The Geometry of Baire Spaces

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Abstract. We introduce the concept of Baire embeddings and we classify them up to $C^{1+\varepsilon}$ conjugacies. We show that two such embeddings are $C^{1+\varepsilon}$-equivalent if and only if they have exponentially equivalent geometries. Next, we introduce the class of IFS-like Baire embeddings and we also show that two Hölder equivalent IFS-like Baire embeddings are $C^{1+\varepsilon}$ conjugate if and only if their scaling functions are the same. In the remaining sections we introduce metric scaling functions and we show that the logarithm of such a metric scaling function and the logarithm of Sullivan's scaling function multiplied by the Hausdorff dimension of the Baire embedding are cohomologous up to a constant. This permits us to conclude that if the Bowen measures coincide for two IFS-like Baire embeddings, then the embeddings are bi-Lipschitz conjugate.

1. introduction

An involved analysis of the geometries of Cantor embeddings and their conjugacies originated in the work of D. S. Sullivan, [5] and [6]; where he introduced the concept of scaling functions. Sullivan presented a classification theorem to the effect that two Cantor sets with bounded geometries are $C^{1+\varepsilon}$ conjugate if and only if their geometries are exponentially equivalent if and only if their scaling functions are the same. This topic was treated in length in [4]; whereas the case of conjugacies between Baire embeddings was treated in [1]. In [1] the case of real-analytic IFS-like Baire embeddings was, in a sense, exhausted. It has been shown there that two real-analytic IFS-like Baire embeddings (one of which must not be essentially affine) are conjugate in a real-analytic fashion if and only if they are bi-Lipschitz conjugate. In terms of scaling functions introduced in [1], it was shown that two C^1 conjugate Baire embeddings have the same scaling functions. In this paper, developing the approach in [4], we analyse in greater detail the geometries and conjugacies between Baire embeddings. In this setting the concept of exponential geometries looses its significance and meaning - already we have that the lengths of the first level sets converge to zero. Instead we assume that the embeddings are Hölder equivalent. We prove

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that two Baire embeddings are $C^{1+\varepsilon}$ conjugate if and only if their geometries are exponentially equivalent. The proof requires a more refined geometrical analysis and in fact, more notions describing the geometry of Baire embeddings than was needed in the case of Cantor embeddings. Following [1] we then introduce dual Cantor sets and scaling functions defined on these sets. We then prove that two Hölder equivalent Baire embeddings are $C^{1+\varepsilon}$ conjugate if and only if they have the same scaling functions. Note that in [1] it was only shown that if the scaling functions are the same then the Baire embeddings are bi-Lipschitz equivalent. In the last part of the paper, our starting point is the natural Bowen measure induced by a given Baire embedding. We first associate to such a measure its counterpart (not simply via reflection) on the dual symbol space and then to the this dual measure, the reciprocal of its Jacobian; which we call the metric scaling function of the original Baire embedding. We show that the logarithm of such a metric scaling function and the logarithm of Sullivan's scaling function multiplied by the Hausdorff dimension of the Baire embedding are cohomologous up to a constant. This implies that two Baire embeddings with the same Bowen measure are bi-Lipschitz equivalent. It is known (Theorem 3.1 in [1]) that in the case of real analytic Baire embeddings (one of which must not be essentially affine) this conjugacy is real-analytic.

2. Preliminary notations and definitions

Let us recall that two topological dynamical systems $T\colon X \to X$ and $S\colon Y \to Y$ are *topologically conjugate* if there exists a homeomorphism $h\colon X \to Y$ such that $S \circ h = h \circ T$, that is, if the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ Y & \xrightarrow{S} & Y \end{array}$$

Denote $I := 2\mathbb{N} - 1$. Consider $\Sigma = (2\mathbb{N} - 1)^{\mathbb{N}} = I^{\mathbb{N}}$ with the product topology. Then the following are homeomorphic

$$\Sigma \cong \mathscr{N} \cong (\mathbb{R} - \mathbb{Q}) \cap [0, 1]$$

where, $\mathscr{N}$ is Baire space, viz. $\mathbb{N}^{\mathbb{N}}$ with the product topology. Let us define

$$\Sigma^* := \bigcup_{k=0}^{\infty} I^k \cup (\bigcup_{k=0}^{\infty} I^k \times 2\mathbb{N}).$$

Denote $I^* := \bigcup_{k=0}^{\infty} I^k$ and $G^* := \bigcup_{k=0}^{\infty} I^k \times 2\mathbb{N}$. One can think of the I^* as the intervals that remain at a finite level and the G^* as the corresponding gaps that are left out. Then

$$\Sigma^* := I^* \cup G^*$$

If $\omega \in I^*$, then $C_\omega := h([\omega])$ and I_ω is the closed convex hull of C_ω. On the other hand if $\omega \in G^*$, where $\omega = \tau j$ for $\tau \in I^*, j \in 2\mathbb{N}$, then I_ω is the gap between $I_{\tau(j-1)}$ and $I_{\tau(j+1)}$.

Definition 2.1. *Let $h\colon \Sigma \to \mathbb{R}$ be a homeomorphism onto its image such that*

(a) $\omega < \tau \Rightarrow h(\omega) < h(\tau)$ *i.e. h is order-preserving*

(b) $h(\Sigma)$ *is bounded*

(c) $h\colon \Sigma \to \mathbb{R}$ *is uniformly continuous, meaning that*

$$\lim_{n\to\infty} \sup_{\omega \in I^n} \{\operatorname{diam}(h([\omega]))\} = 0 \tag{2.1}$$

(d) $\overline{h([i])} \cap \overline{h([j])} = \varnothing$ *for all $i \neq j$*

Call such an h a Baire embedding. Denote by $\mathcal{H}$ the class of all such Baire embeddings.

Notation 2.2. *Let us set the following notations:*

(1) *For every $\omega \in I^n$,*

$$\operatorname{diam}(I_\omega) := \operatorname{diam}(h([\omega])). \tag{2.2}$$

(2) $D_n(h) := \sup_{\omega \in I^n} \operatorname{diam}(h([\omega]))$

(3) *For $n \geq 0$, $E_n := \cup_{\omega \in I^n} I_\omega$ and denote $C := h(\Sigma) = \cap_{n=0}^{\infty} E_n$. Here C is the embedding of our Baire space into $\mathbb{R}$.*

(4) $\Delta := \operatorname{diam} h(\Sigma)$

(5) *Denote the (closed) convex hull of a set A by $co(A)$. Therefore for $\omega \in I^*$, we have that $I_\omega =: co(C_\omega) =: co(h([\omega]))$.*

Remark 2.3. *Observe that conditions (a) and (d) imply that $I_i \cap I_j = \varnothing$ for $i \neq j$ and thus it follows by induction that if $\omega, \tau \in I^*$ are incomparable, then $I_\omega \cap I_\tau = \varnothing$. Moreover in such a circumstance, $\sup I_\omega < \inf I_\tau$ if $\omega < \tau$. Also observe that condition (c), viz. (2.1) can be expressed as*

$$\lim_{n\to\infty} D_n(h) = 0.$$

Lastly, we would like to note that by translating and scaling the map h, we may assume without loss of generality that the $co(h(\Sigma)) = [0, 1]$.

Notation 2.4. *For every $\omega \in I^*$ let $I_{\omega\infty} =: h(\omega\infty) := \lim_{i\to\infty} h([\omega i])$ be the right-hand endpoint of I_ω. In particular, $h(\infty) = h(\varnothing\infty)$ is the right hand endpoint of $I_\varnothing$.*

We now prove the following

Proposition 2.5. $\overline{h(\Sigma)} = h(\Sigma) \cup \{h(\omega\infty) : \omega \in I^*\}$.

Proof. The inclusion $\subseteq$ is clear. In order to prove the reverse inclusion let us fix $x \in \overline{h(\Sigma)}$. Then there exists a sequence $(\omega^{(n)})_{n\ge1}$ of elements $\omega^{(n)} \in \Sigma$ such that $x = \lim_{n\to\infty} h(\omega^{(n)})$. For each $k \in \mathbb{N}$, define

$$E_k(x) := \{\omega^{(n)}|_k : n \ge k\} \text{ and let } E_0(x) := \varnothing.$$

Note that if $\tau \in E_{k+1}(x)$ then there exists $\gamma \in E_k(x)$ such that $\tau|_k = \gamma$. So, $E_k(x)$ is a rooted tree with vertex $E_0(x)$. Now consider two cases: First, suppose that there exists $k \in \mathbb{N}$ such that $E_k(x)$ has infinitely many elements. Then put

$$q := \min\{k \in \mathbb{N} : E_k(x) \text{ has infinitely many elements}\}.$$

The set $E_{q^{-1}}(x)$ is finite and non-empty although it might be equal to the singleton $\{\varnothing\}$.Thus there exists $\tau \in E_{q^{-1}}(x) \subseteq I^*$ and an infinite sequence $(\omega_q^{(n_j)})_{j\ge1}$ of distinct elements of I such that $\tau\omega_q^{(n_j)} = \omega^{(n_j)}|_q$ for all $j \in \mathbb{N}$. Therefore

$$x = \lim_{j\to\infty} h(\omega^{(n_j)}) \in \limsup_{j\to\infty} h(\omega^{(n_j)}|_q) = \limsup_{j\to\infty} h(\tau\omega_q^{(n_j)}) = h(\tau\infty).$$

Thus $x = h(\tau\infty)$ and we are done in the first case.
Next, suppose that the set $E_k(x)$ is finite for every $k \in \mathbb{N}$. Since, as we have mentioned before, these sets form a rooted tree with vertex $E_0(x)$, König's Lemma yields the existence of an infinite word $\omega \in \Sigma$ such that $\omega|_k \in E_k(x)$ for all $k \in \mathbb{N}$. Therefore, there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that $\omega|_k = \omega^{(n_k)}|_k$ for all $k \in \mathbb{N}$. Thus $|h(\omega^{(n_k)}) - h(\omega)| \le D_k(h)$ and thus condition (2.1) yields that $x = \lim_{k\to\infty} h(\omega^{(n_k)}) = h(\omega)$. We are done. $\square$

Now given $h_1, h_2 \in \mathcal{H}$, consider the map $h_2 \circ h_1^{-1} : h_1(\Sigma) \to h_2(\Sigma)$. By Proposition 2.5 the formula

$$H_{1,2}(x) = \begin{cases} h_2 \circ h_1^{-1}(x) & \text{if } x \in h_1(\Sigma) \\ h_2(\omega\infty) & \text{if } x = h_1(\omega\infty) \text{ and } \omega \in I^*, \end{cases}$$

defines an extension $H_{1,2}(x) : \overline{h_1(\Sigma)} \to \overline{h_2(\Sigma)}$ of $h_2 \circ h_1^{-1}$ from the closure of $h_1(\Sigma)$ to the closure of $h_2(\Sigma)$. We shall now prove the following

Proposition 2.6. *If $h_1, h_2 \in \mathcal{H}$, then $H_{1,2} : \overline{h_1(\Sigma)} \to \overline{h_2(\Sigma))}$ is a homeomorphism.*

Proof. Since $H_{2,1} \circ H_{1,2} = \mathrm{Id}_{\overline{\mathrm{h_1(\Sigma)}}}$ and $H_{1,2} \circ H_{2,1} = \mathrm{Id}_{\overline{\mathrm{h_2(\Sigma)}}}$, it suffices to show that $H_{1,2} : \overline{h_1(\Sigma)} \to \overline{h_2(\Sigma))}$ is continuous. We shall prove the continuity of $H_{1,2}$ at every point $x \in \overline{h_1(\Sigma)}$. Suppose first that $x \in h_1(\Sigma)$, i.e. $x = h_1(\omega)$ for some $\omega \in \Sigma$. For every $y \in I_\omega$, let $n_y \ge 1$ be the largest integer $n \ge 1$ such that

$y \in I_{\omega|_n}$. We then have that $\lim_{y\to x} n_y = +\infty$ and therefore it follows from (2.2) combined with continuity of $h_2 : \Sigma \to \mathbb{R}$ that

$$\lim_{\substack{y\to x \\ y\in\overline{h_1(\Sigma)}}} H_{1,2}(y) = h_2(\omega) = h_2 \circ h_1^{-1}(x) = H_{1,2}(x).$$

We are done in this case. Next, suppose that $x \in \overline{h_1(\Sigma)} - h_1(\Sigma)$. Then, in view of Proposition 2.5 there exists $\omega \in I^*$ such that $x = h_1(\omega\infty)$. For every $y \in \overline{h_1([\omega])} - \{x\}$ there exists a unique $j_y \in I$ such that $y \in \overline{h_1([\omega j_y])}$. In addition $\lim_{y\to x} = +\infty$. Therefore we have that

$$\lim_{y\to x} H_{1,2}(y) \in \limsup_{y\to x} \overline{h_2([\omega j_y])} = \limsup_{j\to\infty} \overline{h_2([\omega j])} = h_2(\omega\infty) = H_{1,2}(x).$$

Thus $\lim_{y\to x} H_{1,2}(y) = x$ and we are done. $\square$

Notation 2.7. *Let $\kappa = \sup\{|I_j| : j \in \mathbb{N}\}$. Notice that*

$$\kappa \le \max\{|I_1|, \Delta - |I_1|\} < \Delta = 1 \tag{2.3}$$

Definition 2.8. *$A : \mathbb{N} \to [0,1]$ is said to be a probability vector if and only if $\sum_{n=0}^{\infty} A(n) = 1$. Given $c \ge 1$ two probability vectors $A : \mathbb{N} \to [0,1]$ and $B : \mathbb{N} \to [0,1]$ are said to be c-equivalent, $A \sim_c B$, if $\forall n \ge 1$,*

$$c^{-1}B(n) \le A(n) \le cB(n) \ .$$

Given $\omega \in I^*$ define $A(\omega) : \mathbb{N} \to [0,1]$ by

$$A(\omega)(j) :=: A_j(\omega) :=: \frac{|I_{\omega j}|}{|I_\omega|}$$

Note that each $A(\omega)$ is a probability vector with no zero entries.

