MULTIFRACTAL ANALYSIS FOR CONFORMAL GRAPH DIRECTED MARKOV SYSTEMS

MARIO ROY AND MARIUSZ URBAŃSKI

ABSTRACT. We derive the multifractal analysis of the conformal measure (or equivalently, the invariant measure) associated to a family of weights imposed upon a (multi-dimensional) graph directed Markov system (GDMS) using balls as the filtration. This analysis is done over a subset of J which is often large. In particular, it coincides with the limit set when the GDMS under scrutiny satisfies a boundary separation condition. It also applies to more general situations such as real or complex continued fractions.

1. INTRODUCTION

Multifractal formalism origins from physics and mathematics (among others, see [6], [2], [3] and [4]). In this latter paper, strong hints of parallels between multifractal theory and the theory of statistical physics were suggested. Some of the first rigorous mathematical results on multifractals can be found in [1] and [15]. Since then, many papers have been written on this subject (for instance, see [10], [11], [12] and [14]). In particular, Pesin [13] developed a general framework in which multifractal formalism can be derived.

We now briefly describe our setting. Let μ be a Borel probability measure on a metric space X. The measure μ is said to have local dimension α at a point $x \in X$ if

$$\lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha.$$

For each number $\alpha \geq 0$, let $X_\mu(\alpha)$ be the set of points $x \in X$ where the measure μ has local dimension α, and let $f_\mu(\alpha)$ be the Hausdorff dimension of the set $X_\mu(\alpha)$. The map $\alpha \mapsto f_\mu(\alpha)$ is called the (fine Hausdorff) multifractal spectrum of the measure μ.

The multifractal analysis of equilibrium states for a natural potential function and a natural family of Hölder continuous weights was performed in [5] for infinite conformal iterated function systems and in [9] for infinite conformal graph directed Markov systems. In both cases, the authors used at every point in the limit set of the given system the natural filtration generated by the initial blocks of the word that encodes the point. In other terms, the analysis was carried out using cylinders.

Aiming to give the multifractal analysis a transparent geometrical meaning, we shall derive in the sequel the multifractal analysis for cofinitely regular graph directed Markov systems (GDMSs) using as the filtration a base of balls centred at the given point. Until now, the question of the analysis of balls, which has already been solved in the case of finite systems, remained open for infinite systems.

We conduct our analysis over the full set of parameters on which it can be expected to hold. Moreover, we perform it on a large and dynamically significant subset J_r of the limit set J. In fact, under a mild boundary separation condition (for example, the separation condition) the sets J_r and J coincide. An additional geometric flavor of our analysis results from the fact that we concentrate on a geometrically meaningful family of Hölder continuous weights. All our results apply to a large class of GDMSs, one- and multi-dimensional alike, including real and complex continued fractions.

Let us describe the content of the paper more precisely. In section 2, we recall the basic definition of conformal graph directed Markov system. In section 3, we describe the Hölder families of weights F and $F_{q,t}$ we shall work with and study the properties of the pressure $P(q,t)$ and temperature $T(q)$ they determine. In section 4, we carry out the multifractal analysis of the conformal measure m_F (or equivalently, the invariant measure μ_F) associated to a family of weights F. This analysis is done over a subset J_r of J and conducted by means of balls. In particular, we show that for each α there is an auxiliary measure that witnesses the Hausdorff dimension of the set $J_{r,\mu}(\alpha)$ and that the $f_{r,\mu}(\alpha)$ curve is the Legendre transform of the temperature function $T(q)$. In section 5, we derive the multifractal analysis of the conformal measure m_F (or equivalently, the invariant measure μ_F) under additional conditions on the GDMS. In subsection 5.1, we observe that $J_r = J$ for all GDMSs which satisfy a boundary separation condition. Real continued fractions with the digit 1 deleted are an important example of such systems. In subsection 5.2, we derive the multifractal analysis over J under three conditions and show that there are families of one-dimensional conformal iterated function systems that meet these three conditions. Real continued fractions (with or without the digit 1) are a good example of such a family.

2. Preliminaries on Graph Directed Markov Systems

Let us first describe the setting of conformal graph directed Markov systems introduced in [9]. Graph directed Markov systems are based on a directed multigraph (V, E, i, t) and an associated incidence matrix A. The multigraph consists of a finite set V of vertices, a countable (finite or infinite) set of edges, and two functions $i, t : E \to V$ that indicate for each directed edge $e \in E$ its initial vertex $i(e)$ and its terminal vertex $t(e)$, respectively. The matrix $A : E \times E \to \{0, 1\}$ is an edge incidence matrix and thus tells which edges may follow a given edge. Moreover, it respects the multigraph, that is, if $A_{e_1 e_2} = 1$ then $t(e_1) = i(e_2)$. It is thereafter natural to define the set of all one-sided infinite A-admissible words

$$E_A^\infty = \{\omega \in E^\infty \mid A_{\omega_i \omega_{i+1}} = 1, \forall i \in I\!N\}.$$

The set of all subwords of E_A^∞ of length $n \in I\!N$ will be denoted by E_A^n, whereas the set of all finite subwords will be denoted by $E_A^* = \cup_{n \in I\!N} E_A^n$. From a dynamical point of view, we will consider the left shift map $\sigma : E_A^\infty \to E_A^\infty$ which drops the first letter of each word.

A graph directed Markov system (GDMS) consists of a directed multigraph and an edge incidence matrix together with a set of non-empty compact subsets $\{X_v\}_{v \in V}$ of a common Euclidean space $I\!R^d$, a number $0 < s < 1$, and for every $e \in E$ a one-to-one contraction $\varphi_e : X_{t(e)} \to X_{i(e)}$ with Lipschitz constant at most s.

A GDMS is called iterated function system (IFS) provided that $A_{ef} = 1$ if and only if $t(e) = i(f)$ and that V is a singleton.

For $\omega \in E_A^n$, $n \in I\!N$, we define

$$\varphi_\omega := \varphi_{\omega_1} \circ \varphi_{\omega_2} \circ \cdots \circ \varphi_{\omega_n} : X_{t(\omega)} \to X_{i(\omega)}.$$

Note that the functions i and t extend naturally to E_A^* by setting $i(\omega) := i(\omega_1)$ and $t(\omega) = t(\omega_{|\omega|})$. The main object of our interest will be the limit set J of S. This set is the image of the symbolic space E_A^∞ under a coding map π. Indeed, given any $\omega \in E_A^\infty$, the sets $\varphi_{\omega|_n}(X_{t(\omega|_n)})$ form a decreasing sequence of non-empty compact sets whose diameters converge to zero. Therefore their intersection

$$\bigcap_{n=1}^{\infty} \varphi_{\omega|_n}(X_{t(\omega|_n)})$$

is a singleton, and we denote its element by $\pi(\omega)$. This defines the coding map $\pi : E_A^\infty \to X$, where $X := \bigoplus_{v \in V} X_v$ is the disjoint union of the compact sets X_v. Clearly, π is a continuous function when E_A^∞ is equipped with the topology generated by the cylinders $[e]_n = \{\omega \in E_A^\infty \mid \omega_n = e\}$, $e \in E$, $n \in I\!N$. Hence the limit set of the GDMS S is

$$J = \pi(E^\infty) = \bigcup_{\omega \in E^\infty} \bigcap_{n=1}^{\infty} \varphi_{\omega|n}(X_{t(\omega|_n)}).$$

Recall also (cf. section 4.2 in [9]) that a GDMS $S = \{\varphi_e\}_{e \in E}$ is called conformal (and thereafter a CGDMS) if the following conditions are satisfied.

(i) For every $v \in V$, the set X_v is a compact, connected subset of $I\!R^d$ which is the closure of its interior (i.e. $X_v = \overline{\text{Int}_{I\!R^d}(X_v)}$);

(ii) (Open set condition (OSC)) For all $e, f \in E$, $e \neq f$,

$$\varphi_e(\text{Int}(X_{t(e)})) \cap \varphi_f(\text{Int}(X_{t(f)})) = \emptyset;$$

(iii) For every vertex $v \in V$, there exists an open connected set W_v such that $X_v \subset W_v \subset I\!R^d$ and such that for every $e \in E$ with $t(e) = v$, the map φ_e extends to a C^1 conformal diffeomorphism of W_v into $W_{i(e)}$;

(iv) (Cone property) There exist $\gamma, l > 0$ such that for every $v \in V$ and every $x \in X_v$ there is an open cone $\text{Con}(x, \gamma, l) \subset \text{Int}(X_v)$ with vertex x, central angle γ, and altitude l;

(v) There are two constants $L \geq 1$ and $\alpha > 0$ such that

$$\left| |\varphi_e'(y)| - |\varphi_e'(x)| \right| \leq L \|(\varphi_e')^{-1}\|^{-1} \|y - x\|^\alpha$$

for every $e \in E$ and every pair of points $x, y \in W_{t(e)}$, where $|\varphi'_e(x)|$ denotes the norm of the derivative of φ_e at x and $\|(\varphi'_e)^{-1}\|$ is the supremum norm taken over $W_{i(e)}$.

Remark 2.1. *According to Proposition 4.2.1 in [9], condition (v) is automatically satisfied (with $\alpha = 1$) when $d \geq 2$. This condition is also fulfilled if $d = 1$, the alphabet E is finite and all the φ_e's are of class $C^{1+\varepsilon}$.*

The following useful fact has been proved in Lemma 4.2.2 of [9].

Lemma 2.2. *For all $\omega \in E_A^*$ and all $x, y \in W_{t(\omega)}$ we have*

$$\left| \log|\varphi'_\omega(y)| - \log|\varphi'_\omega(x)| \right| \leq L(1-s)^{-1}\|y-x\|^\alpha.$$

An immediate consequence of this lemma is the famous bounded distortion property.

(v') (Bounded Distortion Property (BDP)) There exists a constant $K \geq 1$ such that

$$|\varphi'_\omega(y)| \leq K|\varphi'_\omega(x)|$$

for every $\omega \in E_A^*$ and every $x, y \in W_{t(\omega)}$.

Recall that a CGDMS S satisfy the Strong Open Set Condition (SOSC) if $J \cap \text{Int}(X) \neq \emptyset$, that is, $\cup_{v \in V}(J_v \cap \text{Int}(X_v)) \neq \emptyset$. Recall further that a matrix A is finitely primitive if there exists a finite set $\Omega \subset E_A^*$ of words of the same length such that for all $e, f \in E$ there is a word $\omega \in \Omega$ for which $e\omega f \in E_A^*$. From this point on we assume that all the systems we deal with satisfy those two properties.

Infinite systems naturally break into two main classes called irregular and regular systems. This dichotomy can be determined from the existence of a conformal measure or, equivalently, the existence of a zero of the topological pressure function. Recall that the topological pressure $P(t)$, $t \geq 0$, is defined as follows. For every $n \in \mathbb{N}$, set

$$P^{(n)}(t) = \sum_{\omega \in E_A^n} \|\varphi'_\omega\|^t,$$

where $\|\varphi'_\omega\| := \sup_{x \in X_{t(\omega)}} |\varphi'_\omega(x)|$. Then

$$P(t) = \lim_{n \to \infty} \frac{1}{n} \log P^{(n)}(t) = \inf_{n \in \mathbb{N}} \frac{1}{n} \log P^{(n)}(t).$$

If the function $\tilde{\zeta} : E_A^\infty \to \mathbb{R}$ is given by the formula

$$\tilde{\zeta}(\omega) = \log|\varphi'_{\omega_1}(\pi(\sigma(\omega)))|,$$

then $P(t) = P(t\tilde{\zeta})$, where $\mathrm{P}(t\tilde{\zeta})$ is the classical topological pressure of the function $t\tilde{\zeta}$ when E is finite (so the space E^∞ is compact), and is understood in the sense of [5] and [9] when E is infinite. The finiteness parameter θ of the system is defined by $\inf\{t \geq 0 : P^{(1)}(t) < \infty\} = \inf\{t \geq 0 : P(t) < \infty\}$. It is easy to show that the pressure function is non-increasing on $[0, \infty)$, that it is (strictly) decreasing, continuous and convex on $[\theta, \infty)$, and that $P(d) \leq 0$. Of course, $P(0) = \infty$ if and only if E is infinite. The following characterization of the Hausdorff dimension $\mathrm{HD}(J)$ of the limit set J was proved in [9], Theorem 4.2.13. For every $F \subset E$, we write $S|_F$ for the subsystem $\{\varphi_e\}_{e \in F}$ of S, and J_F for the limit set of $S|_F$.

