)\colon n \geq l(\eta),\, x \in E,\, w \in f^{-n}(x)\Big\}$$

covers $K_r^k(U)$ and

$$\operatorname{diam}\left(f_w^{-n}\left(B(x,\delta)\right)\right) \leq K2\delta|(f^n)'(w)|_\tau^{-1} \leq \eta$$

for all n, x and w as above. Also

$$\begin{aligned}\sum_{n=l(\eta)}^{\infty}\sum_{x\in E}\sum_{w\in f^{-n}(x)}\operatorname{diam}_{\tau}^{t}\left(f_w^{-n}(B(x,\delta))\right) \le \\ &\le \sum_{n=l(\eta)}^{\infty}\sum_{x\in E}\sum_{w\in f^{-n}(x)}(2K\delta)^t|(f^n)'(w)_\tau|^{-t} \\ &\le (2K\delta)^t\sum_{n=l(\eta)}^{\infty}\sum_{e\in E}\exp\left(\frac{1}{2}P(t)n\right) \\ &= (2K\delta)^t\#E\frac{\exp(P(t)l(\eta)/2)}{1-\exp\left(P(t)/2\right)}.\end{aligned}$$

Since $\lim_{\eta\to 0} l(\eta)=+\infty$, we thus conclude from this that the Hausdorff measure $\mathcal{H}_t\left(K_r^k(U)\right)=0$. Thus, the formula

$$K_r(U)=\bigcup_{k=0}^{\infty}K_r^k(U),$$

yields $\mathcal{H}_t\left(K_r(U)\right)=0$. Hence $\operatorname{HD}\left(K_r(U)\right)\le t$ and the proof is finished. □

We now need the following standard auxiliary fact.

Lemma 6.2. *Let $F\subseteq\mathbb{C}$ be a closed set, let $R\in(0,\infty)$, and let K be a closed subset of $B(0,R)\backslash F$. Then, there exists a smooth C^∞ function $g\colon\mathbb{C}\to[0,+\infty)$ with the following tow properties:*

(a) $\mathbf{1}_F\le g\le\mathbf{1}$ *and*
(b) $g|_K\equiv 0$.

The main technical result of this section is the following.

Proposition 6.3. *If $f\colon\mathbb{C}\to\hat{\mathbb{C}}$ is a dynamically regular function, $t>\frac{\varrho}{\alpha_1+\tau}$ and U is an arbitrary open subset of $\mathbb{C}$ intersecting the Julia set $\mathcal{J}(f)$, then $\bar{P}_U^c(t)<P(t)$.*

Proof. Since $\lim_{n\to\infty} m_t(U_n^c)=0$, there exists $q\ge 1$ so large that

$$m_t(U_q^c)\le\frac{1}{5}||\varrho_t||_\infty^{-1},\ \text{where } \varrho_t=\frac{d\mu_t}{dm_t}.$$

Let $R>0$ be so large that $m_t\left(B^c(0,R)\right)<\frac{1}{8}m_t(U_q^c)$. Let K be a compact subset of $B(0,R)\backslash U_q^c$ such that

$$m_t\left(B(0,R)\backslash\left(U_q^c\cup K\right)\right)<\frac{1}{8}m_t(U_q^c).$$

Finally, let g be the function associated to the triple R, K, and $F = U_q^c$, according to Lemma 6.2. Then,

$$\begin{aligned}
\int_{\mathcal{J}(f)} g dm_t &\le \\
&\le \int_{B^c(0,R)} g dm_t + \int_{U_q^c \cap B(0,R)} g dm_t + \int_K g dm_t + \int_{B(0,R)\setminus(K\cup U_q^c)} g dm_t \\
&= \int_{B^c(0,R)} g dm_t + \int_{U_q^c \cap B(0,R)} g dm_t + \int_{B(0,R)\setminus(K\cup U_q^c)} g dm_t \\
&\le \frac{1}{8} m_t(U_q^c) + m_t(U_q^c) + \frac{1}{8} m_t(U_q^c) \\
&= \frac{5}{4} m_t(U_q^c) \\
&\le \frac{1}{4} ||\varrho_t||_\infty^{-1}.
\end{aligned}$$

