Analytic families of semihyperbolic generalized polynomial-like mappings

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Abstract. We show that the Hausdorff dimension of Julia sets in any analytic family of semihyperbolic generalized polynomial-like mappings depends in a real-analytic manner on the parameter. For the proof we introduce abstract weakly regular analytic families of conformal graph directed Markov systems, we show that the Hausdorff dimension of limit sets in such families is real-analytic, and we associate to each analytic family of semihyperbolic GPLs a weakly regular analytic family of conformal graph directed Markov systems with the Hausdorff dimension of the limit sets equal to the Hausdorff dimension of the Julia sets of the corresponding semihyperbolic GPLs.

1. Introduction

The behavior of the pressure function of a semihyperbolic GPL has been studied in [6] and [9] (comp. also [4], [5] and [9]. The approach in [6] was to associate to a given semihyperbolic GPL a Hofbauer tower whereas in [9] a conformal graph directed Markov system in the sense of [2] was associated. The pressure function was shown to be real-analytic on some interval $(0, u)$ with $u > \mathrm{HD}(J(f))$, and the phase transition phenomenon (breaking-down real analyticity) was observed for some GPLs in [6]. In the present paper we deal with analytic families of semihyperbolic GPLs and, as the main result, we prove that the Hausdorff dimension of Julia sets in these families depends in a real-analytic manner on the parameter. Our approach is to define first abstract weakly regular analytic families of conformal graph directed Markov systems and to show that the Hausdorff dimension of limit sets in such families is real-analytic. This is done in Sections 2-4. In Section 5, summarizing the appropriate parts from [9], the construction of associating to each semihyperbolic GPL a conformal graph directed Markov system is described. It is proved in Section 6 that each analytic family of semihyperbolic GPLs gives rise to an analytic family of conformal graph directed Markov systems. Section 7 is devoted to the main step in the proof that this family is weakly regular. The proof of weak regularity is then completed in Section 7 and this simultaneously completes the proof of Theorem 6.6, the main result of this paper.

I would like to add that some assumptions appearing in this paper can be certainly weakened. For instance the set U in the definition of analytic families of semihyperbolic GPLs may depended "continuously" on parameter, and parabolic points can be allowed. We have worked in such, slightly more restrictive, setting for the ease and clarity of exposition.

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2. Conformal Graph Directed Function Systems

In this section we begin our study of graph directed Markov systems culminating in Section 4 with the proof of real analyticity of the Hausdorff dimension function of limit sets of a weakly regularly analytic family of strongly regular conformal graph directed Markov systems. Let us recall the definition of these systems taken from [2]. Graph directed Markov systems are based upon a directed multigraph and an associated incidence matrix, (V, E, i, t, A). The multigraph consists of a finite set V of vertices and a countable (either finite or infinite) set of directed edges E and two functions $i, t : E \to V$. For each edge e, $i(e)$ is the initial vertex of the edge e and $t(e)$ is the terminal vertex of e. The edge goes from $i(e)$ to $t(e)$. Also, a function $A : E \times E \to \{0, 1\}$ is given, called an incidence matrix. The matrix A is an edge incidence matrix. It determines which edges may follow a given edge. So, the matrix has the property that if $A_{uv} = 1$, then $t(u) = i(v)$. We will consider finite and infinite walks through the vertex set consistent with the incidence matrix. Thus, we define the set of infinite admissible words E_A^∞ on an alphabet A,

$$E_A^\infty = \{\omega \in E^\infty : A_{\omega_i\omega_{i+1}} = 1 \text{ for all } i \geq 1\},$$

by E_A^n we denote the set of all subwords of E_A^∞ of length $n \geq 1$, and by E_A^* we denote the set of all finite subwords of E_A^∞. We will consider the left shift map $\sigma : E_A^\infty \to E_A^\infty$ defined by dropping the first entry of ω. Sometimes we also consider this shift as defined on words of finite length. Given $\omega \in E^*$ by $|\omega|$ we denote the length of the word ω, i.e. the unique n such that $\omega \in E_A^n$. If $\omega \in E_A^\infty$ and $n \geq 1$, then

$$\omega|_n = \omega_1 \dots \omega_n.$$

A Graph Directed Markov System (GDMS) consists of a directed multigraph and incidence matrix together with a set of non-empty compact metric spaces $\{X_v\}_{v\in V}$, a number s, $0 < s < 1$, and for every $e \in E$, a 1-to-1 contraction $\phi_e : X_{t(e)} \to X_{i(e)}$ with a Lipschitz constant $\leq s$. Briefly, the set

$$\Phi = \{\phi_e : X_{t(e)} \to X_{i(e)}\}_{e\in E}$$

is called a GDMS. We now describe its limit set. For each $\omega \in E_A^*$, say $\omega \in E_A^n$, we consider the map coded by ω,

$$\phi_\omega = \phi_{\omega_1} \circ \dots \circ \phi_{\omega_n} : X_{t(\omega_n)} \to X_{i(\omega_1)}.$$

For $\omega \in E_A^\infty$, the sets $\{\phi_{\omega|_n}(X_{t(\omega_n)})\}_{n\geq 1}$ form a descending sequence of non-empty compact sets and therefore $\bigcap_{n\geq 1} \phi_{\omega|_n}(X_{t(\omega_n)}) \neq \emptyset$. Since for every $n \geq 1$, $\mathrm{diam}(\phi_{\omega|_n}(X_{t(\omega_n)})) \leq s^n \mathrm{diam}(X_{t(\omega_n)}) \leq s^n \max\{\mathrm{diam}(X_v) : v \in V\}$, we conclude that the intersection

$$\bigcap_{n\geq 1} \phi_{\omega|_n}(X_{t(\omega_n)})$$

is a singleton and we denote its only element by $\pi(\omega)$. In this way we have defined the coding map π:

$$\pi = \pi_\Phi : E_A^\infty \to X := \bigoplus_{v\in V} X_v$$

from E^∞ to $\bigoplus_{v\in V} X_v$, the disjoint union of the compact sets X_v. The set

$$J = J_\Phi = \pi(E_A^\infty)$$

will be called the limit set of the GDMS Φ. We call a GDMS conformal (CGDMS) if the following conditions are satisfied.

(4a) For every vertex $v \in V$, X_v is a compact connected subset of a Euclidean space $\mathbb{R}^d$ (the dimension d common for all $v \in V$) and $X_v = \overline{\mathrm{Int}(X_v)}$.

(4b) *(Open set condition)(OSC)* For all $a, b \in E$, $a \neq b$,

$$\phi_a(\mathrm{Int}(X_{t(a)}) \cap \phi_b(\mathrm{Int}(X_{t(b)}) = \emptyset.$$

(4c) For every vertex $v \in V$ there exists an open connected set $W_v \supset X_v$ such that for every $e \in I$ with $t(e) = v$, the map ϕ_e extends to a C^1 conformal diffeomorphism of W_v into $W_{i(e)}$.

(4d) *(Cone property)* There exist $\gamma, l > 0$, $\gamma < \pi/2$, such that for every $x \in X \subset \mathbb{R}^d$ there exists an open cone $\mathrm{Con}(x, \gamma, l) \subset \mathrm{Int}(X)$ with vertex x, central angle of measure γ, and altitude l.

(4e) There are two constants $L \geq 1$ and $\alpha > 0$ such that

$$\big| |\phi_e'(y)| - |\phi_e'(x)| \big| \leq L \|(\phi_e')^{-1}\|^{-1} \|y - x\|^\alpha$$

for every $e \in I$ and every pair of points $x, y \in X_{t(e)}$, where $|\phi_\omega'(x)|$ means the norm of the derivative.

We proved in [2] the following remarkable result.

Proposition 2.1. *If $d \geq 2$ and a family $\Phi = \{\phi_e\}_{e\in I}$ satisfies conditions (4a) and (4c), then it also satisfies condition (4e) with $\alpha = 1$.*

As a rather straightforward consequence of (4e) we proved in [2] the following.

Lemma 2.2. *If $\Phi = \{\phi_e\}_{e\in I}$ is a CGDMS, then for all $\omega \in E^*$ and all $x, y \in W_{t(\omega)}$, we have*

$$\big|\log |\phi_\omega'(y)| - \log |\phi_\omega'(x)|\big| \leq \frac{L}{1-s} \|y - x\|^\alpha.$$

As a straightforward consequence of (4e) we get the following.

(4f) (Bounded distortion property). There exists $K \geq 1$ such that for all $\omega \in E^*$ and all $x, y \in X_{t(\omega)}$

$$|\phi_\omega'(y)| \leq K |\phi_\omega'(x)|.$$

It was proved in [2] that for each $t \geq 0$ the following limit exists (can be equal to $+\infty$).

$$\mathrm{P}(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in E_A^n} ||\phi_\omega'||^t.$$

This number is called the topological presuure of the parameter t. In [2] a second useful parameter asscociate with a CGDMS has been introduced. Namely

$$\theta(\Phi) = \inf\{t : \mathrm{P}(t) < +\infty\} = \sup\{t : \mathrm{P}(t) = +\infty\}.$$

Let $\mathcal{F}in(E)$ denote the family of all finite subsets of E. The following characterization of $h_\Phi = \mathrm{HD}(J_\Phi)$ (denoted also by h_E), the Hausdorff dimension of the limit set J_Φ, being a variant of Bowen's formula, was proved in [2] as Theorem 4.2.13.

Theorem 2.3. *If the a CGDMS Φ is finitely irreducible, then*

$$\mathrm{HD}(J_\Phi) = \inf\{t \geq 0 : \mathrm{P}(t) < 0\} = \sup\{h_F : F \in \mathcal{F}in(I)\} \geq \theta(\Phi).$$

If $\mathrm{P}(t) = 0$, then t is the only zero of the function $\mathrm{P}(t)$, $t = \mathrm{HD}(J)$ and the system Φ is called regular.

In fact it was assumed in [2] that the system Φ is finitely primitive but the proof can be easily improved to this slightly more general setting. It will be convenient for us to make use of the following definitions.

Definition 2.4. *A CGDMS is said to be strongly regular if there exists $t \geq 0$ such that $0 < \mathrm{P}(t) < \infty$.*

A family $\{\phi_i\}_{i\in F}$ is said to be a cofinite subsystem of a system of $\Phi = \{\phi_i\}_{i\in E}$ if $F \subset E$ and the difference $E \setminus F$ is finite.

Definition 2.5. *A CGDMS is said to be cofinitely regular if each of its cofinite subsystems is regular.*

The following fact relating all these three notions can be found in [2].

Proposition 2.6. *Each cofinitely regular system is strongly regular and each strongly regular system is regular.*

Note that the system Φ is strongly regular if and only if $\mathrm{HD}(J_\Phi) > \theta(\Phi)$.

3. Analyticity of Perron-Frobenius Operators

The Section 2.6 from [2] devoted to the issue of anallyticity of Perron-Frobenius operators and topological pressure, unfortunately contains sensless and erroruness statements. Fortunately these errors are correctable and, since this Section 2.6 is critically important for us in this paper as well as for future references, we undertake here the task of presenting it in a correct coherent way. Recall that for $\omega, \tau \in E_A^\infty$, we define $\omega \wedge \tau \in E_A^\infty \cup E_A^*$ to be the longest initial block common to both ω and τ. We say that a function $g : E_A^\infty \to \mathbb{R}$ is Hölder continuous with an exponent $\alpha > 0$ if

$$v_\alpha(f) := \sup_{n\geq 1}\{V_{\alpha,n}(f)\} < \infty,$$

where

$$v_{\alpha,n}(f) = \sup\{|f(\omega) - f(\tau)|e^{\alpha(n-1)} : \omega, \tau \in E_A^\infty \text{ and } |\omega \wedge \tau| \geq n\}.$$

For every $\alpha > 0$ let $\mathcal{K}_\alpha$ be the set of all complex-valued Hölder continuous (not necessarily bounded and allowing $-\infty$ value with the convention that $e^{-\infty} = 0$ and $-\infty - (-\infty) = 0$) functions on E_A^∞. Set

$$\mathcal{K}_\alpha^s := \left\{\rho \in \mathcal{K}_\alpha : \sum_{e\in E} \exp\bigl(\sup\bigl(\mathrm{Re}\rho|_{[e]}\bigr)\bigr) < +\infty\right\}.$$

Each member of $\mathcal{K}_\alpha^s$ is called an α-Hölder summable potential. We start with the following little fact.

Lemma 3.1. *For every $R > 0$ there exists $M = M_R \geq 1$ (M_R increases with R) such that if $|z - \xi| \leq R$, then $|\mathrm{e}^\xi - \mathrm{e}^z| \leq M\mathrm{e}^{\mathrm{Re}z}|z - \xi|$.*

Proof. Looking at the Taylor's series expansion of the exponential function about 0, we see that there exists a constant $M \geq 1$ such $|\varepsilon^w - 1| \leq M|w|$, if $|w| \leq R$. Hence $|e^\xi - e^z| = |\varepsilon^z||e^{z-\xi} - 1| \leq e^{\mathrm{Re}z}M|z - \xi|$. $\square$

Our next result is this.

