Relations between stable dimension and the preimage counting function on basic sets with overlaps

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Abstract

In this paper we study non-invertible hyperbolic maps f and the relation between the stable dimension (i.e the Hausdorff dimension of the intersection between local stable manifolds of f and a given basic set Λ) and the preimage counting function of the map f restricted to the fractal set Λ. The case of diffeomorphisms on surfaces was considered in [5] where thermodynamic formalism was used to study the stable/unstable dimensions. In the case of endomorphisms, the non-invertibility generates new phenomena and new difficulties due to the overlappings coming from the different preimages of points, and also due to the variations of the number of preimages belonging to Λ (as compared to [7]). We show that, if the number of preimages belonging to Λ of any point is less or equal than a continuous function $\omega(\cdot)$ on Λ, then the stable dimension at every point is larger or equal than the zero of the pressure function $t \to P(t\Phi^s - \log \omega(\cdot))$. As a consequence we obtain that, if d is the maximum value of the preimage counting function on Λ and if there exists $x \in \Lambda$ with the stable dimension at x equal to the zero t_d of the pressure function $t \to P(t\Phi^s - \log d)$, then the number of preimages in Λ of any point y is equal to d, and the stable dimension is t_d everywhere on Λ. This has further consequences to estimating the stable dimension for non-invertible skew products with overlaps in fibers.

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1 Introduction

Relations between Hausdorff dimension and the zero of topological pressure have been first found in the case of rational maps of one variable, by Bowen ([2]) and Ruelle ([11]).

Theorem (Ruelle)**.** *Let f be a rational map which is hyperbolic on its Julia set $J(f)$. Then the Hausdorff dimension of $J(f)$ is equal to the zero of the pressure function $t \to P(t\Phi^u)$, where $\Phi^u(z) := -\log |Df(z)|, z \in J(f)$. In particular the Hausdorff dimension of the Julia set depends real analytically on parameters when the parameters (i.e the map f) are perturbed holomorphically.*

And for surface diffeomorphisms, Manning and McCluskey proved in [5] that:

Theorem (Manning, McCluskey)**.** *Let Λ be a basic set for a $\mathcal{C}^1$ axiom A diffeomorphism $f : M^2 \to M^2$ with a $(1,1)$ splitting $T_\Lambda = E^s \oplus E^u$. Then $HD(W^s(x) \cap \Lambda) = t_s$ and $HD(W^u(x) \cap \Lambda) = t_u$, where t_s, t_u are the unique zeros of the pressure functions $t \to P(t\Phi^s)$ and respectively $t \to P(t\Phi^u)$. Moreover t_s depends continuously on f in the $\mathcal{C}^1$ topology on diffeomorphisms.*

In [7], Mihailescu and Urbanski studied the Hausdorff dimension of the intersection between local stable manifolds and basic sets for **non-invertible** holomorphic maps of several variables. Here the multidimensional setting and the fact that the map is non-invertible generate new phenomena and obstacles. In [13], Simon studied a certain class of skew products exhibiting a type of transversality condition giving that the attractor Λ is the union of smooth curves that intersect each other in at most one point and that at this point the angle between their tangents is greater than a positive constant, if their first preimages are different.

Transversality type conditions were studied also in [14]. In [9] we introduced a different form of transversality, for parametrized families of skew products in order to prove a Bowen type formula for the stable dimension for almost all parameters. In that paper there are many examples which satisfy this transversality condition including some skew products with iterated function systems in their base and examples from higher dimensional complex dynamics. However we do not know if transversality (in any form) is generic in some way. Also in [12], Schmeling studied attractors for the Belykh family depending on three parameters; there exists an open subset of parameters for which the corresponding maps are not injective and we have a bifurcation picture of invertibility according to the parameters. There are also many other examples of hyperbolic noninvertible maps; for instance holomorphic maps on $\mathbb{P}^2\mathbb{C}$ obtained from perturbations of hyperbolic product maps $(P(z), Q(w))$, or skew products $(P(z), Q(z, w))$ (we will talk about these in the end), solenoids with overlaps, or the family of horseshoes with overlaps introduced by Bothe ([1]). Bothe proved in fact that the set of such non-invertible horseshoes with overlaps has non-empty interior in some sense.

Our paper answers to the case when the transversality condition is not present or, even if it is present, to the case of those parameters for which we do not have necessarily a Bowen type equation for the stable dimension. In particular our work gives estimates for the stable dimension based on the number of preimages that points in the basic set Λ, have in Λ. This allows us flexibility in choosing continuous functions which bound the number of preimages and thus, it allows to use thermodynamical formalism of equilibrium states in Corollary 1 in order to prove a rigidity type result about the stable dimension.

Moreover in this paper we **do not** assume in general that Λ is an attractor (unlike in [13] or [12]), instead Λ is just a **basic set** (as defined below). First let us remind some definitions:

Definition 1. a) Let X be a compact metric space and $f : X \to X$ a continuous map. For a point x from X, we say that a point $y \in X$ is an f**-preimage** of x if $f(y) = x$; we will call such a point y also a **1-preimage** of x. If $f^k(z) = x$ for some $z \in X, k \geq 1$, then we say that z is a **k-preimage** of x.

b) We will say that a finite sequence $C = (x, x_{-1}, \ldots, x_{-n}), n \geq 1$ is a **finite prehistory** of x if $f(x_{-n}) = x_{-n+1}, \ldots, f(x_{-1}) = x$; in this case we will say that n is the length of C and x_{-n} will be called the **final preimage** associated to C.

c) We will say that an infinite sequence $C = (x, x_{-1}, \ldots)$ is a **full prehistory** (or simply a **prehistory**) of x if we have $f(x_{-i-1} = x_{-i}, i \geq 0$. We will assume that notationally $x = x_0$.

d) A full prehistory of x will also be denoted by $\hat{x} = (x, x_{-1}, x_{-2}, \ldots)$. The space of all prehistories from X is denoted by $\hat{X}$ and we have the shift map $\hat{f}(\hat{x}) = (f(x), x, x_{-1}, \ldots)$. The map $\hat{f}$ is a homeomorphism on $\hat{X}$. The pair $(\hat{X}, \hat{f})$ is called **the natural extension** (or **inverse limit**) of (X, f).

It can be remarked that $\hat{X}$ has a compact metric space structure ([6] for more on these notions).

Definition 2. a) Let $f : U \to M$ be a smooth (say $\mathcal{C}^2$) map defined on an open set U in a smooth Riemannian manifold M. Consider also a **basic set** Λ for f, i.e a compact subset of U with the following properties:

1) $f(\Lambda) = \Lambda$ and f is transitive on Λ.

