

# GEOMETRIC RIGIDITY FOR THE CLASS $\mathcal{S}$ OF TRANSCENDENTAL MEROMORPHIC FUNCTIONS

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ABSTRACT. We consider all the transcendental meromorphic functions from the class  $\mathcal{S}$  whose Julia set is a Jordan curve. We show that then the Julia set is either a straight line or its Hausdorff dimension is strictly larger than 1.

## 1. INTRODUCTION

Suppose  $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$  is a meromorphic function and that  $J(f)$  is a Jordan curve. In 1919 Fatou proved in [3] that if  $f$  is rational then either it is a circle/line or it has no tangents on a dense subset. Later on it has been proved ([2], [9], comp. [8]) that the second alternative in the hyperbolic case is much stronger: the Hausdorff dimension of  $J(f)$  is strictly larger than 1 (thus by the topological exactness every non-empty open subset of  $J(f)$  has Hausdorff dimension larger than 1). The case when parabolic point is allowed was covered in [10]. Relaxing the Jordan curve hypothesis further results have been obtained in [8], [10], [11] and [4]. All of this in the rational case. In the landscape of transcendental functions an analogous result have been proved in [5]. For the class  $\mathcal{S}$  it gives our result under additional assumption that there are no rationally indifferent periodic points. In the current paper we prove the straight line/fractal dichotomy in its fullest strength for the whole class  $\mathcal{S}$ . We do not consider cases. We do not use either any knowledge about the dynamics of inner functions. Instead, we associate to our transcendental map a conformal iterated function system, in the sense of [6] and [7], and apply the results proved there. Beyond the class  $\mathcal{S}$  the theorem in general fails. D. Hamilton in [4] has constructed meromorphic functions that are not in the class  $\mathcal{B}$ , whose Julia sets are rectifiable Jordan curves, but do not form straight lines.

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## 2. THE THEOREM

**Theorem 2.1.** *Suppose  $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$  is a transcendental meromorphic function in class  $\mathcal{S}$ . If the Julia set  $J(f)$  of  $f$  is a Jordan curve, then it is either a straight line or its Hausdorff dimension is strictly larger than 1.*

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*Proof.* Let  $A_0$  and  $A_1$  be the two connected components of  $\hat{\mathbb{C}} \setminus J(f)$ . Since our function is in the class  $\mathcal{S}$  for each  $i = 0, 1$  there exists  $a_i \in \overline{A_i} \setminus \{\infty\}$  such that  $f^2(a_i) = a_i$  and either  $a_i$  is an attracting fixed point for  $f^2$  or  $a_i$  is a rationally indifferent fixed point for  $f^2$  and  $A_i$  is its basin of immediate attraction. In order to work with Euclidean derivatives we change the system of coordinates by a Möbius transformation so that  $\infty$  is sent to a finite point and, moreover, the image of the whole Julia set is contained in the complex plane  $\mathbb{C}$ . If one of the points  $a_0$  or  $a_1$  is a parabolic point, denote it by  $\omega$ . Otherwise, let  $\omega$  be an arbitrary periodic point in  $J(f)$ . Let  $A$  be one (arbitrary) of the sets  $A_0$  or  $A_1$ . Replacing  $f$  by its sufficiently high iterate, we may assume without loss of generality that  $f(\omega) = \omega$  and  $f(A) = A$ . For every  $k \geq 1$  denote by  $\hat{\gamma}_k$  the only arc in  $J(f)$  containing  $\omega$  and with endpoints in  $f^{-k}(\omega) \setminus \{\omega\}$ . Fix  $k \geq 1$  so large that

$$(2.1) \quad \bigcap_{j=0}^{\infty} f^{-j}(\hat{\gamma}_k) = \{\omega\}.$$

Set

$$\gamma = \overline{J(f) \setminus \hat{\gamma}_k}.$$

It follows from our assumptions and Theorem (ii) in [1] that there exists a closed topological disk  $X$  contained in  $\mathbb{C}$  with the following properties:

- (a)  $\gamma \subset X$
- (b) The boundary of  $X$  is a piecewise smooth Jordan curve without cusps containing both endpoints of  $\gamma$ .
- (c) There exists an open simply connected set  $V$  disjoint from the postcritical set of  $f$ .
- (d) If  $f_*^{-n}$  is a holomorphic inverse branch of  $f^n$  defined on  $V$  such that  $f_*^{-n}(X) \cap \text{Int}X \neq \emptyset$ , then  $f_*^{-n}(X) \subset X$  and  $f_*^{-n}(V) \subset V$ .