Lemma 3.2. *If $\rho \in \mathcal{K}_\alpha^s$, then $\varepsilon^\rho \in \mathrm{H}_\alpha$ and $||e^\rho||_\alpha \leq (1 + M_{v_\alpha(\rho)}v_\alpha(\rho))e^{\sup(\mathrm{Re}\rho)}$.*

Proof. It follows immediately from the definition of $\mathcal{K}_\alpha^s$ that $r\rho \in C_b(E_A^\infty)$ and $||e^\rho||_\infty \leq e^{\sup(\mathrm{Re}\rho)}$. Putting $R = v_\alpha(\rho)$, we get from Lemma 3.1 that if $\omega, \tau \in E_A^\infty$ and $\omega_1 = \tau_1$, then

$$|e^{\rho(\omega)} - e^{\rho(\tau)}| \leq M_R e^{\mathrm{Re}(\rho(\tau))}|\rho(\omega) - \rho(\tau)| \leq M_R e^{\sup(\mathrm{Re}\rho)} v_\alpha(\rho)\kappa^{|\omega\wedge\tau|}.$$

We are done. $\square$

Given $e \in E$ and $g : E_A^\infty \to \mathbb{C}$ define the mapping $g \circ e : E_A^\infty \to \mathbb{C}$ by the formula

$$g \circ e(\omega) = \begin{cases} g(e\omega) & \text{if } A_{e\omega_1} = 1 \\ -\infty & \text{if } A_{e\omega_1} = 0 \end{cases}$$

if $g \in \mathcal{K}_\alpha^s$, and

$$g \circ e(\omega) = \begin{cases} g(e\omega) & \text{if } A_{e\omega_1} = 1 \\ 0 & \text{if } A_{e\omega_1} = 0 \end{cases}$$

otherwise. As an immediate consequence of this definition, we have the following.

Lemma 3.3. *If $g : E_A^\infty \to \mathbb{C}$ is a Hölder continuous function, then so is $g \circ e : E_A^\infty \to \mathbb{C}$ (for all $e \in E$) and $v_\alpha(g \circ e) \le v_\alpha(g)$. If in addition, $g \in \mathrm{H}_a$, then also $g \circ e \in \mathrm{H}_a$ and $||g \circ e||_\alpha \le ||g||_\alpha$. If $g \in \mathcal{K}_\alpha^s$, then* $\sup(\mathrm{Re}(g \circ e)) = \sup\big(\mathrm{Re}(g|_{[e]})\big)$.

Let us prove the following.

Lemma 3.4. *If $k, l \in \mathrm{H}_\alpha$, then $kl \in \mathrm{H}_\alpha$ and $||kl||_\alpha \le 3||k||_\alpha ||l||_\alpha$.*

Proof. Obviously

$$||kl||_\infty \le ||k||_\infty ||l||_\infty. \tag{3.1}$$

Now fix $\omega, \tau \in E_A^\infty$ with $\omega_1 = \tau_1$. Then

$$\begin{aligned}
|kl(\omega) - kl(\tau)| &= |k(\omega)(l(\omega) - l(\tau)) + l(\tau)(k(\omega) - k(\tau))| \\
&\le |k(\omega)||l(\omega) - l(\tau)| + |l(\tau)||k(\omega) - k(\tau)| \\
&\le ||k||_\infty v_\alpha(l)\kappa^{|\omega\wedge\tau|} + ||l||_\infty v_\alpha(k)\kappa^{|\omega\wedge\tau|} \\
&\le 2||k||_\alpha ||l||_\alpha \kappa^{|\omega\wedge\tau|}.
\end{aligned}$$

Hence, $v_\alpha(kl) \le 2||k||_\alpha||l||_\alpha$, and we complete the proof by combining this with (3.1) □

Now for every $\rho \in \mathcal{K}_a^s$ and every $e \in E$ define the operator $A_{\rho,e} : C_b(E_A^\infty) \to C_b(E_A^\infty)$ by the formula

$$A_{\rho,e}(g) = e^{\rho\circ e} g \circ e.$$

As an immediate consequence of Lemma 3.3, Lemma 3.2, Lemma 3.4 and the increasing property of the function $R \mapsto M_R$ coming from Lemma 3.1, we get the following.

Lemma 3.5. *If $\rho \in \mathcal{K}_a^s$ and $e \in E$, then the operator $A_{\rho,e}$ preserves the Banch space* H_α *and* $||A_{\rho,e}||_\alpha \le 3(1 + M_{v_\alpha(\rho)} v_\alpha(\rho)) \exp\big(\sup\big(\mathrm{Re}\rho|_{[e]}\big)\big)$.

Now notice that the function v_α is a pseudo-norm on the vector space $\mathcal{K}_\alpha$. So, it induces a pseudo-metric on $\mathcal{K}_\alpha$ $(v_\alpha(l-k))$, and this pseudo-metric restricted to $\mathcal{K}_\alpha^s$ induces a topology on $\mathcal{K}_\alpha^s$, which will be called in the sequel the α-Hölder topology. By $B_\alpha(\rho, r) = \{\theta \in \mathcal{K}_\alpha^s : v_\alpha(\theta - \rho) < r\}$ we denote the balls generated by the pseudo-norm v_α. We shall prove the following.

Lemma 3.6. *For every $e \in E$ the function $\rho \mapsto A_{\rho,e} \in L(\mathrm{H}_\alpha)$, defined on $\mathcal{K}_\alpha^s$, is continuous.*

Proof. Note that $\mathcal{K}_\alpha^s$ is closed with respect to additions of functions and with respect to multiplication by scalars ≥ 1. Fix $\rho \in \mathcal{K}_\alpha^s$ an consider an arbitrary $\theta \in B_\alpha(\rho, 1)$. Then, in view of, Lemma 3.5, we have that

$$||A_{\theta,e} - A_{\rho,e}||_\alpha = ||A_{\theta-\rho,e}||_\alpha \le 3(1+M_1)\exp\big(\sup\big((\mathrm{Re}\theta - \mathrm{Re}\rho)|_{[e]}\big)\big) \le 3(1+M_1)\exp(v_\alpha(\theta-\rho)).$$

We are done. □

Now fix $\rho \in \mathcal{K}_\alpha^s$ and notice that for every $g \in C_b$, the function

$$\mathcal{L}_\rho(g) = \sum_{e\in E} e^{\rho\circ e} g\circ e = \sum_{e\in E} A_{\rho,e}(g)$$

is well-defined, belongs to C_b and $||\mathcal{L}_\rho(g)||_\infty \le \sum_{e\in E}\exp\bigl(\sup\bigl(\mathrm{Re}\rho|_{[e]}\bigr)\bigr)||g||_\infty$. We have therefore defined the operator $\mathcal{L}_\rho$ acting continuously on C_b with

$$||\mathcal{L}_\rho||_\infty \le \sum_{e\in E}\exp\bigl(\sup\bigl(\mathrm{Re}\rho|_{[e]}\bigr)\bigr) < \infty.$$

It follows from Lemma 3.5 that the operator $\mathcal{L}_\rho$ preserves the Banch space H_α and

$$||\mathcal{L}_\rho||_\alpha \le 3(1 + M_{v_\alpha(\rho)}v_\alpha)(\rho)\sum_{e\in E}\exp\bigl(\sup\bigl(\mathrm{Re}\rho|_{[e]}\bigr)\bigr).$$

Passing directly to the analitycity issues, we shall prove the following.

Lemma 3.7. *Suppose that Λ is an open subset of $\mathbb{C}$, that for every $\lambda \in \Lambda$, $\rho_\lambda \in \mathcal{K}_\alpha^s$ and that the function $\lambda \mapsto \rho_\lambda(\omega) \in \mathbb{C}$, $\lambda \in \Lambda$, is holomorphic for all $\omega \in E_A^\infty$. If the function $\lambda \mapsto \mathcal{L}_{\rho_\lambda} \in L(\mathrm{H}_\alpha)$, $\lambda \in \Lambda$, is continuous and*

$$\Sigma(\Lambda) := \sum_{e\in E}\exp\bigl(\sup\{\mathrm{Re}\rho_\lambda\circ e : \lambda \in \Lambda\}\bigr) < +\infty,$$

then the map $\lambda \mapsto \mathcal{L}_{\rho_\lambda} \in L(\mathrm{H}_\alpha)$ is holomorphic throughout Λ.

Proof. Let $\gamma \subset \Lambda$ be a simple closed contractible in Λ rectifiable curve. Fix $g \in \mathrm{H}_\alpha$ and $\omega \in E_A^\infty$. Since $\Sigma(\Lambda)$ is finite, it follows from Lemma 3.5 and the Weierstrass M-test that the series $\lambda \mapsto \mathcal{L}_{\rho_\lambda}g(\omega)$ converges absolutely uniformly in $\mathbb{C}$. Therefore the functiom $\lambda \mapsto \mathcal{L}_{\rho_\lambda}g(\omega) \in \mathbb{C}$, $\lambda \in \Lambda$, is holomorphic. Hence by Cauchy's Theorem $\int_\gamma \mathcal{L}_{\rho_\lambda}g(\omega)d\lambda = 0$. Since the function $t \mapsto g \in \mathcal{L}_{\rho_\lambda}g \in \mathrm{H}_\alpha$ is continuous, the integral $\int_\gamma \mathcal{L}_{\rho_\lambda}g(\omega)d\lambda$ exists, and for every $\omega \in E^\infty$, we have that $\int_\gamma \mathcal{L}_{\rho_\lambda}gd\lambda(\omega) = \int_\gamma \mathcal{L}_{\rho_\lambda}g(\omega)d\lambda = 0$. Hence $\int_\gamma \mathcal{L}_{\rho_\lambda}gd\lambda = 0$. Now, since the function $\lambda \mapsto \mathcal{L}_{\rho_\lambda} \in L(\mathrm{H}_\alpha)$ is continuous, the integral $\int_\gamma \mathcal{L}_{\rho_\lambda}d\lambda$ exists, and for every $g \in \mathrm{H}_\alpha$, $\int_\gamma \mathcal{L}_{\rho_\lambda}d\lambda(g) = \int_\gamma \mathcal{L}_{\rho_\lambda}gd\lambda = 0$. Thus $\int_\gamma \mathcal{L}_{\rho_\lambda}d\lambda = 0$ and, by Morera's theorem, the map $\lambda \mapsto \mathcal{L}_{\rho_\lambda} \in L(\mathrm{H}_\alpha)$ is holomorphic throughout Λ. We are done. $\square$

The main result of this section is now concluded as follows.

Theorem 3.8. *Suppose that Λ is an open subset of $\mathbb{C}$ and that the function $\lambda \mapsto \rho_\lambda \in \mathcal{K}_\alpha^s$, $\lambda \in \Lambda$, is continuous. If the function $\lambda \mapsto \rho_\lambda(\omega) \in \mathbb{C}$, defined on Λ is holomorphic for every $\omega \in E_A^\infty$, then the function $\lambda \mapsto \mathcal{L}_{\rho_\lambda} \in L(\mathrm{H}_\alpha)$ is also holomorphic.*

Proof. Fix $\lambda_0 \in \Lambda$. According to Lemma 3.7 it suffices to demonstrate that there exists $\delta > 0$ such that $\Sigma(B(\lambda_0,\delta)) < +\infty$ and the function $\lambda \mapsto \mathcal{L}_{\rho_\lambda} \in L(\mathrm{H}_\alpha)$, $\lambda \in B(\lambda_0,\delta)$, is continuous. Since by the Weierstrass M-test and Lemma 3.5 along with Lemma 3.6, the former implies the latter, we are only to prove the former. Indeed, since the function

$\lambda \mapsto \rho_\lambda \in \mathcal{K}_\alpha^s$, $\lambda \in \Lambda$, is continuous, there exists $\delta > 0$ so small that $v_\alpha(\rho_\lambda - \rho_{\lambda_0}) < 1$ whenever $|\lambda - \lambda_0| < \delta$. We then have for all $\lambda \in B(\lambda_0, \delta)$ and all $a \in E$, that

$$
\begin{aligned}
3\big(1 + M_{v_\alpha\left(\rho_\lambda\right)} v_\alpha(\rho_\lambda)\big) \exp\big(\sup\big(\mathrm{Re}\rho_\lambda|_{[a]}\big)\big) \leq & \\
&\leq 3\big(1 + M_{v_\alpha(\rho_{\lambda_0}))+1} v_\alpha\big(\rho_{\lambda_0})\big) + 1\big) \exp\big(\sup\big(\mathrm{Re}\rho_{\lambda_0}|_{[a]}\big) + 1\big) \\
&= 3e\big(1 + M_{v_\alpha(\rho_{\lambda_0})+1} v_\alpha\big(\rho_{\lambda_0})\big) + 1\big) \exp\big(\sup\big(\mathrm{Re}\rho_{\lambda_0}|_{[a]}\big)\big).
\end{aligned}
$$

Therefore, since $\lambda_0 \in \mathcal{K}_\alpha^s$, we have

$$
\Sigma(B(\lambda_0, \delta)) \leq 3e\left(1 + M_{v_\alpha\left(\rho_{\lambda_0}\right)+1} v_\alpha(\rho_{\lambda_0}) + 1\right) \sum_{a \in E} \exp\big(\sup\big(\mathrm{Re}\rho_{\lambda_0}|_{[a]}\big)\big) < \infty.
$$

We are done. $\square$

4. Dimension analyticity in Graph Directed Markov Systems

In this section we bring up the issue of real analyticity of the Hausdorff dimension function of a weakly regularly analytic family of strongly regular conformal graph directed Markov systems. Our central idea is to embed, the naturally arising, real-analytic family of Perron-Frobenius operators into a family, which by applying Thorem 3.8 from the previous section, can be proven to be analytic. Then one uses the perturbation theory (Kato-Rellich Theorem) for linear operators, a version of Bowen's formula, and the Inverse Function Theorem to conclude the proof. Let $\Lambda \subset \mathbb{C}^d$, $d \geq 1$, be an open susbset of $\mathbb{C}^d$. Let $\{\Phi_\lambda\}_{\lambda \in \Lambda}$ be a family of CGDMS with the same set V of vertices, the same set E of edges, the same finitely irreducible incidence matrix A and the same seed pairs $\{(X_v, W_v)\}_{v \in V}$ with all $W_w \subset \mathbb{C}$. Fix $\lambda_0 \in \Lambda$ and for every $\omega \in E_A^\infty$ consider the function $\psi_\omega : \Lambda \to \mathbb{C}$ given by the formula

$$
\psi_\omega(\lambda) = \frac{(\phi_{\omega_1}^\lambda)'(\pi_\lambda(\sigma\omega))}{(\phi_{\omega_1}^{\lambda_0})'(\pi_{\lambda_0}(\sigma\omega))}, \tag{4.1}
$$

where $\pi_\lambda = \pi_{\Phi_\lambda} : E_A^\infty \to X$ is the coding map induced by the CGDMS Φ_λ. The family $\{\Phi_\lambda\}_{\lambda \in \Lambda}$ is said to be analytic if

(a) for every $e \in E$ and every $x \in X_{t(e)}$, the function $\lambda \mapsto \phi_e^\lambda(x)$, $\lambda \in \Lambda$, is analytic.