Now, since the function g is bounded and Hölder continuous, it follows from Theorem 6.5 in [9] that there exists $s \ge q$ such that

$$\|\mathcal{L}_t^s g - \varrho_t \int g dm_t\|_\infty \le \frac{1}{4}.$$

Consequently

$$\|\mathcal{L}_t^s \mathbf{1}_{U_s^c}\|_\infty \le \|\mathcal{L}_t^s \mathbf{1}_{U_q^c}\|_\infty \le ||\mathcal{L}_t^s g||_\infty \le ||\varrho_t||_\infty \int g dm_t + \frac{1}{4} \le \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Hence, for any $n \ge 0$ and any $z \in \mathcal{J}(f)$, we get that

$$\begin{aligned}
& e^{-P(t)(n+1)s} \mathcal{L}_t^{(n+1)s} \mathbf{1}_{U^c_{(n+1)s}}(z) = \\
&= \exp(-P(t)(n+1)s) \mathcal{L}_t^{ns}\big(\mathcal{L}_t^s \mathbf{1}_{U^c_{(n+1)s}}(z)\big) \\
&= \exp(-P(t)(n+1)s) \sum_{w\in f^{-ns}(z)} |(f^{ns})'(w)|_\tau^{-t} \mathcal{L}_t^s \mathbf{1}_{U^c_{(n+1)s}}(w) \\
&= \exp(-P(t)(n+1)s) \sum_{w\in f^{-ns}(z)} \Big(|(f^{ns})'(w)|_\tau^{-t} \cdot \sum_{x\in f^{-s}(w)\cap U^c_{(n+1)s}} |(f^s)'(x)|_\tau^{-t}\Big) \\
&= \exp(-P(t)(n+1)s) \sum_{w\in f^{-ns}(z)\cap U^c_{ns}} \Big(|(f^{ns})'(w)|_\tau^{-t} \sum_{x\in f^{-s}(w)\cap U^c_s} |(f^s)'(x)|_\tau^{-t}\Big) \\
&= e^{-P(t)ns} \sum_{w\in f^{-ns}(z)} |(f^{ns})'(w)|_\tau^{-t} \mathbf{1}_{U^c_{ns}}(w) \Big(e^{-P(t)s} \sum_{x\in f^{-s}(w)} |(f^s)'(x)|_\tau^{-t} \mathbf{1}_{U^c_s}(x)\Big) \\
&\le e^{-P(t)ns} \sum_{w\in f^{-ns}(z)} \Big(|(f^{ns})'(w)|_\tau^{-t} \mathbf{1}_{U^c_{ns}}(w) \| e^{-P(t)s} \mathcal{L}_t^s \mathbf{1}_{U^c_s}\|_\infty\Big) \\
&\le ||e^{-P(t)ns} \mathcal{L}_t^{ns} \mathbf{1}_{U^s_n}||_\infty ||e^{-P(t)s} \mathcal{L}_t \mathbf{1}_{U^c_s}||_\infty \\
&\le \frac{1}{2} ||e^{-P(t)ns} \mathcal{L}_t^{ns} \mathbf{1}_{U^c_n}||_\infty.
\end{aligned}$$

Therefore,

$$||\exp(-P(t)(n+1)s)\mathcal{L}_t^{(n+1)s}\mathbf{1}_{U^c_{(n+1)s}}||_\infty \le \frac{1}{2}||e^{-P(t)ns}\mathcal{L}_t^{ns}\mathbf{1}_{U_n^s}||_\infty.$$