Definition 2.9. *$h \in \mathcal{H}$ is said to be of bounded geometry provided there exists $c \ge 1$ such that $A(\omega) \sim_c A(\varnothing)$ for all $\omega \in I^*$ and $A_i(h,\omega) \le cA_j(h,\omega)$ whenever $|i-j| \le 1$. Denote by $\mathcal{H}_b$ the class of Baire embeddings with bounded geometry. If more than one Baire embedding is considered, write $A_h(\omega)$ for $A(\omega)$ and $A_j(h,w)$ for $A_j(\omega)$.*

Recall that we had introduced the notation $I_{\omega\infty}$ to denote the right-hand endpoint of the interval I_ω. Let us now prove a straightforward but useful fact.

Lemma 2.10. *Let $h \in \mathcal{H}$ be of bounded geometry. There exists a constant $c^* \ge 1$ such that for all $\omega \in I^*$, if $x \in I_{\omega i}$, $y \in I_{\omega j}$ and $|j-i| \ge 2$, then*

$$\mathrm{diam}(I_{\omega i} \cup I_{\omega j}) \le c^*|y-x|.$$

Note that i or j can be ∞, and it does not hurt to assume that $|\infty - \infty| \ge 2$.

Proof. Since h is of bounded geometry there exists a $c \geq 1$ such that $\frac{|I_{\omega i}|}{|I_{\omega(i+1)}|} \leq c$ and $\frac{|I_{\omega(j-1)}|}{|I_{\omega j}|} \leq c$. Then $|I_{\omega i}| \leq c|I_{\omega(i+1)}| \leq c|y-x|$ since x and y are not in $I_{\omega(i+1)}$. Similarly, $|I_{\omega(j-1)}| \leq c|I_{\omega j}| \leq c|y-x|$. Thus, we have that

$$\begin{aligned}\operatorname{diam}(I_{\omega i} \cup I_{\omega j}) &\leq |I_{\omega i}| + |I_{\omega j}| + |y-x| \\ &\leq (2c+1)|y-x|,\end{aligned}$$

which completes the proof. □

Definition 2.11. *$h_1, h_2 \in \mathcal{H}$ are said to have weakly equivalent geometries, $h_1 \sim_{wk} h_2$, if*

$$\lim_{n\to\infty} \sup_{\omega\in I^n} \sup_{j\in\mathbb{N}} \left\{ \left| \frac{A_j(h_2,\omega)}{A_j(h_1,\omega)} - 1 \right| \right\} = 0$$

Notation 2.12. *Given $\omega \in I^*$ and $i, j \in \mathbb{N}$, let*

$$Q_h(\omega; i, j) :=: \frac{A_j(h,\omega)}{A_i(h,\omega)} :=: \frac{|I_{\omega j}|}{|I_{\omega i}|}$$

If it is clear which homeomorphism we are dealing with, we will frequently drop the subscript h and will write $Q(\omega; i, j)$ for $Q_h(\omega; i, j)$.

Definition 2.13. *$h_1, h_2 \in \mathcal{H}$ are said to have equivalent geometries, $h_1 \sim h_2$, if*

$$\lim_{n\to\infty} \sup_{\omega\in I^n} \sup_{i,j\in\mathbb{N}} \left\{ \left| \frac{Q_{h_2}(\omega; i, j)}{Q_{h_1}(\omega; i, j)} - 1 \right| \right\} = 0,$$

that is, $\forall \omega \in I^*$

$$\lim_{i,j\to\infty} \left| \frac{Q_{h_2}(\omega; i, j)}{Q_{h_1}(\omega; i, j)} - 1 \right| = 0$$

and

$$\lim_{n\to\infty} \sup_{\omega\in I^n} \left| \frac{A_1(h_2,\omega)}{A_1(h_1,\omega)} - 1 \right| = 0. \tag{2.4}$$

Notation 2.14. *Given $\omega \in I^*$ and $j \in \mathbb{N}$ let*

$$I_{\omega j+} := \overline{\cup_{k\geq j} I_{\omega k}} \ .$$

In particular note that $I_{\omega 1+} = I_\omega$.

For all $i, j \in \mathbb{N}$ *let*

$$I_{\omega[i,j]} := \bigcup_{k=\min\{i,j\}}^{\max\{i,j\}} I_{\omega k}$$

be the convex hull containing $I_{\omega i}$ *and* $I_{\omega j}$. *All intervals of the form* $I_{\omega[i,j]}$ *will be called* ω*-intervals.*

Definition 2.15. $h_1, h_2 \in \mathcal{H}$ *are said to have weakly exponentially equivalent geometries,* $h_1 \sim_{wex} h_2$, *if*

$$\liminf_{n\to\infty} \inf_{\omega\in I^n} \inf_{j\in\mathbb{N}} \left\{ \frac{\log |\frac{A_j(h_2,\omega)}{A_j(h_1,\omega)} - 1|}{\min\{\log |I^1_\omega|, \log |I^2_\omega|\}} \right\} > 0. \tag{2.5}$$

This condition means that there exist $c \geq 1$ *and* $\delta > 0$ *such that* $\forall \omega \in I^*$, $\forall i, j \in \mathbb{N}$

$$\left| \frac{A_j(h_2,\omega)}{A_j(h_1,\omega)} - 1 \right| \leq c \min\{|I^1_{\omega j+}|^\delta, |I^2_{\omega j+}|^\delta\}. \tag{2.6}$$

We note that if we wish to be more explicit about the constant δ, *we may write* $h_1 \sim_{wex(\delta)} h_2$ *for* $h_1 \sim_{wex} h_2$.

Definition 2.16. $h_1, h_2 \in \mathcal{H}$ *are said to have exponentially equivalent geometries,* $h_1 \sim_{ex} h_2$, *if*

$$\liminf_{n\to\infty} \inf_{\omega\in I^n} \inf_{i,j\in\mathbb{N}} \frac{\log \left| \frac{Q_{h_2}(\omega;i,j)}{Q_{h_1}(\omega;i,j)} - 1 \right|}{\min\left\{ \log \left| I^1_{\omega[i,j]} \right|, \log \left| I^2_{\omega[i,j]} \right| \right\}} > 0$$

and

$$\liminf_{n\to\infty} \inf_{\omega\in I^n} \frac{\log |\frac{A_1(h_2,\omega)}{A_1(h_1,\omega)} - 1|}{\min\{\log |I^1_\omega|, \log |I^2_\omega|\}} > 0.$$

These conditions equivalently mean that there exist $c \geq 1$ *and* $\delta > 0$ *such that* $\forall \omega \in I^*$, $\forall i, j \in \mathbb{N}$

$$\left| \frac{Q_{h_2}(\omega;i,j)}{Q_{h_1}(\omega;i,j)} - 1 \right| \leq c \min\left\{ \left| I^1_{\omega[i,j]} \right|^\delta, \left| I^2_{\omega[i,j]} \right|^\delta \right\} \tag{2.7}$$

and

$$\left| \frac{A_1(h_2,\omega)}{A_1(h_1,\omega)} - 1 \right| \leq c \min\{|I^1_\omega|^\delta, |I^2_\omega|^\delta\}. \tag{2.8}$$

We note that if we wish to be more explicit about the constant δ, we may write $h_1 \sim_{ex(\delta)} h_2$ for $h_1 \sim_{ex} h_2$.

It is straightforward to see that the relation of being (weakly) exponentially equivalent geometries is reflexive and transitive. Noting that $x^{-1} - 1 = x^{-1}(1 - x)$, it is also easy to see that this relation is symmetric. Thus they are equivalence relations. Let us now prove that our weak notions are implied by their stronger counterparts.

Lemma 2.17. *If the geometries of h_1, h_2 are equivalent, then they are weakly equivalent.*

Proof. The proof follows directly from the following formula and the definition of equivalent geometry.

$$\left|\frac{A_j(h_2,\omega)}{A_j(h_1,\omega)} - 1\right| = \left|\frac{\frac{A_j(h_2,\omega)}{A_1(h_2,\omega)}}{\frac{A_j(h_1,\omega)}{A_1(h_1,\omega)}} \cdot \frac{A_1(h_2,\omega)}{A_1(h_1,\omega)} - 1\right| = \left|\frac{Q_{h_2}(\omega;1,j)}{Q_{h_1}(\omega;1,j)} \cdot \frac{A_1(h_2,\omega)}{A_1(h_1,\omega)} - 1\right|$$

□

Lemma 2.18. *If the geometries of h_1, h_2 are exponentially equivalent, then they are weakly exponentially equivalent.*

Proof. Let u be either 1 or 2.

$$\begin{aligned}
\left|\frac{A_j(h_2,\omega)}{A_j(h_1,\omega)} - 1\right| &= \left|\frac{Q_{h_2}(\omega;1,j)}{Q_{h_1}(\omega;1,j)} \cdot \frac{A_1(h_2,\omega)}{A_1(h_1,\omega)} - 1\right| \\
&= \left| \left(\frac{Q_{h_2}(\omega;1,j)}{Q_{h_1}(\omega;1,j)} - 1\right)\left(\frac{A_1(h_2,\omega)}{A_1(h_1,\omega)} - 1\right) + \right. \\
&\qquad \left. + \left(\frac{Q_{h_2}(\omega;1,j)}{Q_{h_1}(\omega;1,j)} - 1\right) + \left(\frac{A_1(h_2,\omega)}{A_1(h_1,\omega)} - 1\right) \right| \\
&\le c^2|I^u_{\omega 1+}|^\delta |I^u_\omega|^\delta + c|I^u_{\omega 1+}|^\delta + c|I^u_\omega|^\delta \\
&\le c^2|I^u_\omega|^{2\delta} + 2c|I^u_\omega|^\delta \\
&\le 3c^2|I^u_\omega|^\delta.
\end{aligned}$$

□

Definition 2.19. *$h_1, h_2 \in \mathcal{H}$ are called $C^{1+\varepsilon}$-equivalent, $h_1 \sim^{1+\varepsilon} h_2$, for $0 < \varepsilon < 1$, if there exists an increasing $C^{1+\varepsilon}$ diffeomorphism ϕ from a neighbourhood of $co(h_1(\Sigma))$ onto a neighbourhood of $co(h_2(\Sigma))$ such that $\phi|_{h_1(\Sigma)} = h_2 \circ h_1^{-1}$. Similarly, $h_1, h_2 \in \mathcal{H}$ are called C^1-equivalent, $h_1 \sim^{1+0} h_2$, if there exists an increasing C^1 diffeomorphism ϕ from a neighbourhood of $co(h_1(\Sigma))$ onto a neighbourhood of $co(h_2(\Sigma))$ such that $\phi|_{h_1(\Sigma)} = h_2 \circ h_1^{-1}$. Again, $h_1, h_2 \in \mathcal{H}$ are called C^{1+1}-equivalent, $h_1 \sim^{1+1} h_2$, if there exists an increasing C^{1+1} diffeomorphism* *(i.e. one whose first derivative is Lipschitz) ϕ from a neighbourhood of $co(h_1(\Sigma))$ onto a neighbourhood of $co(h_2(\Sigma))$ such that $\phi|_{h_1(\Sigma)} = h_2 \circ h_1^{-1}$.*

The composition $h_2 \circ h_1^{-1} : [0,1] \to [0,1]$ is called the natural conjugacy from $h_1(\Sigma)$ to $h_2(\Sigma)$. Each conjugacy class of the relation $\sim^{1+\varepsilon}$ is called a $C^{1+\varepsilon}$-structure on Σ.

Proposition 2.20. *If $h_1, h_2 \in \mathcal{H}$ and $h_1 \sim^{1+0} h_2$, then h_1 and h_2 have equivalent geometries, i.e. $h_1 \sim h_2$.*

Proof. Let $\phi : U_1 \to U_2$ be a C^1 extension of $h_2 \circ h_1^{-1}$. Let $M_{D\phi}$ be the modulus of continuity of $D\phi$. By assumption, $|D\phi(x)| \geq A > 0$ for all $x \in U_1$. Then

$$\begin{aligned}
\left| \frac{A_j(h_2,\omega)}{A_j(h_1,\omega)} - 1 \right| = \left| \frac{|I^2_{\omega j}|/|I^2_\omega|}{|I^1_{\omega j}|/|I^1_\omega|} - 1 \right| &= \left| \frac{|I^2_{\omega j}|/|I^1_{\omega j}|}{|I^2_\omega|/|I^1_\omega|} - 1 \right| \\
&= \left| \frac{|\phi(I^1_{\omega j})|/|I^1_{\omega j}|}{|\phi(I^1_\omega)|/|I^1_\omega|} - 1 \right| \\
&= \left| \frac{\phi'(x_{\omega j})}{\phi'(x_\omega)} - 1 \right| \text{ for some } x_{\omega j} \in I^1_{\omega j},\ x_\omega \in I^1_\omega\ . \\
&= \frac{|\phi'(x_{\omega j}) - \phi'(x_\omega)|}{|\phi'(x_\omega)|} \\
&\leq \frac{1}{A} \left| \phi'(x_{\omega j}) - \phi'(x_\omega) \right| \\
&\leq \frac{1}{A} M(|x_{\omega j} - x_\omega|) \\
&\leq A^{-1} M(|I_\omega|) \\
&\leq A^{-1} M D_{|\omega|}(h_1)
\end{aligned}$$

Thus taking $j = 1$, the last requirement of equivalent geometries viz.(2.4) is satisfied. Let us deal with the first one. We have

$$
\begin{aligned}
\left|\frac{Q_{h_2}(\omega;i,j)}{Q_{h_1}(\omega;i,j)}-1\right| = \left|\frac{\frac{A_j(h_2,\omega)}{A_i(h_2,\omega)}}{\frac{A_j(h_1,\omega)}{A_i(h_1,\omega)}}-1\right| &= \left|\frac{|I^2_{\omega j}|/|I^2_{\omega i}|}{|I^1_{\omega j}|/|I^1_{\omega i}|}-1\right| \\
&= \left|\frac{|I^2_{\omega j}|/|I^1_{\omega j}|}{|I^2_{\omega i}|/|I^1_{\omega i}|}-1\right| \\
&= \left|\frac{|\phi(I^1_{\omega j})|/|I^1_{\omega j}|}{|\phi(I^1_{\omega i})|/|I^1_{\omega i}|}-1\right| \\
&= \left|\frac{\phi'(x_{\omega j})}{\phi'(x_{\omega i})}-1\right| \text{ for some } x_{\omega j}\in I^1_{\omega j},\ x_{\omega i}\in I^1_{\omega i}\ . \\
&= \frac{|\phi'(x_{\omega j})-\phi'(x_\omega)|}{\phi'(x_{\omega i})} \\
&\le A^{-1}M(\left|I_{\omega[i,j]}\right|)
\end{aligned}
$$