Theorem.

$$\mathrm{HD}(J) = \inf\{t \geq 0 : P(t) \leq 0\} = \sup\{\mathrm{HD}(J_F) : F \subset E \text{ is finite}\} \geq \theta.$$

If $P(t) = 0$, then t is the only zero of the function $P(t)$ and $t = \mathrm{HD}(J)$.

A system S was called regular provided there is some $t \geq 0$ such that $P(t) = 0$. In fact, a system is regular if and only if it admits a conformal measure. Recall that a Borel probability measure m is said to be t-conformal provided $m(J) = 1$ and for every $\omega \in E_A^*$ and for every Borel set $B \subset X_{t(\omega)}$

$$m(\varphi_\omega(B)) = \int_B |\varphi_\omega'|^t \, dm,$$

and for all incomparable words $\omega, \tau \in E_A^*$

$$m\Big(\varphi_\omega(X_{t(\omega)}) \cap \varphi_\tau(X_{t(\tau)})\Big) = 0.$$

There are natural subclasses of regular systems. Among others, a system is called cofinitely regular provided every non-empty cofinite subsystem $S' = \{\varphi_e\}_{e \in E'}$ (i.e. E' is a cofinite subset of E) is regular. A finite system is clearly cofinitely regular, and it was shown in [9], Theorem 4.3.4 that an infinite system is cofinitely regular exactly when the pressure is infinite at the finiteness parameter.

Theorem. An infinite system S is cofinitely regular if and only if $P(\theta) = \infty \Leftrightarrow P^{(1)}(\theta) = \infty \Leftrightarrow \{t \geq 0 : P(t) < \infty\} = (\theta, \infty) \Leftrightarrow \{t \geq 0 : P^{(1)}(t) < \infty\} = (\theta, \infty)$.

Throughout the rest of this paper, all systems under investigation are assumed to be cofinitely regular.

3. Hölder families of functions, Pressure and Temperature

We adopt the notational convention that families of functions shall be denoted by uppercase letters, while their members will be denoted by lowercase letters. Moreover, functions and measures associated with the symbolic space E_A^∞ will wear a tilde $\sim$, with the notable exception of the shift map σ.

Throughout this paper, let $S = \{\varphi_e : X_{t(e)} \to X_{i(e)} \,|\, e \in E\}$ be a cofinitely regular CGDMS satisfying the Strong Open Set Condition (SOSC) and having an underlying finitely primitive edge incidence matrix A. Let θ be the finiteness parameter of S. Let $u > \theta$. Let

$$\Psi = \left\{ \psi_e : X_{t(e)} \to \mathbb{R} \,\middle|\, e \in E \right\}$$

be a bounded Hölder family of functions. Let β be the order of that family. Hölder of order β means that (cf. section 3.1 in [9])

$$V_\beta(\Psi) := \sup_{n \in \mathbb{N}} V_n(\Psi) < \infty,$$

where

$$V_n(\Psi) = \sup_{\omega \in E_A^n} \sup_{x,y \in X_{t(\omega)}} \left| \psi_{\omega_1}(\varphi_{\sigma\omega}(x)) - \psi_{\omega_1}(\varphi_{\sigma\omega}(y)) \right| e^{\beta(n-1)}.$$

Bounded simply means that

$$\|\Psi\| := \sup_{e \in E} \|\psi_e\| < \infty.$$

Denote by $\tilde{\psi} : E_A^\infty \to \mathbb{R}$ the potential function (also called amalgamated function) induced by the family Ψ, which is defined by

$$\tilde{\psi}(\omega) = \psi_{\omega_1}(\pi(\sigma\omega)).$$

According to Lemma 3.1.3 in [9], the function $\tilde{\psi}$ is bounded and Hölder continuous of order β. Let also

$$\mathrm{Log} = \left\{ \log |\varphi_e'| : X_{t(e)} \to \mathbb{R} \,\middle|\, e \in E \right\}.$$

By Lemma 4.2.2 in [9], the family Log is Hölder of order $\alpha \log s$. Moreover, for any $t > \theta$, the family tLog is summable, that is

$$\sum_{e \in E} \left\| \exp(t \log \|\varphi_e'\|) \right\| < \infty.$$

In particular, uLog is summable. Denote by $\tilde{\zeta} : E_A^\infty \to \mathbb{R}$ the amalgamated function induced by the family Log, that is,

$$\tilde{\zeta}(\omega) = \log |\varphi_{\omega_1}'(\pi(\sigma\omega))|.$$

Then $\tilde{\zeta}$ is a summable Hölder continuous function of order $\alpha \log s$. It follows from the above definitions and properties that the family $F = \Psi + u$Log, that is, $F = \{f_e : X_{t(e)} \to \mathbb{R} \,|\, e \in E\}$, where

$$f_e = \psi_e + u \log |\varphi_e'|,$$

is a summable Hölder family of functions of order $\gamma = \max\{\beta, \alpha \log s\}$. The amalgamated function $\tilde{f} : E_A^\infty \to \mathbb{R}$ induced by the family F satisfies

$$\tilde{f} = \tilde{\psi} + u \, \tilde{\zeta}$$

and is a summable Hölder continuous function of order γ. Note also that F and $\tilde{f}$ are bounded above by $\sup \Psi := \sup_{e \in E} \sup_{x \in X_{t(e)}} \psi_e(x)$. Recall that the topological pressure $P(F)$ of F is defined by

$$P(F) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in E_A^n} \exp\Big(\sup_{x \in X_{t(\omega)}} \sum_{i=1}^n f_{\omega_i}(\varphi_{\sigma^i \omega}(x)) \Big).$$

By considering the family $F - P(F)$, we may assume without loss of generality that $P(F) = 0$. Equivalently, $P(\tilde{f}) = 0$ by Proposition 3.1.4 in [9], where

$$P(\tilde{f}) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(\tilde{f})$$

with

$$Z_n(\tilde{f}) = \sum_{\omega \in E_A^n} \exp\Big(\sup_{\tau \in [\omega]} \sum_{i=0}^{n-1} \tilde{f}(\sigma^i \tau) \Big).$$

Since S is a CGDMS with an underlying finitely primitive matrix and F is a summable Hölder family of functions, Theorem 3.2.3 and Proposition 4.2.5 in [9] assert that there exists a unique F-conformal measure m_F supported on J. In other words, for every $\omega \in E_A^*$ and for every Borel set $B \subset X_{t(\omega)}$

$$m_F(\varphi_\omega(B)) = \int_B \exp\Big(S_\omega(F) - P(F)|\omega| \Big) dm_F = \int_B \exp(S_\omega(F)) dm_F$$

and for all incomparable words $\omega, \tau \in E_A^*$

$$m_F\Big(\varphi_\omega(X_{t(\omega)}) \cap \varphi_\tau(X_{t(\tau)}) \Big) = 0.$$

Recall that $S_\omega(F) : X_{t(\omega)} \to \mathbb{R}$ is simply the ergodic sum

$$S_\omega(F)(x) = \sum_{i=1}^n f_{\omega_i}(\varphi_{\sigma^i \omega}(x)).$$

Moreover, $m_F = \tilde{m}_{\tilde{f}} \circ \pi^{-1}$, where $\tilde{m}_{\tilde{f}}$ is the unique eigenmeasure of the conjugate Perron-Frobenius operator $\mathcal{L}_{\tilde{f}}^*$. The existence and the uniqueness of $\tilde{m}_{\tilde{f}}$ is guaranteed by Corollary 2.7.5(a) in [9]. The measure $\tilde{m}_{\tilde{f}}$ is also a Gibbs state for $\tilde{f}$ according to Corollary 2.7.5(b). Furthermore, $\tilde{f}$ admits a unique σ-invariant Gibbs state $\tilde{\mu}_{\tilde{f}}$ according to Corollary 2.7.5(c). This Gibbs state is completely ergodic. It is also the unique equilibrium measure for $\tilde{f}$ by Theorem 2.2.9. As required by that theorem, note that $\tilde{f} \in L^1(\tilde{\mu}_{\tilde{f}})$, for $\tilde{\zeta} \in L^1(\tilde{\mu}_{\tilde{f}})$ as shown

in the following calculation:

$$\begin{aligned}
\int |\tilde{\zeta}|\, d\tilde{\mu}_{\tilde{f}} &= \sum_{e\in E} \int_{[e]} |\tilde{\zeta}|\, d\tilde{\mu}_{\tilde{f}} = \sum_{e\in E} \int_{[e]} \Big|\log|\varphi_e'(\pi(\sigma\omega))|\Big|\, d\tilde{\mu}_{\tilde{f}}(\omega) \\
&\leq \sum_{e\in E} \int_{[e]} \Big(\log K - \log\|\varphi_e'\|\Big) d\tilde{\mu}_{\tilde{f}} \\
&= \log K - \sum_{e\in E} \log\|\varphi_e'\|\, \tilde{\mu}_{\tilde{f}}([e]) \\
&\leq \log K - C \sum_{e\in E} \log\|\varphi_e'\| \exp\big(\sup f_e\big) \\
&\leq \log K - C \sum_{e\in E} \log\|\varphi_e'\| \exp\big(\sup\psi_e + u\sup\log|\varphi_e'|\big) \\
&\leq \log K - Ce^{\|\Psi\|} \sum_{e\in E} \log\|\varphi_e'\| \exp\big(u\log K + u\log\|\varphi_e'\|\big) \\
&\leq \log K - CK^u e^{\|\Psi\|} \sum_{e\in E} \|\varphi_e'\|^u \log\|\varphi_e'\| \\
&= \log K - CK^u e^{\|\Psi\|} \sum_{e\in E} \|\varphi_e'\|^{u-\delta} \|\varphi_e'\|^{\delta} \log\|\varphi_e'\| \\
&\leq \log K - Ce^{\|\Psi\|+u\log K} B \sum_{e\in E} \|\varphi_e'\|^{u-\delta} \\
&< \infty,
\end{aligned} \tag{3.1}$$

where $0 < \delta < u - \theta$ and $-\infty < B := \inf\{\|\varphi_e'\|^{\delta} \log\|\varphi_e'\| : e \in E\} < 0$, and where we used the bounded distortion property (explaining the presence of $K \geq 1$), the fact that $\tilde{m}_{\tilde{f}}$ is a Gibbs state for $\tilde{f}$ (hence the presence of $C \geq 1$), that $P(\tilde{f}) = 0$, and that $\lim_{x\to 0^+} x^{\delta}\log x = 0$ (by means of B).

Since any two Gibbs states for $\tilde{f}$ are boundedly equivalent, the measures $\tilde{\mu}_{\tilde{f}}$ and $\tilde{m}_{\tilde{f}}$ are boundedly equivalent and thus the measures $\mu_F := \tilde{\mu}_{\tilde{f}} \circ \pi^{-1}$ and $m_F = \tilde{m}_{\tilde{f}} \circ \pi^{-1}$ are boundedly equivalent. For this reason, μ_F is called the S-invariant version of m_F.