The family $\{\Phi_\lambda\}_{\lambda \in \Lambda}$ is called weakly regular analytic if in addition the following hold.

(b) Φ^{λ_0} is strongly regular.
(c) There exist a function $\kappa : E \to (0, +\infty)$ and a constant $C_1 \geq 1$ such that

$$
|(\phi_{\omega_1}^\lambda)'(\pi_\lambda(\sigma(\omega)))| \leq C_1 \exp(-\kappa(\omega_1))
$$

for all $\lambda \in \Lambda$ and all $\omega \in E_A^\infty$.

In order to formulate our last condition required for weakly regular analyticity, we shall prove first the following.

Lemma 4.1. *Suppose that $\{\Phi_\lambda\}_{\lambda\in\Lambda}$ is an analytic family of CGDMS. For every vertex $v \in V$ fix $x_v \in \mathrm{Int}X_v$. Then the family $\{\lambda \mapsto \phi_\omega^\lambda(x_{t(\omega)})\}_{\omega\in E_A^*}$ consists of holomorphic maps of on Λ and this family is normal. Also, the family $\{\lambda \mapsto \pi_\lambda(\omega)\}_{\omega\in E_A^\infty}$ consists of holomorphic maps on Λ and is normal.*

Proof. Since all the maps $(\lambda, z) \mapsto \phi_e^\lambda(z)$, $(\lambda, z) \in \Lambda\times\mathrm{Int}X_{t(e)}$, $e \in E$ are holomorphic, all the maps $\lambda \mapsto \phi_\omega^\lambda(x_{t(\omega)})$, $\omega \in E_A^*$ are also holomorphic. Since their ranges are all contained in the bunded set $\bigcup_{v\in V} \mathrm{Int}X_v$, the family $\{\lambda \mapsto \phi_\omega^\lambda(x_{t(\omega)})\}_{\omega\in E_A^*}$ is normal. Therefore, since for every $\omega \in E_A^\infty$, the sequence of functions $\big(\lambda \mapsto \phi_{\omega|_n}^\lambda(x_{t(\omega|_n)})\big)_{n=1}^\infty$ defined on Λ converges pointwise to $\pi_\lambda(\omega)$, we conclude that each function $\lambda \mapsto \pi_\lambda(\omega)$, defined on Λ, is holomorphic. Since the range of all these functions is in the bunded set $\bigcup_{v\in V} X_v$, the family $\{\lambda \mapsto \pi_\lambda(\omega)\}_{\omega\in E_A^\infty}$ is normal. We are done. $\square$

Now we can complete the definition of weak analytic regularity by demanding that

(d) For every $\omega \in E_A^\infty$ there is well-defined $\log\psi_\omega : \Lambda \to \mathbb{C}$, a holomorphic branch of logarithm of ψ_ω such that $\log\psi_\omega(\lambda_0) = 0$ and the family of functions

$$\left\{\lambda \mapsto \frac{1}{\kappa(\omega_1)}\log\psi_\omega(\lambda)\right\}_{\omega\in E_A^\infty}$$

is bounded and, consequently, normal.

Let $h_\lambda = \mathrm{HD}\big(J_{\Phi^\lambda}\big)$ be the Hausdorff dimension of the limit set of the CGDMS Φ^λ. The goal of this section is to prove the following.

Theorem 4.2. *If $\{\Phi_\lambda\}_{\lambda\in\Lambda}$ is a weakly regularly analytic family of CGDMS, then the family of functions $\lambda \mapsto h_\lambda = \mathrm{HD}\big(J_{\Phi^\lambda}\big)$ is real-analytic throughout Λ.*

This theorem was formulated in [7] only for iterated function systems and, what is more important, assuming that the function κ is constant, actually assuming even a little bit more. There was no proof provided in [7] but a one-line indiction of how to make up the proof based on [12] and [11]. Concluding, since this theorem is central for us in this paper, since it is of interest itself, and since there is no written proof of even of its earlier weaker version, we have decided to provide here a self-contained proof of this theorem. The general strategy of the this proof is based on Theorem 3.8 from the previous section. Now, we start it as follows. For every $z = z_1, z_2, \dots, z_d \in \mathbb{C}^d$ and $r_1, r_2, \dots, r_d > 0$ let

$$D_d(z; r_1, r_2, \dots, r_d) = \{(w_1, w_2, \dots, w_d) \in \mathbb{C}^d : |w_j - z_j| < r_j \text{ for all } 1 \le j \le d\}.$$

In the case when all radii r_j are equal, say to r, we will frequently write shrtly $D_d(z; r)$ for $D_d(z; r_1, r, \dots, r)$. Fix $r > 0$ so small that $D_d(\lambda_0; r) \subset \Lambda$. For the ease of exposition we assume now that $d = 1$ i.e. that Λ is an open subset of the complex plane $\mathbb{C}$. Because of item (d) above, for every $\omega \in E^\infty$, the function $\log\psi_\omega$ expands in its Taylor series on

$D_1(\lambda_0;r)$:

$$\log\psi_\omega(\lambda)=\sum_{n=0}^\infty a_n(\omega)(\lambda-\lambda_0)^n \tag{4.2}$$

and, by item (d) again, there exists a constant $M_1>0$, independent of $\lambda\in\Lambda$ and $\omega\in E_A^\infty$, such that

$$|\log\psi_\omega(\lambda)|\le M_1\kappa(\omega_1). \tag{4.3}$$

Hence, applying Cauchy's estimates, we get for every $n\ge 0$ that

$$|a_n(\omega)|\le\frac{M_1\kappa(\omega_1)}{r^n}. \tag{4.4}$$

For every $\lambda=x+iy\in D(\lambda_0;r)$, we have from (4.2) that

$$\begin{aligned}\mathrm{Re}\log\psi_\omega(\lambda)&=\sum_{p,q=0}^\infty\mathrm{Re}\Big(a_{p+q}(\omega)\binom{p+q}{q}i^q\Big)(x-\mathrm{Re}\lambda_0)^p(y-\mathrm{Im}\lambda_0)^q\\&=\sum_{p,q=0}^\infty c_{p+q}(\omega)\binom{p+q}{q}i^q(x-\mathrm{Re}\lambda_0)^p(y-\mathrm{Im}\lambda_0)^q,\end{aligned} \tag{4.5}$$

where, due to (4.4),

$$|c_{p+q}(\omega)|\le\binom{p+q}{q}|a_{p+q}(\omega)|\le 2^{p+q}|a_{p+q}(\omega)|\le 2^{p+q}M_1\kappa(\omega_1)r^{-(p+q)}.$$

Hence, employing the embedding $\mathbb{C}\to\mathbb{C}^2=\mathbb{C}\times\mathbb{C}$, $x+iy\mapsto(x,y)$, we see that $\mathrm{Re}\log\psi_\omega$ extends by the same power series expansion $\sum_{p,q=0}^\infty c_{p+q}(\omega)\binom{p+q}{q}i^q(x-\mathrm{Re}\lambda_0)^p(y-\mathrm{Im}\lambda_0)^q$, $(x,y)\in\mathbb{C}^2$, to a complex-valued analytic function on the polydisk $D_2(\lambda_0;r/4)$. Keep the same symbol $\mathrm{Re}\log\psi_\omega$ for this extension and note that

$$|\mathrm{Re}\log\psi_\omega|\le 4M^1\kappa(\omega_1)\ \text{on}\ D_2(\lambda_0;r/4). \tag{4.6}$$

Define the potential $\zeta_\omega:D_2(\lambda_0;r/4)\to\mathbb{C}$ by the formula

$$\zeta_\omega(\lambda)=\mathrm{Re}\log\psi_\omega(\lambda)+\log\left|(\phi_{\omega_1}^{\lambda_0})'(\pi_{\lambda_0}(\sigma\omega))\right|.$$

Fix $t_0>\theta\big(\Phi^{\lambda_0}\big)$ and put

$$\overline{r}=\min\left\{r/4,\big(t_0-\theta\big(\Phi^{\lambda_0}\big)\big)/2\right\}.$$

Note that for all $(\lambda,t)\in D_{\mathbb{R}^3}((\lambda_0,t_0);\overline{r}):=D_{\mathbb{R}^2}(\lambda_0;\overline{r})\times D_{\mathbb{R}}(t_0,\overline{r})$, we have by (4.1) that

$$\exp(t\zeta_\omega(\lambda))=|\psi_\omega(\lambda)|^t|(\phi_{\omega_1}^{\lambda_0})'(\pi_{\lambda_0}(\sigma\omega))|^t=|(\phi_{\omega_1}^{\lambda})'(\pi_\lambda(\sigma\omega))|^t. \tag{4.7}$$

Now, our goal is to prove the following result announced at the begining of this section.

Lemma 4.3. *There exists $r_2\in(0,\overline{r})$ such that the family of potentials*

$$(\lambda,t)\mapsto t\zeta_{(\cdot)}(\lambda):E_A^\infty\to\mathbb{C},\ \ (\lambda,t)\in D_3((\lambda_0,t_0);\overline{r}),$$

satisfies the hypothesis of Theorem 3.8.

Proof. Indeed, obviously for every $\omega \in E_A^\infty$, the function $(\lambda, t) \mapsto t\zeta_\omega(\lambda)$, $(\lambda, t) \in D_3((\lambda_0, t_0); \overline{r})$ is holomorphic. Using (4.6) we get for all $\omega \in E_A^\infty$ and all $(\lambda, t) \in D_3((\lambda_0, t_0); \overline{r})$ that

$$\begin{aligned}
|\exp(t\zeta_\omega(\lambda))| &= \exp\Big(\mathrm{Re}\big(t\mathrm{Re}\log\psi_\omega(\lambda) + t\log\big|(\phi_{\omega_1}^{\lambda_0})'(\pi_{\lambda_0}(\sigma\omega))\big|\big)\Big) \\
&= \exp\big(\mathrm{Re}\big(t\mathrm{Re}\log\psi_\omega(\lambda)\big)\big)\,\big|(\phi_{\omega_1}^{\lambda_0})'(\pi_{\lambda_0}(\sigma\omega))\big|^{\mathrm{Re}t} \\
&\le \exp\big(|t||\mathrm{Re}\log\psi_\omega(\lambda)|\big)\,\big|(\phi_{\omega_1}^{\lambda_0})'(\pi_{\lambda_0}(\sigma\omega))\big|^{\mathrm{Re}t} \\
&\le \exp(4M_1\kappa(\omega_1)|t|)\,\big|(\phi_{\omega_1}^{\lambda_0})'(\pi_{\lambda_0}(\sigma\omega))\big|^{\mathrm{Re}t} \\
&\le \exp\big(4M_1(t_0 + \overline{r})\kappa(\omega_1)\big)\,\big|(\phi_{\omega_1}^{\lambda_0})'(\pi_{\lambda_0}(\sigma\omega))\big|^{\theta(\Phi^{\lambda_0}) + \overline{r}}.
\end{aligned} \tag{4.8}$$

Now, it follows from item (d) of weakly regular analyticity (equicontinuity and $\log\psi_\omega(\lambda_0) = 0$) that if $r_2 \in (0, \overline{r})$ is taken sufficiently small and λ appearing in formula (4.3) is restricted to the disk $D_1(\lambda_0; r_2)$, then we can have $M_1 > 0$ as small as we wish, for example

$$M_1 \le \frac{1}{8}\overline{r}(t_0 + \overline{r})^{-1},$$

Inserting this inequality to (4.8) and using condition (c), we get

$$|\exp(t\zeta_\omega(\lambda))| \le C_1^{(\overline{r}/2)}\,\big|(\phi_{\omega_1}^{\lambda_0})'(\pi_{\lambda_0}(\sigma\omega))\big|^{\theta(\Phi^{\lambda_0}) + \frac{\overline{r}}{2}}$$

for all $\omega \in E^\infty$ and all $(\lambda, t) \in D_3((\lambda_0, t_0); r_2)$. Therefore

$$\sum_{e \in E} ||\exp\big(t\zeta|_{[e]}\big)||_\infty \le C_1^{(\overline{r}/2)} \sum_{e \in E} ||(\phi_e^{\lambda_0})'||^{\theta(\Phi^{\lambda_0}) + \frac{\overline{r}}{2}} < +\infty.$$