We are done. ☐

Lemma 2.21. *If $h \in \mathcal{H}_b$, then*

$$\limsup_{n\to\infty}\frac{1}{n}\log D_n(h) < 0.$$

Proof. Take $c \ge 1$ such that $A(\omega) \sim_c A(\varnothing)$ for all $\omega \in I^*$. Then for all $\omega \in I^*$ and for all $j \in 2\mathbb{N}-1$, we have

$$\frac{\sum_{i\in\mathbb{N}\setminus\{j\}}|I_{\omega i}|}{|I_\omega|} = \sum_{i\ne j}\frac{|I_{\omega i}|}{|I_\omega|} \ge \sum_{i\ne j}c^{-1}\frac{|I_i|}{|\Delta|} \ge c^{-1}(1-\frac{|I_i|}{\Delta}) \ge c^{-1}(1-\kappa\Delta^{-1}) > 0.$$

Note that the last inequality follows from (2.3). Therefore,

$$\frac{|I_{\omega j}|}{|I_\omega|} = 1-\sum_{i\ne j}\frac{|I_{\omega i}|}{|I_\omega|} \le 1-c^{-1}(1-\kappa\Delta^{-1}) \tag{2.9}$$

i.e.

$$|I_{\omega j}| \le (1-c^{-1}(1-\kappa\Delta^{-1}))(|I_\omega|)$$

Thus by a straightforward induction,

$$|I_\tau| \le (1-c^{-1}(1-\kappa\Delta^{-1}))^{|\tau|}$$

for $\tau \in I^*$. We are done. ☐

We would like to note an immediate consequence of this lemma and the definition of weakly exponentially equivalent geometries, see (2.6), viz. if $h_1, h_2 \in \mathcal{H}_b$

with $h_1 \sim_{wex} h_2$, then we have that

$$\limsup_{n\to\infty} \frac{1}{n} \log \sup_{\omega\in I^n} \sup_{j\in\mathbb{N}} \left\{ \left| \frac{A_j(h_2,\omega)}{A_j(h_1,\omega)} - 1 \right| \right\} < 0. \tag{2.10}$$

3. $C^{1+\varepsilon}$ EQUIVALENCE

In this section we prove our first main result, viz. that two Baire embeddings are $C^{1+\delta}$ conjugate if and only if these have $(1+\delta)$-equivalent geometries. Besides being interesting in itself, this result will play a central role in the next section.

Theorem 3.1. *Let $h_1, h_2 \in \mathcal{H}_b$. Fix $\delta > 0$. Then $h_1 \sim^{1+\delta} h_2 \Leftrightarrow h_1 \sim_{ex(\delta)} h_2$.*

Proof. ($\Rightarrow$) Essentially the same computation from the first part of the proof of Proposition 2.20, gives us

$$\left| \frac{A_j(h_2,\omega)}{A_j(h_1,\omega)} - 1 \right| \le A^{-1} \left| \phi'(x_{\omega j}) - \phi'(x_\omega) \right| \le cA^{-1} |x_{\omega j} - x_\omega|^\delta \le cA^{-1} |I^1_\omega|^\delta.$$

Likewise

$$\left| \frac{A_j(h_1,\omega)}{A_j(h_2,\omega)} - 1 \right| \le cA^{-1} |I^2_\omega|^\delta.$$

Thus the requirement (2.8) is satisfied. For the second requirement we now repeat the same computation from the second part of the Proposition 2.20 to get

$$\left| \frac{Q_{h_2}(\omega; i,j)}{Q_{h_1}(\omega; i,j)} - 1 \right| \le A^{-1} |\phi'(x_{\omega j}) - \phi'(\omega i)| \le A^{-1} c |x_{\omega i} - x_{\omega j}|^\delta \le cA^{-1} |I^1_{\omega[i,j]}|^\delta$$

Likewise

$$\left| \frac{Q_{h_1}(\omega; i,j)}{Q_{h_2}(\omega; i,j)} - 1 \right| \le cA^{-1} |I^2_{\omega[i,j]}|^\delta$$

Thus we are done with the first part of our implication.

($\Leftarrow$) For all $a, b \in \overline{h_1(\Sigma)},\ a \ne b$, set

$$R(a,b) = \frac{H_{1,2}(b) - H_{1,2}(a)}{b-a}$$

and for every interval (closed, open or clopen) $J \subseteq \mathbb{R}$ with endpoints $a, b \in \overline{h_1(\Sigma)}$, set

$$R(J) = \frac{H_{1,2}(b) - H_{1,2}(a)}{b-a} = \frac{|H_{1,2}(J)|}{|J|}$$

Fix $\delta > 0$ and $c \geq 1$ coming from (2.7) and (2.8). We shall prove the following:

$$\left|\frac{R(I^1_{\omega|_n})}{R(I^1_{\omega|_m})} - 1\right| \leq c_1 |I^1_{\omega|_m}|^\delta \leq c_2 e^{-\theta\delta m}, \tag{3.1}$$

with the same constants $c_1, c_2 \geq 1$ for all $\omega \in \Sigma^*(= I^* \cup G^*)$ and also for all $\theta \leq m \leq n \leq |\omega|$.

Indeed, suppose first that $n = m+1$. Then

$$\begin{aligned}
\frac{R(I^1_{\omega|_{m+1}})}{R(I^1_{\omega|_m})} - 1 = \frac{\frac{\left|H_{1,2}\left(I^1_{\omega|_{m+1}}\right)\right|}{\left|I^1_{\omega|_{m+1}}\right|}}{\frac{\left|H_{1,2}\left(I^1_{\omega|_m}\right)\right|}{\left|I^1_{\omega|_m}\right|}} - 1 &= \frac{\left|I^2_{\omega|_{m+1}}\right| / \left|I^1_{\omega|_{m+1}}\right|}{\left|I^2_{\omega|_m}\right| / \left|I^1_{\omega|_m}\right|} - 1 \\
&= \frac{\left|I^2_{\omega|_{m+1}}\right| / \left|I^2_{\omega|_m}\right|}{\left|I^1_{\omega|_{m+1}}\right| / \left|I^1_{\omega|_m}\right|} - 1 \\
&= \frac{A_{\omega_{m+1}}(h_2, \omega|_{m+1})}{A_{\omega_{m+1}}(h_1, \omega|_{m+1})} - 1.
\end{aligned}$$

It therefore follows from Lemma 2.18 that

$$\left|\frac{R(I^1_{\omega|_{m+1}})}{R(I^1_{\omega|_m})} - 1\right| \leq c_3 |I^1_{\omega|_m}|^\delta,$$

with some universal constant $c_3 \geq 1$. In other words, we have

$$\frac{R(I^1_{\omega|_{m+1}})}{R(I^1_{\omega|_m})} = 1 + c(\omega|_{m+1}) |I_{\omega|_m}|^\delta \tag{3.2}$$

with some constant $c(\omega|_{m+1}) \in [-c_3, c_3]$.

Now come back to the general case of arbitrary $m, n \geq 0$. Set

$$\theta := -\log\left(1 - c^{-1}(1 - \kappa\Delta^{-1})\right) > 0 \qquad (cf. (2.9)).$$

We may assume without loss of generality that $n \geq m+1$. Using (3.2), we then get

$$\frac{R(I^1_{\omega|_n})}{R(I^1_{\omega|_m})} = \prod_{j=m}^{n-1} \frac{R(I^1_{\omega|_{j+1}})}{R(I^1_{\omega|_j})} = \prod_{j=m}^{n-1} \left(1 + c(\omega|_{j+1}) |I^1_{\omega|_j}|^\delta\right).$$

Hence

$$\left|\log\frac{R(I^1_{\omega|_n})}{R(I^1_{\omega|_m})}\right| = \left|\sum_{j=m}^{n-1}\log(1+c(\omega|_{j+1})|I^1_{\omega|j}|^\delta)\right| \le \sum_{j=m}^{n-1}|c(\omega|_{j+1})||I^1_{\omega|_j}|^\delta$$
$$\le \sum_{j=m}^{n-1} c|I^1_{\omega|_j}|^\delta$$
$$\le c\sum_{j=m}^{n-1}|I^1_{\omega|_j}|^\delta.$$

But by (2.9), $|I^1_{\omega|_j}| \le e^{-\theta(j-m)}|I^1_{\omega|_m}|$. Therefore,

$$\left|\log\frac{R(I^1_{\omega|_n})}{R(I^1_{\omega|_m})}\right| \le c|I^1_{\omega|_j}|^\delta\sum_{j=m}^{\infty}e^{-\theta\delta(j-m)}$$
$$= c|I^1_{\omega|_m}|^\delta\sum_{i=0}^{\infty}e^{-\theta\delta(i)}$$
$$= c(1-e^{-\theta\delta})^{-1}|I^1_{\omega|_m}|^\delta$$
$$\le c\Delta^\delta(1-e^{-\theta\delta})^{-1}.$$

Thus there exists a constant $c_4 \ge 1$ depending only on the number $c\Delta^\delta(1-e^{-\theta\delta})^{-1}$, such that

$$\left|\frac{R(I^1_{\omega|_n})}{R(I^1_{\omega|_m})}-1\right| \le c_4\log\frac{R(I^1_{\omega|_n})}{R(I^1_{\omega|_m})}$$
$$\le \Delta c\; c_4(1-e^{-\theta\delta})^{-1}|I^1_{\omega|_m}|^\delta$$
$$\le \Delta^{1+\delta}c\; c_4(1-e^{-\theta\delta})^{-1}e^{-\theta\delta m}.$$

The formula (3.1) is proved.

Now we shall show that there exists a constant $c_5 \ge 1$ such that for all $\omega \in I^*$ and all ω-intervals Δ, Γ with $\mathrm{diam}(\Delta\cup\Gamma)$ small enough,

$$\left|\frac{R(\Delta)}{R(\Gamma)}-1\right| \le c_5\big(\mathrm{diam}(\Delta\cup\Gamma)\big)^\delta \tag{3.3}$$

Indeed, set $\Delta = I_{\omega[i_1,i_2]}$ and $\Gamma = I_{\omega[j_1,j_2]}$. Let $a = \min\{i_1, i_2\}$ and $b = \max\{j_1, j_2\}$. Then $\Delta \cup \Gamma \subseteq I_{\omega[a,b]}$ and $\mathrm{diam}(\Delta \cup \Gamma) = |I_{\omega[a,b]}|$. Now using (2.7) we can write

$$\begin{aligned}
\frac{R(\Delta)}{R(\Gamma)} &= \frac{\sum_{k=j_1}^{j_2} |I^2_{\omega k}| / \sum_{k=j_1}^{j_2} |I^1_{\omega k}|}{\sum_{k=i_1}^{i_2} |I^2_{\omega k}| / \sum_{k=i_1}^{i_2} |I^1_{\omega k}|} \\
&= \frac{\sum_{k=j_1}^{j_2} |I^2_{\omega k}| / \sum_{k=i_1}^{i_2} |I^2_{\omega k}|}{\sum_{k=j_1}^{j_2} |I^1_{\omega k}| / \sum_{k=i_1}^{i_2} |I^1_{\omega k}|} \\
&= \frac{\sum_{k=j_1}^{j_2} A_k(h_2,\omega) / \sum_{k=i_1}^{i_2} A_k(h_2,\omega)}{\sum_{k=j_1}^{j_2} A_k(h_1,\omega) / \sum_{k=i_1}^{i_2} A_k(h_1,\omega)} \\
&= \frac{\sum_{k=j_1}^{j_2} Q_{h_2}(\omega;a,k) / \sum_{k=i_1}^{i_2} Q_{h_2}(\omega;a,k)}{\sum_{k=j_1}^{j_2} Q_{h_1}(\omega;a,k) / \sum_{k=i_1}^{i_2} Q_{h_1}(\omega;a,k)} \\
&= \frac{\sum_{k=j_1}^{j_2} (1+c_k|I^1_{\omega[a,b]}|^\delta) Q_{h_1}(\omega;a,k) / \sum_{k=i_1}^{i_2} (1+c_k|I^1_{\omega[a,b]}|^\delta) Q_{h_1}(\omega;a,k)}{\sum_{k=j_1}^{j_2} Q_{h_1}(\omega;a,k) / \sum_{k=i_1}^{i_2} Q_{h_1}(\omega;a,k)} \qquad (3.4)
\end{aligned}$$

with some $c_k \in [-c, c]$ since $I^1_{\omega[a,k]} \subseteq I^1_{\omega[a,b]}$.

Now assume $|I^1_{\omega[a,b]}| = \mathrm{diam}(\Delta \cup \Gamma)$ to be so small that

$$\max\{2c, 3c + c^2, 3c\}.|I^1_{\omega[a,b]}|^\delta \le 1/2. \qquad (3.5)$$

Then we have that

$$\left|(1 \pm c|I^1_{\omega[a,b]}|^\delta)^{-1}\right| = \left|\sum_{n=1}^{\infty} (\mp c|I^1_{\omega[a,b]}|^\delta)^n\right| = \frac{c|I^1_{\omega[a,b]}|^\delta}{1 \mp c|I^1_{\omega[a,b]}|^\delta} \le 2c|I^1_{\omega[a,b]}|^\delta$$

and we also have that

$$1 - 2c|I^1_{\omega[a,b]}|^\delta \le (1 \pm c|I^1_{\omega[a,b]}|^\delta)^{-1} \le 1 + 2c|I^1_{\omega[a,b]}|^\delta.$$

Now using this observation, we have from (3.4) that

$$\begin{aligned}\frac{R(\Delta)}{R(\Gamma)} &\leq \frac{(1+c_k|I^1_{\omega[a,b]}|^\delta)\sum_{k=j_1}^{j_2} Q_{h_1}(\omega;a,k)/(1+c_k|I^1_{\omega[a,b]}|^\delta)\sum_{k=i_1}^{i_2} Q_{h_1}(\omega;a,k)}{\sum_{k=j_1}^{j_2} Q_{h_1}(\omega;a,k)/\sum_{k=i_1}^{i_2} Q_{h_1}(\omega;a,k)}\\ &= \frac{(1+c|I^1_{\omega[a,b]}|^\delta)}{(1-c|I^1_{\omega[a,b]}|^\delta)}\\ &\leq (1+c|I^1_{\omega[a,b]}|^\delta)(1+2c|I^1_{\omega[a,b]}|^\delta)\\ &\leq 1+c_5|I^1_{\omega[a,b]}|^\delta,\end{aligned}$$

and that

$$\begin{aligned}\frac{R(\Delta)}{R(\Gamma)} &\leq \frac{(1-c_k|I^1_{\omega[a,b]}|^\delta)\sum_{k=j_1}^{j_2} Q_{h_1}(\omega;a,k)/(1+c_k|I^1_{\omega[a,b]}|^\delta)\sum_{k=i_1}^{i_2} Q_{h_1}(\omega;a,k)}{\sum_{k=j_1}^{j_2} Q_{h_1}(\omega;a,k)/\sum_{k=i_1}^{i_2} Q_{h_1}(\omega;a,k)}\\ &= \frac{(1-c|I^1_{\omega[a,b]}|^\delta)}{(1+c|I^1_{\omega[a,b]}|^\delta)}\\ &\leq (1-c|I^1_{\omega[a,b]}|^\delta)(1-2c|I^1_{\omega[a,b]}|^\delta)\\ &\leq 1-c_6|I^1_{\omega[a,b]}|^\delta,\end{aligned}$$

with universal constants $c_5, c_6 \geq 1$.