Now, for every $(q,t) \in \mathbb{R}^2$ define the family

$$F_{q,t} := qF + t\mathrm{Log} = q\Psi + (qu + t)\mathrm{Log},$$

and its corresponding amalgamated function

$$\tilde{f}_{q,t} = q\tilde{f} + t\tilde{\zeta} = q\tilde{\psi} + (qu+t)\tilde{\zeta}.$$

It is easy to see that the $F_{q,t}$'s are Hölder families of functions of order γ (the order of F) and that the $\tilde{f}_{q,t}$'s are Hölder continuous functions of order γ (like $\tilde{f}$). Moreover, $F_{q,t}$ and $\tilde{f}_{q,t}$ are summable if and only if $qu + t > \theta$. Hence for every $(q,t) \in \mathbb{R}^2$ such that $qu + t > \theta$, there exists a unique $F_{q,t}$-conformal measure $m_{F_{q,t}}$ supported on J. This measure is such that $m_{F_{q,t}} = \tilde{m}_{\tilde{f}_{q,t}} \circ \pi^{-1}$, where $\tilde{m}_{\tilde{f}_{q,t}}$ is the unique eigenmeasure of the conjugate Perron-Frobenius operator $\mathcal{L}^*_{\tilde{f}_{q,t}}$. The measure $\tilde{m}_{\tilde{f}_{q,t}}$ is a Gibbs state for $\tilde{f}_{q,t}$. Furthermore, $\tilde{f}_{q,t}$ admits

a unique completely ergodic σ-invariant Gibbs state $\tilde{\mu}_{\tilde{f}_{q,t}}$. This Gibbs state is also the unique equilibrium measure for $\tilde{f}_{q,t}$. As the Gibbs states $\tilde{\mu}_{\tilde{f}_{q,t}}$ and $\tilde{m}_{\tilde{f}_{q,t}}$ are boundedly equivalent, so are the measures $\mu_{F_{q,t}} := \tilde{\mu}_{\tilde{f}_{q,t}} \circ \pi^{-1}$ and $m_{F_{q,t}} = \tilde{m}_{\tilde{f}_{q,t}} \circ \pi^{-1}$. Note that $\tilde{f}_{q,t} \in L^1(\tilde{\mu}_{q,t})$, for $\tilde{\zeta} \in L^1(\tilde{\mu}_{q,t})$ as a calculation similar to (3.1) shows. Furthermore,

$$P\big((qu+t)\mathrm{Log}\big) - |q|\,\|\Psi\| \le P(F_{q,t}) \le P\big((qu+t)\mathrm{Log}\big) + |q|\,\|\Psi\|, \tag{3.2}$$

where $P((qu+t)\mathrm{Log}) = P(qu+t)$ (cf. section 2). We now state fundamental properties of the pressure as a function of the two variables q and t.

Theorem 3.1. *Let $S = \{\varphi_e : X_{t(e)} \to X_{i(e)} \,|\, e \in E\}$ be a cofinitely regular CGDMS satisfying the Strong Open Set Condition (SOSC) and having an underlying finitely primitive edge incidence matrix A. Let θ be the finiteness parameter of S, and let $u > 0$. Then the pressure function $(q,t) \mapsto P(q,t) := P(F_{q,t}) = P(\tilde{f}_{q,t})$, $(q,t) \in \mathbb{R}^2$ satisfies the following properties.*

    (a) *$P(q,t) < \infty$ if and only if $qu + t > \theta$;*
    (b) *If $(q_2-q_1)(\sup \Psi + u \log s) + (t_2-t_1)\log s \le 0$, then $P(q_2,t_2) \le P(q_1,t_1)$. In particular, if $\sup \Psi \le -u \log s$, then $P(q,t)$ is decreasing with respect to both variables $q \in \mathbb{R}$ and $t \in \mathbb{R}$.*
    (c) *$t \mapsto P(q,t)$ is strictly decreasing on $(\theta - qu, \infty)$;*
    (d) *$\lim_{t \to \infty} P(q,t) = -\infty$;*
    (e) *$\lim_{t \to (\theta-qu)^+} P(q,t) = \infty$;*
    (f) *$\frac{\partial P}{\partial t}(q,t) = -\chi_{\tilde{\mu}_{q,t}}$ for every $(q,t) \in \mathbb{R}^2$ such that $qu + t > \theta$, where $\chi_{\tilde{\mu}_{q,t}} := -\int \tilde{\zeta}\, d\tilde{\mu}_{q,t}$ is the Lyapunov exponent of $\tilde{\mu}_{q,t}$;*
    (g) *$t \mapsto P(q,t)$ is convex (and thereby continuous) on the interval $(\theta - qu, \infty)$;*

Proof. (a) This follows from (3.2) and the fact that $P(\bar{t}) := P(\bar{t}\,\mathrm{Log}) < \infty$ if and only if $\bar{t} > \theta$.

(b) Let $q_1 \leq q_2$, $t_1 \leq t_2$. If $q_1 u + t_1 \leq \theta$, then $P(q_1, t_1) = \infty$ and the statement thus holds. So suppose that $q_1 u + t_1 > \theta$. Then the n-th partition function of $\tilde{f}_{q_2, t_2}$ satisfies

$$
\begin{aligned}
Z_n(\tilde{f}_{q_2,t_2}) &= \sum_{\omega \in E_A^n} \exp\Big(\sup_{\tau \in [\omega]} S_n \tilde{f}_{q_2,t_2}(\tau) \Big) \\
&= \sum_{\omega \in E_A^n} \sup_{\tau \in [\omega]} \exp\Big(S_n \tilde{f}_{q_2,t_2}(\tau) \Big) \\
&= \sum_{\omega \in E_A^n} \sup_{\rho \in E_A^\infty : A_{\omega_n \rho_1} = 1} \Big(\exp\Big(S_n(q_2 \tilde{\psi}(\omega \rho)) \Big) |\varphi_\omega'(\pi(\rho))|^{q_2 u + t_2} \Big) \\
&\leq \sum_{\omega \in E_A^n} \sup_{\tau \in [\omega]} \Big(\exp\Big(S_n(q_2 \tilde{\psi})(\tau) \Big) \Big) \|\varphi_\omega'\|^{q_2 u + t_2} \\
&\leq \sum_{\omega \in E_A^n} \sup_{\tau \in [\omega]} \Big(\exp\Big(S_n(q_1 \tilde{\psi})(\tau) + S_n((q_2 - q_1)\tilde{\psi})(\tau) \Big) \Big) \|\varphi_\omega'\|^{q_1 u + t_1} \|\varphi_\omega'\|^{(q_2 - q_1)u + (t_2 - t_1)} \\
&\leq s^{n[(q_2 - q_1)u + (t_2 - t_1)]} K^{q_1 u + t_1} e^{n(q_2 - q_1) \sup \Psi} \sum_{\omega \in E_A^n} \sup_{\tau \in [\omega]} \exp\Big(S_n(q_1 \tilde{\psi})(\tau) \Big) \inf_{x \in X_{t(\omega)}} |\varphi_\omega'(x)|^{q_1 u + t_1} \\
&\leq e^{n(q_2 - q_1) \sup \Psi} s^{n[(q_2 - q_1)u + (t_2 - t_1)]} K^{q_1 u + t_1} \sum_{\omega \in E_A^n} \sup_{\rho \in E_A^\infty : A_{\omega_n \rho_1} = 1} \Big(\exp\Big(S_n(q_1 \tilde{\psi})(\omega \rho) \Big) |\varphi_\omega'(\pi(\rho))|^{q_1 u + t_1} \Big) \\
&= e^{n(q_2 - q_1) \sup \Psi} s^{n[(q_2 - q_1)u + (t_2 - t_1)]} K^{q_1 u + t_1} Z_n(\tilde{f}_{q_1, t_1}).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
P(q_2, t_2) &= \lim_{n \to \infty} \frac{1}{n} \log Z_n(\tilde{f}_{q, t_2}) \\
&\leq (q_2 - q_1) \sup \Psi + [(q_2 - q_1)u + (t_2 - t_1)] \log s + \lim_{n \to \infty} \frac{1}{n} \log Z_n(\tilde{f}_{q_1, t_1}) \\
&= (q_2 - q_1) \sup \Psi + [(q_2 - q_1)u + (t_2 - t_1)] \log s + P(q_1, t_1).
\end{aligned}
\tag{3.3}
$$

Part (b) follows immediately.

(c) Letting $q_1 = q_2 = q$ and $t_1 < t_2$ in (3.3) gives (c).

(d) This also follows from (3.3) by setting $q_1 = q_2 = q$, $t_1 > \theta$ and $t_2 = t$ and letting $t \to \infty$.

(e) Let $t > \theta - qu$. Then

$$
\begin{aligned}
Z_n(\tilde{f}_{q,t}) &\geq \sum_{\omega \in E_A^n} \exp(-nq \inf \Psi) K^{-(qu+t)} \|\varphi_\omega'\|^{qu+t} \\
&= \exp(-nq \inf \Psi) K^{-(qu+t)} \sum_{\omega \in E_A^n} \|\varphi_\omega'\|^{qu+t}.
\end{aligned}
$$

Therefore

$$
P(q, t) \geq -q \inf \Psi + P\big((qu + t)\mathrm{Log}\big) = -q \inf \Psi + P(qu + t).
$$

Thus,

$$\lim_{t \to (\theta - qu)^+} P(q,t) \geq -q \inf \Psi + \lim_{t \to (\theta - qu)^+} P(qu + t) = -q \inf \Psi + P(\theta) = \infty$$

since S is cofinitely regular.

(f) This follows from Proposition 2.6.13 in [9] and the fact that $-\chi_{\tilde{\mu}_{q,t}} = \int \tilde{\zeta} \, d\tilde{\mu}_{q,t} > -\infty$ whenever $qu + t > \theta$ by a calculation similar to (3.1).

(g) This follows immediately from Proposition 2.6.14 in [9]. ∎

Observe that in the proof of part (e) of the above theorem $P(\theta) \geq q \inf \Psi$ suffices to guarantee the existence of a zero for the pressure function $t \mapsto P(q,t)$. The assumption of cofinite regularity on S ensures that the pressure function $t \mapsto P(q,t)$ has a zero for every $q \in \mathbb{R}$.

Corollary 3.2. *For all $q \in \mathbb{R}$ there exists a unique $T(q) \in (\theta - qu, \infty)$ such that $P(q, T(q)) = 0$. The function $T(q)$ is called the temperature function.*

In order to allege notation, let

$$\tilde{f}_q = \tilde{f}_{q,T(q)}, \quad F_q = F_{q,T(q)}, \quad \tilde{m}_q = \tilde{m}_{\tilde{f}_{q,T(q)}}, \quad m_q = m_{F_{q,T(q)}}, \quad \tilde{\mu}_q = \tilde{\mu}_{\tilde{f}_{q,T(q)}}, \quad \mu_q = \mu_{F_{q,T(q)}}.$$

Now, let $q \in \mathbb{R}$. Recall that $\tilde{\zeta} \in L^1(\tilde{\mu}_q)$ and thus $\tilde{f} \in L^1(\tilde{\mu}_q)$. Thereafter, let

$$\alpha(q) = \frac{\int \tilde{f} \, d\tilde{\mu}_q}{-\chi_{\tilde{\mu}_q}(\sigma)} = \frac{\int \tilde{f} \, d\tilde{\mu}_q}{\int \tilde{\zeta} \, d\tilde{\mu}_q}.$$

By the variational principle for pressure (cf. Theorems 2.1.6–2.1.8 in [9]), note that

$$\int \tilde{f} \, d\tilde{\mu}_q \leq P(\tilde{f}) - h_{\tilde{\mu}_q}(\sigma) = -h_{\tilde{\mu}_q}(\sigma) \leq 0. \tag{3.4}$$

Hence $\alpha(q) \geq 0$. Moreover,

$$\alpha(q) \leq \frac{\int |\tilde{\psi} + u\tilde{\zeta}| \, d\tilde{\mu}_q}{\int |\tilde{\zeta}| \, d\tilde{\mu}_q} \leq u + \frac{\int |\tilde{\psi}| \, d\tilde{\mu}_q}{\int |\tilde{\zeta}| \, d\tilde{\mu}_q} \leq u + \frac{\|\Psi\|}{-\log s}.$$

Thus, $0 \leq \alpha(q) < \infty$.

Finally, we study some basic properties of the temperature function $T(q)$.

Theorem 3.3. *The temperature function $q \mapsto T(q)$ exhibits the following properties.*
  (a) *The function $T : \mathbb{R} \to \mathbb{R}$ is real-analytic;*
  (b) *$T(0) = \mathrm{HD}(J)$ while $T(1) = 0$;*
  (c) *$T'(q) = -\alpha(q) < 0$ for all $q \in \mathbb{R}$;*

(d) *The function $q \mapsto T(q)$, $q \in I\!R$, is convex, meaning that $T''(q) \geq 0$ for all $q \in I\!R$. This function is not strictly convex if and only if $\tilde{\mu}_{\tilde{f}}$ is equal to $\tilde{\mu}_{-\mathrm{HD}(J)\tilde{\zeta}}$.*

Proof. (a) By Proposition 2.6.13 in [9], $\frac{\partial P}{\partial t}(q, t) = \int \tilde{\zeta} \, d\tilde{\mu}_{q,t} = -\chi_{\tilde{\mu}_{q,t}}(\sigma) < 0$ for every $(q, t) \in I\!R^2$ such that $qu + t > \theta$. In particular, this is true for all pairs $(q, T(q))$. Since $T(q)$ is uniquely determined by the condition $P(q, T(q)) = 0$, it follows from Theorem 2.6.12 in [9] and the implicit function theorem that T is real-analytic on $I\!R$.