Hence, for every $(\lambda, t) \in D_3((\lambda_0, t_0); r_2)$, we have that $t\zeta_{(\cdot)}(\lambda) \in \mathcal{K}_\alpha^s$. So, in order to prove our lemma we are left to show that for every $(\lambda, t) \in D_3((\lambda_0, t_0); r_2)$, the function $\omega \mapsto t\zeta_\omega(\lambda))$, $\omega \in E^\infty$, is Hölder continuous, and then that the mapping $(\lambda, t) \mapsto t\zeta_{(\cdot)}(\lambda) \in \mathcal{K}_\alpha^s$ is continuous. Applying Lemma 4.2.2 from [2] and Koebe's Distortion Theorem for argument (see Corollary on page 353 of [1]), we conclude that there exists a constant $K_1 \ge 1$ such that $|\log(\phi_\omega^\lambda)'(y)| - \log(\phi_\omega^\lambda)'(x)| \le K_1|y - x|$ for all $\lambda \in \Lambda$, all $\omega \in E_A^\infty$ and all $x, y \in X_{t(\omega)}$. Hence if $\omega, \tau \in E_A^\infty$ and $|\omega \wedge \tau| \ge 1$, then

$$\begin{aligned}
|\log\psi_\omega(\lambda) - \log\psi_\tau(\lambda)| &= \big|\log\big((\phi_{\omega_1}^\lambda)'(\pi_\lambda(\sigma\omega))\big) - \log\big((\phi_{\tau_1}^\lambda)'(\pi_\lambda(\sigma\tau))\big)\big| \\
&\le K_1|\pi_\lambda(\sigma\omega) - \pi_\lambda(\sigma\tau)| \le K_1 s^{|\omega\wedge\tau|-1}\mathrm{diam}(X) \\
&= K_1 s^{-1} s^{|\omega\wedge\tau|}\mathrm{diam}(X).
\end{aligned}$$

Thus, using Cauchy's Estimates again, we conclude that for all $n \ge 0$,

$$|a_n(\omega) - a_n(\tau)| \le K_1 s^{-1}\mathrm{diam}(X) s^{|\omega\wedge\tau|} r^{-n}.$$

Consequently,

$$|c_{p,q}(\omega) - c_{p,q}(\tau)| \le 2^{p+q}|a_{p+q}(\omega) - a_{p+q}(\tau)| \le 2^{p+q}K_1 s^{-1}\mathrm{diam}(X) r^{-(p+q)} s^{|\omega\wedge\tau|}. \tag{4.9}$$

Therefore,

$$|\mathrm{Re}\log\psi_\omega(\lambda) - \mathrm{Re}\log\psi_\tau(\lambda)| \le 4K_1 s^{-1}\mathrm{diam}(X) s^{|\omega\wedge\tau|}.$$

for all $\lambda \in D_2(\lambda_0, r/2)$ and $\omega, \tau \in E_A^\infty$ with $|\omega \wedge \tau| \geq 1$. Hence, using also (4.9), we conclude that

$$|t\zeta_\omega(\lambda) - t\zeta_\tau(\lambda)| \leq 5K_1 s^{-1}\mathrm{diam}(X)s^{|\omega\wedge\tau|}$$

for all $(\lambda, t) \in D_3((\lambda_0, t_0); r_2)$ and all $\omega, \tau \in E_A^\infty$ with $|\omega \wedge \tau| \geq 1$. The proof that the function $\omega \mapsto t\zeta_\omega(\lambda))$ belongs to $\mathcal{K}_a^s$ is complete. This function is obviously continuous on the polydisk $D_3((\lambda_0, t_0); r_2)$ with respect to the variable t. It is therefore sufficient to prove the Lipschitz continuity of the functions $\lambda \mapsto t\zeta_{(\cdot)}(\lambda) \in \mathcal{K}_\alpha^s$ with Lipschitz constants independent of t. In order to do it, fix $\lambda = (\lambda_x, \lambda_y)$ and $\lambda' = (\lambda_x', \lambda_y')$ in $D_2(\lambda_0, r_2)$. Put $a_x = \lambda_x' - \mathrm{Re}\lambda_0$, $a_y = \lambda_y' - \mathrm{Im}\lambda_0$, $b_x = \lambda_x - \mathrm{Re}\lambda_0$ and $b_y = \lambda_y - \mathrm{Im}\lambda_0$. We then have

$$\begin{aligned}
|a_x^p a_y^q - b_x^p b_y^q| &= |a_x^p(a_y^q - b_y^q) + b_y^q(a_x^p - b_x^p)| \\
&\leq |a_x^p||a_y - b_y|\sum_{i=0}^{q-1}|a_y|^i|b_y|^{q-1-i} + |b_y^q||a_x - b_x|\sum_{i=0}^{p-1}|a_x|^i|b_x|^{p-1-i} \\
&\leq \left(q\left(\frac{r}{4}\right)^p\left(\frac{r}{4}\right)^{q-1} + p\left(\frac{r}{4}\right)^q\left(\frac{r}{4}\right)^{p-1}\right)||\lambda' - \lambda|| \\
&\leq \frac{4}{r}(p+q)\left(\frac{r}{4}\right)^p\left(\frac{r}{4}\right)^q||\lambda' - \lambda||.
\end{aligned} \tag{4.10}$$

Now fix $\omega, \tau \in E_A^\infty$ with $\omega_1 = \tau_1$. It follows from (4.5), (4.9) and (4.10) that

$$\begin{aligned}
&\big|\mathrm{Re}\log\psi_\omega(\lambda') - \mathrm{Re}\log\psi_\omega(\lambda) - (\mathrm{Re}\log\psi_\tau(\lambda') - \mathrm{Re}\log\psi_\tau(\lambda))\big| \\
&= \left|\sum_{p,q=0}^\infty (c_{p,q}(\omega) - c_{p,q}(\tau))\big((\lambda_x' - \mathrm{Re}\lambda_0)^p(\lambda_y' - \mathrm{Im}\lambda_0)^q - (\lambda_x - \mathrm{Re}\lambda_0)^p(\lambda_y - \mathrm{Im}\lambda_0)^q)\big)\right| \\
&\leq 4K_1 s^{-1}\mathrm{diam}(X)r^{-1}s^{|\omega\wedge\tau|}||\lambda' - \lambda||\sum_{p,q=0}^\infty (p+q)2^{-(p+q)} \\
&= C||\lambda' - \lambda||s^{|\omega\wedge\tau|},
\end{aligned}$$

where $C = 4K_1 s^{-1}\mathrm{diam}(X)r^{-1}\sum_{p,q=0}^\infty (p+q)2^{-(p+q)}$ is finite. Thus,

$$v_\alpha\big(t\mathrm{Re}\log\psi_{(\cdot)}(\lambda') - t\mathrm{Re}\log\psi_{(\cdot)}(\lambda))\big) \leq C(|t_0| + r_2)r^{-1}||\lambda' - \lambda||.$$

for all $(\lambda, t) \in D_3((\lambda_0, t_0); r_2)$. Since $t\zeta_{(\cdot)}(\lambda') - t\zeta_{(\cdot)}(\lambda) = t\mathrm{Re}\log\psi_{(\cdot)}(\lambda') - t\mathrm{Re}\log\psi_{(\cdot)}(\lambda)$, the proof of continuity of the function $(\lambda, t) \mapsto t\zeta_{(\cdot)}(\lambda) \in \mathcal{K}_\alpha^s$, $(\lambda, t) \in D_3((\lambda_0, t_0); r_2)$, is complete, and the proof of Lemma 4.3 is finished. $\square$

Now set $t_0 = h_{\lambda_0}$, which is larger than $\theta\big(\Phi^{\lambda_0}\big)$ because of strong regularity of the system Φ^{λ_0} and for every $(\lambda, t) \in D_3((\lambda_0, h_{\lambda_0}); r_2)$ put

$$\mathcal{L}_{\lambda,t} = \mathcal{L}_{t\zeta_{(\cdot)}}(\lambda) : \mathrm{H}_\alpha \to \mathrm{H}_\alpha.$$

In view of (4.7) and Theorem 2.4.6 from [2], $e^{\mathrm{P}_{\lambda_0}(h_{\lambda_0})}$, $(t \in B_\mathbb{R}(h_{\lambda_0}, r_2))$ is a simple isolated eigenvalue of the operator $\mathcal{L}_{\lambda,t} : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$. Hence, in view of Lemma 4.3 and Theorem 3.8, Kato-Rellich Perturbation Theorem ([3], Theorem XII.8) is applicable to yield $r_3 \in (0, r_2]$ and a holomorphic function $\gamma : D_3((\lambda_0, h_{\lambda_0}); r_3) \to \mathbb{C}$ such that $\gamma(\lambda_0, h_{\lambda_0}) = e^{\mathrm{P}_{\lambda_0}(h_{\lambda_0})}$ is a simple isolated eigenvalue of $\mathcal{L}_{\lambda,t} : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ with the remainder of the spectrum uniformly separated from $\gamma(\lambda, t)$. In particular there exists $r_4 \in (0, r_3]$ and $\eta > 0$ such that

$$\sigma(\mathcal{L}_{\lambda,t}) \cap D_1(e^{\mathrm{P}_{\lambda_0}(h_{\lambda_0})}, \eta) = \{\gamma(\lambda, t)\} \tag{4.11}$$

for all $(\lambda, t) \in D_3((\lambda_0, h_{\lambda_0}); r_4)$. Since $e^{\mathrm{P}_\lambda(h_\lambda)}$ is the spectral radius $r(\mathcal{L}_{\lambda,h_\lambda})$ of the operator $\mathcal{L}_{\lambda,h_\lambda}$ for all $(\lambda, t) \in B(\lambda_0, r_4) \times B_{\mathbb{R}}(t_0, r_4)$ in view of semi-continuity of the spectral set function (see Theorem 10.20 on p.256 in [8]), taking r_4 appropriately smaller, we also have that $r(\mathcal{L}_{\lambda,t}) \in [0, e^{\mathrm{P}_{\lambda_0}(h_{\lambda_0})} + \eta)$, and along with (4.11), this implies that $e^{\mathrm{P}_\lambda(t)} = \gamma(\lambda, t)$ for all $(\lambda, t) \in B(\lambda_0, r_4) \times B_{\mathbb{R}}(t_0, r_4)$. Consequently, the function $(\lambda, t) \mapsto \mathrm{P}_\lambda(t)$, $(\lambda, t) \in B(\lambda_0, r_4) \times B_{\mathbb{R}}(t_0, r_4)$ is real-analytic. By Bowen's formula (see Theorem 2.3) and strong regularity of the system Φ^{λ_0}, the pressure $\mathrm{P}_{\lambda_0}(h_{\lambda_0}) = 0$. But by Proposition 2.6.13 and Proposition 3.1.4 from [2],

$$\frac{\partial \mathrm{P}}{\partial t}|_{(\lambda_0, h_{\lambda_0})} = \int \log |(\phi^{\lambda_0}_{\omega_1})(\pi_{\lambda_0}(\sigma(\omega)))| d\mu_0(\omega) < 0,$$

where μ_0 is the Gibbs (equilibrium) (see [2] for these concepts in the context of graph directed Markov systems) state of the potential $\omega \mapsto h_{\lambda_0} \log |(\phi^{\lambda_0}_{\omega_1})(\pi_{\lambda_0}(\sigma(\omega)))|$. Consequently, it follows from the Implicit Function Theorem that there exist $r_5 \in (0, r_4]$ and a real-analytic function $t(\lambda)$, $\lambda \in B(\lambda_0, r_5)$ such that $\mathrm{P}_\lambda(t(\lambda)) = 0$ and $t(\lambda_0) = h_{\lambda_0}$. Invoking Theorem 2.3 again, we conclude that $h_\lambda = t(\lambda)$, and the proof of Theorem 4.2 is finished. □

5. Generalized Polynomial-Like Mappings; basics

In this section we recall from [9] and [10] the class of semihyperbolic generalized polynomial-like mappings (GPL) and canonically associate to them conformal graph directed Markov systems in the sense of [2]. For $U \subset \mathbb{C}$, an open Jordan domain with smooth boundary, let $\mathcal{U} := \bigcup_{i \in I} U_i$ be a finite union of open Jordan domains U_i whose closures are pairwise disjoint and are all contained in U. A GPL-map f is a holomorphic map

$$f : \mathcal{U} \to U$$

such that for each $i \in I$ the restriction of f to U_i is a surjective branched covering map having at most one critical (branching) point. The Julia set $J(f)$ of f is defined to be the set of all those points in $\mathcal{U}$ whose all iterates under f are well-defined but each of their neighborhoods has a point which is eventually mapped out of $\mathcal{U}$. Also, define

$$\mathrm{Crit}(f) := \{c : f'(c) = 0\} \quad \text{and} \quad \mathrm{Crit}(J(f)) := J(f) \cap \mathrm{Crit}(f).$$

The index set I is split in the following way.

$I_o := \{i \in I : \overline{U_i} \cap \overline{\bigcup_{n \geq 1} f^n(\mathrm{Crit}(f))} = \emptyset\}$ ('post-critical free indices'),
$I_c := \{i \in I : U_i \cap \mathrm{Crit}(f) \neq \emptyset\}$ ('critical indices'),
$I_r := I \setminus I_c$ ('regular indices').

With this decomposition of the finite index set I, we put

$$\mathcal{U}_o := \bigcup_{i \in I_o} U_i, \;\; \mathcal{U}_c := \bigcup_{i \in I_c} U_i, \;\; \mathcal{U}_r := \bigcup_{i \in I_r} U_i.$$

For every $i \in I_r$ set

$$f_i^{-1} := \left(f|_{U_i}\right)^{-1} : U \to U_i.$$

Definition 5.1. *A GPL-map f is called semihyperbolic if and only if $I_c \subset I_o$.*

Throughout the paper we assume f to be a semihyperbolic GPL-map. The following lemma is immediate.

Lemma 5.2. *If f is a semihyperbolic GPL-map f, then the closure of the forward orbit of Crit(f) is nowhere dense in $J(f)$.*

In order to take fruits of the previous sections we now associate to the semihyperbolic map f a graph directed Markov system.