So, by induction,

$$||e^{-P(t)ns}\mathcal{L}_t^{ns}\mathbf{1}_{U^c_{ns}}||_\infty \le 2^{-n} \tag{6.1}$$

for all $n \ge 0$. Now, for any integer $k \ge 0$, write $k = ns + r$, $0 \le r \le s-1$. Formula (6.1) implies then that

$$\begin{aligned}
||\mathcal{L}_t^k\mathbf{1}_{U_k^c}||_\infty &\le ||\mathcal{L}_t^k\mathbf{1}_{U^c_{ns}}||_\infty \le ||\mathcal{L}_t^r||_\infty||\mathcal{L}_t^{ns}\mathbf{1}_{U^c_{ns}}||_\infty \\
&\le Q_s 2^{-n}e^{P(t)ns} \\
&\le Q_s 2^{-\frac{k-r}{s}}e^{P(t)r}e^{-P(t)k} \\
&= Q_s 2^{\frac{r}{s}}e^{-P(t)r}2^{-\frac{k}{s}}e^{P(t)k} \\
&\le M_s 2^{-\frac{k}{s}}e^{-P(t)k},
\end{aligned}$$

where $M_s = 2^{\frac{s-1}{s}}Q_s\max\{e^{-P(t)r} \colon 0 \le r \le s-1\}$. Thus

$$\overline{P}_U^c \le \overline{\lim}_{n\to\infty}\frac{1}{k}\log||\mathcal{L}_t^k\mathbf{1}_{U_k^c}||_\infty \le P(t) - \frac{1}{s}\log 2 < P(t).$$

We are done. □

Corollary 6.4. *If $f\colon \mathbb{C} \to \hat{\mathbb{C}}$ is a dynamically regular function and U is an arbitrary open subset of $\mathbb{C}$ intersecting the Julia set $\mathcal{J}(t)$, then* $\mathrm{HD}(K_r(U)) < \mathrm{HD}(\mathcal{J}_r)$.

Proof. We know that the topological pressure $P(t)$ is finite for all $t > \frac{\varrho}{\alpha_1+\tau}$. We also know (see theorem 8.3 in [9]) that $P(\mathrm{HD}(\mathcal{J}_r)) = 0$. Since in addition the function $t \to \underline{P}_U^c(t) \le P(t)$ is continuous (as convex) throughout $\left(\frac{\varrho}{\alpha_1+\tau}, +\infty\right)$, we therefore conclude from Proposition 6.3 that there exists $t \in \left(\frac{\varrho}{\alpha_1+\tau}, \mathrm{HD}(\mathcal{J}_r)\right)$ such that $\underline{P}_U^c(t) < 0$. Lemma 6.1 then yields $\mathrm{HD}(K_r(U)) \le t < \mathrm{HD}(\mathcal{J}_r)$. □

Corollary 6.5. *If $f\colon \mathbb{C} \to \hat{\mathbb{C}}$ is a dynamically regular meromorphic function of divergence type, then each nice set $U \in \mathcal{U}$ gives rise to a strongly regular IFS. In particular, the function $f\colon \mathbb{C} \to \hat{\mathbb{C}}$ is strongly N–regular.*

Proof. Let $S_U = \{\varphi_e\}_{e\in E}$ be the conformal IFS generated by the nice set U. Fix one $b \in E$ and let $S_{U,b} = \{\varphi_e\}_{e\in E\setminus\{b\}}$. Then $\mathrm{HD}(\mathcal{J}_{S_{U,b}}) \le \mathrm{HD}(K_r(\varphi_b(U))) < \mathrm{HD}(\mathcal{J})) = \mathrm{HD}(\mathcal{J}_{S_U})$, where the inequality sign " $<$ " follows from Corollary 6.4 and the equality sign " $=$ " comes from Theorem 3.4. The system S_U is thus strongly regular because of Theorem 4.3.10 from [7]. □

7. Real Analyticity of Hausdorff Dimension for Dynamically Regular Meromorphic Functions

Taking fruits of the previous section, in the present one, fairly short, we provide concrete examples of analytic families of meromorphic functions that satisfy the hypotheses of Theorem 1.1. As an ultimate consequence, the Hausdorff dimension of their radial Julia sets varies in a real–analytic fashion. The following theorem already stated in the introduction.