(b) Since S is regular, we have $P(\mathrm{HD}(J)\mathrm{Log}) = 0$, which means that $T(0) = \mathrm{HD}(J)$. Moreover, since $\tilde{f}_{1,0} = \tilde{f}$ and $P(F) = P(\tilde{f}) = 0$ by assumption, we deduce that $P(1, 0) = P(\tilde{f}_{1,0}) = P(\tilde{f}) = 0$, and from the uniqueness of $T(1)$ it follows that $T(1) = 0$.

(c) It follows from the fact that $P(q, T(q)) = 0$ for all $q \in I\!R$ and from Proposition 2.6.13 in [9] that

$$0 = \frac{dP}{dq}(q, T(q)) = \frac{\partial P}{\partial q}(q, T(q)) + \frac{\partial P}{\partial t}(q, T(q)) \cdot T'(q) = \int \tilde{f} \, d\tilde{\mu}_q - \chi_{\tilde{\mu}_q}(\sigma) T'(q).$$

Hence

$$T'(q) = -\frac{\int \tilde{f} \, d\tilde{\mu}_q}{-\chi_{\tilde{\mu}_q}(\sigma)} = -\alpha(q).$$

Having already observed that $\alpha(q) \geq 0$, we thus know that $T'(q) \leq 0$. In order to prove that $T'(q) < 0$, we need to show that $\int \tilde{f} \, d\tilde{\mu}_q \neq 0$. But since $\int \tilde{f} \, d\tilde{\mu}_q \leq -h_{\tilde{\mu}_q}(\sigma) \leq 0$ by (3.4), it suffices to show that $h_{\tilde{\mu}_q}(\sigma) > 0$. This follows immediately from Theorem 2.5.2 in [9].

(d) Lemma 4.9.5 in [9] (with Δ_1 replaced by $I\!R$) and its proof carry over to the current setting without any change. ∎

4. Multifractal analysis of the conformal measure m_F over a subset of J

Let $S = \{\varphi_e : X_{t(e)} \to X_{i(e)} \,|\, e \in E\}$ be a cofinitely regular CGDMS satisfying SOSC and having an underlying edge incidence matrix A which is finitely primitive. Let θ be the finiteness parameter of S. Let also F be a family of functions of the form $F = \Psi + u\mathrm{Log}$ such that $P(F) = 0$, where Ψ is a bounded Hölder family of functions and $u > \theta$.

We shall now develop the multifractal analysis of the conformal measure m_F (or equivalently, the invariant measure μ_F) associated to the family F. We shall conduct this analysis by means of balls and we shall restrict ourselves to a subset J_r of the limit set J of S. As we shall see later, J_r is often a fairly large subset of J. By definition, J_r is the set of points of J which are coded by the set of infinite admissible words

$$E_r^\infty = \Big\{ \omega \in E_A^\infty \,\Big|\, \limsup_{n \to \infty} \mathrm{dist}\Big(\pi(\sigma^n \omega), \partial X_{i(\sigma^n \omega)} \Big) > 0 \Big\}.$$

The words in this set code points of the limit set that behave tamely when a multifractal analysis is carried out using balls, for infinitely many of their iterates are positively separated

from the boundary of the phase space. The conformality of the measure m_F can then be used at those iterates to estimate the local dimension of m_F from above. Before going any further, we observe that E_r^∞ is a set of full measure.

Lemma 4.1. *For every ergodic, σ-invariant Borel probability measure $\tilde{\mu}$ on E_A^∞ with* supp $\tilde{\mu} = E_A^\infty$, *we have $\tilde{\mu}(E_r^\infty) = 1$.*

Proof. Let $\tilde{\mu}$ be an ergodic, σ-invariant Borel probability measure on E_A^∞ with supp $\tilde{\mu} = E_A^\infty$. Observe that E_r^∞ is completely σ-invariant, that is, $\sigma^{-1}(E_r^\infty) = E_r^\infty = \sigma(E_r^\infty)$. Thus, by ergodicity of $\tilde{\mu}$, we have that $\tilde{\mu}(E_r^\infty)$ is 0 or 1. We shall now show that this latter possibility always prevails. Since S satisfies SOSC, there exists $x \in J_v \cap \mathrm{Int}(X_v)$ for some $v \in V$. Let $\omega \in E_A^\infty$ be such that $\pi(\omega) = x$. Let also $0 < r < \mathrm{dist}(x, \partial X_v)$. Note that for any $\tau \in E_A^*$, we have $\pi([\tau]) \subset \varphi_\tau(X_{t(\tau)})$. Since $\varphi_{\omega|_k}(X_{t(\omega_k)}) \subset B(\pi(\omega), r) = B(x, r)$ for all $k \in \mathbb{N}$ large enough, we obtain that $[\omega|_k] \subset \pi^{-1}(B(x, r))$ for some $k \in \mathbb{N}$. Then $\tilde{\mu}(\pi^{-1}(B(x, r))) \geq \tilde{\mu}([\omega|_k]) > 0$ since supp $\tilde{\mu} = E_A^\infty$. It follows from Birkhoff's Ergodic Theorem that the set of infinite admissible words whose iterates' images visit infinitely many times the ball $B(x, r)$ has measure 1, that is,

$$\tilde{\mu}\Big(\Big\{\tau \in E_A^\infty \,\Big|\, \sigma^n \tau \in \pi^{-1}(B(x, r)) \text{ for infinitely many } n\text{'s}\Big\}\Big) = 1.$$

Therefore $\tilde{\mu}(E_r^\infty) = 1$. (The same conclusion can be drawn by means of Poincaré's Recurrence Theorem.) ∎

This lemma tells us that $\tilde{\mu}_{\tilde{f}}(E_r^\infty) = 1$ and $\tilde{\mu}_{\tilde{f}_{q,t}}(E_r^\infty) = 1$ for all $(q, t) \in \mathbb{R}^2$ such that $qu + t > \theta$. In particular, $\tilde{\mu}_q(E_r^\infty) = 1$ for all $q \in \mathbb{R}$. This implies immediately that $\mu_F(J_r) = \tilde{\mu}_{\tilde{f}} \circ \pi^{-1}(\pi(E_r^\infty)) = 1$, that is, the set E_r^∞ is a set of full μ_F-measure. Similarly, $\mu_{F_{q,t}}(J_r) = 1$ for all $(q, t) \in \mathbb{R}^2$ such that $qu + t > \theta$. In particular, $\mu_q(J_r) = 1$ for all $q \in \mathbb{R}$.

In fact, J_r contains a rich family of subsets of full measure. To define these subsets, we proceed as follows. For every $\omega \in E_A^\infty$ and $r \geq 0$, let $\{n_j(\omega, r)\}$ be the increasing sequence of all positive integers n such that $\sigma^n \omega \in \pi^{-1}(X_{i(\sigma^n \omega)} \setminus B(\partial X_{i(\sigma^n \omega)}, r))$. This sequence may be empty, non-empty and finite, or infinite depending on ω and r. However, for every $\omega \in E_r^\infty$ the sequence $\{n_j(\omega, r)\}$ is infinite for every $0 \leq r < r_{max}(\omega)$, where

$$r_{max}(\omega) := \limsup_{n \to \infty} \mathrm{dist}\Big(\pi(\sigma^n \omega), \partial X_{i(\sigma^n \omega)}\Big) > 0.$$

Now, for every $R \geq 0$ define the completely invariant set

$$E_{rr}^\infty(R) := \Big\{\omega \in E_r^\infty \,|\, R < r_{max}(\omega) \text{ and } \lim_{j \to \infty} \frac{S_{n_{j+1}(\omega,R)}\tilde{\zeta}(\omega) - S_{n_j(\omega,R)}\tilde{\zeta}(\omega)}{S_{n_j(\omega,R)}\tilde{\zeta}(\omega)} = 0\Big\}.$$

We claim that these subsets of E_r^∞ have all full measure.

Lemma 4.2. *For every $R \geq 0$ small enough, we have that $\tilde{\mu}(E_{rr}^\infty(R)) = 1$ for all ergodic, σ-invariant Borel probability measure $\tilde{\mu}$ on E_A^∞ with* supp $\tilde{\mu} = E_A^\infty$ *and such that $\int \tilde{\zeta}\, d\tilde{\mu} > -\infty$.*

Proof. Let $0 \leq R < D$, where $D = \sup_{x \in J} \text{dist}(x, \partial X) := \sup_{v \in V} \sup_{x \in J_v} \text{dist}(x, \partial X_v)$. SOSC guarantees that $D > 0$. Let $\tilde{\mu}$ be an ergodic, σ-invariant Borel probability measure on E_A^∞ with supp $\tilde{\mu} = E_A^\infty$ and such that $\int \tilde{\zeta} \, d\tilde{\mu} > -\infty$. Since $E_{rr}^\infty(R)$ is completely σ-invariant, the ergodicity of $\tilde{\mu}$ forces $\tilde{\mu}(E_{rr}^\infty(R))$ to equal 0 or 1. We shall now prove that this latter possibility always prevails. Since $0 \leq R < D$, there is $x \in J_v \backslash B(\partial X_v, (R+D)/2)$ for some $v \in V$. Let $\omega \in E_A^\infty$ be such that $\pi(\omega) = x$. Then there is some $k \in \mathbb{N}$ such that $\pi([\omega|_k]) \subset \varphi_{\omega|_k}(X_{t(\omega_k)}) \subset B(\pi(\omega), (D-R)/2) = B(x, (D-R)/2)$, or equivalently $[\omega|_k] \subset \pi^{-1}B(x, (D-R)/2)$. Then

$$\tilde{\mu}\big(\pi^{-1}(X_v \backslash B(\partial X_v, R))\big) \geq \tilde{\mu}\big(\pi^{-1}B(x, (D-R)/2)\big) \geq \tilde{\mu}([\omega|_k]) > 0$$

since supp $\tilde{\mu} = E_A^\infty$. Applying Birkhoff's Ergodic Theorem twice (once with the characteristic function of the set $\pi^{-1}(X_v \backslash B(\partial X_v, R))$ and once with the potential function $\tilde{\zeta}$), we obtain that the set

$$\left\{ \omega \in E_r^\infty \mid R < r_{max}(\omega), \lim_{j \to \infty} \frac{n_{j+1}(\omega, R)}{n_j(\omega, R)} = 1 \text{ and } \lim_{j \to \infty} \frac{S_{n_j(\omega, R)}\tilde{\zeta}(\omega)}{n_j(\omega, R)} = \int \tilde{\zeta} \, d\tilde{\mu} = -\chi_{\tilde{\mu}}(\sigma) \right\}$$

has measure 1. Writing n_j instead of $n_j(\omega, R)$ to allege notation, we have for every ω in this set that

$$
\begin{aligned}
\lim_{j \to \infty} \frac{S_{n_{j+1}}\tilde{\zeta}(\omega) - S_{n_j}\tilde{\zeta}(\omega)}{S_{n_j}\tilde{\zeta}(\omega)} &= \lim_{j \to \infty} \frac{n_{j+1}}{n_j} \frac{\frac{1}{n_{j+1}}\big(S_{n_{j+1}}\tilde{\zeta}(\omega) - S_{n_j}\tilde{\zeta}(\omega)\big)}{\frac{1}{n_j}S_{n_j}\tilde{\zeta}(\omega)} \\
&= \lim_{j \to \infty} \frac{\frac{1}{n_{j+1}}S_{n_{j+1}}\tilde{\zeta}(\omega) - \frac{n_j}{n_{j+1}} \cdot \frac{1}{n_j}S_{n_j}\tilde{\zeta}(\omega)}{\frac{1}{n_j}S_{n_j}\tilde{\zeta}(\omega)} \\
&= \frac{-\chi_{\tilde{\mu}}(\sigma) - 1 \cdot (-\chi_{\tilde{\mu}}(\sigma))}{-\chi_{\tilde{\mu}}(\sigma)} = 0.
\end{aligned}
$$

It follows immediately that $\tilde{\mu}(E_{rr}^\infty(R)) = 1$. ∎

This lemma reveals that $\tilde{\mu}_{\tilde{f}}(E_{rr}^\infty(R)) = 1$ and $\tilde{\mu}_{\tilde{f}_{q,t}}(E_{rr}^\infty(R)) = 1$ for all $(q, t) \in \mathbb{R}^2$ such that $qu + t > \theta$. In particular, $\tilde{\mu}_q(E_{rr}^\infty(R)) = 1$ for all $q \in \mathbb{R}$. This implies immediately that $\mu_F(\pi(E_{rr}^\infty(R))) = 1$, that is, the set $E_{rr}^\infty(R)$ is a set of full μ_F-measure for all $R \in \mathbb{R}$. Consequently, $\mu_{F_{q,t}}(\pi(E_{rr}^\infty(R))) = 1$ for all $(q, t) \in \mathbb{R}^2$ such that $qu + t > \theta$. In particular, $\mu_q(\pi(E_{rr}^\infty(R))) = 1$ for all $q \in \mathbb{R}$.