We now form an iterated function system in the sense of [7]. It is defined to consist of all holomorphic inverse branches  $f_*^{-n} : V \rightarrow \mathbb{C}$  such that  $f_*^{-n}(X) \cap \text{Int}X \neq \emptyset$  and  $f^k(f_*^{-n}(X)) \cap \text{Int}X \neq \emptyset$  for all  $k = 1, 2, \dots, n-1$ . We parameterize all such inverse branches by a countable alphabet  $I$  and denote them by  $\phi_i$ ,  $i \in I$ . It follows immediately from this definition that  $\phi_i(\text{Int}X) \cap \phi_j(\text{Int}X) = \emptyset$  whenever  $i \neq j$ . Along with properties (a)-(d) this implies that  $S = \{\phi_i : X \rightarrow X\}_{i \in I}$  is a conformal iterated function system in the sense of [7]. Let  $J^*$  be the limit set of  $S$ . Using (2.1) note that

$$J^* = \gamma \setminus \bigcup_{n=0}^{\infty} f^{-n}(\{\omega\} \cup E),$$

where  $E$  is the countable set of essential singularities of  $f$  (for original  $f$  the set  $E$  is a singleton, say  $e$ , but for an iterate  $f^k$ , the entire set  $\{e\} \cup f^{-1}(\{e\}) \cup f^{-2}(\{e\}) \cup \dots \cup f^{-(k-1)}(e)$  consists of essential singularities of  $f^k$ ). In particular

$$\text{HD}(J^*) = \text{HD}(\gamma) = \text{HD}(J(f)) := h.$$

Now suppose that  $\text{HD}(J(f)) = 1$ . So,  $\text{HD}(J^*) = 1$ , and it follows from Theorem 4.5.1 and Theorem 4.5.11 in [7] that  $\text{H}^1(J^*) < +\infty$ . Since  $\text{H}^1(J^*) = \text{H}^1(\gamma) > 0$ , it follows from Theorem 4.5.10 in [7] (with  $d = 1$  and  $X$  replaced by  $\gamma$ ) that  $m := (\text{H}^1(\gamma))^{-1} \text{H}^1|_{\gamma}$  is a 1-conformal measure for the system  $S$ , meaning that

$$m(\phi_i(A)) = \int_A |\phi'_i| dm$$

for every Borel set  $A \subset \gamma$  and

$$m(\phi_i(\gamma) \cap \phi_j(\gamma)) = 0$$

whenever  $i \neq j$ . Theorem 4.4.7 in [7] yields then (it is in fact much stronger than we need) the existence of a unique Borel probability measure  $\mu$  on  $\gamma$  with the following properties:

- (e)  $\mu(J^*) = 1$ ,
- (f)  $\mu$  and  $m$  are equivalent with positive and continuous Radon-Nikodym derivatives.
- (g) (invariance) For every Borel set  $A \subset \gamma$ ,  $\sum_{i \in I} \mu(\phi_i(A)) = \mu(A)$ .

Now consider two Riemann mappings  $R_0 : \overline{\mathbb{D}^1} \rightarrow \overline{A_0}$  and  $R_1 : \hat{\mathbb{C}} \setminus \mathbb{D}^1 \rightarrow \{A_1\}$  such that  $R_0(1) = R_1(1) = \omega$  (since  $J(f)$  is a Jordan curve,  $R_0$  and  $R_1$  are uniquely defined respectively on the on closed disks  $\overline{\mathbb{D}^1}$  and  $\hat{\mathbb{C}} \setminus \mathbb{D}^1$  due to Caratheodory's theorem). Define two continuous maps

$$g_0 := R_0^{-1} \circ f \circ R_0 : \overline{\mathbb{D}^1} \setminus R_0^{-1}(E) \rightarrow \overline{\mathbb{D}^1} \quad \text{and} \quad g_1 := R_1^{-1} \circ f \circ R_1 : (\hat{\mathbb{C}} \setminus \mathbb{D}^1) \setminus R_1^{-1}(E) \rightarrow \hat{\mathbb{C}} \setminus \mathbb{D}^1.$$