Proposition 5.3. *Let f be a semihyperbolic GPL-map. Then there exists a finitely primitive order 1 CGDM-system Φ_f with $J_{\Phi_f} \subset J(f)$ such that*

$$J_{\Phi_f} \cap \mathcal{U}_o = J(f) \cap \mathcal{U}_o \setminus \bigcup_{n\ge 0} f^{-n}\big(\bigcap_{k\ge 0} f^{-k}(\mathcal{U}_r)\big).$$

In addition $\mathrm{HD}\big(J_{\Phi_f}\big) = \mathrm{HD}(J(f))$. *Denote the corresponding incidence matrix by A.*

Proof. We take I_o to be the set of vertices. The conformal univalent contractions of our system are given as follows. For every $i \in I_o$ fix an open topological disk $\tilde{U}_i$ with smooth boundary which contains U_i and whose closure is disjoint from the closure of the postcritical set $\bigcup_{n\ge 1} f^n(\mathrm{Crit}(f))$. Of course we can always take U_i for $\tilde{U}_i$ but we will need in the next section such a more general construction. By the definition of the sets $\tilde{U}_i$, for each vertex $i \in I_o$ all the holomorphic inverse branches of any iterate of f are well-defined on a fixed neighbourhood W_i of the closure of $\tilde{U}_i$. Hence, for each $j \in I_o$ and $n \ge 1$ we consider all the holomorphic inverse branches $f_*^{-n} : \tilde{U}_j \to U$ of f^n such that $f_*^{-n}(\tilde{U}_j) \subset \tilde{U}_k$ for some $k \in I_o$ and $f^i(f_*^{-n}(\tilde{U}_j)) \cap \mathcal{U}_o = \emptyset$ for all $i = 1, 2, \dots, n-1$. We then write $\phi_e : \tilde{U}_{t(e)} \to \tilde{U}_{i(e)}$ for $f_*^{-n} : \tilde{U}_j \to \tilde{U}_k$, where $t(e) = j$ and $i(e) = k$. Also, we define $||e|| := n$. Now, let

$$\Phi^f := \{\phi_e : \overline{\tilde{U}_{t(e)}} \to \overline{\tilde{U}_{i(e)}}\}_{e\in E_f},$$

where E_f is some countable auxiliary set parametrizing the family Φ^f. Note that the set I_o of vertices is finite, whereas in general the set E_f of edges is infinite. The equality we immediately obtain from the construction of Φ_f is that

$$J_{\Phi^f} \cap \mathcal{U}_o = J(f) \cap \mathcal{U}_o \setminus \bigcup_{n\ge 0} f^{-n}\big(\bigcap_{k\ge 0} f^{-k}(\mathcal{U}_r)\big)$$

and the limit set J_{Φ^f} is independent of the admissible choice of the disks $\tilde{U}_j$, $j \in I_o$. We remark that the cone condition is satisfied, since for each $i \in I_0$ the boundaries of the disks

$\overline{\tilde{U}_i}$ are smooth. Also, the open set condition follows immediately from the construction of Φ^f, noting that the elements of Φ^f are inverse branches of forward iterates of f. Finally, since for each pair $j,k \in I_o$ there is a holomorphic inverse $f_{i,k}^{-1}$ of f defined on $\tilde{U}_k$ and mapping $\tilde{U}_k$ into $\tilde{U}_j$ and $f_{i,k}^{-1} : \tilde{U}_k \to \tilde{U}_j$ is in Φ_f, we conclude that the system Φ_f is primitive of order 1. We are left to show equality of dimensions. Indeed, it follows from Theorem 2.1 in [9] and continuity of the pressure function $\mathrm{P}_f(t)$ appearing there (by the item (6) of the definition of the pressure function in [9], this function is convex and hence continuous) that $\mathrm{P}_f(\mathrm{HD}(J(f)) = 0$. It then follows from Lemma 4.6 in [9] that $\mathrm{P}_{\Phi_f}(\mathrm{HD}(J(f)) = 0$. Consequaently $\mathrm{HD}\big(J_{\Phi_f}\big) = \mathrm{HD}(J(f))$ by Theorem 2.3. $\square$

6. Analytic Families of Semihyperbolic GPLs

Definition 6.1. *Let Λ be an open open subset of $\mathbb{C}^d$ with some $d \geq 1$. A family $\{f_\lambda : \mathcal{U}_\lambda \to U\}_{\lambda\in\Lambda}$ of semihyperbolic GPLs is called analytic provided that the following conditions are satisfied.*

(a) *The index sets I, I_o, I_c and I_r are the same for all $\lambda \in \Lambda$.*
(b) *For every $i \in I$ the map $\lambda \mapsto \partial U_{\lambda,i} \in \mathcal{D}(\overline{U})$, $\lambda \in \Lambda$ is continuous, where $\mathcal{D}(\overline{U})$ is the space of all compact subsets of $\overline{U}$ endowed with the Hausdorff metric. Consequently the map $\lambda \mapsto \overline{U_{\lambda,i}}$ is continuous and also the set $(\Lambda * U)_i = \bigcup_{\lambda\in\Lambda}\{\lambda\} \times U_{\lambda,i} \subset \Lambda \times \mathbb{C}$ is open.*
(c) *Put $\Lambda * U = \bigcup_{i\in I}(\Lambda * U)_i$. The map $F : \Lambda * U \to U$ given by the formula $F(\lambda, z) = f_\lambda(z)$ is holomorphic.*
(d) *For every $i \in I_c$ and every $\lambda \in \Lambda$ denote by $c_{\lambda,i}$ the only critical point of the map f_λ in $U_{\lambda,i}$. The order of critical points $c_{\lambda,i}$ is assumed to be independent of λ and is denoted by $q_i \geq 2$.*
(e) *For every $i \in I_o$ there exists an open disk $U_i \subset U$ containing closures of all sets $U_{\lambda,i}$, $\lambda \in \Lambda$ and such that*

$$\overline{U_i} \cap \overline{\bigcup_{\lambda\in\Lambda}\bigcup_{n\geq 1} f_\lambda^n(Crit(f_\lambda))} = \emptyset.$$

As an immediate consequence of items (c), (d) and the Implicit Function Theorem, we get the following.

Lemma 6.2. *For every $i \in I_c$ the map $\lambda \mapsto c_{\lambda_i} \in U$, $\lambda \in \Lambda$, is holomorphic.*

As an immediate consequence of item (b) above and the fact (one of the requirements in the definition of GPL's) for all $\lambda \in \Lambda$, $\bigcup_{\in\in I} \overline{U_{\lambda,i}} \subset U$, we get the following.

Lemma 6.3. *For every $\gamma \in \Lambda$ there exists $R_\gamma > 0$ such that*

$$\mathrm{dist}\left(\partial U, \bigcup_{i\in I}\bigcup_{\lambda\in B(\gamma,R_\gamma)} U_{\lambda,i}\right) > 0.$$

Also, for every compact set $\Gamma \subset \Lambda$, the union $\bigcup_{i\in I}\bigcup_{\lambda\in\Gamma}\overline{U_{\lambda,i}}$ is a compact subset of U and, in particular,

$$\mathrm{dist}\left(\partial U, \bigcup_{i\in I}\bigcup_{\lambda\in\Gamma}\overline{U_{\lambda,i}}\right) > 0.$$

Let us state the following obvious but crucial for us lemma.

Lemma 6.4. *If $\lambda_0 \in \Lambda$, $i \in I_c$, and $Q \subset U$ is an open simply connected set such that $f_{\lambda_0,i}(c_{\lambda_0,i}) \notin Q$, then there exists $r > 0$ such that if $f^{-1}_{\lambda_0,i,*} : Q \to \mathbb{C}$ is a holomorphic inverse branch of $f_{\lambda_0,i}$, then for every $\lambda \in B(\lambda_0, r)$ there exists $f^{-1}_{\lambda,i,*} : Q \to \mathbb{C}$ a unique holomorphic inverse branch of $f_{\lambda,i}$ such that the map $(\lambda, z) \mapsto f^{-1}_{\lambda,i,*}(z)$, $(\lambda, z) \in B(\lambda_0, r)\times Q$, is analytic.*

We will need in the sequel a better description of derivatives of these inverse branches. For each $i \in I_c$ and every $\lambda \in \Lambda$ there exists a holomorphic map $H_{\lambda,i} : U_{\lambda,i} \to \mathbb{C}$ such that

$$f_\lambda(z) = (z - c_{\lambda,i})^{q_i} H_{\lambda,i}(z) + f_\lambda(c_{\lambda,i}) \quad\text{and}\quad H_{\lambda,i}(c_{\lambda,i}) \neq 0.$$

Obviously the map $(\lambda, z) \mapsto H_{\lambda,i}(z)$ is holomorphic on $(\Lambda * U)_i$. also

$$\begin{aligned} f'_\lambda(z) &= q_i(z - c_{\lambda,i})^{q_i-1}H_{\lambda,i}(z) + (z - c_{\lambda,i})^{q_i}H'_{\lambda,i}(z)\\ &= (z - c_{\lambda,i})^{q_i-1}\big(q_iH_{\lambda,i}(z) + H'_{\lambda,i}(z)(z - c_{\lambda,i})\big). \end{aligned}$$

Since $c_{\lambda,i}$ is the only point in $U_{\lambda,i}$ where the derivative f'_λ vanishes,

$$q_iH_{\lambda,i}(z) + H'_{\lambda,i}(z)(z - c_{\lambda,i}) \neq 0 \tag{6.1}$$

for all $z \in U_{\lambda,i}$. Now, let $f^{-1}_{\lambda,i,*} : Q \to \mathbb{C}$ be a holomorphic inverse branch of $f_{\lambda,i}$ defined on an open simply connected set $Q \subset U \setminus \{f_\lambda(c_{\lambda,i})\}$. A straightforward calculation shows that

$$\big((f^{-1}_{\lambda,i,*})'(z)\big)^{q_i} = (z - f_\lambda(c_{\lambda,i}))^{1-q_i}G_{\lambda,i}\big(f^{-1}_{\lambda,i,*}(z)\big) \tag{6.2}$$

for all $z \in Q$, where

$$G_{\lambda,i}(w) = H^{q_i-1}_{\lambda,i}(w)\big(q_iH_{\lambda,i}(w) + H'_{\lambda,i}(w)(w - c_{\lambda,i})\big)^{-q_i},\ w \in U_{\lambda,i}.$$

Since, by (6.1), $G_{\lambda,i}$ does not vansish throughout $U_{\lambda,i}$ and since the set $U_{\lambda,i}$ is simply connected, there exists $\log_\lambda G_{\lambda,i} : U_{\lambda,i} \to \mathbb{C}$, a holomorphic branch of logarithm of $G_{\lambda,i}$. Clearly, as long as Λ is simply connected (we can always assure this by decreasing Λ to a round neighborhood of a frozen point), we can choose these branches so that the following holds.

Lemma 6.5. *For every $i \in I_c$ the function $(\lambda, z) \mapsto \log_\lambda G_{\lambda,i}(z)$, $(\lambda, z) \in (\Lambda * U)_i$, is holomorphic.*

The main result of this section and the ultimate goal of the paper is this.

Theorem 6.6. *If Λ is an open open subset of $\mathbb{C}^d$ with some $d \geq 1$ and $\{f_\lambda : \mathcal{U}_\lambda \to U\}_{\lambda\in\Lambda}$ is an analytic family of semihyperbolic GPLs, then the function $\lambda \mapsto \mathrm{HD}(J(f_\lambda))$ is real-analytic.*

Naturally we want to apply the machinery of weakly analytic families of conformal graph directed Markov systems developed in previous sections. Indeed, Let $\Phi^\lambda = \Phi_{f_\lambda}$, $\lambda \in \Lambda$, be the CGDMS resulting from Proposition 5.3 where the role of the disks $\tilde{U}_i$, $i \in I_o$, there, is played by the current disks U_i, $i \in I_o$. Thus, the seed sets of all the systems Φ^λ, $\lambda \in \Lambda$, are the same. Note that because of condition (e) above, each element $\phi_e^\lambda : U_{\lambda,t(e)} \to U$, $e \in E_\lambda := E_{f_\lambda}$, extends uniquely to an inverse holomorphic branch of $f^{||e||}$ defined on $U_{t(e)}$. This extension will be denoted by the same symbol ϕ_e^λ. Our goal now is to reparametrize the sets E_λ so that all the corresponding incidence matrices coincide and all the systems Φ^λ, $\lambda \in \Lambda$ form a weakly regular analytic family. Indeed, fix $\gamma \in \Lambda$ and consider a map $\phi_e^\gamma \in \Phi^\gamma$, $e \in E_\gamma$. Suppose first that $\phi_e^\gamma\big(U_{t(e)}\big) \cap \mathcal{U}_{\gamma,c} = \emptyset$. Then

$$\phi_{\gamma,e} = f_{\gamma,e_1}^{-1} \circ f_{\gamma,e_2}^{-1} \circ \dots f_{\gamma,e_n}^{-1} : U_{e_{n+1}} = U_{t(e)} \to U_{i(e)} = U_{e_1},$$

with some $n \geq 1$, $e_2, e_3 \dots, e_n \in I_r$, $e_1 \in I_r \cap I_o$, and $e_{n+1} \in I_o$. Then for every $\lambda \in \Lambda$ the map

$$f_{\lambda,e_1}^{-1} \circ f_{\lambda,e_2}^{-1} \circ \dots f_{\lambda,e_n}^{-1} : U_{e_{n+1}} = U_{t(e)} \to U_{i(e)} = U_{e_1}$$

belongs to Φ^λ and there is exactly one vertex $e_\lambda \in E_\lambda$ such that $f_{\lambda,e_1}^{-1} \circ f_{\lambda,e_2}^{-1} \circ \dots f_{\lambda,e_n}^{-1} = \phi_{e_\lambda}^\lambda$. It obviously follows from condition (b) of Definition 6.1 that the map $(\lambda, z) \mapsto \phi_{e_\lambda}^\lambda(z)$, $(\lambda, z) \in \Lambda \times U_{e_{n+1}}$, is analytic. If

$$\phi_e^\gamma\big(U_{t(e)}\big) \cap \mathcal{U}_{\gamma,c} \neq \emptyset, \quad \text{say} \quad \phi_e^\gamma\big(U_{t(e)}\big) \subset U_{\gamma,k}, \ k \in I_c,$$

then $f_\gamma \circ \phi_e^\gamma = f_{\gamma,e_1}^{-1} \circ f_{\gamma,e_2}^{-1} \circ \dots f_{\gamma,e_n}^{-1} : U_{e_{n+1}} \to U_{e_1}$ with some $n \geq 0$, $e_1, e_2, e_3 \dots, e_n \in I_r$, and $e_{n+1} \in I_o$. Taking a sufficiently small ball, say $B(\gamma, R)$, around γ, we may assume without loss of generality that the set Λ is simply connected. We claim that there exists a unique family of maps $\{\tilde{\phi}_\lambda : U_{e_{n+1}} \to U_{e_1}\}_{\lambda\in\Lambda}$ with the following properties holding for all $\lambda \in \Lambda = B(\gamma, R)$.