2) there exists an open neighbourhood V of Λ such that $\Lambda = \bigcap\limits_{n \in \mathbb{Z}} f^n(V)$.

b) We say that f is **hyperbolic** on Λ if there exists a continuous splitting of the tangent bundle over $\hat{\Lambda}$, $T_{\hat{\Lambda}}(M)$, as $T_{\hat{x}}M = E^s_x \oplus E^u_{\hat{x}}$, where $T_{\hat{x}}M = \{(\hat{x}, v), v \in T_xM\}$; the subspaces $E^s_x, E^u_{\hat{x}}$ are invariant, i.e $Df_x(E^s_x) \subset E^s_{fx}, Df_x(E^u_{\hat{x}}) \subset E^u_{\hat{f}\hat{x}}$ and the derivative of f contracts, respectively expands uniformly on E^s_x and $E^u_{\hat{x}}$.

c) If f is hyperbolic on Λ there exist **local stable, and local unstable manifolds**, namely $W^s_r(x, f) := \{y \in U, d(f^i y, f^i x) \leq r, i \geq 0\}$, and respectively

$W^u_r(\hat{x}, f) = \{y \in U,$ there exists a full prehistory of y in $\Lambda, \hat{y},$ s.t $d(y_{-j}, x_{-j}) \leq r, j \geq 0\}$. They will also be denoted by $W^s_r(x)$ and respectively $W^u_r(\hat{x})$ when no confusion upon f may arise.

d) We mean by **stable dimension** at $x \in \Lambda$ the Hausdorff dimension $HD(W^s_r(x) \cap \Lambda)$; it will be denoted by $\delta^s(x)$.

The sets $W^s_r(x), W^u_r(\hat{x})$ have indeed the structure of manifolds of dimensions equal to the respective dimensions of $E^s_x, E^u_{\hat{x}}$; the local unstable manifolds depend in general on the whole prehistories, whereas the local stable manifolds depend only on their base point. In general for a non-invertible map we may have infinitely many local unstable manifolds passing through a point $x \in \Lambda$ (as was proved in [10]); this is complicating further the situation.

Also, let us notice that we work with general basic sets as defined above, i.e intersections of $f^n(V)$ for all $n \in \mathbb{Z}$, and not just with attractors (which require only intersections of $f^n(V)$ for $n \geq 0$). Clearly, any attractor Λ, for which there exists a neighbourhood V so that $f(V) \subset V$, is also a basic set.

Coming back to the hyperbolic non-invertible higher dimensional case, in [7] we showed that

Theorem. *Assume f is a holomorphic endomorphism on $\mathbb{P}^2\mathbb{C}$ and that f is hyperbolic on a basic set Λ of unstable index 1; suppose also that the critical set of f, $\mathcal{C}_f$ does not intersect Λ, and that each point x from Λ has at least d f-preimages in Λ. Then $HD(W^s_r(x) \cap \Lambda) \leq t^s_0$, where t^s_0 is the unique zero of the function $t \to P(t \log |Df_s(y) - \log d)$. Therefore this estimate is independent of $x \in \Lambda$.*

In the conformal hyperbolic non-invertible case (for instance for hyperbolic holomorphic maps on projective spaces), the situation is different than in the diffeomorphism case; this is due to the non-existence of inverse iterates and also to the fact that when taking forward images of balls centered at different preimages of the same point, these images may overlap. Also since the number of preimages belonging to Λ of a point x can vary when x ranges in Λ, it follows that the multiplicities of covers from [7] are not constant so we cannot apply the successive elimination process from [7].

To illustrate some of the new phenomena/difficulties that appear in the non-invertible case, in [7] we proved that

Theorem (Behavior of endomorphisms at perturbation)**.** *Given the map $f_\varepsilon(z,w) = (z^2 + a\varepsilon z + b\varepsilon w + c + d\varepsilon zw + e\varepsilon w^2, w^2)$, there exist small positive constants $c(a,b,d,e)$ and $\varepsilon(a,b,c,d,e)$ such that, for $b \neq 0, 0 \neq |c| < c(a,b,d,e)$ and $0 < \varepsilon < \varepsilon(a,b,c,d,e)$ we have that f_ε is injective on its basic set Λ_ε close to $\Lambda := \{p_0(c)\} \times S^1$ (where $p_0(c)$ is the attracting fixed point for $z \to z^2 + c$).*

In particular there exists a positive constant $\alpha(c)$ such that $HD(W^s_r(y, f_\varepsilon) \cap \Lambda_\varepsilon) > \alpha(c)$ for all $\varepsilon > 0$ small enough and all $y \in \Lambda_\varepsilon$.

This result implies that the stable dimension for f_ε does not depend real analytically (not even continuously) on the parameters when we perturb the map $f(z,w) = (z^2 + c, w^2)$, since the stable dimension of f relative to Λ is equal to zero (as the intersection $W^s_r(x,f) \cap \Lambda$ consists of only one point). But, as the previous Theorem proves, $HD(W^s_r(y, f_\varepsilon) \cap \Lambda_\varepsilon) > \alpha(c) > 0$, for all $\varepsilon > 0$ small.

Therefore, for non-invertible maps the situation is significantly different and the methods from the one variable case or from the diffeomorphism case do not apply in general.

One must be careful also about the different preimages belonging to Λ, whose number may vary. Locally near Λ a point x may have a constant number of f-preimages, but some of these preimages may not be in Λ. However we need for the estimate of stable dimension only those preimages from Λ, since Λ is f-invariant.

We will employ in the sequel maps of the following type:

Definition 3. Let M be a smooth (say $\mathcal{C}^2$) Riemannian manifold and $f : U \to M$ a smooth finite-to-one map defined on an open set U of M. Assume that f is hyperbolic on the basic set $\Lambda \subset U$ and that f is conformal on stable manifolds. Also suppose that the critical set $\mathcal{C}_f$ of f does not intersect Λ. We will say that f is then a **c-hyperbolic map** on Λ.

Definition 4. Let f be a c-hyperbolic function on a basic set Λ, and let an arbitrary point x from Λ. We shall denote by $d(x)$ the number of f-preimages of x belonging to Λ and the function $d(\cdot)$ will be called the **preimage counting function** on Λ.

We remark that the number of preimages $d(x)$ may vary when x ranges in Λ. This is bringing as we mentioned, additional significant difficulties for the estimate of the stable dimension.

In Theorem 1 we will prove that if the function $d(\cdot)$ is smaller or equal than a locally constant function $\omega(\cdot)$ on Λ, then the stable dimension $\delta^s(x)$ at any point $x \in \Lambda$ is larger or equal than the unique zero t_ω of the pressure function $t \to P(t\Phi^s - \log \omega)$, where $\Phi^s(y) := \log |Df_s(y)|, y \in \Lambda$.

We refine this result in Theorem 2 for the case when ω is any continuous function on Λ; this is a large extension of the class of maps for which we can estimate the stable dimension and it is improving the estimate from [7].

Then in Corollary 1 we shall prove that if there exists at least a point $x \in \Lambda$ where the stable dimension $\delta^s(x)$ is equal to the unique zero t_d of the pressure function $t \to P(t\Phi^s - \log d)$, where d is the maximum value for $d(\cdot)$ on Λ, then $d(\cdot)$ is identically equal to d on Λ, and the stable dimension will be t_d everywhere on Λ. In Corollary 2 we obtain an estimate for the stable dimension of fractal sets in the fibers of some non-invertible hyperbolic skew products, having finite IFS (iterated function systems) in the base, and related to [9]. And in Corollaries 3, 4 we give cases when the stable dimension is non-zero.