Theorem 1.2. *Suppose that $f\colon \mathbb{C} \to \hat{\mathbb{C}}$ is a dynamically regular meromorphic function of divergence type which belongs to class $\mathcal{S}$. If $\Lambda \subseteq \mathbb{C}$ is an open set, $\{f_\lambda\}_{\lambda\in\Lambda}$ is an analytic family (in the sense of Section 5) of meromorphic functions, and $f_{\lambda_0} = f$ for some $\lambda_0 \in \Lambda$, then the function $\Lambda \ni \lambda \mapsto \mathrm{HD}(\mathcal{J}_r(f_\lambda))$ is real–analytic in some open neighborhood of λ_0 contained in Λ.*

Proof. Since our family is analytic, for every $a_{\lambda_0} \in \mathrm{sing}(f_{\lambda_0}^{-1})$ there exists a meromorphic function $\lambda \mapsto a_\lambda \in \mathrm{sing}(f_\lambda^{-1})$ defined on some sufficiently small neighborhood of λ_0. Furthermore, the analyticity of the family $\{f_\lambda\}_{\lambda\in\Lambda}$ applied again entails the functions $\{\lambda \mapsto f_\lambda^n(a_\lambda)\}_{n=0}^\infty$ to form a normal family on some sufficiently small neighborhood of λ_0 for every point a_{λ_0} in $\mathrm{sing}(f_{\lambda_0}^{-1})$. Therefore, see Lemma 9.3 in [9], there exists a holomorphic motion $H\colon \Gamma_{\lambda_0} \times \mathcal{J}_{\lambda_0} \to \hat{\mathbb{C}}$ over some neighborhood $\Gamma_{\lambda_0} \subseteq \Lambda$ of λ_0, such that $H_\lambda(\mathcal{J}_{\lambda_0}) = \mathcal{J}_\lambda$ and $H_\lambda \circ f_{\lambda_0} = f_{\lambda_0} \circ H_\lambda$ on $\mathcal{J}_{\lambda_0}$ for all $\lambda \in \Gamma_{\lambda_0}$. Since also the meromorphic function $f_{\lambda_0} : \mathbb{C} \to \hat{\mathbb{C}}$, as dynamically regular is tame, and since, by Corollary 6.5, it is strongly N–regular, invoking Theorem 1.1 completes the proof. □

Notice that, unlike in [9], we did not have to assume in this theorem anything technical like that our family $\{f_\lambda\}_{\lambda\in\Lambda}$ is of uniformly balanced growth or of bounded deformation.

8. Appendix; Conformal Graph Directed Markov Systems

Graph directed Markov systems are based on a directed multigraph (V, I, i, t) and an associated incidence matrix $A : I \times I \to \{0, 1\}$. The multigraph consists of a finite set V of vertexes, a countable (finite or infinite) set of edges, frequently called an alphabet, and two functions $i, t : I \to V$ that indicate for each directed edge $i \in I$ its initial vertex $i(e)$ and its terminal vertex $t(e)$, respectively. The matrix A is an edge incidence matrix and thus tells which edges may follow a given edge. Moreover, it respects the multigraph, that is, if $A_{ab} = 1$ then $t(a) = i(b)$. It is thereafter natural to define the set of all one-sided infinite A-admissible words

$$I_A^\infty := \{\omega \in I^\infty \,|\, A_{\omega_i\omega_{i+1}} = 1, \forall\, i \in \mathbb{N}\}.$$

The set of all subwords of I_A^∞ of length $n \in \mathbb{N}$ will be denoted by I_A^n, whereas the set of all finite subwords will be denoted by $I_A^* = \cup_{n\in\mathbb{N}} I_A^n$. A graph directed Markov system(GDMS) S consists of a directed multigraph (V, I, i, t) and an edge incidence matrix A, together with a set of non-empty compact subsets $\{X_v\}_{v\in V}$ of a Euclidean space $\mathbb{R}^d$, a number $0 < s < 1$, and for every $a \in I$, a one-to-one contraction $\phi_a : X_{t(a)} \to X_{i(a)}$ with Lipschitz constant not exceeding s. A GDMS is called *iterated function system* (IFS) provided that V is a singleton and the matrix $A : E \times E \to \{0, 1\}$ takes on the value 1 only.

The graph directed Markov system S is said to be conformal provided that the following conditions are satisfied.