- (f) $f_\lambda^{n+1} \circ \tilde{\phi}_\lambda = \mathrm{Id}$,
- (g) The map $\tilde{\phi}_{(\cdot)} : \Lambda \times U_{e_{n+1}} \to \mathbb{C}$ is analytic.
- (h) $\tilde{\phi}_\gamma = \phi_e^\gamma$.
- (i) $f_\lambda \circ \tilde{\phi}_\lambda = f_{\lambda,e_1}^{-1} \circ f_{\lambda,e_2}^{-1} \circ \dots f_{\lambda,e_n}^{-1}$.

Put $\psi_\lambda = f_{\lambda,e_1}^{-1} \circ f_{\lambda,e_2}^{-1} \circ \dots f_{\lambda,e_n}^{-1} : U_{e_{n+1}} \to U_{e_1}$ and note that, as above, the map $(\lambda, z) \mapsto \psi_\lambda(z)$, is analytic. Let $0 \leq R_* \leq R$ be the largest radius such that the family satisfying conditions (f)-(i) is defined for all $\lambda \in B(\gamma, R_*)$. Seeking contradiction suppose that $R_* < R$. Consider an arbitrary point $\mu \in \Lambda$ such that $||\mu - \gamma|| = R_*$. Since $f_\mu(\mathrm{Crit}(f_\mu)) \cap \overline{\psi_\mu(U_{e_{n+1}})} = \emptyset$, all the inverse branches $f_{\mu,k,*} : Q \to U$ are well-defined on some open simply connected set Q containing $\overline{\psi_\mu(U_{e_{n+1}})}$. Applying Lemma 6.4 with $\lambda_0 = \mu$ results in all the branches $f_{\lambda,k,*}^{-1} : Q \to U$ for all $\lambda \in B(\mu, r_\mu)$ with some $r_\mu > 0$. One can assume $r_\mu > 0$ to be so small that $\psi_\lambda(U_{e_{n+1}}) \subset Q$ for all $\lambda \in B(\mu, r_\mu)$. Then for every $\lambda \in B(\mu, r_\mu) \cap B(\gamma, R_*)$ there exists $*(\lambda)$, indicating a holomorphic branch of $f_{\mu,k}^{-1}$, such that $\tilde{\phi}_\lambda = f_{\lambda,k,*(\lambda)}^{-1} \circ \psi_\lambda$. Since the map $F_* : B(\gamma, R_*) \times U_{e_{n+1}} \to U$, given by the formula $F_*(\lambda, z) = \tilde{\phi}_\lambda(z)$, is holomorphic, the index $*(\lambda)$ is constant on $B(\mu, r_\mu) \cap B(\gamma, R_*)$, say equal to $*$. Thus the formula

$$F_\mu(\lambda, z) = \begin{cases} F_*(\lambda, z) & \text{if } \lambda \in B(\gamma, R_*) \\ f_{\lambda,k,*}^{-1} \circ \psi_\lambda(z) & \text{if } \lambda \in B(\mu, r_\mu) \end{cases}$$

defines a holomorphic function on $B(\gamma, R_*) \cup B(\mu, r_\mu)$. If now μ_1 and μ_2 are two arbitrary points in Λ such that $||\mu_2 - \gamma|| = ||\mu_1 - \gamma|| = R_*$ and $B(\mu_1, r_{\mu_1}) \cap B(\mu_2, r_{\mu_2}) \neq \emptyset$, then $B(\mu_1, r_{\mu_1}) \cap B(\mu_2, r_{\mu_2}) \cap B(\gamma, R_*) \neq \emptyset$ and

$$F_{\mu_2}|_{(B(\mu_1,r_{\mu_1})\cap B(\mu_2,r_{\mu_2})\cap B(\gamma,R_*))\times U_{e_{n+1}}} = F_{\mu_1}|_{(B(\mu_1,r_{\mu_1})\cap B(\mu_2,r_{\mu_2})\cap B(\gamma,R_*))\times U_{e_{n+1}}}.$$

Since F_{μ_1} and F_{μ_2} are holomorphic, we therefore conclude that

$$F_{\mu_2}|_{(B(\mu_1,r_{\mu_1})\cap B(\mu_2,r_{\mu_2}))\times U_{e_{n+1}}} = F_{\mu_1}|_{(B(\mu_1,r_{\mu_1})\cap B(\mu_2,r_{\mu_2}))\times U_{e_{n+1}}}.$$

Thus the formula

$$F(\lambda, z) = \begin{cases} F_*(\lambda, z) & \text{if } \lambda \in B(\gamma, R_*) \\ F_\mu(\lambda, z) & \text{if } ||\mu - \gamma|| = R_* \text{ and } \lambda \in B(\mu, r_\mu) \end{cases}$$

defines a holomorphic function on the open connected set $Z = B(\gamma, R_*) \cup \bigcup_{||\mu-\gamma||=R_*} B(\mu, r_\mu)$. Obviously the family $\{\tilde{\phi}_\lambda := F(\lambda, \cdot)\}_{\lambda \in Z}$ satisfies the conditions (f)-(i). Since the sphere $\{\mu \in \Lambda : ||\mu - \gamma|| = R_*\}$ is compact, there exists $R' \in (R_*, R]$ such that $B(\gamma, R') \subset Z$. This contradiction finishes the proof of the equality $R_* = R$.

Thus, each map $\tilde{\phi}^\lambda$ is a member of Φ^λ, i.e. there exists a unique element $e_\lambda \in E_\lambda$ such that $\tilde{\phi}^\lambda = \phi_{e_\lambda}^\lambda$ and the map $(\lambda, z) \mapsto \phi_{e_\lambda}^\lambda(z)$ is analytic. Summarizing the two cases above, we have thus defined a function $e \mapsto e_\lambda$ from E_γ to E_λ with the following properties.

(j) $e_\gamma = e$.
(k) The map $e \mapsto e_\lambda$ from E_γ to E_λ is bijective.
(l) The map $(\lambda, z) \mapsto \phi_{e_\lambda}^\lambda(z)$ from $U_{e_{n+1}}$ to $\mathbb{C}$ is analytic.

We now set $E = E_\gamma$ and identify the elements of E_λ with those of E via the bijective map $e \mapsto e_\lambda$. We have thus proved the following.

Lemma 6.7. *The family $\{\Phi^\lambda\}_{\lambda \in \Lambda}$ is analytic.*

Our aim now is to show that the family $\{\Phi^\lambda\}_{\lambda \in \Lambda}$ is weakly regular analytic. For every $\omega = \omega_1\omega_2 \ldots \omega_n \in I_r^*$ and every $\lambda \in \Lambda$, put

$$f_{\lambda,\omega}^{-1} = f_{\lambda,\omega_1}^{-1} \circ f_{\lambda,\omega_2}^{-1} \circ \ldots f_{\lambda,\omega_n}^{-1} : U \to U_{\lambda,\omega_1}.$$

We start with the following.

Lemma 6.8. *For every compact set $\Gamma \subset \Lambda$ and every compact set $V \subset U$ there exist $\alpha > 0$ and $C_1 > 0$ such that*

$$\left|\left(f_{\lambda,\omega}^{-1}\right)'(z)\right| \le C_1 e^{-\alpha||\omega||}$$

for all $\lambda \in \Gamma$, all $\omega \in I_r^*$ and all $z \in V$.

Proof. By the condition of (b) of Definition 6.1, the union

$$W = \bigcup_{i\in I_r} \bigcup_{\lambda\in\Gamma} \overline{U_{\lambda,i}}$$

is a compact subset of U. There thus exists $\kappa \in (0,1)$ such that all the maps $f_{l,i}^{-1} : V \to U_{l,i} \subset W \subset U$, $i \in I_r$, decrease hyperbolic distances on V and W (viewed as compact subsets of the hyperbolic surface U) by the constant factor κ. It follows from this and compactness of V and W that there exist $\beta < 1$ and $q \ge 1$ such that $\left|\left(f_{\lambda,\tau}^{-1}\right)'(z)\right| \le \beta$ for all $(\lambda,\tau,z) \in \Gamma \times I_r^q \times W$. Hence, noting that $\sup\left\{\left|\left(f_{\lambda,i}^{-1}\right)'(z)\right| : (\lambda,i,z) \in \Gamma \times I_r \times W\right\} < +\infty$, our lemma follows. □

Decreasing all the sets U_i, $i \in I_o$, slightly to keep all the requirements imposed on them in Definition 6.1 and to be compactly contained in the original sets U_i, in the same way as Lemma 6.8, we prove the following.

Lemma 6.9. *For every compact set $\Gamma \subset \Lambda$ there exist $\beta > 0$ and $C_2 > 0$ such that*

$$\left|\left(\phi_\omega^\lambda\right)'(z)\right| \le C_2 e^{-\beta|\omega|}$$

for all $\lambda \in \Gamma$, all $\omega \in E_A^*$ and all $z \in U_{t(\omega)}$.

We end this section with the following two results.

Lemma 6.10. *If Γ is a compact subset of Λ, then for every $i \in I_c$ the closure of the set $\Gamma_i^* := \bigcup_{\lambda\in\Gamma} \bigcup_{\substack{\omega\in E_A^*\\ \omega_{1,1}=i}} \{\lambda\} \times \phi_\omega^\lambda\big(U_{\lambda,t(\omega)}\big)$ is a compact subset of $\bigcup_{\lambda\in\Lambda}\{\lambda\} \times U_{\lambda,i}$.*

Proof. Consider an arbitrary sequence $(\lambda_n, x_n)_{n=1}^\infty \in \Gamma_i^*$. Since Γ is compact, we can assume without loss of generality that the sequence $(\lambda_n)_{n=1}^\infty$ converges in Γ, say to λ_∞. Since $\overline{U}$ is compact, we can assume further that the sequence $(x_n)_{n=1}^\infty$ converges, say to some point $x_\infty \in \overline{U}$. Since $x_n \in U_{\lambda_n,i}$ for all $n \ge 1$, it follows from condition (b) of Definition 6.1 that $x_\infty \in \overline{U_{\lambda_\infty,i}}$. We are to show that $x_\infty \in U_{\lambda_\infty,i}$. Since the set I_o is finite, we may assume without loss of generality that $f_{\lambda_n}(x_n) \in U_{\lambda_n,k}$ for some $k \in I_o$ and all $n \ge 1$. Suppose now for the contrary that $x_\infty \in U_{\lambda_\infty,i}$. Then $F(\lambda_\infty, x_\infty) \in \partial U$ and therefore

$$\lim_{n\to\infty} \mathrm{dist}(f_{\lambda_n}(x_n), \partial U) = \lim_{n\to\infty} \mathrm{dist}(F(\lambda_n, x_n), \partial U) = \mathrm{dist}(F(\lambda_\infty, x_\infty), \partial U) = 0$$

contrary to the fact that $f_{\lambda_n}(x_n) \in U_{\lambda_n,k}$ and $\mathrm{dist}\Big(\bigcup_{j\in I_o} \bigcup_{\lambda\in\Gamma} U_{\lambda_j}, \partial U\Big) > 0$ (see Lemma 6.3). We are done. □

As an immediate consequence of Lemma 6.10 and Lemma 6.10, we get the following.

Lemma 6.11. *For every compact subset Γ of Λ the number*

$$M_\Gamma = \max_{i\in I_c}\{\sup\{|\log_\lambda G_{\lambda_i}(z)| : z \in \Gamma_i^*\}\}$$

is finite.

7. The ψ function

We shall show in this section that the condition (d) of the definition of weakly regular analyticity is satisfied for the family $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ of CGDMS defined in the previous section. As a direct consequence of the last sentence of Lemma 4.1, we get the following.

Lemma 7.1. *The family $\{\lambda \mapsto f_{\lambda,\tau}^{-1}(\pi_\lambda(\omega)) : \omega \in E_A^\infty, \tau \in I_r^*\}$ consists of holomorphic maps and is normal (since bounded).*

Fix $\gamma \in \Lambda$ (called λ_0 in Section 4). Recall that we have defined for every $\omega \in E_A^\infty$, the ψ function by the formula

$$\psi_\omega(\lambda) = \frac{(\phi_{\omega_1}^\lambda)'(\pi_\lambda(\sigma\omega))}{(\phi_{\omega_1}^\gamma)'(\pi_\gamma(\sigma\omega))}.$$

The goal of this section is to prove the following.