Thus we have shown that

$$\left|\frac{R(\Gamma)}{R(\Delta)} - 1\right| \leq \max\{c_5, c_6\}|I^1_{\omega[a,b]}|^\delta$$

and we are done with the proof of (3.3).

Now it follows from (3.1) that for every $\omega \in I^{\mathbb{N}}$, the limit of the sequence $(R(I^1_{\omega|_n}))_{n=1}^\infty$ exists, is finite and non-zero. We define it to be

$$H^*_{1,2}(h_1(\omega)) := \lim_{n\to\infty} R(I^1_{\omega|_n}) \cdot$$

In fact, we can further observe from (3.1) that

$$\left|\frac{H^*_{1,2}(h_1(\omega))}{R(I^1_{\omega|_n j})} - 1\right| \leq c'_1|I^1_{\omega|_n}|^\delta \quad \text{for every } \omega \in I^{\mathbb{N}}, j \in \mathbb{N}, n \in \mathbb{N}. \tag{3.6}$$

Whereas, noting that $I^1_{\omega 1+} = I^1_\omega$, it follows from (3.3) that

$$\left|\frac{H^*_{1,2}(h_1(\omega\infty))}{R(I^1_\omega)} - 1\right| \leq c'_5|I^1_\omega|^\delta \quad \text{for every } \omega \in I^*. \tag{3.7}$$

Likewise, it follows from (3.3) that for every $\omega \in I^*$, the limit of the sequence $(R(I^1_{\omega n+}))_{n=1}^\infty$ exists, is finite and non-zero. We define this limit to be

$$H^*_{1,2}(h_1(\omega\infty)) := \lim_{n\to\infty} R(I^1_{\omega n+}) \cdot$$

Thus, looking at Proposition 2.5, we see that we have defined a function $H^*_{1,2} : \overline{h_1(\Sigma)} \to (0,+\infty)$. It readily follows from (3.1) and (3.3) that

$$0 < \inf\{H^*_{1,2}\} \le \sup\{H^*_{1,2}\} < +\infty \tag{3.8}$$

and that

$$\begin{cases} \left|\frac{H^*_{1,2}(h_1(\omega))}{R(I^1_{\omega|_n})} - 1\right| \le c_1'|I^1_{\omega|_n}|^\delta \\ \\ \left|\frac{H^*_{1,2}(h_1(\tau\infty))}{R(\Delta)} - 1\right| \le c_5'(\mathrm{diam}(\Delta\cup\{h_1(\tau\infty)\}))^\delta \end{cases} \tag{3.9}$$

for all $\omega \in I^{\mathbb{N}}$, $\tau \in I^*$, $n \in \mathbb{N}$ and a τ-interval Δ. Note that it also follows from (3.1), (3.3) and Lemma 2.10 that

$$\left|\frac{H^*_{1,2}(h_1(\omega))}{H^*_{1,2}(h_1(\tau))} - 1\right| \le \tilde{c_1}|I^1_{\omega\wedge\tau[a,b]}|^\delta \le \tilde{c}|h_1(\omega) - h_1(\tau)|^\delta \tag{3.10}$$

for all $\omega, \tau \in I^{\mathbb{N}}$ with $|a-b| \ge 2$; where $a := \min\{\omega_{|\omega\wedge\tau|+1}, \tau_{|\omega\wedge\tau|+1}\}$ and $b := \max\{\omega_{|\omega\wedge\tau|+1}, \tau_{|\omega\wedge\tau|+1}\}$ and also that

$$\left|\frac{H^*_{1,2}(h_1(\tau\infty))}{H^*_{1,2}(h_1(\tau\omega))} - 1\right| \le \tilde{c_5'}|I^1_{\tau\omega_1+}|^\delta \tag{3.11}$$

for all $\tau \in I^*$ and $\omega \in I^{\mathbb{N}}$.

Now observe that the complement of the Baire set $\overline{h_1(\Sigma)}$ in the interval $[0,1]$ has all its connected components (frequently referred to as gaps of $\overline{h_1(\Sigma)}$) of the following form, viz. the gap between $I^1_{\omega n}$ and $I^1_{\omega(n+2)}$ in I^1_ω, $\omega \in I^*$, $n \in I := 2\mathbb{N}-1$ that has endpoints $h_1(\omega n\infty)$ and $h_1(\omega(n+2)1^\infty)$ and is denoted by $I^1_{\omega(n+1)}$. Denote these endpoints respectively by $a_\omega(n)$ and $b_\omega(n)$. We first extend $H^*_{1,2}$ to each gap $[a_\omega(n), b_\omega(n)]$ as follows.

Take an arbitrary $t \in \mathbb{R}$ and extend $H^*_{1,2}$ to a function $H^*_{(1,2);t} : [a_\omega(n), b_\omega(n)] \to \mathbb{R}$ by demanding that

(a) $H^*_{(1,2);t}$ is linear on $[a_\omega(n), \frac{a_\omega(n)+b_\omega(n)}{2}]$ and on $[\frac{a_\omega(n)+b_\omega(n)}{2}, b_\omega(n)]$

(b) $H^*_{(1,2);t}(a_\omega(n)) = H^*_{1,2}(a_\omega(n))$

(c) $H^*_{(1,2);t}(b_\omega(n)) = H^*_{1,2}(b_\omega(n))$
and
(d) $H^*_{(1,2);t}(\frac{a_\omega(n)+b_\omega(n)}{2}) = t.$

Now an elementary computation gives that

$$\frac{\int\limits_{a_\omega(n)}^{b_\omega(n)} H^*_{(1,2);t}(x)dx}{b_\omega(n)-a_\omega(n)} = \frac{\frac{1}{2}\big(b_\omega(n)-a_\omega(n)\big)\Big(\frac{H^*_{(1,2);t}(a_\omega(n))+t}{2}+\frac{H^*_{(1,2);t}(b_\omega(n))+t}{2}\Big)}{b_\omega(n)-a_\omega(n)} \tag{3.12}$$
$$= \frac{1}{4}\Big(H^*_{(1,2)}(a_\omega(n)) + H^*_{(1,2)}(b_\omega(n)) + 2t\Big)$$

Thus there exists a unique $t \in \mathbb{R}$ such that

$$\frac{\int\limits_{a_\omega(n)}^{b_\omega(n)} H^*_{(1,2)}(x)dx}{b_\omega(n)-a_\omega(n)} = R(a_\omega(n), b_\omega(n)) \;\left(= \frac{H_{(1,2)}(b_\omega(n)) - H_{(1,2)}(a_\omega(n))}{b_\omega(n)-a_\omega(n)}\right) \tag{3.13}$$

Set $H^*_{1,2} = H^*_{(1,2);t}$ on $[a_\omega(n), b_\omega(n)]$ with this unique t. It follows from (3.12) and (3.13) that

$$\frac{t}{R(a_\omega(n), b_\omega(n))} = 2 - \frac{1}{2}\left(\frac{H^*_{1,2}(a_\omega(n))}{R(a_\omega(n), b_\omega(n))}\right) + \frac{1}{2}\left(\frac{H^*_{1,2}(b_\omega(n))}{R(a_\omega(n), b_\omega(n))}\right).$$

Equivalently, we have that

$$\frac{t}{R(a_\omega(n), b_\omega(n))} - 1 = -\frac{1}{2}\left(\frac{H^*_{1,2}(a_\omega(n))}{R(a_\omega(n), b_\omega(n))} - 1\right) - \frac{1}{2} - \frac{1}{2}\left(\frac{H^*_{1,2}(b_\omega(n))}{R(a_\omega(n), b_\omega(n))} - 1\right). \tag{3.14}$$

Now in view of (3.9),

$$\left|\frac{H^*_{1,2}(b_\omega(n))}{R(I^1_{\omega(n+2)})} - 1\right| \le c_5 |I^1_{\omega(n+2)}|^\delta$$

and in view of (2.10) and bounded geometry of h_1, we have that

$$\left|\frac{R(I^1_{\omega(n+2)})}{R(a_\omega(n), b_\omega(n))} - 1\right| \le c_5 |I^1_{\omega(n+2)}|^\delta \le c_5(1+c)^\delta |I^1_{\omega(n+1)}|^\delta = c_5(1+c)^\delta \big(b_\omega(n) - a_\omega(n)\big)^\delta.$$

Hence we have that

$$\left|\frac{H^*_{1,2}(b_\omega(n))}{R(a_\omega(n),b_\omega(n))}-1\right|\le c_9\big(b_\omega(n)-a_\omega(n)\big)^\delta, \tag{3.15}$$

with some universal constant $c_9\ge 1$.

Likewise in view of (3.9),

$$\left|\frac{H^*_{1,2}(a_\omega(n))}{R(I^1_{\omega n})}-1\right|\le c_5|I^1_{\omega n}|^\delta$$

and in view of (3.3) and bounded geometry of h_1, we have that

$$\left|\frac{R(I^1_{\omega n})}{R(a_\omega(n),b_\omega(n))}-1\right|\le c_5|I^1_{\omega[n,n+1]}|^\delta\le c_5(1+c)^\delta|I^1_{\omega(n+1)}|^\delta=c_5(1+c)^\delta\big(b_\omega(n)-a_\omega(n)\big)^\delta.$$

Hence, we have that

$$\left|\frac{H^*_{1,2}(b_\omega(n))}{R(a_\omega(n),b_\omega(n))}-1\right|\le c_{10}\big(b_\omega(n)-a_\omega(n)\big)^\delta, \tag{3.16}$$

with some universal constant $c_{10}\ge 1$.

Combining this and (3.15) with (3.12), we get that

$$\left|\frac{t}{R(a_\omega(n),b_\omega(n))}-1\right|\le \frac{1}{2}(c_9+c_{10})\big(b_\omega(n)-a_\omega(n)\big)^\delta. \tag{3.17}$$

It follows from this that $t>0$ if $b_\omega(n)-a_\omega(n)=|I^1_{\omega(n+1)}|$ is small enough.

Now since for every $x\in I^1_{\omega(n+1)}$, the point $H^*_{1,2}(x)$ belongs to the convex hull of $H^*_{1,2}(a_\omega(n)),H^*_{1,2}(b_\omega(n))$ and t; we then have that (3.15), (3.16) and (3.17) taken together yield

$$\left|\frac{H^*_{1,2}(x)}{R(a_\omega(n),b_\omega(n))}-1\right|\le c_{11}\big(b_\omega(n)-a_\omega(n)\big)^\delta, \tag{3.18}$$

with some universal constant $c_{11}\ge 1$. In fact with a possible bigger constant c_{12}, we have

$$\left|\frac{H^*_{1,2}(x)}{H^*_{1,2}(a_\omega(n))}-1\right|,\left|\frac{H^*_{1,2}(x)}{H^*_{1,2}(b_\omega(n))}-1\right|\le c_{12}\big(b_\omega(n)-a_\omega(n)\big)^\delta. \tag{3.19}$$

It therefore follows from (3.8) that on U, the complement in $[0,1]$ of some collection of finitely many gaps, we have that

$$
(3.20) \qquad 0 \quad < \quad m := \inf\{H^*_{1,2}|_U\} \quad \leq \quad M := \sup\{H^*_{1,2}|_U\} \quad < \quad +\infty.
$$

Our aim now is to show that $H^*_{1,2}|_U$ is Hölder continuous with exponent δ. To do so let us come back to our gap $(a_\omega(n), b_\omega(n))$. Due to our definition of $H^*_{1,2}$ throughout the interval $\left[a_\omega(n), \frac{a_\omega(n)+b_\omega(n)}{2}\right]$, the absolute value of the slope of $H^*_{1,2}$ on this interval is equal to $\frac{2\left(t-H^*_{1,2}(a_\omega(n))\right)}{b_\omega(n)-a_\omega(n)}$. Therefore using (3.19) and (3.20), we get for all $a_\omega(n) \leq x \leq y \leq \frac{a_\omega(n)+b_\omega(n)}{2}$ that

$$
(3.21) \qquad \begin{cases} \left|H^*_{1,2}(y) - H^*_{1,2}(x)\right| = \dfrac{2|t - H^*_{1,2}(a_\omega(n))|}{b_\omega(n) - a_\omega(n)} \cdot |y-x| \\ \qquad\qquad\qquad\qquad \leq H^*_{1,2}(a_\omega(n))c_{12}\dfrac{|y-x|(b_\omega(n)-a_\omega(n))^\delta}{b_\omega(n)-a_\omega(n)} \\ \qquad\qquad\qquad\qquad = c_{12}H^*_{1,2}(a_\omega(n))|y-x|(b_\omega(n)-a_\omega(n))^{\delta-1} \\ \qquad\qquad\qquad\qquad \leq c_1 M|y-x|^\delta. \end{cases}
$$

Similarly, (3.21) holds for all $\frac{a_\omega(n)+b_\omega(n)}{2} \leq x \leq y \leq b_\omega(n)$. Now suppose that $a_\omega(n) \leq x \leq \frac{a_\omega(n)+b_\omega(n)}{2} \leq y \leq b_\omega(n)$. Then

$$
\frac{a_\omega(n)+b_\omega(n)}{2} - x \ , \ y - \frac{a_\omega(n)+b_\omega(n)}{2} \ \leq \ y - x \ ,
$$

and applying (3.21), we get

$$
\begin{aligned}
\left|H^*_{1,2}(y) - H^*_{1,2}(x)\right| &\leq \left|H^*_{1,2}(y) - H^*_{1,2}\left(\frac{a_\omega(n)+b_\omega(n)}{2}\right)\right| + \\
&\qquad + \left|H^*_{1,2}\left(\frac{a_\omega(n)+b_\omega(n)}{2}\right) - H^*_{1,2}(x)\right| \\
&\leq c_1 M\left(y - \frac{a_\omega(n)+b_\omega(n)}{2}\right)^\delta + \\
&\qquad + c_1 M\left(\frac{a_\omega(n)+b_\omega(n)}{2} - x\right)^\delta \\
&\leq 2c_1 M|y-x|^\delta.
\end{aligned}
$$

Thus we have that

$$
(3.22) \qquad \left|H^*_{1,2}(y) - H^*_{1,2}(x)\right| \leq 2c_1 M|y-x|^\delta
$$

holds for all $x, y \in [a_\omega(n), b_\omega(n)]$.