(a) Each set X_v, $v \in V$, is compact connected and $X_v = \overline{\mathrm{Int}(X_v)}$, where the closure and interior are taken with respect to the Euclidean space $\mathbb{R}^d$.
(b) There exists an open connected set $W \supset X$ such that for every $i \in I$ the map ϕ_i extends to a C^1 conformal diffeomorphism of W into W.

(c) (Cone property) There exists $\gamma, l > 0$, $\gamma < \pi/2$, such that for every $v \in V$ and every $x \in X_v \subset \mathbb{R}^d$ there exists an open cone $\mathrm{Con}(x, \gamma, l) \subset \mathrm{Int}(X_V)$ with vertex x, central angle of measure γ, and altitude l.

(d) There are two constants $L \geq 1$ and $\alpha > 0$ such that

$$\big||\phi_i'(y)| - |\phi_j'(x)|\big| \leq L\|(\phi_i')^{-1}\|^{-1}\|y - x\|^\alpha$$

for every $i \in I$ and every pair of points $x, y \in X_{t(i)}$, where $|\phi_i'(x)|$ denotes the norm (equivalently, the scaling factor) of the derivative $\phi_i'(x)$.

Usually, but let us emphasize that not always in this paper, one assumes the following.

(e) (Open set condition; OSC) For all $a, b \in I$, $\neq b$,

$$\phi_a(\mathrm{Int}(X_{t(a)})) \cap \phi_b(\mathrm{Int}(X_{t(b)})) = \emptyset.$$

For every $\omega = \omega_1\omega_2 \ldots \omega_n \in I_A^n$, $n \geq 1$, set $t(\omega) = t(\omega_n)$ and $i(\omega) = i(\omega_n)$, and then define the composition

$$\phi_\omega := \phi_{\omega_1} \circ \phi_{\omega_2} \circ \ldots \phi_{\omega_n} : X_{t(\omega)} \to X_{i(\omega)}.$$

It is proved in [7] that if $d \geq 2$ and a family $S = \{\phi_i\}_{i \in I}$ satisfies conditions (b) and (d), then it also satisfies condition (f) with $\alpha = 1$. Condition (f) in turn implies the so called bounded distortion property, which says that

$$\frac{||\phi_\omega'(y)||}{||\phi_\omega'(y)||} \leq K$$

with some constant $K > 0$, all $\omega \in \bigcup_{n \geq 1} I^n$, and all $x, y \in X_{t(\omega)}$. If $\omega \in I_A^* \cup I_A^\infty$ and $n \in \mathbb{N}$ does not exceed the length of ω, we denote by $\omega|_n$ the word $\omega_1\omega_2 \ldots \omega_n$. Since, given $\omega \in I_A^\infty$, the diameters of the compact sets $\phi_{\omega|_n}(X_{t(\omega_n)})$, $n \in \mathbb{N}$, converge to zero and since these sets form a decreasing family, the set

$$\bigcap_{n=1}^\infty \phi_{\omega|_n}(X_{t(\omega_n)})$$

is a singleton, and we denote its element by $\pi_S(\omega)$, or, occasionally, by π_I. This defines the coding map

$$\pi_S : I_A^\infty \to \mathbb{R}^d.$$

Clearly, π_S is a continuous function when I_A^∞ is equipped with the Tichonov topology. The main object of our interest will be the limit set

$$J_S = \pi_S(I_A^\infty).$$

Observe that J_S satisfies the natural invariance equality,

$$J_S = \bigcup_{i \in I} \phi_i(J_S).$$

Recall that a matrix A is *finitely primitive* if there exists a finite set $\Gamma \subset I_A^*$ of words of the same length such that for all $a, b \in I$ there is a word $\omega \in \Gamma$ for which $a\omega b \in I_A^*$. If the matrix A is finitely primitive, we also say that the system S is finitely primitive. Each iterated function system is of course finitely primitive; take $\Gamma = \emptyset$. We now recall the definition of *topological pressure*. Given $t \geq 0$, the following limit exists:

$$\mathrm{P}_S(t) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in I_A^n} \|\phi_\omega'\|_\infty^t.$$

This limit exists since the sequence involved is subadditive; it is called the topological pressure of the parameter t. I will be occasionally denoted by $\mathrm{P}_I(t)$. Let

$$\theta_S := \inf\{t \geq 0 : P_S(t) < +\infty\}.$$

The basic fact for pursuing the study of fractal properties of limit sets J_S is the following theorem, of Bowen type, proved in [7]. We call the number

$$\inf\{t \geq 0 : \mathrm{P}_S(t) \leq 0\}$$

Bowen's parameter of the system S, and we denote it by $b(S)$. Its geometrical meaning is signified by the following.