As an immediate corollary, we obtain that the completely invariant set

$$E_{rr}^\infty = \bigcup_{R > 0} E_{rr}^\infty(R)$$

is a set of full measure.

Corollary 4.3. *For every ergodic, σ-invariant Borel probability measure $\tilde{\mu}$ on E_A^∞ with supp $\tilde{\mu} = E_A^\infty$ and such that $\int \tilde{\zeta} \, d\tilde{\mu} > -\infty$, we have $\tilde{\mu}(E_{rr}^\infty) = 1$.*

We now recall a few basic definitions from multifractal analysis.

Let μ be a Borel probability measure on X. The pointwise or local dimension $d_\mu(x)$ of μ at $x \in X$ is the power law behaviour (if any) of $\mu(B(x, r))$ for small $r > 0$, that is,

$$d_\mu(x) = \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}.$$

We further define the lower and upper dimensions of μ at $x \in X$ by

$$\underline{d}_\mu(x) = \liminf_{n \to \infty} \frac{\log \mu(B(x, r))}{\log r}$$

and

$$\overline{d}_\mu(x) = \limsup_{n \to \infty} \frac{\log \mu(B(x, r))}{\log r},$$

respectively. Denote the set of points of J_r at which the local dimension of a measure μ is equal to α by

$$J_{r,\mu}(\alpha) = \{x \in J_r \mid d_\mu(x) = \alpha\}.$$

Denote the Hausforff dimension of $J_{r,\mu}(\alpha)$ by

$$f_{r,\mu}(\alpha) = \mathrm{HD}(J_{r,\mu}(\alpha)).$$

Now for every $\alpha \geq 0$, let

$$E_r^\infty(\alpha) = \Big\{\omega \in E_r^\infty : \lim_{n \to \infty} \frac{S_n \tilde{f}(\omega)}{S_n \tilde{\zeta}(\omega)} = \alpha\Big\}.$$

We shall now prove that for every $q \in I\!R$ the measure μ_q confers full measure to the set of points of J_r where the local dimension of the measure m_F is $\alpha(q)$.

Theorem 4.4. *The following statements hold.*
   (a) *For every $\alpha \geq 0$, we have $\pi(E_r^\infty(\alpha) \cap E_{rr}^\infty) \subset J_{r,m_F}(\alpha)$;*
   (b) *$\tilde{\mu}_q(E_r^\infty(\alpha(q)) \cap E_{rr}^\infty) = 1$ for all $q \in I\!R$;*
   (c) *$\mu_q(J_{r,m_F}(\alpha(q))) = 1$ for all $q \in I\!R$.*

Proof. (a) Let $x \in \pi(E_r^\infty(\alpha) \cap E_{rr}^\infty)$. Then there is some $\omega \in E_r^\infty(\alpha) \cap E_{rr}^\infty$ such that $\pi(\omega) = x$. Therefore $\omega \in E_{rr}^\infty(R)$ for some $0 < R < \min_{v \in V} \mathrm{dist}(X_v, \partial W_v)$. Let $\{n_j\}_{j \in I\!N} := \{n_j(\omega, R)\}_{j \in I\!N}$ be the increasing sequence of all n's such that $\pi(\sigma^n \omega) \in X_{t(\omega_n)} \backslash B(\partial X_{t(\omega_n)}, R)$. Let $0 < r \leq K^{-1} R|\varphi'_{\omega|_{n_1}}(\pi(\sigma^{n_1}\omega))|$. Let $j \in I\!N$ be the unique natural number so that

$$K^{-1} R|\varphi'_{\omega|_{n_{j+1}}}(\pi(\sigma^{n_{j+1}}\omega))| < r \leq K^{-1} R|\varphi'_{\omega|_{n_j}}(\pi(\sigma^{n_j}\omega))|.$$

Since $B(\pi(\sigma^{n_j}\omega), R) \subset W_{t(\omega_{n_j})}$, the conformality of the generators of the system ensures that

$$B(x, r) \subset B\Big(\pi(\omega), K^{-1} R|\varphi'_{\omega|_{n_j}}(\pi(\sigma^{n_j}\omega))|\Big) \subset \varphi_{\omega|_{n_j}}\Big(B(\pi(\sigma^{n_j}\omega), R)\Big),$$

where the last inclusion follows from relation (4.22) in [9]. Moreover, every $y \in J_{t(\omega_{n_j})}$ admits a $\tau \in [\omega|_{n_j}]$ such that $y = \pi(\sigma^{n_j}\tau)$, and for such y and τ we have $S_{\omega|_{n_j}}F(y) = S_{n_j}\tilde{f}(\tau)$. The conformality of m_F, the fact that $B(\pi(\sigma^{n_j}\omega), R) \subset \text{Int}(X_{t(\omega_{n_j})})$, and the OSC then give

$$
\begin{aligned}
m_F(B(x,r)) &\le m_F\Big(\varphi_{\omega|_{n_j}}(B(\pi(\sigma^{n_j}\omega), R))\Big) \\
&\le m_F\Big(\varphi_{\omega|_{n_j}}(\text{Int}(X_{t(\omega_{n_j})}))\Big) \\
&= m_F\Big(\varphi_{\omega|_{n_j}}(\text{Int}(X_{t(\omega_{n_j})})) \cap J_{i(\omega)}\Big) \\
&= m_F\Big(\varphi_{\omega|_{n_j}}\big(\text{Int}(X_{t(\omega_{n_j})}) \cap J_{t(\omega_{n_j})}\big)\Big) \\
&\le \exp\Big(\sup_{y \in \text{Int}(X_{t(\omega_{n_j})}) \cap J_{t(\omega_{n_j})}} S_{\omega|_{n_j}}F(y)\Big)\, m_F\Big(\text{Int}(X_{t(\omega_{n_j})}) \cap J_{t(\omega_{n_j})}\Big) \quad (4.1) \\
&\le \exp\Big(\sup_{\tau \in [\omega|_{n_j}]} S_{n_j}\tilde{f}(\tau)\Big) \\
&\le B(\tilde{f})\exp\Big(S_{n_j}\tilde{f}(\omega)\Big),
\end{aligned}
$$

where $B(\tilde{f})$ is a constant of bounded variation for $\tilde{f}$ (see the bounded variation principle for Hölder continuous potentials, Lemma 2.3.1 in [9]).

On the other hand, the conformality of the generators of the system guarantees that

$$
B(x,r) \supset B\big(\pi(\omega), K^{-1}R|\varphi'_{\omega|_{n_{j+1}}}(\pi(\sigma^{n_{j+1}}\omega))|\big) \supset \varphi_{\omega|_{n_{j+1}}}\big(B(\pi(\sigma^{n_{j+1}}\omega), K^{-2}R)\big).
$$

Moreover, every $y \in J_{t(\omega_{n_{j+1}})}$ admits a $\tau \in [\omega|_{n_{j+1}}]$ such that $y = \pi(\sigma^{n_{j+1}}\tau)$, and for such y and τ we have $S_{\omega|_{n_{j+1}}}F(y) = S_{n_{j+1}}\tilde{f}(\tau)$. Then the conformality of the measure m_F leads to

$$
\begin{aligned}
m_F(B(x,r)) &\ge m_F\Big(\varphi_{\omega|_{n_{j+1}}}(B(\pi(\sigma^{n_{j+1}}\omega), K^{-2}R))\Big) \\
&\ge m_F\Big(\varphi_{\omega|_{n_{j+1}}}\big(B(\pi(\sigma^{n_{j+1}}\omega), K^{-2}R) \cap J_{t(\omega_{n_{j+1}})}\big)\Big) \\
&\ge \exp\Big(\inf_{y \in J_{t(\omega_{n_{j+1}})}} S_{\omega|_{n_{j+1}}}F(y)\Big)\, m_F\Big(B(\pi(\sigma^{n_{j+1}}\omega), K^{-2}R) \cap J_{t(\omega_{n_{j+1}})}\Big) \\
&\ge \exp\Big(\inf_{\tau \in [\omega|_{n_{j+1}}]} S_{n_{j+1}}\tilde{f}(\tau)\Big)\, m_F\Big(B(\pi(\sigma^{n_{j+1}}\omega), K^{-2}R)\Big) \quad (4.2) \\
&\ge B(\tilde{f})\exp(S_{n_{j+1}}\tilde{f}(\omega))\, M_F(K^{-2}R) \\
&= M_F(K^{-2}R)B(\tilde{f})\,\exp(S_{n_{j+1}}\tilde{f}(\omega)),
\end{aligned}
$$

where

$$
M_F(a) = \inf_{y \in \overline{J}} m_F(B(y,a)) > 0
$$

and $B(\tilde{f})$ is a constant of bounded variation for $\tilde{f}$ (see the bounded variation principle for Hölder continuous potentials, Lemma 2.3.1 in [9]).

From the definition of n_j, we also have that

$$\log(K^{-1}R) + \log|\varphi'_{\omega|_{n_{j+1}}}(\pi(\sigma^{n_{j+1}}\omega))| < \log r \le \log(K^{-1}R) + \log|\varphi'_{\omega|_{n_j}}(\pi(\sigma^{n_j}\omega))| < 0.$$
(4.3)

As μ_F and m_F are boundedly equivalent, we deduce from (4.2) and (4.3) that

$$
\begin{aligned}
d_{m_F}(x) = \lim_{r\to 0}\frac{\log m_F(B(x,r))}{\log r} \ &\le\ \lim_{j\to\infty}\frac{S_{n_{j+1}}\tilde f(\omega) + \log(M_F(K^{-2}R)B(\tilde f))}{\log(K^{-1}R) + \log|\varphi'_{\omega|_{n_j}}(\pi(\sigma^{n_j}\omega))|}\\
&=\ \lim_{j\to\infty}\frac{S_{n_{j+1}}\tilde f(\omega) + \log(M_F(K^{-2}R)B(\tilde f))}{\log(K^{-1}R) + S_{n_j}\tilde\zeta(\omega)}\\
&=\ \lim_{j\to\infty}\frac{\frac{S_{n_{j+1}}\tilde f(\omega)}{S_{n_{j+1}}\tilde\zeta(\omega)} + \frac{\log(M_F(K^{-2}R)B(\tilde f))}{S_{n_{j+1}}\tilde\zeta(\omega)}}{\frac{\log(K^{-1}R)}{S_{n_{j+1}}\tilde\zeta(\omega)} + \frac{S_{n_j}\tilde\zeta(\omega)}{S_{n_{j+1}}\tilde\zeta(\omega)}}\\
&=\ \lim_{j\to\infty}\frac{\frac{S_{n_{j+1}}\tilde f(\omega)}{S_{n_{j+1}}\tilde\zeta(\omega)} + \frac{\log(M_F(K^{-2}R)B(\tilde f))}{S_{n_{j+1}}\tilde\zeta(\omega)}}{\frac{\log(K^{-1}R)}{S_{n_{j+1}}\tilde\zeta(\omega)} + \left(1 + \frac{(S_{n_{j+1}} - S_{n_j})\tilde\zeta(\omega)}{S_{n_j}\tilde\zeta(\omega)}\right)^{-1}}\\
&=\ \frac{\alpha + 0}{0 + (1+0)^{-1}} = \alpha
\end{aligned}
$$

since $\omega \in E_r^\infty(\alpha) \cap E_{rr}^\infty(R)$.