2 Main results for the non-invertible case.

For the rest of the paper, we will work with a c-hyperbolic mapping f on a basic set Λ. We recall that by $d(x)$ we denoted the number of f-preimages of x belonging to the fixed basic set Λ; $d(\cdot)$ is called the **preimage counting function** associated to f and Λ. It is important to know first some simple topological properties of $d(\cdot)$.

Lemma 1. *Let f be a c-hyperbolic map on a basic set Λ. Then the preimage counting function $d(\cdot)$ is upper semi-continuous and bounded on Λ.*

Proof. Indeed let us take a point $x \in \Lambda$ and a sequence x_n converging towards x in Λ. Then let an integer value d' such that for any n large enough, there are at least d' f-preimages of x_n denoted by $y(n,1), \ldots, y(n,d')$ in Λ. By taking eventually a subsequence of $(x_n)_n$, it happens that the respective preimages will accumulate to certain points $y(1), \ldots, y(d')$ in Λ. Also, since the critical set $\mathcal{C}_f$ does not intersect Λ there exists a positive ε_0 such that the mutual distances between $y(1), \ldots, y(d')$ are larger than ε_0. Now, since f is continuous on Λ it follows that $y(1), \ldots y(d')$ are different preimages of x, hence $d(x) \geq d'$. Since this is true for any subsequence $(x_n)_n$ conveging to x, it implies that $d(\cdot)$ is upper semicontinuous on Λ. And since Λ is compact this means that $d(\cdot)$ is bounded.

□

We will prove now the first Theorem of the paper about the case when the preimage counting function is bounded above by a locally constant function. After this we shall state and prove also the theorem in the case when the preimage counting function is bounded above by a continuous function; the idea of proof is essentially the same, but in the first case it is easier to see the method of proof.

Theorem 1. *Let a smooth function $f : U \to M$ defined on an open set of a smooth Riemannian manifold M and assume that f is c-hyperbolic on a basic set $\Lambda \subset U$. Assume that there exists a locally constant function ω on Λ such that $d(x) \leq \omega(x), x \in \Lambda$. Then $\delta^s(x) \geq t_\omega$, where t_ω is the unique zero of the pressure function $t \to P(t\Phi^s - \log \omega)$.*

Proof. Let us fix a point $x \in \Lambda$ and denote by $W := W^s_r(x) \cap \Lambda$. Let also $\varepsilon > 0$ small. Since Λ is compact, we can cover it with a finite number of balls $B(z_1, \varepsilon/2), \dots B(z_k, \varepsilon/2)$. From the transitivity property of f on Λ, it follows that for all $j \in \{1, \dots k\}$, there exists $m_j = m_j(\varepsilon)$ such that any local unstable manifold of type $W^u_\varepsilon(\hat{y})$ intersects the set $f^{-m_j}(W) \cap \Lambda$, for all $\hat{y} \in \hat{\Lambda}, y \in B(z_j, \varepsilon/2)$.

Since f is locally bi-Lipschitz near Λ (f being smooth), we obtain that $HD(W) = HD(f^{-m_j}W \cap \Lambda)$. Take an arbitrary number $t > \delta^s(x)$; then there exists a covering $\{U_i\}_{i \in I_j}$ of $f^{-m_j}W \cap \Lambda$ such that

$$\sum_{i \in I_j} (\text{diam} U_i)^t < \frac{1}{2k} \tag{1}$$

Then we consider the union $I := \bigcup\limits_{j=1}^{k} I_j$. Thus we obtain a collection of sets $U_i, i \in I$ such that any local unstable manifold $W^u_\varepsilon(\hat{y})$ intersects at least one such U_i and from (1) we obtain

$$\sum_{i \in I} (\text{diam} U_i)^t < \frac{1}{2} \tag{2}$$

Now consider $i \in I$ and suppose that $\text{diam} U_i > 0$. We can assume in fact that U_i is contained in a local stable manifold. Let us introduce a type of tubular unstable set used in [8] for the inverse pressure: for a finite prehistory $C = (x, x_{-1}, \dots, x_{-n})$ of x in Λ, define

$$\Lambda(C, \varepsilon) := \{y \in U, \text{there exists a prehistory of } y, (y, y_{-1}, \dots, y_{-n}), \text{s.t } d(y_{-j}, x_{-j}) < \varepsilon, j = 0, \dots, n\}$$

By **stable diameter** of $\Lambda(C, \varepsilon)$ we will understand the diameter of the intersection $\Lambda(C, \varepsilon) \cap W^s_r(x)$.

We will detail now how to take some special prehistories C of points in U_i. For a point $y \in U_i$, consider a prehistory C of y in Λ of length n such that if $C = (y, \dots, y_{-n})$, then n is the largest integer such that $\varepsilon |Df^n_s(y_{-n})| > \text{diam} U_i$. We will call such a prehistory C a **maximal prehistory** relative to U_i and its length will be denoted also by $n(C)$. Obviously we cannot have just any length for such a maximal prehistory, so let us denote by $n_{i1}, \dots, n_{iq_i}$ all the different lengths of U_i-maximal prehistories. From construction it is clear that $U_i \subset \Lambda(C, \varepsilon)$ for C as above.

Now let us denote the set of U_i-maximal prehistories by $\mathcal{C}_i$ and let us assume that $\mathcal{F}_i$ is a minimal set of points of type $y_{-n(C)}$ for $C \in \mathcal{C}_i$, such that for any $C \in \mathcal{C}_i$, there exists $z \in \mathcal{F}_i$ with $y_{-n(C)} \in B_{n(C)}(z, \varepsilon)$ (where in general $B_m(z, \varepsilon)$ denotes the Bowen ball, i.e the set of points whose orbits are within ε distance of the orbit of z up to order m).

Denote the corresponding set of prehistories from $\mathcal{C}_i$ ending with the points of $\mathcal{F}_i$, by $\mathcal{C}^*_i$. Hence $\mathcal{C}^*_i \subset \mathcal{C}_i, i \in I$. If $z \in \mathcal{F}_i$, we will denote also by $n(z)$ the length of the corresponding prehistory $C \in \mathcal{C}^*_i$ having z as final preimage.

Without loss of generality we may assume that the preimage counting function is equal to ω and thus locally constant, this giving in fact the case when the stable dimension is minimal under the assumption $d(\cdot) \leq \omega(\cdot)$ on Λ . Since $d(\cdot)$ takes only finitely many values on Λ, we denote them by $d_1, \ldots, d_p$. In this setting, denote by $V_j := \{z \in \Lambda, d(z) = d_j\}, j = 1, \ldots, p$; thus these sets are closed and mutually disjointed. In general, V_j's may be taken to be the level sets of the locally constant map ω. Assume that $d(V_j, V_k) > \varepsilon_0 > 0, j \neq k$, for some positive constant ε_0. Since the critical set of f does not intersect Λ, different f-preimages of any arbitrary point $x \in \Lambda$ are at a positive distance apart; this distance may be assumed to be larger than ε_0 too.