Theorem 8.1. *If the conformal graph directed Markov system S is finitely primitive and satisfies the Open Set Condition, then*

$$\mathrm{HD}(J_S) = b(S) = \sup\{\mathrm{HD}(J_F) : F \ \ \textit{is a finite subset of} \ \ I\} \geq \theta_S.$$

We recall from [7] that the system S is *regular* if there is $t \geq 0$ such that $\mathrm{P}_S(t) = 0$, it is called *strongly regular* if there is $t \geq 0$ such that $0 < \mathrm{P}_S(t) < +\infty$, and it is called *cofinitely (hereditarily) regular* if $\mathrm{P}_S(\theta_S) = +\infty$. The basic properties of these concepts, proved in [7], are these.

Fact 8.2. *If the conformal graph directed Markov system S is finitely primitive and satisfies the Open Set Condition, then the following hold.*

(a) *Each cofinitely regular system is strongly regular and each strongly regular system is regular.*
(b) *If the system S is regular, then there is a unique $h \geq 0$ such that $\mathrm{P}_S(h) = 0$. Then $h_S := h > 0$ and $h_S = \mathrm{HD}(J_S)$.*
(c) *The system S is cofinitely regular if and only if the series*

$$\sum_{i\in I} \|\phi_i'\|_\infty^{\theta_S}$$

diverges.

Fix $t \geq 0$. A Borel probability measure m on $\mathbb{R}^d$ is called *t-conformal* with respect to the iterated function system S provided that $m(J_S) = 1$,

$$m(\phi_i(A)) = \int_A |\phi_i'|^t \, dm$$

for every every $i \in I$ and every Borel set $A \subset J_i := \pi(\{\omega \in I_A^\infty : A_{i\omega_1} = 1\})$, and

$$m(\phi_i(J_i) \cap \phi_j(J_j)) = 0$$

whenever $i \neq j$ and both i and j are in I. Note that if S is an iterated function system, then $J_i = J_S$ for all $i \in I$. Furthermore,

Fact 8.3. *If the conformal graph directed Markov system S is finitely primitive and satisfies the Open Set Condition, then the following hold.*

(a) *For every $t \geq 0$ there exists at most one t-conformal measure.*

(b) *For any t a t-conformal measure exists if and only if the system S is regular. If it is regular, then such a t is unique, in fact $t = h_S = \mathrm{HD}(J_S)$, and we denote the corresponding h_S-conformal measure by m_S. We refer to it as the conformal measure of the system S.*

(c) *If the system S is regular, then there exists a unique Borel probability S-invariant measure μ_S on J_S absolutely continuous with respect to m_S. S-invariance means that for every Borel set $A \subset X$,*

$$\mu_S(A) = \sum_{i \in I} \mu_S(\phi_i(A \cap J_i)).$$

The measure μ_S is ergodic and equivalent to m_S. The corresponding Radon-Nikodym derivative $\frac{d\mu_S}{dm_S}$ is a log bounded function on J_S and has a real-analytic extension on each set X_v, $v \in V$.

(d) *The h_S-dimensional Hausdorff measure $\mathrm{H}_{h_S}(J_S)$ of J_S is always finite while the h_S-dimensional packing measure $\mathrm{P}_{h_S}(J_S)$ of J_S is positive whenever $J \cap \mathrm{Int}(X) \neq \emptyset$, i.e. whenever the so called Strong Open Set Condition is satisfied.*

(e) *If either the h_S-dimensional Hausdorff or packing measure of J_S is positive, then the system S_I is regular and the normalized version of the h_S-dimensional Hausdorff or packing measure on J_S coincides with m_S.*

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