Now fix x, y in the same (out of finitely many) connected component of U. Let $\omega \in I^*$ be the longest word such that $x, y \in I^1_\omega$. If $x, y \in I^1_{\omega n}$ with the same $n \in 2\mathbb{N}$ (i.e. in the same gap), we are done by (3.22). So suppose that $x \in I^1_{\omega i}$ and $y \in I^1_{\omega j}$ with $i, j \in \mathbb{N}$, $i \neq j$. We may assume without loss of generality that $i < j$. Consider two cases:

Firstly, suppose that $j - i \geq 2$. In view of the first part of (3.9) and (3.18), we get that

$$\left|\frac{H^*_{1,2}(x)}{R(I^1_{\omega i})} - 1\right| \leq c_{13}|I^1_{\omega i}|^\delta \quad \text{and} \quad \left|\frac{H^*_{1,2}(y)}{R(I^1_{\omega j})} - 1\right| \leq c_{13}|I^1_{\omega j}|^\delta,$$

where $c_{13} = \max\{c_1, c_{12}\}$. Now applying (3.3) and the inequalities above, we obtain that

$$\left|\frac{H^*_{1,2}(y)}{H^*_{1,2}(x)} - 1\right| \leq c_{14}(\mathrm{diam}(I^1_{\omega i} \cup I^1_{\omega j}))^\delta \leq c_{15}|y - x|^\delta, \tag{3.23}$$

where there are constants $c_{14}, c_{15} \geq 1$. Note that we use Lemma 2.10 for the last inequality.

Secondly, suppose that $j = i + 1$. We will have 2 subcases, viz. when $I^1_{\omega i}$ is an interval and $I^1_{\omega(i+1)}$ is a gap and vice versa.

Subcase 1. $I^1_{\omega i}$ is an interval and $I^1_{\omega(i+1)}$ is a gap.

In this case, we have an interval followed by a gap. Let a be the right-hand end-point of the interval $I^1_{\omega i}$, i.e. $a = h(\omega i \infty)$. Since $I^1_{\omega(i+1)}$ is a gap we have that

$$|H^*_{1,2}(y) - H^*_{1,2}(a)| \leq c|y - a|^\delta. \tag{3.24}$$

Now let $x \in I_{\omega in}$ for some $n \in \mathbb{N}$. There are two subcases: either x belongs to a gap or to $h_1(\Sigma)$. If x belongs to a gap, then let x' be the right-hand end-point of the gap. If $x \in h_1(\Sigma)$, set $x' = x$. In either case we have from (3.22),

$$|H^*_{1,2}(x) - H^*_{1,2}(x')| \leq c^1|x - x'|^\delta. \tag{3.25}$$

Since $x' \in h_1(\Sigma)$, we get from (3.11) that

$$|H^*_{1,2}(a) - H^*_{1,2}(x')| \leq c_2|I_{\omega in+}|^\delta = c_2|a - x'|^\delta. \tag{3.26}$$

Now combining (3.24), (3.25) and (3.26) we have that

$$|H^*_{1,2}(y) - H^*_{1,2}(x)| \leq c_4|y-x|^\delta$$

and so we are done in this case.

Subcase 2. $I^1_{\omega i}$ is a gap and $I^1_{\omega(i+1)}$ is an interval.

Now let a denote the right hand end point of the gap $I_{\omega i}$, i.e. $a = h(\omega(i+1)1^\infty)$. Since $I_{\omega i}$ is a gap, for $x \in I_{\omega i}$ we know that

$$|H^*_{1,2}(x) - H^*_{1,2}(a)| \leq c|x-a|^\delta. \tag{3.27}$$

Now let $y \in I_{\omega(i+1)n}$ for some $n \in \mathbb{N}$. Again there are two subcases: either y belongs to a gap or to $h_1(\Sigma)$. If y belongs to a gap, then let y' be the right-hand end-point of the gap. If $y \in h_1(\Sigma)$, set $y' = y$. In either case we have

$$|H^*_{1,2}(y) - H^*_{1,2}(y')| \leq c|y-y'|^\delta. \tag{3.28}$$

Now $y' \in h_1(\Sigma)$, say $y' = h(\omega(i+1)\kappa)$ with $\kappa \in I^{\mathbb{N}}, \kappa_1 = n$. Since $y' \neq a = h(\omega(i+1)1^\infty)$, there must exist some j such that $\kappa_j \neq 1$ and $\kappa_j \in 2\mathbb{N}-1$. Now we would like to use (3.10), where we also use Lemma 2.10. Using notation from the Lemma we now have that $|a-b| = |\kappa_j - 1| \geq 2$, and thus we obtain

$$|H^*_{1,2}(a) - H^*_{1,2}(y')| \leq c|a-y'|^\delta. \tag{3.29}$$

Now combining (3.27), (3.28) and (3.29) we have that

$$|H^*_{1,2}(y) - H^*_{1,2}(x)| \leq c|y-x|^\delta$$

and so we are done in this case.

Thus we have proved the theorem.

$\square$

4. The class of IFS-like Baire embeddings and Scaling functions

In this section we deal with the class of Baire embeddings that give rise to iterated function systems in the sense of [3]. We will call them IFS-like in the sequel.

Definition 4.1. *For the shift map σ recall that the inverse branches were labeled σ_i^{-1}. For every $i \in 2\mathbb{N}-1$ set $\psi_i := h \circ \sigma_i^{-1} \circ h^{-1} : h(\Sigma) \to h([i])$. A Baire embedding $h : \Sigma \to [0,1]$ is IFS-like if each map $\psi_i := h \circ \sigma_i^{-1} \circ h^{-1} : h(\Sigma) \to h([i])$ has a bijective differentiable extension $\phi_i : co(h(\Sigma)) = [0,1] \to I_i = co(h([i]))$ with the following property: $\exists \varepsilon > 0\ \exists L > 0\ \forall i \in 2\mathbb{N}-1\ \forall x, y \in [0,1]$*

$$|\phi_i'(y) - \phi_i'(x)| \le L \cdot \inf\{|\phi_i'(z)| : z \in [0,1]\} \cdot |y-x|^\varepsilon. \tag{4.1}$$

For every $\omega \in (2\mathbb{N}-1)^*$, say $\omega \in (2\mathbb{N}-1)^n$, set

$$\phi_n := \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_n} : [0,1] \to I_\omega$$

Now as in Lemma 4.2.2 in [2], we shall prove the following

Lemma 4.2. *Let h be a Baire-embedding that is IFS-like. Then for every $\omega \in (2\mathbb{N}-1)^*$, there exists $T > 0$ such that for all $x, y \in [0,1]$,*

$$\big|\log|\phi_\omega'(y)| - \log|\phi_\omega'(x)|\big| \le T\cdot$$

Proof. For every $\omega \in (2\mathbb{N}-1)^*$, say $\omega \in (2\mathbb{N}-1)^n$, set $z_k := \phi_{\omega_{n-k+1}} \circ \phi_{\omega_{n-k+2}} \circ \cdots \circ \phi_{\omega_n}(z)$ for $z \in [0,1]$. Put $z_0 = z$. Recall that $D_n(h) := \sup_{\omega \in I^n} \operatorname{diam}(h([\omega]))$. Then for $x, y \in [0,1]$ we have that

$$\begin{aligned}
\big|\log|\phi_\omega'(y)| - \log|\phi_\omega'(x)|\big| &= \Big|\sum_{j=1}^n \log\left(1 + \frac{|\phi_{\omega_j}'(y_{n-j})| - |\phi_{\omega_j}'(x_{n-j})|}{|\phi_{\omega_j}'(x_{n-j})|}\right)\Big| \\
&\le \sum_{j=1}^n \|(\phi_{\omega_j}')^{-1}\| \big|\log|\phi_{\omega_j}'(y_{n-j})| - \log|\phi_{\omega_j}'(x_{n-j})|\big| \\
&\le \sum_{j=1}^n L|y_{n-j} - x_{n-j}|^\varepsilon \qquad \text{by (4.1)} \\
&\le \sum_{j=1}^n L|I_{\sigma^j\omega}|^\varepsilon \\
&\le \sum_{j=1}^\infty (D_{n-j}(h))^\varepsilon \\
&\le \sum_{j=0}^\infty (D_j(h))^\varepsilon =: T < +\infty, \qquad \text{by Lemma 2.21.}
\end{aligned}$$

$\square$

Proposition 4.3. *Suppose $h : \Sigma \to [0,1]$ is an IFS-like Baire embedding. Then there exists $s > 0$ and $c \geq 1$ such that*

$$\|\phi'_\omega\|_\infty \leq c\exp(-s|\omega|),$$

for all $\omega \in (2\mathbb{N}-1)^*$.

Proof. By Lemma 2.21 there exists $s > 0$ and $c_1 \geq 1$ such that

$$|I_\omega| \leq c_1 \exp(-s|\omega|)$$

for all $\omega \in (2\mathbb{N}-1)^*$. Then by Lemma 4.2, we get that

$$\|\phi'_\omega\|_\infty \leq \exp(T)\inf\{|\phi'_i(z)| : z \in [0,1]\} \leq \exp(T)|I_\omega| \leq c_1 \exp(T)\exp(-s|\omega|)$$

□

Lemma 4.4. *Let h be a Baire-embedding that is IFS-like. Then for every $\omega \in (2\mathbb{N}-1)^*$, there exists $s > 0$ and $c \geq 1$ such that for all $x, y \in [0,1]$,*

$$\big|\log|\phi'_\omega(y)| - \log|\phi'_\omega(x)|\big| \leq L\frac{c^\varepsilon}{1-\exp(-\varepsilon s)}\ |y-x|^\varepsilon.$$

Proof. Note that

$$\begin{aligned}|y_{n-j} - x_{n-j}| &= |\phi_{\omega_{j+1}} \circ \phi_{\omega_{j+2}} \circ \cdots \circ \phi_{\omega_n}(x) - \phi_{\omega_{j+1}} \circ \phi_{\omega_{j+2}} \circ \cdots \circ \phi_{\omega_n}(y)| \\ &\leq \|\phi'_{\omega_{j+1}\ldots\omega_n}\| \cdot |x-y| \\ &\leq c\exp(-s(n-j))|x-y|.\end{aligned}$$

Therefore

$$|y_{n-j} - x_{n-j}|^\varepsilon \leq c^\varepsilon \exp(-\varepsilon s(n-j))|x-y|^\varepsilon.$$

Then using this estimate in the inequality from the proof of Lemma 4.2, we have that

$$\begin{aligned}\big|\log|\phi'_\omega(y)| - \log|\phi'_\omega(x)|\big| &\leq \sum_{j=1}^n L|y_{n-j} - x_{n-j}|^\varepsilon \\ &\leq Lc^\varepsilon \sum_{j=1}^n \exp(-\varepsilon s(n-j))|x-y|^\varepsilon \\ &\leq L\frac{c^\varepsilon}{1-\exp(-\varepsilon s)}|x-y|^\varepsilon.\end{aligned}$$

□

Definition 4.5. *We now define the dual Cantor set and the functions $S_n(\omega; j)$. Define $\mathbb{N}^- := \{\ldots, -3, -2, -1\}$. Then the dual Cantor set is defined as $\tilde{\Sigma} :=$*

$(2\mathbb{N}-1)^{\mathbb{N}^-}$. *Now define for every $n \in \mathbb{N}$ the functions $S_n : \tilde{\Sigma} \times \mathbb{N} \to [0,1]$ given by*

$$S_n(\omega; j) = A(\omega|_n)(j) = \frac{|I_{\omega|_n j}|}{|I_{\omega|_n}|} \ ,$$

where $\omega|_n = \omega_{-n} \dots \omega_{-1}$ and $j \in \mathbb{N}$.

Theorem 4.6. *If the map $h : \Sigma \to I$ is IFS-like, then $\exists c > 0, \exists \varepsilon \in (0,1]$ such that*

(a) $\forall \omega \in \tilde{\Sigma}, \forall j \in \mathbb{N}, \forall n \geq 1$

$$\left| \frac{S_{n+1}(\omega; j)}{S_n(\omega; j)} - 1 \right| \leq L |I_{\omega|_n}|^{\varepsilon} \tag{4.2}$$

(b) $\forall \omega \in \tilde{\Sigma}, \forall i, j \in \mathbb{N}, \forall n \geq 1,$

$$\left| \Big(\frac{S_{n+1}(\omega; j)}{S_n(\omega; j)} \Big/ \frac{S_{n+1}(\omega; i)}{S_n(\omega; i)} \Big) - 1 \right| \leq c \left| I_{\omega|_n [i,j]} \right|^{\varepsilon} . \tag{4.3}$$

(c) *[consequence of (a)]*

$$S(\omega; j) := \lim_{n \to \infty} S_n(\omega; j) \quad \textit{exists, and}$$

$$\left| \frac{S(\omega; j)}{S_n(\omega; j)} - 1 \right| \leq c' |I_{\omega|_n}|^{\varepsilon} \ . \tag{4.4}$$

We call this $S(\omega; j)$ the Sullivan scaling function of the Baire embedding h.