Lemma 7.2. *There exists $R_* > 0$ such that for every $\omega \in E_A^\infty$ there exists $\log\psi_\omega : B(\gamma, R_*) \to \mathbb{C}$, a holomorphic branch of logarithm of ψ_ω, such that $\log\psi_\omega(\gamma) = 0$ and the family of functions $\left\{\lambda \mapsto \frac{1}{||\omega_1||}\log\psi_\omega(\lambda)\right\}_{\omega\in E_A^\infty}$ is bounded and, consequently, normal.*

Proof. Since there are only finitely many elements $e \in E$ with $||e|| = 1$, we can assume without loss of generality that $||\omega_1|| \geq 2$. Take $R_1 > 0$ so small that $\overline{B}(\gamma, R_1) \subset \Lambda$. Then $\overline{B}(\gamma, R_1)$ is compact and, by Lemma 6.3, the set

$$W = \bigcup_{i\in I}\bigcup_{\lambda\in\overline{B}(\gamma,R_1)} \overline{U_{\lambda,i}} \subset U$$

is compact. Thus, for every $i \in I_r$ the function $(\lambda, z) \mapsto \left(f_{\lambda,i}^{-1}\right)'(z)$ restricted to $\overline{B}(\gamma, R_1)\times W$ is uniformly continuous. Therefore, since I_r is a finite set and since these functions nowhere vanish, there exists $R_2 \in (0, R_1]$ such that

$$\left|\frac{\left(f_{\lambda,i}^{-1}\right)'(z)}{\left(f_{\gamma,i}^{-1}\right)'(\xi)} - 1\right| < \frac{1}{4} \tag{7.1}$$

for all $i \in I_r$, all $\lambda \in B(\gamma, R_2)$ and all $(z,\xi) \in W$ with $|z-\xi| \le R_2$. In view of Lemma 7.1 there exists $R_3 \in (0, R_2]$ such that

$$\left|f_{\lambda,\tau}^{-1}(\pi_\lambda(\omega)) - f_{\gamma,\tau}^{-1}(\pi_\gamma(\omega))\right| < R_2 \tag{7.2}$$

for all $\omega \in E_A^\infty$, all $\tau \in I_r^*$, and all $\lambda \in B(\gamma, R_3)$. Since

$$\bigcup_{\lambda \in \overline{B}(\gamma,R_3)} \pi_\lambda(E_A^\infty) \subset \bigcup_{\lambda \in \overline{B}(\gamma,R_1)} \pi_\lambda(E_A^\infty) \subset W,$$

and since $f_{\lambda,\tau}^{-1}(W) \subset W$ for all $\tau \in I_r^*$, and all $\lambda \in B(\gamma, R_3)$, combining (7.1) and (7.2), we conclude that

$$\left|\frac{\left(f_{\lambda,i}^{-1}\right)'\left(f_{\lambda,\tau}^{-1}(\pi_\lambda(\omega))\right)}{\left(f_{\gamma,i}^{-1}\right)'\left(f_{\gamma,\tau}^{-1}(\pi_\lambda(\omega))\right)} - 1\right| < \frac{1}{4}$$

for all $i \in I_r$, all $\omega \in E_A^\infty$, all $\tau \in I_r^*$, and all $\lambda \in B(\gamma, R_3)$. In particular, there exists

$$\log\left(\frac{\left(f_{\lambda,i}^{-1}\right)'\left(f_{\lambda,\tau}^{-1}(\pi_\lambda(\omega))\right)}{\left(f_{\gamma,i}^{-1}\right)'\left(f_{\gamma,\tau}^{-1}(\pi_\lambda(\omega))\right)}\right),$$

a holomorphic branch of logarithm of the function

$$\lambda \mapsto \frac{\left(f_{\lambda,i}^{-1}\right)'\left(f_{\lambda,\tau}^{-1}(\pi_\lambda(\omega))\right)}{\left(f_{\gamma,i}^{-1}\right)'\left(f_{\gamma,\tau}^{-1}(\pi_\lambda(\omega))\right)},\ \lambda \in B(\gamma, R_3),$$

whose value at γ is equal to 0. Note that there exists a universal constant $M_1 > 0$, an upper bound of moduli of all these logarithms. Now, if $\omega \in E_A^\infty$ and $\omega_{1,1} \notin I_c$, then we set $(\lambda \in B(\gamma, R_3))$

$$\log \psi_\omega(\lambda) = \sum_{j=1}^{||\omega_1||} \log\left(\frac{\left(f_{\lambda,\omega_{1,j}}^{-1}\right)'\left(f_{\lambda,\overline{\omega}_j}(\pi_\lambda(\sigma(\omega)))\right)}{\left(f_{\gamma,\omega_{1,j}}^{-1}\right)'\left(f_{\gamma,\overline{\omega}_j}(\pi_\gamma(\sigma(\omega)))\right)}\right),$$

where $\overline{\omega}_j = \omega_{1,j+1}\omega_{1,j+2}\ldots\omega_{1,||\omega_1||} \in I_r^{||\omega_1||-j}$ if $1 \le j \le ||\omega_1|| - 1$ and $\overline{\omega}_{1,||\omega_1||} = \emptyset$. So,

$$|\log \psi_\omega(\lambda)| \le M_1 ||\omega_1||.$$

If $\omega_{1,1} \in I_c$, then write $f_\lambda(c_{\lambda,\omega_{1,1}}) = v_\lambda$ and put

$$\log_\omega^{(0)}(\lambda) = \sum_{j=2}^{||\omega_1||} \log\left(\frac{\left(f_{\lambda,\omega_{1,j}}^{-1}\right)'\left(f_{\lambda,\overline{\omega}_j}(\pi_\lambda(\sigma(\omega)))\right)}{\left(f_{\gamma,\omega_{1,j}}^{-1}\right)'\left(f_{\gamma,\overline{\omega}_j}(\pi_\gamma(\sigma(\omega)))\right)}\right),$$

and we have

$$|\log_\omega^{(0)}(\lambda)| \le M_1(||\omega_1|| - 1). \tag{7.3}$$

Write $f \circ \varphi_{\omega_1}^\lambda = f_{\lambda,\tau}^{-1}|_{U_k}$, where $\tau = \omega_2\omega_3\ldots\omega_{|\omega|} \in I_r^{|\omega_1||-1}$ and $k = t(\omega_1) \in I_o$. Put $n = ||\tau|| = |\omega_1|| - 1$. Then $f_\lambda^n(v_\lambda) \notin U_k$. Since W is a compact subset of U and in virtue of item (d) of Definition 6.1,

$$\Delta_1 := \min\left\{\mathrm{dist}(W, \partial U), \mathrm{dist}\left(\bigcup_{i \in I_o} \overline{U_i}, \bigcup_{\lambda \in \Lambda}\bigcup_{j \ge 1} f_\lambda^j(\mathrm{Crit}(f_\lambda))\right)\right\} > 0.$$

Hence, for $j \geq 0$,

$$\operatorname{dist}\big(U_k, \partial U\big), \operatorname{dist}\big(U_k, f_\lambda^j(v_\lambda)\big) \geq \Delta_1 > 0. \tag{7.4}$$

Since all the mappings $f_{\lambda,j}^{-1}$, $j \in I_r$, are conformal homeomorphisms, the moduli of their derivatives restricted to W are uniformly bounded away from zero and infinity. Hence, using (7.4) and Lemma 6.8, we conclude that there exist a universal integer $p \geq 1$ and a constant $\Delta_2 > 0$, such that

$$B\big(f_{\lambda,\tau|_{n-p+1}^n}(U_k)^{-1}, \Delta_2\big) \subset U \tag{7.5}$$

and

$$\begin{aligned} \Delta_1 &\leq \operatorname{dist}\big(f_{\lambda,\tau|_{n-p+1}^n}^{-1}(U_k), f_\lambda^{n-p}(v_\lambda)\big) \leq \operatorname{diam}\big(f_{\lambda,\tau|_{n-p+1}^n}^{-1}(U_k) \cup f_\lambda^{n-p}(v_\lambda)\big) \\ &\leq K_3^{-1} \min\{\kappa, \pi/24\}\Delta_2, \end{aligned} \tag{7.6}$$

where $\kappa > 0$ is so small that

$$\max\left\{\frac{1+\kappa}{(1-\kappa)^3}, \left(\frac{1-\kappa}{(1+\kappa)^3}\right)^{-1}\right\} \leq \sqrt{2}$$

and K_3 comes from Theorem 4.1.2 in [2]. Note that $f_\lambda^{n-p}(v_\lambda) = f_{\lambda,\tau|_{n-p+1}^n}^{-1}(f_\lambda^{n-p}(v_\lambda))$. Put

$$g_\lambda := f_{\lambda,\tau|_{n-p}}^{-1} \quad \text{and} \quad \rho_\lambda := f_{\lambda,\tau|_{n-p+1}^n}^{-1}.$$

Using (7.5) and (7.6) and applying Theorem 4.1.5 from [2], we get that

$$\begin{aligned} &\left|\frac{g_\lambda(\rho_\lambda(\pi_\lambda(\sigma(\omega)))) - v_\lambda}{g_\lambda'(\rho_\lambda(\pi_\lambda(\sigma(\omega))))\big(f_\lambda^{n-p}(v_\lambda) - \rho_\lambda(\pi_\lambda(\sigma(\omega)))\big)} - 1\right| = \\ &= \left|\frac{g_\lambda(\rho_\lambda(\pi_\lambda(\sigma(\omega)))) - g_\lambda\big(f_\lambda^{n-p}(v_\lambda)\big)}{g_\lambda'(\rho_\lambda(\pi_\lambda(\sigma(\omega))))\big(f_\lambda^{n-p}(v_\lambda) - \rho_\lambda(\pi_\lambda(\sigma(\omega)))\big)} - 1\right| \leq \\ &\leq K_3\Delta_2^{-1}|f_\lambda^{n-p}(v_\lambda) - \rho_\lambda(\pi_\lambda(\sigma(\omega)))| \\ &\leq \min\{\kappa, \pi/24\} < 1/3. \end{aligned} \tag{7.7}$$

Let $\log_0 : B(1,1) \to \mathbb{C}$ be the holomorphic branch of logarithm determined by the requirement that $\log_0 1 = 0$. By (7.7) the following composition $\log_\omega^{(1)} : B(\gamma, R_3) \to \mathbb{C}$, given by the formula

$$\log_\omega^{(1)}(\lambda) = \log_0\left(\frac{g_\lambda(\rho_\lambda(\pi_\lambda(\sigma(\omega)))) - v_\lambda}{g_\lambda'(\rho_\lambda(\pi_\lambda(\sigma(\omega))))\big(f_\lambda^{n-p}(v_\lambda) - \rho_\lambda(\pi_\lambda(\sigma(\omega)))\big)}\right)$$

is well define. Furthermore, it follows from (7.7) that

$$||\log_\omega^{(1)}||_\infty \leq M_2 := \sup\{|\log_0(z)| : z \in \overline{B}(1, 1/3)\}. \tag{7.8}$$

The same considerations as those leading us to (7.3) provide us with a holomorphic branch of logarithm $\log_\omega^{(2)} : B(\gamma, R_3) \to \mathbb{C}$ of the function $\lambda \mapsto g_\lambda'(\rho_\lambda(\pi_\lambda(\sigma(\omega))))/g_\gamma'(\rho_\gamma(\pi_\gamma(\sigma(\omega))))$, $\lambda \in B(\gamma, R_3)$, such that

$$||\log_\omega^{(2)}||_\infty \leq M_1(n-p) \leq M_1||\omega_1||. \tag{7.9}$$

it follows from (7.6) and Koebe's Distortion Theorem (the sets U_k are uniformly compactly contained in U) that there exist $R_4 \in (0, R_3]$ sufficiently small, a constant $M_3 > 0$ and $\log_\omega^{(3)} : B(\gamma, R_4) \to \mathbb{C}$, a holomorphic branch of logarithm of the function $\lambda \mapsto f_\lambda^{n-p}(v_\lambda) - \rho_\lambda(\pi_\lambda(\sigma(\omega)))$, such that

$$||\log_\omega^{(3)}||_\infty \le M_3. \tag{7.10}$$

Since

$$
\begin{aligned}
&\frac{g_\lambda(\rho_\lambda(\pi_\lambda(\sigma(\omega)))) - v_\lambda}{g_\gamma(\rho_\gamma(\pi_\gamma(\sigma(\omega)))) - v_\lambda} = \\
&\quad = \frac{g_\lambda(\rho_\lambda(\pi_\lambda(\sigma(\omega)))) - v_\lambda}{g_\lambda'(\rho_\lambda(\pi_\lambda(\sigma(\omega))))\big(f_\lambda^{n-p}(v_\lambda) - \rho_\lambda(\pi_\lambda(\sigma(\omega)))\big)} \cdot \frac{g_\lambda'(\rho_\lambda(\pi_\lambda(\sigma(\omega))))}{g_\gamma'(\rho_\gamma(\pi_\gamma(\sigma(\omega))))} \cdot \\
&\quad\quad \frac{f_\lambda^{n-p}(v_\lambda) - \rho_\lambda(\pi_\lambda(\sigma(\omega)))}{f_\gamma^{n-p}(v_\lambda) - \rho_\gamma(\pi_\gamma(\sigma(\omega)))} \cdot \frac{g_\gamma'(\rho_\gamma(\pi_\gamma(\sigma(\omega))))\big(f_\gamma^{n-p}(v_\lambda) - \rho_\gamma(\pi_\gamma(\sigma(\omega)))\big)}{g_\gamma(\rho_\gamma(\pi_\gamma(\sigma(\omega)))) - v_\lambda},
\end{aligned}
$$

the function $\log_\omega^{(4)}(\lambda) = \log_\omega^{(1)}(\lambda) + \log_\omega^{(2)}(\lambda) + \log_\omega^{(3)}(\lambda) - \log_\omega^{(3)}(\gamma) - \log_\omega^{(1)}(\gamma)$ is a holomorphic branch of logarithm of the function $\lambda \mapsto \big(g_\lambda(\rho_\lambda(\pi_\lambda(\sigma(\omega)))) - v_\lambda\big)/\big(g_\gamma(\rho_\gamma(\pi_\gamma(\sigma(\omega)))) - v_\lambda\big)$, $\lambda \in B(\gamma, R_4)$, and it follows from (7.8)-(7.8) that

$$||\log_\omega^{(4)}||_\infty \le 2M_2 + M_1||\omega_1|| + 2M_3. \tag{7.11}$$

Since $\log_\omega^{(4)}(\gamma) = 2\pi i l$ with some integer l, we thus get that $|2\pi i l| \le M_1||\omega_1|| + 2(M_2 + M_3)$. Therefore, using (7.11) again, we conclude that $\log_\omega^{(5)}(\lambda) := \log_\omega^{(4)}(\lambda) - 2\pi i l$ is a holomorphic branch of logarithm of the function $\lambda \mapsto \big(g_\lambda(\rho_\lambda(\pi_\lambda(\sigma(\omega)))) - v_\lambda\big)/\big(g_\gamma(\rho_\gamma(\pi_\gamma(\sigma(\omega)))) - v_\lambda\big)$ such that

$$\log_\omega^{(5)}(\gamma) = 0 \quad \text{and} \quad ||\log_\omega^{(5)}||_\infty \le 2(M_1||\omega_1|| + 2(M_2 + M_3)). \tag{7.12}$$