Let us take now a point $\xi \in V_1$, hence ξ has d_1 f-preimages denoted by $\xi_1, \ldots, \xi_{d_1}$. These are simple preimages due to the fact that $\mathcal{C}_f \cap \Lambda = \emptyset$. Assume that there exists a sequence of points y from Λ which converges towards ξ, and let $y_1, \ldots, y_{d_1}$ be the d_1 preimages of y. Assume also that $d(\{y_1, \ldots, y_{d_1}\}, \{\xi_1, \ldots, \xi_{d_1}\}) > \alpha > 0$, for all points y in this sequence. Then the points $y_1, \ldots, y_{d_1}$ accumulate (eventually for a subsequence) to some points $y_1^*, \ldots, y_{d_1}^*$ which are preimages of ξ. But due to the condition on the distances between the sets of preimages, it follows that there exists at least a point y_j^* which is not in the set $\{\xi_1, \ldots, \xi_{d_1}\}$. This implies then that ξ has more than d_1 preimages in Λ, hence contradiction.

So each point $\xi \in \Lambda$ has a neighbourhood $V(\xi)$ such that any point $y \in V(\xi)$ has d_1 preimages in Λ close to the preimages $\xi_1, \ldots, \xi_{d_1}$ of ξ. Now, if for any $\eta > 0, \eta << \varepsilon_0$ there exists a point $y(\eta) \in \Lambda$ such that there exists a point $z(\eta) \in B(y(\eta), \eta)$ with the preimages of $z(\eta)$ in Λ far from the preimages of $y(\eta)$ in Λ, then we can take a subsequence of $y(\eta)$ converging towards a point $w \in \Lambda$ which has the property that in any neighbourhood there are points $z(\eta)$ with preimages far from the preimages of w, hence a contradiction with the fact proved earlier. So there exists a positive ε_1 such that if $d(y, z) < \varepsilon_1$, then the preimages of y in Λ are close (i.e closer than $d(y, z) \cdot \sup\limits_{\Lambda} |Df_s|^{-1}$) to the preimages of z in Λ. In this we used implicitly the fact that the preimages of any point from Λ have multiplicity 1, since $\mathcal{C}_f \cap \Lambda = \emptyset$.

In particular, for $C \in \mathcal{C}_i, C = (y, \ldots, y_{-n(C)})$, and $z \in B_{n(C)}(y_{-n(C)}, \varepsilon)$ we have that $f^k(z)$ has the same number of f-preimages in Λ as $f^k(y_{-n(C)})$ and moreover, these preimages are close to the f-preimages of $f^k(y_{-n(C)})$, for $k = 0, \ldots, n(C)$ (namely $\varepsilon \sup\limits_{\Lambda} |Df_s|^{-1}$-close).

Consider now the set of points of the form $y_{-n(C)}$ for some $C \in \mathcal{C}_i$ a U_i-maximal prehistory; from the definition we know that $\mathcal{F}_i$ is minimal and for any $C \in \mathcal{C}_i$ there is a prehistory $C^* = (f^{n(C)}z, \ldots, z) \in \mathcal{C}_i^*$ such that $n(C) = n(C^*)$ and $y_{-n(C)} \in B_{n(C)}(z, \varepsilon)$.

The prehistories in $\mathcal{C}_i^*$ may have different lengths. But if for example $z \in \mathcal{F}_i$ and $f(z) \in V_j$ then there exists $d_j - 1$ other points in $f^{-1}(f(z)) \cap \Lambda$ and these points will generate other prehistories from $\mathcal{C}_i^*$. Due to the above considerations we can assume without loss of generality that the set $\mathcal{F}_i$ is given by prehistories of a single point $y \in U_i$. Also we may assume that these points $y \in U_i$ do not belong to other sets $U_j, j \neq i$.

Let us arrange now the lengths of prehistories from $\mathcal{C}_i^*$ as

$$n_{i,q_i} > n_{i,q_i-1} > \ldots > n_{i,1}$$

Then denote by $\mathcal{F}_{i,n_{i,q_i}}$ the set of points $z \in \mathcal{F}_i$ which correspond to prehistories in $\mathcal{C}_i^*$ of length n_{i,q_i}. Denote also the cardinality of $\mathcal{F}_{i,n_{i,q_i}}$ by $N_{i,n_{i,q_i}}$.

Then let us take the set $\mathcal{F}_{i,n_{i,q_i}-1}$ as the union of $f(\mathcal{F}_{i,n_{i,q_i}})$ and the set of points $z \in \mathcal{F}_i$ which correspond to prehistories of length $n_{i,q_i} - 1$. The cardinality of $\mathcal{F}_{i,n_{i,q_i}-1}$ is denoted by $N_{i,n_{i,q_i}-1}$. We do this until reaching $N_{i,0}$ which is equal to 1, since these are considered as prehistories of a single point y from U_i. We now define:

$$N_{i,n_{i,q_i}}(j_1, \dots, j_{n_{i,q_i}}) := \text{Card}\{z \in \mathcal{F}_{i,n_{i,q_i}}, f(z) \in V_{j_1}, \dots, f^{n_{i,q_i}}(z) \in V_{j_{n_{i,q_i}}}\}$$

and similarly $N_{i,n_{i,q_i}-1}(j_1, \dots, j_{n_{i,q_i}-1}) := \text{Card}\{\zeta \in \mathcal{F}_{i,n_{i,q_i}-1}, f(\zeta) \in V_{j_1}, \dots, f^{n_{i,q_i}-1}(z) \in V_{j_{n_{i,q_i}-1}}\}$, etc.

Then from the above construction we have that

$$\frac{N_{i,n_{i,q_i}}(1, j_2, \dots, j_{n_{i,q_i}})}{d_1} + \dots + \frac{N_{i,n_{i,q_i}}(p, j_2, \dots, j_{n_{i,q_i}})}{d_p} \le N_{i,n_{i,q_i}-1}(j_2, \dots, j_{n_{i,q_i}}) \tag{3}$$

Next we obtain

$$\frac{N_{i,n_{i,q_i}-1}(1, j_3, \dots, j_{n_{i,q_i}})}{d_1} + \dots + \frac{N_{i,n_{i,q_i}-1}(p, j_3, \dots, j_{n_{i,q_i}})}{d_p} \le N_{i,n_{i,q_i}-2}(j_3, \dots, j_{n_{i,q_i}}), \tag{4}$$

and we can combine this inequality with (3). By induction we obtain then that for all $i \in I$,

$$\Sigma_i := \sum_{z \in \mathcal{F}_i} \frac{1}{d_1^{m_1(z)} \cdot \ldots \cdot d_p^{m_p(z)}} \le 1, \tag{5}$$

where for each $z \in \mathcal{F}_i$, $m_1(z)$ represents the number of times that the orbit $z, f(z), \dots, f^{n(z)}z$ hits $V_1, \ldots$, and $m_p(z) :=$ number of times that the above orbit hits V_p. We assumed that the points y chosen inside U_i do not belong to other $U_j, j \ne i$, and that the points of $\mathcal{F}_i$ are preimages (of different orders) of $y \in U_i$.