Proof. (a) The proof follows from the following sequence of inequalities given below,

$$
\begin{aligned}
\left|\frac{S_{n+1}(\omega;j)}{S_n(\omega;j)}-1\right| &= \left|\frac{|I_{\omega|_{(n+1)}j}|/|I_{\omega|_{(n+1)}}|}{|I_{\omega|_n j}|/|I_{\omega|_n}|}-1\right| \\
&= \left|\frac{|\phi_{\omega_{-(n+1)}}(I_{\omega|_n j})|/|I_{\omega|_n}|}{|\phi_{\omega_{-(n+1)}}(I_{\omega|_n})|/|I_{\omega|_n}|}-1\right| \\
&= \left|\frac{\phi'_{\omega_{-(n+1)}}(y_n)}{\phi'_{\omega_{-(n+1)}}(x_n)}-1\right| \\
&\qquad \text{(for some } x_n \in I_{\omega|_n}, y_n \in I_{\omega|_n j} \text{ by the Mean Value Theorem)} \\
&= \frac{\left|\phi'_{\omega_{-(n+1)}}(y_n)-\phi'_{\omega_{-(n+1)}}(x_n)\right|}{\left|\phi'_{\omega_{-(n+1)}}(x_n)\right|} \\
&\le \frac{L\cdot\inf\{|\phi'_{\omega_{-(n+1)}}(z)| : z\in[0,1]\}\cdot|y_n-x_n|^{\varepsilon}}{\left|\phi'_{\omega_{-(n+1)}}(x_n)\right|} \qquad \text{(by (4.1))} \\
&\le L|y_n-x_n|^{\varepsilon} \\
&\le L|I_{\omega|_n}|^{\varepsilon}
\end{aligned}
$$

(b) The proof also follows a very similar strategy to that of part (a).

$$
\begin{aligned}
\left|\left(\frac{S_{n+1}(\omega;j)}{S_n(\omega;j)}\Big/\frac{S_{n+1}(\omega;i)}{S_n(\omega;i)}\right)-1\right| &= \left|\frac{|\phi_{\omega|_{(n+1)}}(I_j)|/|\phi_{\omega|_n}(I_j)|}{|\phi_{\omega|_{(n+1)}}(I_i)|/|\phi_{\omega|_n}(I_i)|}-1\right| \\
&= \left|\frac{\phi'_{\omega_{-(n+1)}}(y)}{\phi'_{\omega_{-(n+1)}}(x)}-1\right| \\
&\qquad \text{(for some } x\in I_{\omega|_n}, y\in I_{\omega|_n j}, \text{by the Mean Value Theorem)} \\
&= \frac{|\phi'_{\omega_{-(n+1)}}(y)-\phi'_{\omega_{-(n+1)}}(x)|}{|\phi'_{\omega_{-(n+1)}}(x)|} \\
&\le L|y-x|^{\varepsilon} \\
&\le L\left|I_{\omega|_n[i,j]}\right|^{\varepsilon}.
\end{aligned}
$$

(c) This follows from the estimate in part (a).

$\square$

Lemma 4.7. *Let us stick to the same set-up in the previous theorem. Then* $\exists c_3 > 0$ *such that* $\forall \omega \in \tilde{\Sigma}, \forall i, j \in \mathbb{N}, \forall n \geq 1$

$$\left| \Big(\frac{S_n(\omega; j)}{S(\omega; j)} \Big/ \frac{S_n(\omega; i)}{S(\omega; i)} \Big) - 1 \right| \leq c_3 |I_{\omega[i,j]}|^{\varepsilon}.$$

Proof. Note that

$$\frac{S_n(\omega; j)}{S_{n+k}(\omega; j)} = \prod_{l=n}^{n+k-1} \frac{S_l(\omega; j)}{S_{l+1}(\omega; j)} \ ,$$

and therefore that

$$\begin{aligned} \frac{S_n(\omega; j)}{S_{n+k}(\omega; j)} \Big/ \frac{S_n(\omega; i)}{S_{n+k}(\omega; i)} &= \prod_{l=n}^{n+k-1} \frac{S_l(\omega; j)}{S_{l+1}(\omega; j)} \Big/ \prod_{l=n}^{n+k-1} \frac{S_l(\omega; i)}{S_{l+1}(\omega; i)} \\ &= \prod_{l=n}^{n+k-1} \frac{S_l(\omega; j)}{S_{l+1}(\omega; j)} \Big/ \frac{S_l(\omega; i)}{S_{l+1}(\omega; i)}. \end{aligned}$$

Therefore we have that

$$\begin{aligned} \left| \log\left(\frac{S_n(\omega; j)}{S_{n+k}(\omega; j)} \Big/ \frac{S_n(\omega; i)}{S_{n+k}(\omega; i)} \right) \right| &= \left| \log\left(\prod_{l=n}^{n+k-1} \frac{S_l(\omega; j)}{S_{l+1}(\omega; j)} \Big/ \frac{S_l(\omega; i)}{S_{l+1}(\omega; i)} \right) \right| \\ &= \Big| \sum_{l=n}^{n+k-1} \log(1 + c_l |I_{\omega|_l[i,j]}|^{\varepsilon}) \Big| \\ &\qquad \text{(for some } c_l \in [-c, c]\text{, where } c \text{ is from (4.3).)} \\ &\leq \sum_{l=n}^{n+k-1} \frac{1}{1 + w_l} |c_l| |I_{\omega|_l[i,j]}|^{\varepsilon} \\ &\qquad \text{(for some } |w_l| < \big| c_l |I_{\omega|_l[i,j]}|^{\varepsilon} \big| \text{ by the Mean Value Theorem.)} \\ &\leq 2|c_l| \sum_{l=n}^{n+k-1} |I_{\omega|_l[i,j]}|^{\varepsilon} \\ &\leq 2|c_l| \sum_{l=n}^{\infty} |I_{\omega|_l[i,j]}|^{\varepsilon} \\ &\leq 2|c_l| \sum_{l=n}^{\infty} |I_{\omega|_n[i,j]}|^{\varepsilon} s^{\varepsilon(l-n)} \\ &\qquad \text{(for some } s < 1\text{, see Lemma 2.21.)} \\ &= 2|c_l| |I_{\omega|_n[i,j]}|^{\varepsilon} (1 - s^{\varepsilon})^{-1} \end{aligned}$$

Thus we now have that

$$\left|\log\left(\frac{S_n(\omega;j)}{S_{n+k}(\omega;j)}\Big/\frac{S_n(\omega;i)}{S_{n+k}(\omega;i)}\right)\right| \le M|I_{\omega|_n[i,j]}|^{\varepsilon},$$

where $M = 2|c_l|(1-s^{\varepsilon})^{-1}$.
Finally noting that for $y \in \mathbb{R}$ we have that $1-2y \le e^y \le 1+2y$ and therefore that $|e^y - 1| \le 2y$ we conclude that

$$\left|\left(\frac{S_n(\omega;j)}{S(\omega;j)}\Big/\frac{S_n(\omega;i)}{S(\omega;i)}\right) - 1\right| \le 2M|I_{\omega[i,j]}|^{\varepsilon}.$$

We are done noting that $c_3 = 2M$. $\square$

Proposition 4.8. *If h_1, h_2 are IFS-like and are $C^{1+\varepsilon}$-equivalent, then $S_{h_1} = S_{h_2}$, i.e. their scaling functions are the same.*

Proof. Notice that we have

$$\left|\frac{S_{h_2}(\omega|_n;j)}{S_{h_1}(\omega|_n;j)} - 1\right| \le \min\left\{|I^{(2)}_{\omega|_n}|^{\delta}, |I^{(1)}_{\omega|_n}|^{\delta}\right\}.$$

This implies that $S_{h_1}(\omega;j) = S_{h_2}(\omega;j)$. $\square$

Proposition 4.9. *If h_1, h_2 are IFS-like, $h_1 \circ h_2^{-1}$ is bi-Hölder continuous and $S_{h_1} = S_{h_2}$, then $h_1 \sim^{1+\varepsilon} h_2$.*

Proof. Let $\omega \in \tilde{\Sigma}$.

$$\begin{aligned}
\left|\frac{A_1(h_2,\omega|_n)}{A_1(h_1,\omega|_n)} - 1\right| &= \left|\frac{S_{h_2}(\omega|_n;j)}{S_{h_1}(\omega|_n;j)} - 1\right| \\
&= \left|\frac{S_{h_2}(\omega|_n;j)}{S_{h_2}(\omega;j)} \cdot \frac{S_{h_1}(\omega;j)}{S_{h_1}(\omega|_n;j)} - 1\right| \\
&\le c_2 \max\left\{\left|I^{(2)}_{\omega|_n}\right|^{\varepsilon}, \left|I^{(1)}_{\omega|_n}\right|^{\varepsilon}\right\} \text{ (using (4.4))} \\
&\le c_2 \min\left\{\left|I^{(1)}_{\omega|_n}\right|^{\eta}, \left|I^{(2)}_{\omega|_n}\right|^{\eta}\right\} \text{ (since } h_1 \circ h_2^{-1} \text{ is Hölder),}
\end{aligned}$$

with some $0 < \eta \le \varepsilon$.

Now using Lemma 4.7, we get that

$$\begin{aligned}
\left|\frac{Q_{h_2}(\omega|_n;i,j)}{Q_{h_1}(\omega|_n;i,j)}-1\right| &= \left|\frac{S_{h_2}(\omega|_n;j)}{S_{h_2}(\omega|_n;i)}\Big/\frac{S_{h_1}(\omega|_n;j)}{S_{h_1}(\omega|_n;i)}-1\right| \\
&= \left|\frac{S_{h_2}(\omega|_n;j)}{S_{h_1}(\omega|_n;j)}\Big/\frac{S_{h_2}(\omega|_n;i)}{S_{h_1}(\omega|_n;i)}-1\right| \\
&= \left|\left(\left|\frac{S_{h_2}(\omega|_n;j)}{S_{h_2}(\omega;j)}\right|\Big/\left|\frac{S_{h_1}(\omega|_n;j)}{S_{h_1}(\omega;j)}\right|\Bigg/\left|\frac{S_{h_2}(\omega|_n;i)}{S_{h_2}(\omega;i)}\right|\Big/\left|\frac{S_{h_1}(\omega|_n;i)}{S_{h_1}(\omega;i)}\right|\right)-1\right| \\
&= \left|\left(\left|\frac{S_{h_2}(\omega|_n;j)}{S_{h_2}(\omega;j)}\right|\Big/\left|\frac{S_{h_2}(\omega|_n;i)}{S_{h_2}(\omega;i)}\right|\Bigg/\left|\frac{S_{h_1}(\omega|_n;j)}{S_{h_1}(\omega;j)}\right|\Big/\left|\frac{S_{h_1}(\omega|_n;i)}{S_{h_1}(\omega;i)}\right|\right)-1\right| \\
&\le c_4\max\left\{\left|I^{(2)}_{\omega|_n[i,j]}\right|^{\varepsilon},\left|I^{(1)}_{\omega|_n[i,j]}\right|^{\varepsilon}\right\} \\
&\le c_4\min\left\{|I^{(2)}_{\omega|_n}|^{\eta},|I^{(1)}_{\omega|_n}|^{\eta}\right\}\ \text{(since } h_1\circ h_2^{-1} \text{ is Hölder).}
\end{aligned}$$

The proof is concluded by invoking Theorem 3.1. □

Thus we can finally state the main result of the section, viz.

Theorem 4.10. *Let h_1, h_2 be IFS-like. If $h_1 \sim^{1+\varepsilon} h_2$, then $S_{h_1} = S_{h_2}$. Conversely, if $S_{h_1} = S_{h_2}$ and $h_1 \circ h_2^{-1}$ is bi-Hölder continuous, then $h_1 \sim^{1+\varepsilon} h_2$.*

Proof. Follows immediately from Propositions 4.8 and 4.9. □

It was proved in [1] (Theorem 4.3) that if $h_1 \sim^{1+0} h_2$, then $S_{h_1} = S_{h_2}$. Note that this also follows easily from our considerations here. Now as an immediate consequence of this and Theorem 4.10 we get the following rigidity result:

Corollary 4.11. *If h_1, h_2 are two IFS-like C^{1+0}-equivalent Baire embeddings, then they are $C^{1+\varepsilon}$-equivalent with some $\varepsilon > 0$.*

5. Gibbs States, Dual measures and Conditional measures

We begin this section by introducing some notation and definitions. In a sense this section is independent of the rest of the paper and we will thus attempt to write it in such a fashion.

Definition 5.1. *Let us make the following conventions:* $\mathbb{N} := \{1, 2, 3 \dots\}$; $\mathbb{N}_0 := \{0, 1, 2, 3 \dots\}$; $\mathbb{N}^- := \{\dots -3, -2, -1\}$ *and* $\mathbb{N}_0^- := \{\dots -3, -2, -1, 0\}$. *Now let I be a countable set and $A : I \times I \to \{0, 1\}$ be a finitely primitive incidence matrix.*

$\Sigma^0 =: \Sigma_A^0 := \{\omega \in I^{\mathbb{N}_0} : A_{\omega_n\omega_{n+1}} = 1, \forall n \ge 0\}$ *will be called the symbol space.*

$\widetilde{\Sigma}^0$:=: $\widetilde{\Sigma^0_A}$:= $\{\omega \in I^{\mathbb{N}_0^-} : A_{\omega_{n-1}\omega_n} = 1, \forall n \leq 0\}$ will be called the dual symbol space.

We also define $\widetilde{\Sigma}$:=: $\widetilde{\Sigma_A}$:= $\{\omega \in I^{\mathbb{N}^-} : A_{\omega_{n-1}\omega_n} = 1, \forall n \leq 1\}$.