Let now $f_{\lambda,*}^{-1} : f_{\lambda,\tau}^{-1}(U_k) \to U_{\lambda,\omega_{1,1}}$ be the holomorphic inverse branch of f determined by the requirement that $f_{\lambda,*}^{-1} \circ f_{\lambda,\tau}^{-1}|_{U_k} = \phi_{\omega_1}^\lambda$. Put $G_\lambda = G_{\lambda,\omega_{1,1}}$ and $q = q_{\omega_{1,1}}$. It follows from (6.2), Lemma 6.10, Lemma 6.11, and (7.12) that the function $\log_\omega^{(6)} : B(\gamma, R_4) \to \mathbb{C}$ given by the formula

$$\lambda \mapsto \frac{1-q}{q} \log_\omega^{(5)}(\lambda) + \frac{1}{q}\big(\log G_\lambda(\pi_\lambda(\sigma(\omega))) - \log G_\lambda(\pi_\gamma(\sigma(\omega)))\big)$$

is a holomorphic branch of logarithm of the function

$$\lambda \mapsto \frac{f_{\lambda,*}^{-1}\big(g_\lambda(\rho_\lambda(\pi_\lambda(\sigma(\omega))))\big)}{f_{\gamma,*}^{-1}\big(g_\gamma(\rho_\gamma(\pi_\gamma(\sigma(\omega))))\big)}$$

and

$$||\log_\omega^{(6)}||_\infty \le 2\left(1 - \frac{1}{q_c}\right)(M_1||\omega_1|| + 2(M_2 + M_3)) + 2M_4,$$

where $M_4 = M_{\overline{B}(\gamma,R_4)}$ is the number coming from Lemma 6.11. Combining this with (7.3), we see that $\log\psi_\omega = \log_\omega^{(0)} + \log_\omega^{(6)}$ and

$$||\log_\omega^{(6)}||_\infty \le 2\left(\left(2 - \frac{1}{q_+}\right) M_1||\omega_1|| + \left(1 - \frac{1}{q_+}\right)(M_2 + M_3) + M_4\right),$$

where $q_+ = \max\{q_i : i \in I_c\}$. We are done. $\square$

8. Conclusion of the Proof of the Main Theorem

In this section we complete the proof of Theorem 6.6. Keeping the notation from the proof of Lemma 7.2 we begin with showing that condition (c) of weak regular analyticity (see the beginning of Section 4 is satisfied for the family $\mathcal{F} = \{\Phi^\lambda\}_{\lambda \in B(\gamma,R_*)}$ from Section 6 with $\kappa(e) = -\beta||e||$ with some constant $\beta > 0$. Putting $q_- = \min\{q_i : i \in I_c\}$, note that it follows from (6.2), (7.7), (7.6) and Lemma 6.10 that

$$\begin{aligned}
&|f_{\lambda,c}^{-1}\big(g_\lambda(\rho_\lambda(\pi_\lambda(\sigma(\omega))))\big)| \le \\
&\qquad \le ||G||_\infty^{1/q} |g_\lambda(\rho_\lambda(\pi_\lambda(\sigma(\omega)))) - v_\lambda|^{\frac{1}{q}-1} \\
&\qquad \le (3/2)^{1-\frac{1}{q}} ||G||_\infty^{1/q} |g_\lambda'(\rho_\lambda(\pi_\lambda(\sigma(\omega))))|^{\frac{1}{q}-1} \left|f_\lambda^{n-p}(v_\lambda) - \rho_\lambda(\pi_\lambda(\sigma(\omega)))\right|^{\frac{1}{q}-1} \\
&\qquad \le (3/2)^{1-\frac{1}{q}} ||G||_\infty^{1/q} \Delta_1^{\frac{1}{q}-1} |g_\lambda'(\rho_\lambda(\pi_\lambda(\sigma(\omega))))|^{\frac{1}{q}-1},
\end{aligned}$$

where $||G||_\infty = \max\{1, \max_{i \in I_c} \sup\{||G_\lambda(z)| : (\lambda, z) \in (\overline{B}(\gamma, R) * U)_i\}\}$. Therefore,

$$\begin{aligned}
|\phi_{\omega_1}^\lambda(\pi_\lambda(\sigma(\omega)))| &= |f_{\lambda,*}^{-1}\big(g_\lambda(\rho_\lambda(\pi_\lambda(\sigma(\omega))))\big)||g_\lambda'(\rho_\lambda(\pi_\lambda(\sigma(\omega))))| \cdot |\rho_\lambda'(\pi_\lambda(\sigma(\omega)))| \\
&\le (2D_1/3)^{\frac{1}{q}-1} |g_\lambda'(\rho_\lambda(\pi_\lambda(\sigma(\omega))))|^{\frac{1}{q}} \cdot |\rho_\lambda'(\pi_\lambda(\sigma(\omega)))| \\
&\le M_6 |g_\lambda'(\rho_\lambda(\pi_\lambda(\sigma(\omega))))|^{\frac{1}{q}},
\end{aligned}$$

where $M_6 = (2D_1/3)^{\frac{1}{q_-}-1} \max^p\{||(f_j^{-1})'||_\infty : j \in I_r\}$. Finally, applying Lemma 6.8, we get

$$\begin{aligned}
|\phi_{\omega_1}^\lambda(\pi_\lambda(\sigma(\omega)))| &\le ||G||_\infty^{\frac{1}{q}} M_6 \exp\left(-\frac{\alpha}{q}(||\omega_1|| - (p+1))\right) \\
&\le ||G||_\infty^{\frac{1}{q_-}} M_6 \exp\left(-\frac{\alpha}{q_-}(||\omega_1|| - (p+1))\right) \qquad (8.1) \\
&\le C_2 e^{-\beta||\omega_1||}
\end{aligned}$$

for all $\omega \in E_A^\infty$ (also those for which $||\omega_1|| = 1$), all $\lambda \in B(\gamma, R_4)$ and some universal constants $C_2 \ge 1$ and $\beta > 0$.

Item (b) of weakly regular analyticity of the family $\mathcal{F}$, i.e. strong regularity of the system Φ^γ has been done in [9]. Indeed, it follows from Lemma 4.5 there with $u = \mathrm{HD}(J(f_\gamma)) = \mathrm{HD}\big(J_{\Phi^\gamma}\big)$ (the latter equality established in Proposition 5.3) and $s = 0$; then $\mathrm{P}(u) = 0$. Now, combining this fact along with Lemma 7.2 (condition (d) of weakly regular analyticity), (8.1) (condition (c)) and Lemma 6.7, we conclude that the familiy $\mathcal{F} = \{\Phi^\lambda\}_{\lambda \in B(\gamma,R_*)}$ is weakly regular analytic. Therefore, Theorem 6.6 follows now immediately from Theorem 4.2 and Proposition 5.3.

9. Examples

We shall describe in this section two classes of analytic families of semihyperbolic generalized polynomial-like mappings. In the first one the critical point and the critical disk vary whereas in the second one, regular maps vary.

Example 1. Suppose that $g : \mathcal{U}_r \to U = B(0,1)$ is a GPL without branching (critical) points, consisting for example of affine maps. Fix an integer $q \geq 2$, a point $v \in J(g)$ and $\xi \in U \setminus \overline{\mathcal{U}_r}$. Take then $R > 0$ so small that $\overline{B}(\xi, 2R \subset U \setminus \overline{\mathcal{U}_r}$. Let $H : B(0,1) \to B(0,1)$ be the conformal homeomorphism of the unit disk $B(0,1)$ onto itself given by the formula $H(z) = \frac{z+v}{1+\overline{v}z}$. Define the maps $f_\lambda : \mathcal{U}_r \cup B(\xi + \lambda_1, \lambda_2) \to B(0,1)$, $\lambda = (\lambda_1, \lambda_2) \in B(0,R) \times (B(0,R) \setminus \{0\})$, by the formula

$$f_\lambda(z) = \begin{cases} g(z) & \text{if } z \in \mathcal{U}_r \\ H\left(\frac{(z-(\xi+\lambda_1))^q}{\lambda_2^q}\right) & \text{if } z \in B(\xi + \lambda_1, \lambda_2). \end{cases}$$

Obviously, all the maps f_λ are generalized polynomial-like mappings. Furthermore, $\xi + \lambda_1$ is the only critical point of f_λ and $f_\lambda(\xi + \lambda_1) = v$. Since $v \in J(g)$, since $f_\lambda(J(g)) = J(g)$, since $B(\xi + \lambda_1, \lambda_2) \subset B(\xi, 2R)$ and since $J(g) \cap B(\xi, 2R) = \emptyset$, all the maps f_λ are semihyperbolic and all conditions (a)-(e) of Definition 6.1 are satisfied. This means that $\{f_\lambda\}_{\lambda \in B(0,R)\times(B(0,R)\setminus\{0\})}$ is an analytic family of semihyperbolic GPLs and, as a consequence of Theorem 6.6, we get that the map $\lambda \mapsto \mathrm{HD}(J(f_\lambda))$, $\lambda \in B(0,R) \times (B(0,R) \setminus \{0\})$, is real-analytic.

Example 2. Consider a hyperbolic GPL $f : \mathcal{U}_r \to U$ with the following two properties.

(a) $f(f(c)) = f(c)$ for every critical point c of f.

(b) If c_1, c_2 are two different critical points of f and $f(c_1) \in U_{i_1}$, $f(c_2) \in U_{i_2}$, then $i_1 \neq i_2$.

For every critical point c of f let $i_c \in I$ be the only element of I such that $f(c) \in I_{i(c)}$. Put $\hat{I} = \{j(c) : c \in \mathrm{Crit}(f)\}$. For every $c \in \mathrm{Crit}(f)$ let $R_{j(c)} : B(0,1) \to U_{j(c)}$ be a conformal homeomorphism sending 0 to $f(c)$. For every $\lambda \in B(0,1)$ consider the map $f_\lambda : \mathcal{U}_r \to \bigcup_{j\in\hat{I}} R_j(B(0,1))$ given by the formula

$$f_l(z) = \begin{cases} f(z) & \text{if } z \notin \bigcup_{j\in\hat{I}} U_j \\ f\big(R_j\big(\lambda^{-1}R_j^{-1}(z)\big)\big) & \text{if } z \in R_j(B(0,\lambda)) \text{ and } j \in \hat{I}. \end{cases}$$

Since for every $\lambda \in B(0,1)$, $\mathrm{Crit}(f_\lambda) = \mathrm{Crit}(f)$ and $f_\lambda(f_\lambda(c)) = f(c) = f_\lambda(c)$ for every $c \in \mathrm{Crit}(f)$, it is straightforward to verify that $\{f_\lambda\}_{\lambda\in B(0,1)}$ is an analytic family of semihyperbolic GPLs and, as a consequence of Theorem 6.6, we get that the map $\lambda \mapsto \mathrm{HD}(J(f_\lambda))$, $\lambda \in B(0,1)$, is real-analytic.

References

[1] E. Hille, Analytic function theory, II. AMS Chelsea Publishing (2002), 2nd Edition.

[2] R. D. Mauldin, M. Urbański, *Graph directed Markov systems: geometry and dynamics of limit sets*, Cambridge Univ. Press (2003).

[3] T. Kato, Perturbation theory for linear operators, Springer (1995).

[4] N. Makarov, S. Smirnov, Phase transition in subhyperbolic Julia sets, Ergod. Th. & Dynam. Sys. 16 (1996), 125-157.

[5] N. Makarov, S. Smirnov, On "thermodynamics" of rational maps, I. Negative spectrum, Comm. Math. Phys. 211 (2000), 705-743.

[6] N. Makarov, S. Smirnov, On "thermodynamics" of rational maps, II. Non-recurrent maps. Preprint.

[7] M. Roy, M. Urbański, Regularity properties of Hausdorff dimension in conformal infinite IFS, Ergod. Th. & Dynam. Sys. 25 (2005), 1961-1983.
[8] W. Rudin, Functional Analysis, McGraw-Hill, Inc. (1991) 2nd Edition.
[9] B. Stratmann, M. Urbański, Real analyticity of topological pressure for parabolically semihyperbolic generalized polynomial-like maps, Indag. Mathem. 14 (2003), 119-134.
[10] B. Stratmann, M. Urbański, Multifractal analysis for parabolically semihyperbolic generalized polynomial-like maps, IP New Studies in Advanced Mathematics 5 (2004), 393-447.
[11] M. Urbański, A. Zdunik, Real analyticity of Hausdorff dimension of finer Julia sets of exponential family, Ergod. Th. & Dynam. Sys.24 (2004), 279-315.
[12] M. Urbański, M. Zinsmeister, Geometry of hyperbolic Julia-Lavaurs sets, Indagationes Math. 12 (2001) 273 - 292.

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