Let us assume also that N is the largest integer $n_{i,j}, 1 \le j \le q_i, i \in I$; since I is finite, it follows that $N < \infty$.

We know from construction of $\mathcal{F}_i$ that any preimage of type $y_{-n(C)}$ for C a maximal prehistory associated to U_i belongs to a Bowen ball of type $B_{n(C)}(z, \varepsilon)$, for some $z \in \mathcal{F}_i$.

Any local unstable manifold of size ε is contained in the union $\underset{C \in \mathcal{C}_i^*}{\cup} \Lambda(C, \varepsilon)$, and we want to extend these prehistories as to obtain in the end a common (or close) length for all of them. More precisely we will extend these prehistories until we reach a length between n and $n + N$, for a large integer n. The idea is the following: let $z \in \mathcal{F}_i$ corresponding to a prehistory $C \in \mathcal{C}_i^*$ of length $n(C)$; then z itself is covered by $\underset{j \in I}{\cup} \underset{C \in \mathcal{C}_j^*}{\cup} \Lambda(C, \varepsilon)$, hence there exists $j \in I$ and a prehistory $D \in \mathcal{C}_j^*$ such that $z \in \Lambda(D, \varepsilon)$. We will concatenate now like in [8] the prehistories C and D and will obtain $\Lambda(CD, \varepsilon) := \{y, \exists (y, \dots, y_{-n(C)})\text{prehistory of } y \ \varepsilon-\text{shadowing } C, \text{and } y_{-n(C)} \in \Lambda(D, \varepsilon)\}$; so we follow the prehistories of preimages until we reach a length between n and $n + N$ for some large n.

To this end, consider the set $\mathcal{S}_n$ of all the multiples $(s, j_1, \dots, j_s, p_1, \dots, p_s)$ such that $s \in \mathbb{N}^*, j_1, \dots, j_s \in I, 1 \le p_k \le q_{j_k}, k = 1, \dots, s$ and $n \le n_{j_1,p_1} + \ldots + n_{j_s,p_s} < n + N$.

For such an element of $\mathcal{S}_n$, we start with a prehistory $C_1 = (\zeta, \dots, \zeta_{-n_{j_1,p_1}}), \zeta \in U_{j_1}$, then we assume $\zeta_{-n_{j_1,p_1}} \in \Lambda(C_2, \varepsilon)$ with C_2 a prehistory of length n_{j_2,p_2} of a point in U_{j_2}, etc. This procedure will give in the end a final preimage $\zeta_{-n_{j_1,p_1}-\dots-n_{j_s,p_s}} \in \Lambda$ and we denote by F_n the set of all such final points obtained by the above procedure.

Since for any $\mathcal{F}_i, i \in I$ we covered all the possible preimages $y_{-n(C)}$ corresponding to maximal U_i-prehistories C in Λ from $\mathcal{C}_i$, it follows that F_n is (n, ε)-spanning for Λ.

For $1 \le k \le q_i$, denote by $\tilde{N}_{ik}(m_1, \dots, m_p)$ the number of elements ξ of $\mathcal{F}_i$ such that $n(\xi) = n_{i,k}$ and so that in the $n_{i,k}$-forward orbit of ξ there are exactly m_1 iterates belonging to $V_1, \dots, m_p$ iterates belonging to V_p. By taking the product of the inequalities from (5) for $j_1, \dots, j_s$, we obtain that

$$\sum_{1 \le p_1 \le q_{j_1}, 1 \le p_s \le q_{j_s}} \sum_{m_1 + \dots + m_p = n_{j_1,p_1}} \frac{\tilde{N}_{j_1 p_1}(m_1, \dots, m_p)}{d_1^{m_1} \dots d_p^{m_p}} \cdot \dots \cdot \sum_{l_1 + \dots + l_p = n_{j_s,p_s}} \frac{\tilde{N}_{j_s p_s}(l_1, \dots, l_p)}{d_1^{l_1} \dots d_p^{l_p}} \le 1 \qquad (6)$$

So, if $P_n(t\Phi^s - \log d(\cdot)) := \inf\{\sum_{z \in F} \exp(S_n(t\Phi^s - \log d(\cdot))(z), F\ (n,\varepsilon)-\text{spanning for } \Lambda\}$ and since F_n is (n, ε)-spanning, we obtain:

$$\begin{aligned}
&P_n(t\Phi^s - \log d(\cdot)) \le \sum_{z \in F_n} \exp(S_n(t\Phi^s - \log d(\cdot))(z) \le \\
&\le \sum_{(s, j_1, \dots, j_s, p_1, \dots, p_s) \in \mathcal{S}_n} \sum_{m_1 + \dots + m_p = n_{j_1,p_1}} \frac{\tilde{N}_{j_1 p_1}(m_1, \dots, m_p)}{d_1^{m_1} \dots d_p^{m_p}} \cdots \sum_{l_1 + \dots + l_p = n_{j_s,p_s}} \frac{\tilde{N}_{j_s p_s}(l_s, \dots, l_p)}{d_1^{l_1} \dots d_p^{l_p}} \cdot \qquad (7) \\
&\cdot (\text{diam} U_{j_1})^t \dots (\text{diam} U_{j_s})^t \le \sum_{s, j_1, \dots, j_s} (\text{diam} U_{j_1})^t \dots (\text{diam} U_{j_s})^t,
\end{aligned}$$

after using (6).

Therefore, by using (2)

$$\begin{aligned}
P_n(t\Phi^s - \log d(\cdot)) &\le \sum_s \sum_{j_1, \dots, j_s} (\text{diam} U_{j_1})^t \dots (\text{diam} U_{j_s})^t = \\
&= \sum_s (\sum_{j \in I} (\text{diam} U_j)^t)^s \le \sum_s (\frac{1}{2})^s < 2
\end{aligned} \qquad (8)$$

But $P(t\Phi^s - \log d(\cdot)) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P_n(t\Phi^s - \log d(\cdot))$. This implies that $t \ge t_{d(\cdot)}$, where $t_{d(\cdot)}$ is the unique zero of the pressure function $t \to P(t\Phi^s - \log d(\cdot))$.

If we have that the preimage counting function $d(\cdot)$ is only smaller or equal than $\omega(\cdot)$ at any point $x \in \Lambda$, it follows that $t \ge t_\omega$ in the same way.