Now let μ be a Borel probability shift-invariant measure on Σ^0_A. For every finite word $\omega \in {\Sigma^0_A}^*$, set $\tilde{\mu}([\omega]) = \mu([\omega])$, where $[\omega]$ on the left-hand side of the equality is treated as a subset of $\widetilde{\Sigma^0_A}$, whereas $[\omega]$ on the right-hand side is treated as a subset of Σ^0_A.
Let us define

$$[\omega|_a^b]_m^{m+b-a} := \{\tau : \tau_{m+i} = \omega_{a+i} \ \forall 0 \leq i \leq b-a\}.$$

Then one can define

$$\tilde{\mu}\Big([\omega|_1^n]_{-m}^{-m+n-1}\Big) := \mu\Big([\omega|_1^n]_0^{n-1}\Big).$$

Note that since the measure μ is shift invariant, $\tilde{\mu}$ extends uniquely to an additive function on the algebra generated by finite-length cylinders. It is also easy to check that the continuity condition is satisfied and thus $\tilde{\mu}$ extends to a (σ-additive) measure on $\widetilde{\Sigma_A}$. Also we note that since μ is measure, $\tilde{\mu}$ is (right) shift-invariant.
Let us denote the space of shift-invariant Borel probability measures on $\widetilde{\Sigma^0_A}$ and Σ^0_A by $\mathcal{M}(\widetilde{\Sigma^0_A})$ and $\mathcal{M}(\Sigma^0_A)$ respectively. Thus we have just defined a map

$$\mathcal{M}(\Sigma^0_A) \ni \mu \mapsto \tilde{\mu} \in \mathcal{M}(\widetilde{\Sigma^0_A});$$

which one can also define in the reverse direction by symmetry, i.e.

$$\mathcal{M}(\widetilde{\Sigma^0_A}) \ni \nu \mapsto \tilde{\nu} \in \mathcal{M}(\Sigma^0_A).$$

Note that the $\tilde{}$ map is an involution, i.e. $\tilde{\tilde{\mu}} = \mu$ and hence is a bijection between $\mathcal{M}(\widetilde{\Sigma^0_A})$ and $\mathcal{M}(\Sigma^0_A)$.
Let $\xi := \{[\omega] : \omega \in \widetilde{\Sigma_A}\}$. One can check that ξ is a measurable partition of $\widetilde{\Sigma^0_A}$, and let $\{\mu_\omega\}_{\omega \in \widetilde{\Sigma_A}}$ be the corresponding canonical system of conditional measures. Then it follows from the Martingale Convergence Theorem that for μ-a.e. $\omega \in \widetilde{\Sigma^0_A}$,

$$\tilde{S}_\mu(\omega) := \tilde{\mu}_{\omega|_{-\infty}^{-1}}(\omega) = \lim_{n\to\infty} \frac{\tilde{\mu}([\omega|_{-n}^0])}{\tilde{\mu}([\omega|_{-n}^{-1}])} \cdot$$

We call $\tilde{S}_\mu : \widetilde{\Sigma_A^0} \to [0,1]$ *the scaling function of the measure* μ. *It is clear from the formula above that* $\tilde{S}_\mu$ *is the inverse of the Jacobian of the right-oriented shift map* $\sigma : \widetilde{\Sigma_A^0} \to \widetilde{\Sigma_A^0}$ *with respect to the measure* $\tilde{\mu}$, *i.e.*

$$\tilde{S}_\mu(\omega) := \Big(\frac{d\tilde{\mu}\circ\sigma}{d\tilde{\mu}}(\omega)\Big)^{-1} =: \big(\mathfrak{J}_\sigma(\tilde{\mu})\big)^{-1}(\omega)\ \cdot \tag{5.1}$$

We also note that for every $\omega \in \widetilde{\Sigma_A}$,

$$\sum_{j\,:\,A_{\omega_{-1}j}=1} \tilde{S}_\mu(\omega j) = 1\ \cdot \tag{5.2}$$

We shall now define partition functions, topological pressure and Gibbs states and state some of their well-known properties (see for e.g.[3]).

Definition 5.2. *Given a function* $f : \Sigma_A^0 \to \mathbb{R}$ *we define the nth partition function by*

$$Z_n(f) := \sum_{\omega\in(\Sigma_A^0)^n} \exp\big(\sup_{\tau\in[\omega]} \sum_{j=0}^{n-1} f(\sigma^j(\tau))\big)\ \cdot$$

Note that we would have the analogous definition for function defined on $\widetilde{\Sigma_A^0}$. *We denote the nth partial orbit sum by*

$$S_n(f) := \sum_{j=0}^{n-1} f\circ\sigma^j.$$

Next we define the topological pressure of f *with respect to the shift map* σ *to be*

$$P(f) := \lim_{n\to\infty} \frac{1}{n}\log Z_n(f) = \inf_{n\in\mathbb{N}} \frac{1}{n}\log Z_n(f)\ \cdot$$

Definition 5.3. *If* $f : \Sigma_A^0 \to \mathbb{R}$ *is a Hölder continuous function, then a Borel probability measure* μ_f *on* Σ_A *is called a Gibbs state for* f, *when there exist constants* $Q \geq 1$ *and* P_{μ_f} *such that for every* $\omega \in \Sigma_A^*$ *and every* $\tau \in [\omega]$ *we have that*

$$Q^{-1} \leq \frac{\mu_f([\omega])}{\exp(S_{|\omega|}f(\tau)) - P_{\mu_f}\cdot|\omega|)} \leq Q\ \cdot \tag{5.3}$$

In addition, if μ_f *is shift-invariant, it is then called an invariant Gibbs state. Also note that we have the analogous definition for functions* $f : \widetilde{\Sigma_A^0} \to \mathbb{R}$. *Notice that* $S_{|\omega|}f(\tau)$ *in the definition refers to the* $|\omega|$*-th partial orbit sum of* f *with respect to the shift and should not be confused with the scaling function* $\tilde{S}_\mu : \widetilde{\Sigma_A^0} \to [0,1]$. *Let us denote the spaces of invariant Gibbs states of* $\widetilde{\Sigma_A^0}$ *and* Σ_A^0 *by* $\mathcal{G}(\widetilde{\Sigma_A^0})$ *and* $\mathcal{G}(\Sigma_A^0)$ *respectively.*

We recall that a Hölder continuous function $f : \widetilde{\Sigma_A^0} \to \mathbb{R}$ *with an exponent* $\beta > 0$ *that satisfies the condition*

$$\sum_{i \in I} \exp(\sup(f|_{[i]})) < \infty$$

is called summable.

Definition 5.4. *A function* $\phi : \widetilde{\Sigma_A^0} \to (0,1)$ *is said to be a Keane function if and only if*

$$\phi \textit{ is Hölder continuous and } \sum_{j\,:\,A_{\omega_{-1}j}=1} \phi(\omega j) = 1, \textit{for all}\, \omega \in \widetilde{\Sigma_A}.$$

Likewise, a function $\psi : \Sigma_A^0 \to (0,1)$ *is said to be a Keane function if and only if*

$$\psi \textit{ is Hölder continuous and } \sum_{j\,:\,A_{j\omega_1}=1} \psi(j\omega) = 1, \textit{for all}\, \omega \in \Sigma_A.$$

We denote the class of Keane functions on $\widetilde{\Sigma_A^0}$ *and* Σ_A^0 *by* $\mathcal{K}(\widetilde{\Sigma_A^0})$ *and* $\mathcal{K}(\Sigma_A^0)$ *respectively.*

Remark 5.5. *We now state the following well-known fact about Gibbs states, viz. If* μ *is a Gibbs state, then* $-\log \mathfrak{J}_\mu$ *is Hölder continuous. Conversely, if* μ *is an invariant Borel probability measure and* $-\log \mathfrak{J}_\mu$ *is Hölder continuous, then* $-\log \mathfrak{J}_\mu$ *is summable and* μ *is a Gibbs state for* $-\log \mathfrak{J}_\mu$*. In other words, the map* $\mu \mapsto -\log \mathfrak{J}_\mu$ *is a bijection between* $\mathcal{G}$ *and* $\mathcal{K}$ *for both* Σ_A^0 *and* $\widetilde{\Sigma_A^0}$*.*

Theorem 5.6. *The following hold:*

(a) *If* $\mu \in \mathcal{G}(\Sigma_A^0)$*, then* $\tilde{\mu} \in \mathcal{G}(\widetilde{\Sigma_A^0})$*.*
(b) *The mapping* $\mathcal{G}(\Sigma_A^0) \ni \mu \mapsto \tilde{\mu} \in \mathcal{G}(\widetilde{\Sigma_A^0})$ *is a bijection.*
(c) *The mapping* $\mu \mapsto \tilde{S}_\mu$ *is a bijection between* $\mathcal{G}(\Sigma_A)$ *and the class of Keane functions* $\mathcal{K}(\widetilde{\Sigma_A^0})$*.*

Proof. Consider $\mu \in \mathcal{G}(\Sigma_A^0)$ and put $\phi = -\log \mathfrak{J}_\mu$.In view of Remark 5.5, in order to prove that $\tilde{S}_\mu$ is a Keane function, it suffices to demonstrate that $\tilde{S}_\mu$ is a nowhere-vanishing Hölder continuous function.
In order to prove that, for every $\omega \in \widetilde{\Sigma_A^0}$ and every $n \geq 1$ set

$$\tilde{S}_n(\omega) := \frac{\tilde{\mu}([\omega|_{-n}^{0}])}{\tilde{\mu}([\omega|_{-n}^{-1}])}\ .$$

Then for every $k \geq 0$,

$$\frac{\tilde{S}_{n+k}(\omega)}{\tilde{S}_n(\omega)} = \frac{\tilde{\mu}([\omega|^0_{-(n+k)}])}{\tilde{\mu}([\omega|^0_{-n}])} \cdot \frac{\tilde{\mu}([\omega|^{-1}_{-n}])}{\tilde{\mu}([\omega|^{-1}_{-(n+k)}])}$$

$$= \frac{\tilde{\mu}([\omega|^0_{-(n+k)}])}{\int\limits_{[\omega|^0_{-(n+k)}]} \exp(-S_k\phi(\tau))d\mu(\tau)} \cdot \frac{\int\limits_{[\omega|^{-1}_{-(n+k)}]} \exp(-S_k\phi(\tau))d\mu(\tau)}{\tilde{\mu}([\omega|^{-1}_{-(n+k)}])}\ , \tag{5.4}$$

where in the second line of this formula we treated $[\omega|^0_{-(n+k)}]$ and $[\omega|^{-1}_{-(n+k)}]$ as subsets of $\widetilde{\Sigma^0_A}$. Now, fix $\hat{\omega} \in \Sigma^0_A$ such that $\hat{\omega}|_0^{n+k} = \omega|^0_{-(n+k)}$. We can then continue (5.4) as follows:

$$\frac{\tilde{S}_{n+k}(\omega)}{\tilde{S}_n(\omega)} = \frac{\tilde{\mu}([\omega|^0_{-(n+k)}])}{\exp(-S_k\phi(\hat{\omega}))\int\limits_{[\omega|^0_{-(n+k)}]} \exp(S_k\phi(\hat{\omega}) - S_k\phi(\tau))d\mu(\tau)} \cdot$$

$$\cdot \frac{\exp(-S_k\phi(\hat{\omega}))\int\limits_{[\omega|^{-1}_{-(n+k)}]} \exp(S_k\phi(\hat{\omega}) - S_k\phi(\tau))d\mu(\tau)}{\tilde{\mu}([\omega|^{-1}_{-(n+k)}])}$$

$$= \frac{\tilde{\mu}([\omega|^0_{-(n+k)}])}{\int\limits_{[\omega|^0_{-(n+k)}]} \exp(S_k\phi(\hat{\omega}) - S_k\phi(\tau))d\mu(\tau)} \cdot \frac{\int\limits_{[\omega|^{-1}_{-(n+k)}]} \exp(S_k\phi(\hat{\omega}) - S_k\phi(\tau))d\mu(\tau)}{\tilde{\mu}([\omega|^{-1}_{-(n+k)}])}\ .$$

Now, since $|S_k\phi(\hat{\omega}) - S_k\phi(\tau)| \leq ce^{-\alpha n}$ with the same universal constant $c > 0$ for all $\tau \in [\omega|^{-1}_{-(n+k)}]$, we can further write

$$\frac{\tilde{S}_{n+k}(\omega)}{\tilde{S}_n(\omega)} = (1 \pm c_2e^{-\alpha n})(1 \pm c_2e^{-\alpha n}) = 1 \pm c_3e^{-\alpha n}\ ,$$

with some numbers $c_1, c_2, c_3 > 0$ bounded above independently of ω, n and k, say by $c > 0$. In other words,

$$\left|\frac{\tilde{S}_{n+k}(\omega)}{\tilde{S}_n(\omega)} - 1\right| \leq ce^{-\alpha n}\ . \tag{5.5}$$

Now an elementary analysis shows that $\tilde{S}_\mu = \lim_{n\to\infty} \tilde{S}_n$ is a nowhere-vanishing function whose logarithm is Hölder continuous function with exponent α.

Finally, the proof of all the items in the theorem follow by applying (5.1) along with Remark 5.5. $\square$

6. Relations between Scaling functions and Gibbs States

We assume that all homeomorphisms $h : \Sigma \to [0,1]$ appearing in this section are mutually Hölder equivalent. All h are assumed to be IFS-like and thus they induce iterated function systems and these are the main concern in this section.