Similarly, we deduce from (4.1) and (4.3) that

$$d_{m_F}(x) \ge \lim_{j\to\infty}\frac{S_{n_j}\tilde f(\omega) + \log B(\tilde f)}{\log(K^{-1}R) + \log|\varphi'_{\omega|_{n_{j+1}}}(\pi(\sigma^{n_{j+1}}\omega))|} = \alpha.$$

Hence $d_{m_F}(x) = \alpha$. This completes the proof of (a).

(b) Let $q \in \mathbb{R}$. According to Birkhoff's Ergodic Theorem and Corollary 4.3, there exists $E_q \subset E_{rr}^\infty$ such that $\tilde\mu_q(E_q) = 1$ and so that for all $\omega \in E_q$ we have

$$\lim_{n\to\infty}\frac{1}{n}S_n\tilde\zeta(\omega) = \int\tilde\zeta d\tilde\mu_q$$

and

$$\lim_{n\to\infty}\frac{1}{n}S_n\tilde f(\omega) = \int\tilde f d\tilde\mu_q.$$

Therefore, for all $\omega \in E_q$ we get

$$\lim_{n\to\infty}\frac{S_n\tilde f(\omega)}{S_n\tilde\zeta(\omega)} = \frac{\int\tilde f d\tilde\mu_q}{\int\tilde\zeta d\tilde\mu_q} = \alpha(q).$$

Hence $E_q \subset E_r^\infty(\alpha(q)) \cap E_{rr}^\infty$ and thus $\tilde{\mu}_q(E_r^\infty(\alpha(q)) \cap E_{rr}^\infty) \geq \tilde{\mu}_q(E_q) = 1$.

(c) Let $q \in \mathbb{R}$. Using part (a) with $\alpha = \alpha(q)$ and part (b), we deduce that

$$\mu_q\Big(J_{r,m_F}(\alpha(q))\Big) \geq \tilde{\mu}_q \circ \pi^{-1}\Big(\pi(E_r^\infty(\alpha(q)) \cap E_{rr}^\infty)\Big) \geq \tilde{\mu}_q\Big(E_r^\infty(\alpha(q)) \cap E_{rr}^\infty\Big) = 1.$$

∎

Let us now remind the reader about the Legendre transform. Let k be a strictly convex function on an interval I (hence $k'' > 0$ wherever this second derivative exists). The Legendre transform of k is the function l defined by $l(p) = \max\{p\,x - k(x)\}$ wherever the maximum exists. It can be proved that the domain of l is either a point, an interval or a half-line. It can further be shown that l is strictly convex and that the Legendre transform is involutive. We then say that the functions k and l form a Legendre transform pair. The following theorem (see [16]) gives a useful characterization of a Legendre transform pair.

Theorem 4.5. *Two strictly convex differentiable functions k and l form a Legendre transform pair if and only if $l(-k'(q)) = k(q) - qk'(q)$.*

We shall now prove that $f_{r,m_F}(\alpha)$ and $T(q)$ form a Legendre transform pair. Recall that $T'(q) = -\alpha(q)$ by Theorem 3.3.

Theorem 4.6. *For every $q \in \mathbb{R}$ we have $f_{r,m_F}(\alpha(q)) = q\alpha(q) + T(q)$. In other terms, $f_{r,m_F}(-T'(q)) = T(q) - qT'(q)$.*

Proof. Using Theorem 4.4(a,b), Theorem 4.4.2 in [9], Theorem 2.2.9 in [9] which guarantees that $\tilde{\mu}_q$ is an (in fact, the unique) equilibrium state for $\tilde{f}_q$, and the fact that $P(q, T(q)) = P(F_q) = P(\tilde{f}_q) = 0$ by definition of the temperature function $T(q)$ in Corollary 3.2, we obtain

$$\begin{aligned}
f_{r,m_F}(\alpha(q)) &= \mathrm{HD}(J_{r,m_F}(\alpha(q))) \geq \mathrm{HD}(\pi(E_r^\infty(\alpha(q)) \cap E_{rr}^\infty)) \geq \mathrm{HD}(\tilde{\mu}_q \circ \pi^{-1}) \\
&= \frac{h_{\tilde{\mu}_q}(\sigma)}{\chi_{\tilde{\mu}_q}(\sigma)} = \frac{P(\tilde{f}_q) - \int \tilde{f}_q \, d\tilde{\mu}_q}{\chi_{\tilde{\mu}_q}(\sigma)} = \frac{-\int \tilde{f}_q \, d\tilde{\mu}_q}{-\int \tilde{\zeta} \, d\tilde{\mu}_q} = \frac{\int (q\tilde{f} + T(q)\tilde{\zeta}) \, d\tilde{\mu}_q}{\int \tilde{\zeta} \, d\tilde{\mu}_q} \\
&= q\frac{\int \tilde{f} \, d\tilde{\mu}_q}{\int \tilde{\zeta} \, d\tilde{\mu}_q} + T(q) = q\alpha(q) + T(q).
\end{aligned}$$

To prove the other inequality, fix $x \in J_{r,m_F}(\alpha(q))$. Then there is $\omega \in E_r^\infty$ such that $\pi(\omega) = x$. Let $0 < R < \min\{K^{-1}, r_{max}(\omega)\}$. Let also $\{n_j\}_{j\in\mathbb{N}}$ be any subsequence of the increasing sequence $\{n_j(\omega, R)\}$ of all n's such that $\pi(\sigma^n\omega) \in X_{t(\omega_n)} \backslash B(\partial X_{t(\omega_n)}, R)$. For every $n \in \mathbb{N}$ we have

$$\varphi_{\omega|_n}\Big(B(\pi(\sigma^n\omega), R)\Big) \subset B\Big(\pi(\omega), KR|\varphi'_{\omega|_n}(\pi(\sigma^n\omega))|\Big).$$

Like (4.2), the conformality of m_q and the bounded variation principle for $\tilde{f}_q$ give

$$m_q\Big(B(x, KR|\varphi'_{\omega|_n}(\pi(\sigma^n\omega))|)\Big) \geq m_q\Big(\varphi_{\omega|_n}\Big(B(\pi(\sigma^n\omega), R) \cap J_{t(\omega_n)}\Big)\Big)$$

$$\geq \exp\Big(\inf_{\tau\in[\omega|_n]} S_n\tilde{f}_q(\tau)\Big)\, m_q\Big(B(\pi(\sigma^n\omega), R)\Big) \qquad (4.4)$$

$$\geq M_{m_q}(R)B(\tilde{f}_q)\exp\Big(S_n\tilde{f}_q(\omega)\Big)$$

where $B(\tilde{f}_q)$ is a constant of bounded variation (see Lemma 2.3.1 in [9]). Hence

$$\frac{\log m_q\Big(B(x, KR|\varphi'_{\omega|_n}(\pi(\sigma^n\omega))|)\Big)}{\log(KR|\varphi'_{\omega|_n}(\pi(\sigma^n\omega))|)} \leq \frac{\log(M_{m_q}(R)B(\tilde{f}_q)) + S_n\tilde{f}_q(\omega)}{\log(KR) + \log|\varphi'_{\omega|_n}(\pi(\sigma^n\omega))|}$$

$$= \frac{\log(M_{m_q}(R)B(\tilde{f}_q)) + qS_n\tilde{f}(\omega) + T(q)S_n\tilde{\zeta}(\omega)}{\log(KR) + S_n\tilde{\zeta}(\omega)} \qquad (4.5)$$

$$= \frac{\frac{\log(M_{m_q}(R)B(\tilde{f}_q))}{S_n\tilde{\zeta}(\omega)} + q\frac{S_n\tilde{f}(\omega)}{S_n\tilde{\zeta}(\omega)} + T(q)}{\frac{\log(KR)}{S_n\tilde{\zeta}(\omega)} + 1}.$$

(So far the estimates are valid for all $n \in I\!\!N$.) Like (4.1), every $y \in J_{t(\omega_{n_j})}$ admits a $\tau \in [\omega|_{n_j}]$ such that $y = \pi(\sigma^{n_j}\tau)$, and for such y and τ we have $S_{\omega|_{n_j}}F(y) = S_{n_j}\tilde{f}(\tau)$. The conformality of m_F, the fact that $B(\pi(\sigma^{n_j}\omega), R) \subset \text{Int}(X_{t(\omega_{n_j})})$, and the OSC then give

$$m_F\Big(B(x, K^{-1}R|\varphi'_{\omega|_{n_j}}(\pi(\sigma^{n_j}\omega))|)\Big) \leq m_F\Big(\varphi_{\omega|_{n_j}}(B(\pi(\sigma^{n_j}\omega), R))\Big)$$

$$\leq m_F\Big(\varphi_{\omega|_{n_j}}(\text{Int}(X_{t(\omega_{n_j})}))\Big)$$

$$= m_F\Big(\varphi_{\omega|_{n_j}}(\text{Int}(X_{t(\omega_{n_j})})) \cap J_{i(\omega)}\Big)$$

$$= m_F\Big(\varphi_{\omega|_{n_j}}\Big(\text{Int}(X_{t(\omega_{n_j})}) \cap J_{t(\omega_{n_j})}\Big)\Big)$$

$$\leq \exp\Big(\sup_{y\in\text{Int}(X_{t(\omega_{n_j})})\cap J_{t(\omega_{n_j})}} S_{\omega|_{n_j}}F(y)\Big) m_F\Big(\text{Int}(X_{t(\omega_{n_j})}) \cap J_{t(\omega_{n_j})}\Big)$$

$$\leq \exp\Big(\sup_{\tau\in[\omega|_{n_j}]} S_{n_j}\tilde{f}(\tau)\Big)$$

$$\leq B(\tilde{f})\exp\Big(S_{n_j}\tilde{f}(\omega)\Big),$$

where $B(\tilde{f})$ is a constant of bounded variation for $\tilde{f}$ (see the bounded variation principle for Hölder continuous potentials, Lemma 2.3.1 in [9]). Thus,

$$\frac{\log m_F\Big(B(x, K^{-1}R|\varphi'_{\omega|_{n_j}}(\pi(\sigma^{n_j}\omega))|)\Big)}{\log(K^{-1}R|\varphi'_{\omega|_{n_j}}(\pi(\sigma^{n_j}\omega))|)} \geq \frac{\log(B(\tilde{f})) + S_{n_j}\tilde{f}(\omega)}{\log(K^{-1}R) + S_{n_j}\tilde{\zeta}(\omega)} \qquad (4.6)$$

Using (4.5) and (4.6), we deduce that

$$
\begin{aligned}
d_{m_q}(x) = \lim_{r \to 0} \frac{\log m_q(B(x,r))}{\log r} \;\; &\leq\;\; \limsup_{j \to \infty} \frac{\log m_q\Big(B(x, KR|\varphi'_{\omega|_{n_j}}(\pi(\sigma^{n_j}\omega))|)\Big)}{\log(KR|\varphi'_{\omega|_{n_j}}(\pi(\sigma^{n_j}\omega))|)} \\[2mm]
&\leq\;\; \limsup_{j \to \infty}\Big(q\frac{S_{n_j}\tilde{f}(\omega)}{S_{n_j}\tilde{\zeta}(\omega)} + T(q)\Big) \\[2mm]
&=\;\; q\limsup_{j \to \infty} \frac{\log(B(\tilde{f})) + S_{n_j}\tilde{f}(\omega)}{\log(K^{-1}R) + S_{n_j}\tilde{\zeta}(\omega)} + T(q) \\[2mm]
&\leq\;\; q\limsup_{j \to \infty} \frac{\log m_F\Big(B(x, K^{-1}R|\varphi'_{\omega|_{n_j}}(\pi(\sigma^{n_j}\omega))|)\Big)}{\log(K^{-1}R|\varphi'_{\omega|_{n_j}}(\pi(\sigma^{n_j}\omega))|)} + T(q) \\[2mm]
&=\;\; q\alpha(q) + T(q).
\end{aligned}
$$

As $d_{m_q}(x) \leq q\alpha(q) + T(q)$ for every $x \in J_{r,m_F}(\alpha(q))$, we deduce that $f_{r,m_F}(\alpha(q)) \leq q\alpha(q) + T(q)$. ∎

All of the above results give us an analog of Theorem 4.9.4 in [9].