But t was taken arbitrarily larger than $HD(W^s_r(x) \cap \Lambda)$, so we obtain the announced inequality

$$HD(W^s_r(x) \cap \Lambda) \ge t_\omega, \forall x \in \Lambda$$

$\square$

Theorem 2. *In the same setting as in Theorem 1, assume that there exists a continuous function ω on Λ such that for any point $z \in \Lambda$, we have $d(z) \le \omega(z)$. Then $\delta^s(x) \ge t_\omega$, for any $x \in \Lambda$, where t_ω is the unique zero of the pressure function $t \to P(t\Phi^s - \log \omega)$.*

Proof. The proof is similar to the one of the previous Theorem. We consider as before the set of U_i-maximal prehistories $\mathcal{C}_i$ and an associated minimal set $\mathcal{F}_i$ of final preimages given by these prehistories (see Definition 1 and previous proof).

Using the fact that the preimage counting function $d(\cdot)$ is upper semicontinuous on Λ we find again that for each point $z \in \Lambda$ there exists a neighbourhood of z such that each point y in this neighbourhood has at most $d(z)$ preimages and they are close to some of the preimages of z (however the point y may have strictly less than $d(z)$ preimages in Λ).

Again we will have that $N_{i0} = 1$ since in the minimal set $\mathcal{F}_i$ we can take only preimages of a point $y \in U_i$ where $\omega(\cdot)$ is largest on U_i. If not, then we can complete the prehistories of y with prehistories of other points but the total number will be the same as if we were considering prehistories of a single point from U_i.

From the continuity of ω on Λ there exists a positive function $\rho(\varepsilon)$ defined for small $\varepsilon > 0$, with the following property:

$$\text{if } y, z \in \Lambda, \text{ and } d(y,z) < \varepsilon, \text{ then } |\omega(y) - \omega(z)| \le \rho(\varepsilon) \tag{9}$$

Since ω is continuous it follows that $\rho(\varepsilon) \to 0$ when $\varepsilon \to 0$, and we can assume that ρ has been taken such that it is an increasing function.

Now we notice that, if $y \in B_n(z, \varepsilon)$, then for any $0 \le j \le n$, $d(f^j y) \le \omega(f^j z) + \rho(\varepsilon)$, since by assumption $d(f^j y) \le \omega(f^j y)$. Thus the number of preimages of $f^j y$, $d(f^j y)$ may differ from $d(f^j z)$ by at most 1, but still $d(f^j y)$ is less or equal than $\omega(f^j z) + \rho(\varepsilon)$, where $\rho(\varepsilon) \underset{\varepsilon \to 0}{\to} 0$.

We take as before the set F_n of final preimages of type $y_{-n_{j_1,p_1} - \ldots n_{j_s,p_s}}$, over all sequences $(s, j_1, \ldots, j_s, p_1, \ldots, p_s)$ such that $j_1, \ldots, j_s \in I$ and $1 \le p_1 \le q_{j_1}, 1 \le p_s \le q_{j_s}$ with $n \le n_{j_1,p_1} + \ldots + n_{j_s,p_s} < n + N$. This set of sequences is denoted again by $\mathcal{S}_n$ as in the proof of Theorem 1.

Now as we mentioned, the preimage counting function is smaller or equal than ω and ω varies with at most $\rho(\varepsilon)$ on a ball of radius ε, thus we can apply this at every iterate (up to order n) for points in a Bowen ball $B_n(z, \varepsilon)$. We will have then the analogues of inequalities (5) and (6), namely:

$$\Sigma_i := \sum_{z \in \mathcal{F}_i} \frac{1}{(\omega(fz) + \rho(\varepsilon)) \ldots (\omega(f^{n(C)} z) + \rho(\varepsilon))} \le 1, \tag{10}$$

where we assumed that $C = (f^{n(C)}(z), \ldots, z)$ is the prehistory from $\mathcal{C}_i^*$ whose final preimage is z, for $z \in \mathcal{F}_i$. We will denote the length $n(C)$ associated to the above C, by $n(z)$.

Since ω is continuous on Λ, it will take only finitely many positive integer values, denoted again by $d_1, \ldots, d_p$ arranged as $d_1 < \ldots < d_p$. And similarly, by taking the product of the inequalities (10) for $j_1, \ldots, j_s$ we shall obtain:

$$\sum_{1\le p_1\le q_{j_1},1\le p_s\le q_{j_s}} \sum_{z\in\mathcal{F}_{j_1},n(z)=n_{j_1,p_1}} \frac{1}{(\omega(fz)+\rho(\varepsilon))\dots(\omega(f^{n(z)}z)+\rho(\varepsilon)} \cdot \dots$$
$$\cdot \sum_{z\in\mathcal{F}_{j_s},n(z)=n_{j_s,p_s}} \frac{1}{(\omega(fz)+\rho(\varepsilon))\dots(\omega(f^{n(z)}z)+\rho(\varepsilon))} \le 1 \tag{11}$$

Then since by construction the set F_n is (n,ε)-spanning for Λ with respect to f (since we cover all final preimages with $\mathcal{F}_i$), we can finish the proof by using (11) in the same way as in the proof of Theorem 1.

Therefore we obtain that $t\ge t(\varepsilon)$ for $\varepsilon>0$ small, with $t(\varepsilon)$ being the unique zero of the pressure function $t\to P_\varepsilon(t\Phi^s-\log(\omega+\rho(\varepsilon)))$, where in general $P_\varepsilon(g):=\limsup_n \frac{1}{n}\log\inf\{\sum_{z\in F}\exp(S_n(g)(z)), F(n,\varepsilon)-$spanning for $\Lambda\}$ for g continuous on Λ.

Let us take now some T arbitrarily larger than t and $\eta>0$ small; then $T>t\ge t(\eta)$. But if $0<\varepsilon<\eta$, we get that $\rho(\varepsilon)\le\rho(\eta)$, so $t\Phi^s-\log(\omega+\rho(\varepsilon))\ge T\Phi^s-\log(\omega+\rho(\eta))$. Now since $t\ge t(\varepsilon)$ for all ε small, it follows that $0\ge P_\varepsilon(t\Phi^s-\log(\omega+\rho(\varepsilon)))\ge P_\varepsilon(T\Phi^s-\log(\omega+\rho(\eta)))$ for all $\varepsilon>0$ small enough. But recalling the definition of the topological pressure $P(g)=\lim_{\varepsilon\to0}P_\varepsilon(g)$, for all g continuous, we obtain that

$$P(T\Phi^s-\log(\omega+\rho(\eta)))\le 0 \tag{12}$$

Now let t_ω be the unique zero of the pressure function $t\to P(t\Phi^s-\log\omega)$. From the continuity of the pressure with respect to the potential, it follows that t_ω is the limit of the zeros of the pressure functions $t\to P(t\Phi^s-log(\omega+\rho(\eta)))$ when η converges to 0. Hence from (12),

$$T\ge t_\omega$$

Therefore since T was chosen arbitrarily larger than t which in turn was chosen arbitrarily larger than $HD(W^s_r(x)\cap\Lambda)$, we obtain the conclusion,

$$HD(W^s_r(x)\cap\Lambda)\ge t_\omega$$

□

We are now ready to prove some consequences of these results.