So consider $\Phi = \{\phi_i\}_{i\in I}$, an iterated function system (IFS); where $I := 2\mathbb{N}-1$. For every $t \in Fin(\Phi)$ consider the potentials $\zeta_t : \Sigma \to \mathbb{R}$ given by the formula

$$\zeta_t(\omega) = -t \log |\phi'_{\omega_0}(\pi(\sigma\omega))| \cdot$$

Let $\mathcal{L}_t : C(\Sigma) \to C(\Sigma)$ be the corresponding Perron-Fröbenius operator. It was proved in [3] that there exists $\hat{m}_t$, a Borel probability measure on Σ such that

$$\mathcal{L}_t^*(m_t) = e^{P(t)} m_t,$$

where $P(t)$ is the topological pressure of the potential ζ_t. Recall from the previous section that there exists a unique shift-invariant Gibbs state μ_t for the potential ζ_t. Furthermore, μ_t and m_t are equivalent with Radon-Nikodym derivatives uniformly bounded above and separated from zero. Let $\tilde{\mu}_t$ be the corresponding measure on $\tilde{\Sigma}$ that was produced in Theorem 5.6 and Remark 5.5. Finally, let $\mu_t^* = \mu_t \circ \pi^{-1}$ and $m_t^* = m_t \circ \pi^{-1}$. The Borel probability measure m_t^* is uniquely determined by the conditions,

$$m_t^*(\phi_i(A)) = \int_A e^{-P(t)} |\phi_i'|^t dm_t^* \,, \quad \forall i \in I$$

and

$$m_t^*(\phi_i([0,1]) \cap \phi_j([0,1])) = 0 \,, \quad \forall i \neq j \in I \cdot$$

Frequently to be more specific and in order to avoid confusion, we will write $\zeta_{\Phi,t}$, $P_\Phi(t)$, $m_{\Phi,t}$, $\mu_{\Phi,t}$, $\tilde{\mu}_{\Phi,t}$, $\mu^*_{\Phi,t}$ and $m^*_{\Phi,t}$ for ζ_t, $P(t)$, m_t, μ_t, $\tilde{\mu}_t$, μ_t^* and m_t^* respectively. We will also use the subscript h rather than Φ if the former was our actual starting point. For example, $\mu_{h,t}$ for $\mu_{\Phi,t}$. We shall prove the following

Proposition 6.1. *If $h_2 \circ h_1^{-1} : h_1(\Sigma) \to h_2(\Sigma)$ is bi-Lipschitz continuous, then*

(a) $\mu_{h_2,t} = \mu_{h_1,t}$ *and* (a') $m_{h_2,t} \asymp m_{h_1,t}$
(b) $P_{h_2,t} = P_{h_1,t}$
(c) $\mu^*_{h_1,t} \circ (h_2 \circ h_1^{-1})^{-1} = \mu^*_{h_2,t}$

(d) $\tilde{\mu}_{h_2,t} = \tilde{\mu}_{h_1,t}$ and $\tilde{S}_{\mu_{h_2,t}} = \tilde{S}_{\mu_{h_1,t}}$

(e) $m^*_{h_1,t} \circ (h_2 \circ h_1^{-1})^{-1} \asymp m^*_{h_2,t}$

Proof. Since $h_2 \circ h_1^{-1}$ is bi-Lipschitz continuous, we have that

$$||(\phi_\omega^{(2)})'|| \asymp ||(\phi_\omega^{(1)})'|| \tag{6.1}$$

for all $\omega \in I^{\mathbb{N}_0}$, where $\{\phi_i^{(2)}\}_{i\in I}$ and $\{\phi_i^{(1)}\}_{i\in I}$ are the iterated function systems induced respectively by h_2 and h_1. It follows from (6.1) that

$$||(\phi_\omega^{(2)})'||^t \asymp ||(\phi_\omega^{(1)})'||^t \tag{6.2}$$

for all $t \geq 0$ and all $\omega \in I^{\mathbb{N}_0}$.

In particular, $Z_n(h_2,t) \asymp Z_n(h_1,t)$ and consequently

$$P_{h_2}(t) := \lim_{n\to\infty} \frac{1}{n} \log Z_n(h_2,t) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(h_1,t) =: P_{h_1}(t). \tag{6.3}$$

Thus property (b) is established.

Now, since for every $\omega \in \Sigma$, $m_{h_i,t}([\omega|_n]) \asymp e^{-nP_{h_i(t)}}||(\phi^{(i)}_{[\omega|_n]})'||^t$ for $i = 1,2$; it follows from (6.2) and (6.3) that property (a') holds. Thus $\mu_{h_2,t} \asymp \mu_{h_1,t}$ and thus, since both these measures are ergodic, they must be equal - hence property (a) holds. Then immediately property (d) holds as well.

Since, $\mu^*_{h_i,t} = \mu_{h_i,t} \circ h_i^{-1}$ for $i = 1,2$, it follows from property (a) that

$$\mu^*_{h_1,t} \circ (h_2 \circ h_1^{-1})^{-1} = \mu_{h_1,t} \circ h_1^{-1} \circ (h_1 \circ h_2^{-1}) = \mu_{h_1,t} \circ h_2^{-1} = \mu_{h_2,t} \circ h_2^{-1} = \mu^*_{h_2,t},$$

which proves property (c). Now property (e) follows from the fact that $m^*_{h_i,t} \asymp \mu^*_{h_i,t}$ for $i = 1,2$. Thus we are done. □

Definition 6.2. *Let*

$$\delta_h := \mathrm{HD}(h(\Sigma)), \textit{where } \mathrm{HD} \textit{ stands for Hausdorff Dimension.}$$

We set

$$\tilde{S}_h := \tilde{S}_{\mu_{h,\delta_h}},$$

and call it the metric scaling function of h. We also consider the function $\check{S}_h : \tilde{\Sigma} \to (0,1)$ given by the formula

$$\check{S}_h(\omega) := S_h(\omega|_{-\infty}^{-1}; \omega_0),$$

and call it the reduced scaling function of h.

Definition 6.3. *We call a Baire embedding h regular if $P(\delta_h) = 0$. We refer to* [3] *for a lengthier exposition of this concept.*

As an immediate consequence of Proposition 6.1 and Theorem 4.10, we get the following

Corollary 6.4. *If $S_{h_2} = S_{h_1}$, then all the properties (a) - (e) from Proposition 6.1 hold; in particular $\tilde{S}_{h_2} = \tilde{S}_{h_1}$ and $\mu_{h_1,\delta_1} = \mu_{h_2,\delta_2}$.*

Hence the scaling function determines uniquely the metric scaling function. The next proposition describes this relation more explicitly.

Definition 6.5. *Two functions $f, g : \Sigma \to \mathbb{R}$ are cohomologous (modulo a constant) in a class $\mathcal{C}$ if there exists a function $u : \Sigma \to \mathbb{R}$ in the class $\mathcal{C}$ such that*

$$g - f = u - u \circ \sigma \quad (+C),$$

where, σ is a shift and C is a constant. We denote this by $f \backsimeq g$. Note that this definition can be modified for our functions to be defined on the appropriate symbol space in question.

Proposition 6.6. *If h is IFS-like, then $\log \tilde{S}_h$ and $\log \check{S}_h^{\delta_h}$ are cohomologous modulo a constant in the class of bounded Hölder continuous functions on $\widetilde{\Sigma}$. This constant is equal to $P(\delta_h)$ ($= 0$ if h is regular). Consequently, $\tilde{\mu}_{h,\delta_h}$ is the Gibbs state of the potential $\log \check{S}_h^{\delta_h}$.*

Proof. Fix $\omega \in \widetilde{\Sigma}$ and $\tau \in \Sigma^*$, where $\tau = \tau_1 \dots \tau_q$. Then

$$\begin{aligned}
\check{S}_h(\omega\tau) := \lim_{n\to\infty} \frac{|I_{\omega|_n\tau}|}{|I_{\omega|_n}|} &= \lim_{n\to\infty} \frac{|I_{\omega|_n\tau_1}|}{|I_{\omega|_n}|} \cdot \frac{|I_{\omega|_n\tau_1\tau_2}|}{|I_{\omega|_n\tau_1}|} \cdots \frac{|I_{\omega|_n\tau_1\dots\tau_q}|}{|I_{\omega|_n\tau_1\dots\tau_{q-1}}|} \\
&= \lim_{n\to\infty} \frac{|I_{\omega|_n\tau_1}|}{|I_{\omega|_n}|} \cdot \lim_{n\to\infty} \frac{|I_{\omega|_n\tau_1\tau_2}|}{|I_{\omega|_n\tau_1}|} \cdots \lim_{n\to\infty} \frac{|I_{\omega|_n\tau_1\dots\tau_q}|}{|I_{\omega|_n\tau_1\dots\tau_{q-1}}|} \\
&= \check{S}_h(\omega\tau_1)\check{S}_h(\omega\tau_1\tau_2)\cdots\check{S}_h(\omega\tau)\ . \qquad (6.4)
\end{aligned}$$

Likewise, putting $\mu^* = \mu^*_{h,\delta_h}$ and $\mu = \mu_{h,\delta_h}$, we have that

$$\begin{aligned}
\tilde{S}_h(\omega\tau) &:= \lim_{n\to\infty} \frac{\mu^*(\phi_{\omega|_n\tau}(X))}{\mu^*(\phi_{\omega|_n}(X))} = \lim_{n\to\infty} \frac{\mu([\omega|_n\tau])}{\mu([\omega|_n])} \\
&= \lim_{n\to\infty} \frac{\mu([\omega|_n\tau_1])}{\mu([\omega|_n])} \cdot \frac{\mu([\omega|_n\tau_1\tau_2])}{\mu([\omega|_n\tau_1])} \cdots \frac{\mu([\omega|_n\tau_1\dots\tau_q])}{\mu([\omega|_n\tau_1\dots\tau_{q-1}])} \\
&= \lim_{n\to\infty} \frac{\mu([\omega|_n\tau_1])}{\mu([\omega|_n])} \cdot \lim_{n\to\infty} \frac{\mu([\omega|_n\tau_1\tau_2])}{\mu([\omega|_n\tau_1])} \cdots \lim_{n\to\infty} \frac{\mu([\omega|_n\tau_1\dots\tau_q])}{\mu([\omega|_n\tau_1\dots\tau_{q-1}])} \\
&= \tilde{S}_h(\omega\tau_1)\tilde{S}_h(\omega\tau_1\tau_2)\cdots\tilde{S}_h(\omega\tau)\ . \qquad (6.5)
\end{aligned}$$

Now using the Bounded Distortion Property we get that

$$\mu^*(\phi_{\omega|_n\tau}(X)) \le K^{\delta_h} e^{-P(\delta_h)(n+q)} ||(\phi_{\omega|_n\tau})'||^{\delta_h} \le K^{\delta_h} e^{-P(\delta_h)(n+q)} |I_{\omega|_n\tau}|^{\delta_h}$$

and

$$\mu^*(\phi_{\omega|_n\tau}(X)) \geq K^{-\delta_h} e^{-P(\delta_h)(n+q)} ||(\phi_{\omega|_n\tau})'||^{\delta_h} \geq K^{-\delta_h} e^{-P(\delta_h)(n+q)} |I_{\omega|_n\tau}|^{\delta_h};$$

and likewise

$$\mu^*(\phi_{\omega|_n}(X)) \leq K^{\delta_h} e^{-P(\delta_h)n} |I_{\omega|_n}|^{\delta_h}$$

and

$$\mu^*(\phi_{\omega|_n}(X)) \geq K^{-\delta_h} e^{-P(\delta_h)n} |I_{\omega|_n}|^{\delta_h} \ .$$

Combining the last four formulae along with the definition parts of (6.4) and (6.5), we obtain

$$K^{-2\delta_h} \leq \frac{\tilde{S}_h(\omega\tau)}{\check{S}_h^{\delta_h}(\omega\tau)\exp(-P(\delta_h)q)} \leq K^{2\delta_h} \ . \tag{6.6}$$

Combining this with (6.4) and (6.6), we obtain that for all $\omega \in \widetilde{\Sigma}$ and for all $q \geq 1$ we have

$$\left| \sum_{j=0}^{q-1} \log \tilde{S}_h(\sigma^j\omega) - \sum_{j=0}^{q-1} \log \check{S}_h^{\delta_h}(\sigma^j\omega) - P(\delta_h)q \right| \leq 2\delta_h \log K \ .$$

This means that condition (5) from Theorem 2.2.7 in [3] is satisfied and thus our proposition follows directly from this theorem.

□

Theorem 6.7. *Suppose h_1, h_2 are regular IFS-like Baire embeddings and that $\delta_{h_1} = \delta_{h_2} =: \delta$. Then $\mu_{h_1,\delta} = \mu_{h_2,\delta}$ if and only if $\log \check{S}_{h_1} \backsimeq \log \check{S}_{h_2}$. If either of these two conditions hold, then h_1 and h_2 are bi-Lipschitz equivalent.*

Proof. For the forward direction, note that $\mu_{h_1,\delta} = \mu_{h_2,\delta}$ implies that $\tilde{\mu}_{h_1,\delta} = \tilde{\mu}_{h_2,\delta}$ (since the map $\mu \mapsto \tilde{\mu}$ is bijective, as shown in Theorem 5.6) and this in turn implies $\tilde{S}_{h_1} = \tilde{S}_{h_2}$ by definition of $\tilde{S}$. So we have that $\log \tilde{S}_{h_1} = \log \tilde{S}_{h_2}$ and thus $\delta_{h_1} = \delta_{h_2} =: \delta$ and Proposition 6.6 give us that $\log \check{S}_{h_1} \backsimeq \log \check{S}_{h_2}$.
For the reverse direction, we again use $\delta_{h_1} = \delta_{h_2} =: \delta$ and Proposition 6.6 to give us that $\log \check{S}_{h_1} \backsimeq \log \check{S}_{h_2}$ implies $\log \tilde{S}_{h_1} \backsimeq \log \tilde{S}_{h_2}$. Now by Theorem 2.2.7 of [3] we have that $\tilde{\mu}_{h_1,\delta} = \tilde{\mu}_{h_2,\delta}$ and thus $\mu_{h_1,\delta} = \mu_{h_2,\delta}$.
Finally notice that h_1, h_2 being regular means that $P(\delta) = 0$ and thus from the definition of Gibbs state, i.e. (5.3), we have that

$$\mu_{h_1,\delta}([\omega]) \asymp |I_\omega^{(1)}|^\delta$$

and that

$$\mu_{h_2,\delta}([\omega]) \asymp |I_\omega^{(2)}|^\delta.$$

Thus $\mu_{h_1,\delta} = \mu_{h_2,\delta}$ would imply that $|I_\omega^{(1)}| \asymp |I_\omega^{(2)}|$, viz. that $h_1$and h_2 are bi-Lipschitz. □

References

[1] P. Hanus, M. Urbański, Rigidity of Infinite One-dimensional Iterated Function Systems, Real Analysis Exchange, Vol. 24 (1), 1998/99, 123-136.

[2] R. D. Mauldin, M. Urbański, Dimensions and measures in infinite iterated function systems, Proc. London Math. Soc. (3) 73 (1996) 105-154.

[3] D. Mauldin, M. Urbański, Graph Directed Markov Systems: Geometry and Dynamics of Limit Sets, Cambridge Univ. Press (2003).

[4] F. Przytycki, M. Urbański, Fractals in the Plane - Ergodic Theory Methods, to be published by Cambridge Univ. Press.

[5] D. S. Sullivan, Quasiconformal Homeomorphisms in Dynamics, Topology and Geometry, Proc. International Congress of Mathematicians, A.M.S. (1986), 1216-1228.

[6] D. S. Sullivan, Differentiable Structures on Fractal-like Sets determined by Intrinsic Scaling Functions on Dual Cantor Sets, The Mathematical Heritage of Hermann Weyl, A.M.S. Proc. Symp. Pure Math. 48 (1987), 15-23.

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