Theorem 4.7. *Let $S = \{\varphi_e : X_{t(e)} \to X_{i(e)} \,|\, e \in E\}$ be a cofinitely regular CGDMS satisfying SOSC and having an underlying finitely primitive edge incidence matrix A. Let θ be the finiteness parameter of S. Suppose that $h_{\tilde{\mu}_q}(\sigma)/\chi_{\tilde{\mu}}(\sigma) > \theta$. Then the following statements hold.*

(a) *The number $d_{\mu_F}(x)$ exists for μ_F-a.e. $x \in J_r$ and*

$$
d_{\mu_F}(x) = \frac{\int \tilde{f}\, d\tilde{\mu}_{\tilde{f}}}{\int \tilde{\zeta}\, d\tilde{\mu}_{\tilde{f}}}.
$$

(b) *The function $T : \mathbb{R} \to \mathbb{R}$ is real-analytic, $T(0) = \mathrm{HD}(J)$, and $T'(q) < 0$, $T''(q) \geq 0$ for all $q \in \mathbb{R}$.*

(c) *For every $q \in \mathbb{R}$, we have $f_{r,\mu_F}(-T'(q)) = f_{r,\mu_F}(\alpha(q)) = q\alpha(q) + T(q) = T(q) - qT'(q)$. That is, $f_{r,\mu_F}(\alpha)$ and $T(q)$ form a Legendre pair of functions.*

(d) *If $\tilde{\mu}_{\tilde{f}} \neq \tilde{\mu}_{\mathrm{HD}(J)\tilde{\zeta}}$ or, equivalently, if $\tilde{f}$ and $\mathrm{HD}(J)\tilde{\zeta}$ are not cohomologous modulo any constant, then the function $\alpha \mapsto f_{r,\mu_F}(\alpha)$, $\alpha \in (\alpha_1, \alpha_2)$ is real-analytic, where the interval (α_1, α_2), $0 \leq \alpha_1 \leq \alpha_2 \leq \infty$, is the range of $-T'(q)$. Otherwise, $T'(q) = \mathrm{HD}(J)$ for every $q \in \mathbb{R}$.*

Proof. (a) Using Theorem 3.3(b), notice that $\mu_1 = \mu_F$. Thus, by Theorem 4.4(c), we have $\mu_F(J_{r,m_F}(\alpha(1))) = 1$. Since $\alpha(1) = \int \tilde{f}\, d\tilde{\mu}_{\tilde{f}} / \int \tilde{\zeta}\, d\tilde{\mu}_{\tilde{f}}$ and since the measures μ_F and m_F are boundedly equivalent, we hence have that

$$
\mu_F\Big(J_{r,m_F}\Big(\int \tilde{f}\, d\tilde{\mu}_{\tilde{f}} / \int \tilde{\zeta}\, d\tilde{\mu}_{\tilde{f}}\Big)\Big) = 1.
$$

Part (b) is essentially Theorem 3.3. Part (c) corresponds to Theorem 3.3 and Theorem 4.6. Finally, part (d) is a consequence of Lemma 4.9.5 in [9] (with $\Delta_1 = I\!\!R$) and parts (c) and (b) of the present theorem. ∎

5. Multifractal analysis over J

5.1. **Multifractal analysis over J under the Boundary Separation Condition.** If S satisfies the Boundary Separation Condition (BSC), that is, if

$$\text{dist}(\partial X, \overline{\cup_{i \in I} \varphi_i(X)}) > 0,$$

then $E_r^\infty = E_A^\infty$ and thus $J_r = J$. Thus, section 4 gives us the multifractal analysis over J. Indeed, denoting the set of points of J at which the local dimension of a measure μ is equal to α by $J_\mu(\alpha)$ and the Hausdorff dimension of $J_\mu(\alpha)$ by $f_\mu(\alpha)$, Theorem 4.7 reduces to the following.

Theorem 5.1. *Let $S = \{\varphi_e : X_{t(e)} \to X_{i(e)} \,|\, e \in E\}$ be a cofinitely regular CGDMS satisfying SOSC and BSC, and having an underlying finitely primitive edge incidence matrix A. Let θ be the finiteness parameter of S. Suppose that $h_{\tilde{\mu}_q}(\sigma)/\chi_{\tilde{\mu}}(\sigma) > \theta$. Then the following statements hold.*

(a) *The number $d_{\mu_F}(x)$ exists for μ_F-a.e. $x \in J$ and*

$$d_{\mu_F}(x) = \frac{\int \tilde{f} \, d\tilde{\mu}_{\tilde{f}}}{\int \tilde{\zeta} \, d\tilde{\mu}_{\tilde{f}}}.$$

(b) *The function $T : I\!\!R \to I\!\!R$ is real-analytic, $T(0) = \text{HD}(J)$, and $T'(q) < 0$, $T''(q) \geq 0$ for all $q \in I\!\!R$.*

(c) *For every $q \in I\!\!R$, we have $f_{\mu_F}(-T'(q)) = f_{\mu_F}(\alpha(q)) = q\alpha(q) + T(q) = T(q) - qT'(q)$. That is, $f_{\mu_F}(\alpha)$ and $T(q)$ form a Legendre pair of functions.*

(d) *If $\tilde{\mu}_{\tilde{f}} \neq \tilde{\mu}_{\text{HD}(J)\tilde{\zeta}}$ or, equivalently, if $\tilde{f}$ and $\text{HD}(J)\tilde{\zeta}$ are not cohomologous modulo any constant, then the function $\alpha \mapsto f_{\mu_F}(\alpha)$, $\alpha \in (\alpha_1, \alpha_2)$ is real-analytic, where the interval (α_1, α_2), $0 \leq \alpha_1 \leq \alpha_2 \leq \infty$, is the range of $-T'(q)$. Otherwise, $T'(q) = \text{HD}(J)$ for every $q \in I\!\!R$.*

An important family of CGDMSs (in fact, conformal IFSs (CIFSs)) which satisfies the boundary separation condition are real continued fractions with the digit 1 deleted.

Example 5.2. *Let $-1/4 \leq \varepsilon < 0$. Set $X = [-\varepsilon, 3/4]$. Let $S = \{\varphi_n : X \to X \,|\, n \in I\!\!N \backslash \{1\}\}$, where*

$$\varphi_n(x) = \frac{1}{n + x}.$$

Then S is a cofinitely regular CIFS which satisfies both the strong open set condition and the boundary separation condition.

5.2. **Multifractal analysis over J under other conditions.** We shall now prove that $f_{m_F}(\alpha)$ and $T(q)$ form a Legendre transform pair under some conditions. Recall that $T'(q) = -\alpha(q)$ by Theorem 3.3.

Theorem 5.3. *Suppose there exists a countable set $J_0 \subset J$ such that for every $x \in J \backslash J_0$ there are $\omega \in E_A^\infty$ with $\pi(\omega) = x$, a constant $C = C(x, \omega) > 0$, an increasing subsequence $\{n_j\}_{j \in \mathbb{N}} = \{n_j(x, \omega)\}_{j \in \mathbb{N}}$ of natural numbers and a sequence $\{r_{n_j}\}_{j \in \mathbb{N}} = \{r_{n_j}(x, \omega)\}_{j \in \mathbb{N}}$ of positive real numbers such that*

(1) $m_F(B(x, r_{n_j})) \leq C \exp(S_{n_j} \tilde{f}(\omega))$ *for all $j \in \mathbb{N}$;*
(2) $1 > r_{n_j} \geq \tilde{C} \|\varphi'_{\omega|_{n_j}}\|$ *for all $j \in \mathbb{N}$;*
(3) $\lim_{j \to \infty} r_{n_j} = 0$.

Then $f_{m_F}(\alpha(q)) = q\alpha(q) + T(q)$ for all $q \in \mathbb{R}$.

Proof. Clearly, $f_{m_F}(\alpha(q)) \geq f_{r,m_F}(\alpha(q)) = q\alpha(q) + T(q)$ using Theorem 4.6. We shall now prove the other inequality. Since J_0 is countable, it is sufficient to show that $d_{m_q}(x) \leq q\alpha(q) + T(q)$ for every $x \in J_{m_F}(\alpha(q)) \backslash J_0$. Accordingly, fix $x \in J_{m_F}(\alpha(q)) \backslash J_0$. Let ω, C, $\{n_j\}_{j \in \mathbb{N}}$ and $\{r_{n_j}\}_{j \in \mathbb{N}}$ be as above. For every $n \in \mathbb{N}$ we have

$$\varphi_{\omega|_n}\big(B(\pi(\sigma^n\omega), R)\big) \subset B\big(\pi(\omega), KR|\varphi'_{\omega|_n}(\pi(\sigma^n\omega))|\big).$$

Like in (4.4), the conformality of m_q and the bounded variation principle for $\tilde{f}_q$ give

$$m_q\big(B(x, KR|\varphi'_{\omega|_n}(\pi(\sigma^n\omega))|)\big) \geq M_{m_q}(R)B(\tilde{f}_q)\exp\big(S_n\tilde{f}_q(\omega)\big)$$

where $B(\tilde{f}_q)$ is a constant of bounded variation (see Lemma 2.3.1 in [9]). Hence, as in (4.5),

$$\frac{\log m_q\big(B(x, KR|\varphi'_{\omega|_n}(\pi(\sigma^n\omega))|)\big)}{\log(KR|\varphi'_{\omega|_n}(\pi(\sigma^n\omega))|)} \leq \frac{\frac{\log(M_{m_q}(R)B(\tilde{f}_q))}{S_n\tilde{\zeta}(\omega)} + q\frac{S_n\tilde{f}(\omega)}{S_n\tilde{\zeta}(\omega)} + T(q)}{\frac{\log(KR)}{S_n\tilde{\zeta}(\omega)} + 1}. \tag{5.1}$$

So far the estimates are valid for all $n \in \mathbb{N}$. Now, by assumptions (1) and (2) we obtain for all $j \in \mathbb{N}$

$$\begin{aligned}
\frac{\log m_F(B(x, r_{n_j}))}{\log r_{n_j}} &\geq \frac{\log(C) + S_{n_j}\tilde{f}(\omega)}{\log r_{n_j}} \\
&\geq \frac{\log(C) + S_{n_j}\tilde{f}(\omega)}{\log(C) + \log\|\varphi'_{\omega|_{n_j}}\|} \\
&\geq \frac{\log(C) + S_{n_j}\tilde{f}(\omega)}{\log(C) + \log|\varphi'_{\omega|_{n_j}}(\omega)|} \\
&= \frac{\log(C) + S_{n_j}\tilde{f}(\omega)}{\log(C) + S_{n_j}\tilde{\zeta}(\omega)}.
\end{aligned} \tag{5.2}$$

Using (5.1) and (5.2), we deduce that

$$
\begin{aligned}
\underline{d}_{m_q}(x) = \liminf_{r \to 0} \frac{\log m_q(B(x,r))}{\log r} &\leq \limsup_{j \to \infty} \frac{\log m_q\Big(B(x, KR|\varphi'_{\omega|n_j}(\pi(\sigma^{n_j}\omega))|)\Big)}{\log(KR|\varphi'_{\omega|n_j}(\pi(\sigma^{n_j}\omega))|)} \\
&\leq \limsup_{j \to \infty}\Big(q\frac{S_{n_j}\tilde{f}(\omega)}{S_{n_j}\tilde{\zeta}(\omega)} + T(q)\Big) \\
&= q\limsup_{j \to \infty} \frac{\log(C) + S_{n_j}\tilde{f}(\omega)}{\log(C) + S_{n_j}\tilde{\zeta}(\omega)} + T(q) \\
&\leq q\limsup_{j \to \infty} \frac{\log m_F(B(x, r_{n_j}))}{\log r_{n_j}} + T(q) \\
&= q\alpha(q) + T(q).
\end{aligned}
$$

As $\underline{d}_{m_q}(x) \leq q\alpha(q) + T(q)$ for every $x \in J_{m_F}(\alpha(q)) \setminus J_0$ and J_0 is countable, we deduce that $f_{m_F}(\alpha(q)) \leq q\alpha(q) + T(q)$. ∎

In the framework of Theorem 5.3, Theorem 4.7 reduces to the following.