More precisely we consider first what happens if there exists a point $x\in\Lambda$ such that the stable dimension at x is the smallest one possible.

Corollary 1. *Assume that f is c-hyperbolic on a basic set Λ and that the preimage counting function $d(\cdot)$ reaches a maximum value of d on Λ. If there exists a point $x\in\Lambda$ such that $\delta^s(x)=t_d$, where t_d is the unique zero of the pressure function $t\to P(t\Phi^s-\log d)$, then $d(y)=d,\forall y\in\Lambda$. And hence the stable dimension at every point of Λ is equal to t_d.*

Proof. We know that there exists a point $x\in\Lambda$ with $\delta^s(x)=t_d$. Assume that there exists an open set $V\subset\Lambda$ such that $d(y)\le d-1, y\in V$ and let also W open inside V such that $\bar W\subset V$.

Then we can take a Lipschitz function Ψ with $\Psi(z) = d - 1, z \in \bar{W}$, $\Psi(z) = d$, for z outside V, and $d - 1 \leq \Psi \leq d$ on $V \setminus \bar{W}$. Thus we have $d(y) \leq \Psi(y), y \in \Lambda$.

But then from Theorem 2, it follows that $\delta^s(x) \geq t_\Psi$, where t_Ψ denotes the unique zero of the function $t \to P(t\Phi^s - \log \Psi)$; hence, since $\delta^s(x) = t_d$, it means that $t_d \geq t_\Psi$. But also $\Psi \leq d$ on Λ so $t_\Psi \geq t_d$, therefore $t_d = t_\Psi$.

Let us consider now an equilibrium measure μ_d for the Holder continuous potential $t_d\Phi^s - \log d$. So, from the definition of equilibrium measures ([4]) and since $P(t_d\Phi^s - \log d) = 0$, we have:

$$\int (t_d\Phi^s - \log d) d\mu_d + h_{\mu_d} = 0, \tag{13}$$

where h_μ denotes in general the metric entropy of the f-invariant probability measure μ on Λ.

But then from the Variational Principle applied to the potential $t_\Psi\Phi^s - \log \Psi$ ([4]), we obtain

$$\int (t_\Psi\Phi^s - \log \Psi) d\mu_d + h_{\mu_d} \leq P(t_\Psi\Phi^s - \log \Psi) = 0 \tag{14}$$

Recall also that we proved above that $t_d = t_\Psi$, so consequently:

$$\int (t_d\Phi^s - \log \Psi) d\mu_d + h_{\mu_d} \leq \int (t_d\Phi^s - \log d) d\mu_d + h_{\mu_d} = 0$$

This implies that

$$\int \log \Psi d\mu_d \geq \int (\log d) d\mu_d \tag{15}$$

On the other hand μ_d is an equilibrium measure, hence it is positive on open sets, since any open set contains a Bowen ball and thus one can use the estimates for the equilibrium measures on Bowen balls similar to the ones for homeomorphisms from [4]. These estimates were proved in [4] (Lemma 20.3.4) for homeomorphisms with specification; we know however that hyperbolicity implies specification ([4]). For the case of noninvertible maps they follow by using the lift to the inverse limit $\hat{\Lambda}$. Indeed let us denote by $B_n(z, \varepsilon, f) := \{w \in \Lambda, d(f^i z, f^i w) < \varepsilon, i = 0 \ldots n-1\}$ a Bowen ball relative to $f|_\Lambda$, by $B_n(\hat{z}, \varepsilon, \hat{f}) := \{\hat{w} \in \hat{\Lambda}, d(\hat{f}^i \hat{z}, \hat{f}^i \hat{w}) < \varepsilon, i = 0 \ldots n-1\}$ a Bowen ball relative to $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$, and by $\pi : \hat{\Lambda} \to \Lambda$ the canonical projection. Then we have that there exists a $k = k(\varepsilon) \geq 1$ such that $\hat{f}^k(\pi^{-1} B_n(y, \varepsilon, f)) \subset B_{n-k}(\hat{f}^k \hat{y}, 2\varepsilon, \hat{f}) \subset \hat{\Lambda}$. Then if ϕ is some Holder potential and μ_ϕ denotes the equilibrium state of ϕ with $\hat{\mu}_\phi$ its unique lifting to $\hat{\Lambda}$, it follows that $\mu_\phi(B_n(y, \varepsilon, f)) = \hat{\mu}_\phi(\hat{f}^k \pi^{-1}(B_n(y, \varepsilon, f))) \leq \hat{\mu}_\phi(B_{n-k}(\hat{f}^k \hat{y}, 2\varepsilon, \hat{f})) \leq A_{2\varepsilon} e^{S_{n-k}\phi(f^k y) - (n-k)P(\phi)} \leq \tilde{A}_\varepsilon e^{S_n \phi(y) - nP(\phi)}$, for any $y \in \Lambda, n \geq 1$, for some positive constant $\tilde{A}_\varepsilon$ depending on ε, ϕ, f. Conversely we have $\pi(B_n(\hat{y}, \varepsilon, \hat{f})) \subset B_n(y, \varepsilon, f)$, thus we get the lower bound for $\mu_\phi(B_n(y, \varepsilon, f))$ from the lower bound for $\hat{\mu}_\phi(B_n(\hat{y}, \varepsilon, \hat{f}))$, since $\hat{f}$ is an expansive homeomorphism with specification on $\hat{\Lambda}$. In conclusion the estimates

$$B_\varepsilon e^{S_n \phi(y) - nP(\phi)} \leq \mu_\phi(B_n(y, \varepsilon, f)) \leq A_\varepsilon e^{S_n \phi(y) - nP(\phi)}, \forall y \in \Lambda, n \geq 1,$$

hold also for hyperbolic noninvertible maps f on Λ. Thus we showed indeed that the equilibrium measure μ_d (of the potential $t_d\Phi^s - \log d$) is positive on Bowen balls, hence also on non-empty open sets.

Coming back to the proof of the Corollary, we chose Ψ with $0 \le \Psi \le d$ on Λ, and $\log \Psi \le \log(d-1) < \log d$ on W. Hence if μ_d is positive on open sets, we obtain then a contradiction with (15).

Hence the preimage counting function $d(\cdot)$ must be equal to d on a dense set in Λ. But recall that $d(\cdot)$ is upper semicontinuous (Lemma 1), therefore $d(y) = d, \forall y \in \Lambda$.

□

Finally we can apply the above theorems for the case of hyperbolic skew products with overlaps in the stable fibers and having a finite IFS (iterated function system) in the base.

Let us consider a finite union of compact sets $X_1, \ldots, X_m$ in an open set $S \subset \mathbb{R}^l$ and denote by $X := X_1 \cup \ldots \cup X_m$. Consider also a continuous expanding topologically transitive function $f : X \to X$. Assume also that f is injective on each X_i and that $f(X_i) = X(i,1) \cup \ldots \cup X(i, m_i), i = 1, \ldots, m$, where $X(i,j)$ are sets from the same collection $\{X_1, \ldots, X_m\}$.