Thus, the Schwartz Reflection Principle allows us to extend  $g_0$  and  $g_1$  respectively to  $\mathbb{C} \setminus R_0^{-1}(E)$  and  $\mathbb{C} \setminus R_1^{-1}(E)$ . Then the iterated function system  $S$  lifts up to the two respective systems  $S_0 = \{\phi_i^0\}_{i \in I}$  and  $S_1 = \{\phi_i^1\}_{i \in I}$  formed by respective inverse branches of iterates of  $g_0$  and  $g_1$ . Fix  $j \in \{0, 1\}$ . Note that the normalized Lebesgue measure  $\lambda_j$  on  $R_j^{-1}(\gamma)$  is a conformal measure for the system  $S_j$ . Again, by Theorem 4.4.7 in [7], this system has a unique invariant measure  $\mu_j$  equivalent to  $\lambda_j$ . But  $\mu \circ R_j$  is also  $S_j$ -invariant and, by Riesz Theorem, is equivalent to  $\lambda_j$ . Thus,  $\mu_j = \mu \circ R_j$ . Hence,

$$(2.2) \quad \mu_1 = \mu \circ R_1 = \mu \circ R_0 \circ (R_0^{-1} \circ R_1) = \mu_0 \circ R_0^{-1} \circ R_1 = \mu_0 \circ (R_1^{-1} \circ R_0)^{-1}.$$

For every  $z \in S^1$  put

$$D_j(z) = \frac{d\mu_j}{d\lambda}(z).$$

In view of Theorem 6.1.3 from [7] the function  $z \mapsto D_j(z)$  has a real-analytic extension onto a neighborhood of  $R_j^{-1}(\gamma)$  in  $\mathbb{C}$ . Let

$$F_j(z) = \int_1^z D_j(t) d\lambda(t),$$

where the integration is taken along the unit circle arc from 1 to  $z$  against the Lebesgue measure  $\lambda$  on  $S^1$ . Formula (2.2) and  $R_1^{-1} \circ R_0(1) = 1$  then give for every  $z \in R_1^{-1}(\gamma)$  that

$$F_0(z) = F_1(R_1^{-1} \circ R_0(z)).$$

Since both functions  $F_1$  and  $F_0$  are invertible (as  $D_j$  is positive on  $R_j^{-1}(\gamma)$ ), we conclude that  $R_1^{-1} \circ R_0 = F_1^{-1} \circ F_0$  is real analytic on  $R_0^{-1}(\gamma)$ . Thus,  $R_1^{-1} \circ R_0$  has a holomorphic extension  $\psi$  on an open neighborhood  $U$  of  $R_0^{-1}(\gamma)$  in  $\mathbb{C}$ . The formula

$$T(z) = \begin{cases} R_0(z) & \text{if } z \in \overline{\mathbb{D}^1} \cap U \\ \psi(z) & \text{if } z \in (\hat{\mathbb{C}} \setminus \mathbb{D}^1) \cap U \end{cases}$$

thus defines a holomorphic map from  $U$  into  $\mathbb{C}$  mapping  $R_0^{-1}(\gamma)$  onto  $\gamma$ . Therefore,  $\gamma$  is a real-analytic curve, and topological exactness of  $f : J(f) \rightarrow J(f)$  implies that  $J(f)$  itself is a real-analytic curve. So, by the Schwartz Reflection Principle  $R_0$  extends to an entire bijective map of  $\mathbb{C}$  onto  $\mathbb{C}$ . Thus,  $R_0$  is an affine map ( $z \mapsto az + b$ ), and  $J(f) = R_0(S^1)$  is a geometric circle. We are done.  $\square$

Note that because of Theorem in [1] (where the hypothesis of having two completely invariant domains can be weakened by requiring that the second iterate has two completely invariant domains) our assumption that the Julia set  $J(f)$  is a Jordan curve is equivalent to require that  $f^2$  has two completely invariant domains.

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