Theorem 5.4. *Let $S = \{\varphi_e : X_{t(e)} \to X_{i(e)} \,|\, e \in E\}$ be a cofinitely regular CGDMS satisfying SOSC and BSC, and having an underlying finitely primitive edge incidence matrix A. Let θ be the finiteness parameter of S. Suppose that the conditions of Theorem 5.3 are fulfilled. Suppose also that $h_{\tilde{\mu}_q}(\sigma)/\chi_{\tilde{\mu}}(\sigma) > \theta$. Then the following statements hold.*

(a) *The number $d_{\mu_F}(x)$ exists for μ_F-a.e. $x \in J_r$ and*

$$
d_{\mu_F}(x) = \frac{\int \tilde{f}\, d_{\tilde{\mu}_{\tilde{f}}}}{\int \tilde{\zeta}\, d_{\tilde{\mu}_{\tilde{f}}}}.
$$

(b) *The function $T : \mathbb{R} \to \mathbb{R}$ is real-analytic, $T(0) = \mathrm{HD}(J)$, and $T'(q) < 0$, $T''(q) \geq 0$ for all $q \in \mathbb{R}$.*

(c) *For every $q \in \mathbb{R}$, we have $f_{\mu_F}(-T'(q)) = f_{\mu_F}(\alpha(q)) = q\alpha(q) + T(q) = T(q) - qT'(q)$. That is, $f_{\mu_F}(\alpha)$ and $T(q)$ form a Legendre pair of functions.*

(d) *If $\tilde{\mu}_{\tilde{f}} \neq \tilde{\mu}_{\mathrm{HD}(J)\tilde{\zeta}}$ or, equivalently, if $\tilde{f}$ and $\mathrm{HD}(J)\tilde{\zeta}$ are not cohomologous modulo any constant, then the function $\alpha \mapsto f_{\mu_F}(\alpha)$, $\alpha \in (\alpha_1, \alpha_2)$ is real-analytic, where the interval (α_1, α_2), $0 \leq \alpha_1 \leq \alpha_2 \leq \infty$, is the range of $-T'(q)$. Otherwise, $T'(q) = \mathrm{HD}(J)$ for every $q \in \mathbb{R}$.*

There are interesting families of CGDMSs (in fact, even of CIFSs) which satisfy the conditions imposed in Theorem 5.3. Among others, let us mention real continued fractions over the interval $[0,1]$.

Example 5.5. *Let $X = [0, 1]$. Let $S = \{\varphi_n : X \to X \mid n \in I\!N\}$, where*

$$\varphi_n(x) = \frac{1}{n + x}.$$

Then S is a cofinitely regular CIFS which satisfies the SOSC and conditions (1)–(3) of Theorem 5.3 (but not the boundary separation condition). Moreover, writing

$$x = \frac{1}{x_1 + \frac{1}{x_2 + \dots}}$$

we note that $J_r = \{x \in [0, 1] : \liminf_{n \to \infty} x_n < \infty\}$ and thus $J_r^c = \{x \in [0, 1] : \lim_{n \to \infty} x_n = \infty\}$.

In fact, there is a larger class of one-dimensional CIFSs for which conditions (1)–(3) of Theorem 5.3 are fulfilled. By one-dimensional, we simply mean that X is a subinterval of $I\!R$.

Theorem 5.6. *Let $S = \{\varphi_n : X \to X \mid n \in I\!N\}$ be a one-dimensional CIFS satisfying the following properties.*

(i) *$\|\varphi_n'\|$ is comparable to $\|\varphi_{n+1}'\|$, that is, there exists a constant $C \geq 1$ such that*

$$C^{-1} \leq \frac{\|\varphi_{n+1}'\|}{\|\varphi_n'\|} \leq C, \qquad \forall \, n \in I\!N.$$

(ii) *The generators φ_n, $n \in I\!N$, of S either all preserve orientation (i.e. $\varphi_n' > 0$ for all $n \in I\!N$) or all reverse orientation (i.e. $\varphi_n' < 0$ for all $n \in I\!N$).*

(iii) *Either $\varphi_{n+1} > \varphi_n$ for all $n \in I\!N$ or $\varphi_{n+1} < \varphi_n$ for all $n \in I\!N$.*

Then all three conditions in Theorem 5.3 are met.

Proof. It follows from the bounded distortion property and (i) that

$$(CK)^{-1} \leq \frac{|\varphi_{n+1}(X)|}{|\varphi_n(X)|} \leq CK, \qquad \forall \, n \in I\!N. \tag{5.3}$$

Moreover, a rather straightforward calculation shows that the amalgamated function $\tilde{\zeta}$ satisfies

$$\sup_{\omega, \tau \in E_A^\infty : |\omega_1 - \tau_1| = 1} |\tilde{\zeta}(\omega) - \tilde{\zeta}(\tau)| \leq \log(CK) := D.$$

Now, let

$$J_0 = \left\{ \pi(\tau) \,\middle|\, \sigma^k \tau = 1^\infty \text{ for some } k \in I\!N \right\}.$$

Clearly, J_0 is countable. Let $x \in J \backslash J_0$ and $\omega \in I\!N^\infty$ such that $\pi(\omega) = x$. Let n_j be the j-th letter in the word ω which is not a 1. Thus, $\omega_{n_j} \neq 1$. Let $r_{n_j} = |\varphi_{\omega_{n_j}}(X)|/(CK)$. Therefore $r_{n_j} \leq \min\{|\varphi_{\omega_{n_j}-1}(X)|, |\varphi_{\omega_{n_j}}(X)|, |\varphi_{\omega_{n_j}+1}(X)|\}$ by (5.3) and hence

$$B(x, r_{n_j}) \cap J \subset \bigcup_{k=\omega_{n_j}-1}^{\omega_{n_j}+1} \varphi_{\omega|_{n_j-1}k}(X) \cap J.$$

Then, since $P(F) = 0$,

$$
\begin{aligned}
m_F\big(B(x, r_{n_j})\big) &\leq \sum_{k=\omega_{n_j}-1}^{\omega_{n_j}+1} m_F\big(\varphi_{\omega|_{n_j}-1 k}(X)\big) \\
&\leq \sum_{k=\omega_{n_j}-1}^{\omega_{n_j}+1} \int_J \exp\big(S_{\omega|_{n_j}-1 k} F(y) - n_j P(F)\big) dm_F \\
&\leq \sum_{k=\omega_{n_j}-1}^{\omega_{n_j}+1} \exp\big(\sup_{y\in J} S_{\omega|_{n_j}-1 k} F(y)\big) \\
&= \sum_{k=\omega_{n_j}-1}^{\omega_{n_j}+1} \exp\big(\sup_{\tau\in[\omega|_{n_j}-1 k]} S_{n_j} \tilde{f}(\tau)\big) \\
&\leq B(\tilde{f}) \sum_{k=\omega_{n_j}-1}^{\omega_{n_j}+1} \exp\big(S_{n_j} \tilde{f}(\omega|_{n_j-1} k \omega|_{n_j+1}^\infty)\big) \\
&= B(\tilde{f}) \sum_{k=\omega_{n_j}-1}^{\omega_{n_j}+1} \exp\big(S_{n_j-1} \tilde{f}(\omega|_{n_j-1} k \omega|_{n_j+1}^\infty) + \tilde{f}(k \omega|_{n_j+1}^\infty)\big) \\
&= B(\tilde{f}) \sum_{k=\omega_{n_j}-1}^{\omega_{n_j}+1} \exp\big(S_{n_j-1} \tilde{f}(\omega|_{n_j-1} k \omega|_{n_j+1}^\infty) - S_{n_j-1}\tilde{f}(\omega)\big) \exp\big(S_{n_j-1}\tilde{f}(\omega)\big) \\
&\qquad\qquad\qquad\qquad \cdot \exp\big(\tilde{f}(k\omega|_{n_j+1}^\infty) - \tilde{f}(\omega|_{n_j}^\infty)\big) \exp\big(\tilde{f}(\omega|_{n_j}^\infty)\big) \\
&\leq B(\tilde{f}) \sum_{k=\omega_{n_j}-1}^{\omega_{n_j}+1} B(\tilde{f}) \exp\big(S_{n_j-1}\tilde{f}(\omega)\big) \exp(|k - \omega_{n_j}|D) \exp(\tilde{f}(\omega|_{n_j}^\infty)) \\
&\leq 3e^D (B(\tilde{f}))^2 \exp\big(S_{n_j}\tilde{f}(\omega)\big).
\end{aligned}
$$

Thus, condition (1) in Theorem 5.3 is satisfied. Note also that condition (2) is fulfilled since

$$
r_{n_j} = \frac{1}{CK} |\varphi_{\omega_{n_j}}(X)| \geq \frac{1}{CK^2} \|\varphi'_{\omega_{n_j}}\|.
$$

Finally, condition (3) is obviously satisfied, as $\sum_{j\in\mathbb{N}} |\varphi_{\omega_{n_j}}(X)| \leq |X| < \infty$ and hence $r_{n_j} \to 0$.
∎

References

[1] R. Cawley and R.D. Mauldin, *Multifractal decompositions of Moran fractals*, Adv. Math. **92** (1992), 196–236.

[2] U. Frisch and G. Parisi, *On the singularity structure of fully developed turbulence.* appendix ato U. Frisch, Fully developed turbulence and intermittency, in: *Turbulence and predictability in geophysical*

fluid dynamics and climate dynamics (Proc. Int. Sch. Phys., "Enrico Fermi", Course LXXXVIII), North-Holland (Amsterdam, 1985), 84–88.

[3] P. Grassberger and I. Procaccia, *Characterization of strange attractors*, Phys. Lett. **50** (1983), 346–349.

[4] T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia and B.J. Shraiman, *Fractal measures and their singularities: The characterization of strange sets*, Phys. Rev. **A33** (1986), 1141–1151.

[5] P. Hanus, R.D. Mauldin and M. Urbański, *Thermodynamic formalism and multifractal analysis of conformal infinite iterated function systems*, Acta Math. Hungarica **96** (2002), 27–98.

[6] B.B. Mandelbrot, *Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier*, J. Fluid Mech. **62** (1974), 331–358.

[7] R. D. Mauldin and M. Urbański, *Dimensions and measures in infinite iterated function systems*, Proc. London Math. Soc. **73** (3) (1996), no. 1, 105–154.

[8] R. D. Mauldin, M. Urbański, *Conformal iterated function systems with applications to the geometry of continued fractions*, Trans. Amer. Math. Soc. **351** (1999), 4995–5025.

[9] R. D. Mauldin, M. Urbański, *Graph Directed Markov Systems: Geometry and Dynamics of Limit Sets*, Cambridge 2003.

[10] L. Olsen, *A multifractal formalism*, Adv. Math **116** (1995), 82–196.

[11] L. Olsen, *Multifractal geometry*, Proceedings Fractal Geometry and Stochastics II, Greifswald, Germany Aug. 1998. Progress in Probability, Vol. **46** (Editors C. Bandt, S. Graf, and M. Zähle), 1–37. Birkhäuser, 2000.

[12] N. Patzschke, *Self-conformal multifractal measures*, Adv. Appl. Math. **19** (1997), 486–513.

[13] Y.B. Pesin, *Dimension Theory in Dynamical Systems* (Chicago, 1997).

[14] Y.B. Pesin and H. Weiss, *The multifractal analysis of equilibrium measures for conformal expanding maps and Markov Moran geometric constructions*, J. Sta. Phys. **86** (1997), 233–275.

[15] D. Rand, *The singularity spectrum for cookie-cutters*, Erg. Th. Dyn. Sys. **9** (1989), 155–186.

[16] R. Rockafellar, *Convex Analysis*, Princeton University Press (1970).

MARIO ROY, GLENDON COLLEGE, YORK UNIVERSITY, 2275 BAYVIEW AVENUE, TORONTO, CANADA, M4N 3M6;
mroy@gl.yorku.ca

MARIUSZ URBAŃSKI, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, DENTON, TX 76203-1430, USA;
urbanski@unt.edu
Webpage: www.math.unt.edu/∼urbanski