The source-model for this is the case of an expanding map $f : I_1 \cup \ldots \cup I_m \to I_1 \cup \ldots \cup I_m$, with $I_1, \ldots, I_m$ compact subintervals in $[0,1]$, such that $f(I_j)$ is a union of some of the same subintervals, i.e $f(I_j) = I(j,1) \cup \ldots \cup I(j, m_j), j = 1, \ldots, m$.

We also take functions $g(x,y) : X \times \tilde{W} \to X \times \tilde{W}$, with $\tilde{W} \subset \mathbb{R}^k$ a neighbourhood of the closure of an open set W, such that g is smooth (say $\mathcal{C}^2$) in (x,y), and such that for every $x \in X$, the function $g(x,\cdot) : W \to W$ is contracting uniformly in x, and it is injective and conformal. We shall denote the function $g(x,\cdot)$ also by g_x; due to the contraction, $g_x(\bar{W})$ is strictly contained in V.

We then take the compact f-invariant set $X^* := \{y \in X, f^j y \in X, j \ge 0\}$ and for each $x \in X^*$, let us consider the fiber $\Lambda_x := \bigcap_{n \ge 0} \bigcup_{z \in f|_{X^*}^{-n}(x)} g_{f^n z} \circ \ldots \circ g_z(\bar{W})$. Then define

$$\Lambda := \bigcup_{x \in X^*} \Lambda_x,$$

(see for example [9] for a similar type of skew products).

Λ is an invariant set for the skew product $F(x,y) = (f(x), g(x,y))$ defined on $X^* \times W$, and because of the expansion on X^* and the contraction on vertical fibers, F is hyperbolic on Λ. Thus we see that F is c-hyperbolic on Λ. The local stable manifolds of F are contained in the vertical fibers $\{x\} \times W, x \in X^*$. We call then (F, Λ) a **c-hyperbolic skew-product pair**.

The important thing to notice here is that we allow the images $g_y(W)$, coming from **different preimages** y of a point $x \in X^*$ **to overlap**.

In this case we can apply the above Theorem 1. Indeed for each j with $1 \le j \le m$, we know that a point $z \in X^* \cap X_j$ has at most q_j preimages in Λ, where q_j is the number of subsets $X_i, 1 \le i \le m$ such that $f(X_i) \supset X_j$. Then we have that the preimage counting function associated to F and Λ is smaller or equal than a locally constant function ω given by $\omega(x,y) := q_j$ if $x \in X_j, 1 \le j \le m$. However points in $\Lambda \cap (\{x\} \times W)$ may have strictly less than q_j F-preimages in Λ for $x \in X_j \cap X^*$. Thus we obtain the following Corollary which gives a lower estimate for the stable dimension:

Corollary 2. *Let a c-hyperbolic skew-product pair (F, Λ) as above. Then the stable dimension of Λ, i.e the Hausdorff dimension of the fibers $\Lambda_x, x \in X^*$, is larger or equal than the unique zero of the pressure function $t \to P(t\Phi^s - \log \omega)$, where $\omega|_{(X^* \cap X_j) \times W} = q_j, 1 \le j \le m$.*

Corollary 1 has also the following consequence:

Corollary 3. *In the setting of Corollary 2, if $f(I_j)$ contains all subintervals $I_1, \ldots, I_m, 1 \le j \le m$, it follows that F is m-to-1 on Λ if and only if $\exists z \in \Lambda$ with $\delta^s(z) = 0$. In this case we obtain $\delta^s(y) = 0, \forall y \in \Lambda$.*

Proof. Since $f(I_j) \supset I_1 \cup \ldots \cup I_m$, for all $j, 1 \le j \le m$, we can model the dynamics of f on X^* after the one-sided shift on m symbols Σ_m^+, whose topological entropy is equal to $\log m$. Also let us notice that the topological entropy of F on Λ is equal to the topological entropy of f on X^* since on vertical fibers we have contractions which do not add to the entropy. So $h_{top}(F|_\Lambda) = h_{top}(f|X^*) = \log m$.

From Corollary 1 and [7], it follows also that F is m-to-1 if and only if $\delta^s(z) = t_m$ for some point $z \in \Lambda$. So if F is m-to-1 on Λ, then we have $\delta^s(z) = t_m$; but $t_m = 0$ since $P(0 - \log m) = h_{top}(f|_{X^*}) - \log m = \log m - \log m = 0$.

Conversely, $P(0 \cdot \Phi^s - \log m) = h_{top}(f|_{X^*}) - \log m = 0$, so $t_m = 0$ as being the unique zero of the pressure. Thus if there exists a point $z \in \Lambda$ with $\delta^s(z) = 0$, then $\delta^s(z) = t_m$. Hence from Corollary 1 we obtain that F is m-to-1 on Λ.

□

Corollary 4. *In the setting of Corollary 2, assume that $f(I_j)$ contains all the subintervals $I_1, \ldots, I_m$ for $j = 1, \ldots, m$ and that there exists $x \in X^*$ such that $g_\xi(W) \cap g_\zeta(W) = \emptyset$ for some 1-preimages $\xi, \zeta \in X^*$ of x, with $\xi \ne \zeta$. Then it follows that $\delta^s(z) > 0, \forall z \in \Lambda$.*

Proof. Since $h_{top}(F|_\Lambda) = h_{top}(f|_{X^*}) = \log m$, we have that $P(0 \cdot \Phi^s - \log m) = h_{top}(f|_{X^*}) - \log m = 0$, hence $t_m = 0$. Now if there would exist a point $z \in \Lambda$ with $\delta^s(z) = 0$, then $\delta^s(z) = t_m$ and from Corollary 1 we have that F is m-to-1 on Λ. But since g_y is injective for all $y \in X^*$ and since there exist 1-preimages ξ, ζ of x so that $g_\xi(W) \cap g_\zeta(W) = \emptyset$, we conclude that F cannot be m-to-1 on Λ. Therefore

$$\delta^s(z) > 0, \forall z \in \Lambda$$

□

Remark: The dynamics of f on X^* can be modeled in general after shifts of finite type. Indeed we define the matrix $A = (a_{ij})_{1 \le i \le m, 1 \le j \le m}$ with $a_{ij} = 1$ if and only if $f(I_i) \supset I_j$. We can assume that A is irreducible so that σ_A becomes topologically transitive on Σ_A. The entropy of this dynamical system is equal to $\log |\lambda_A|$, where $|\lambda_A|$ is the spectral radius of A (equal to the maximum eigenvalue of A). A minimal (n, ε)-spanning set is obtained from cylinders of rank $n + p$, where p depends only on ε ([4]). This spanning set can be used then to estimate the pressure of the function $t\Phi^s - \log \omega$ where $\omega(z) = q_j$ for $z \in \Lambda \cap ((X^* \cap X_j) \times W)$ and $q_j := \sum_{1 \le i \le m} a_{ij}, 1 \le j \le m$.

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