ERGODIC THEORY OF PARABOLIC HORSESHOES

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Abstract. In this paper we develop the ergodic theory for a horseshoe map f which is uniformly hyperbolic, except at one parabolic fixed point ω and possibly also on $W^s(\omega)$. We call f a parabolic horseshoe map. In order to analyze dynamical and geometric properties of such horseshoes, by making use of induced maps, we establish, in the context of σ-finite measures, an appropriate version of the variational principle for continuous potentials on subshifts of finite type. Staying in this setting, we propose a concept of σ-finite equilibrium states (each "old" probability equilibrium state is a σ-finite equilibrium state). We then study the unstable pressure function $t \mapsto P(-t \log |Df|E^u|)$, the corresponding finite and σ-finite equilibrium states and their associated conditional measures. The main idea is to relate the pressure function to the pressure of an embedded parabolic iterated function system and to apply the developed theory of the symbolic σ-finite thermodynamic formalism. We prove, in particular, an appropriate form of the Bowen-Ruelle-Manning-McCluskey formula, the existence of exactly two σ-finite ergodic conservative equilibrium states for the potential $-t^u \log |Df|E^u|$ (where t^u denotes the unstable dimension), one of which is the Dirac δ-measure supported at the parabolic fixed point and the other being atomless. We also show that the conditional measures of this atomless equilibrium state on unstable manifolds, are equivalent to (finite and positive) packing measures, whereas the Hausdorff measures vanish. As an application of our results we obtain a classification for the existence of a generalized physical measure, as well as, a criteria implying the non-existence of an ergodic measure of maximal dimension.

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1. Introduction

1.1. **Motivation.**

Even mild parabolic features in one dimensional non-invertible dynamics have a profound impact on the dynamical and geometric character of the reference dynamical systems. Our goal in this paper is to understand whether in the case of higher dimensional systems the presence of, even weakly, parabolic points deeply affects the dynamics and, especially the geometry of the corresponding invariant set. We test this issue on one of the simplest, at least in our opinion, examples of parabolic higher dimensional systems. Namely, we introduce the concept of a parabolic horseshoe, and investigate it in detail throughout the paper. Even though parabolic horseshoes can be derived by slightly perturbing a hyperbolic horseshoe at one of its fixed points (see Example 1 in Section 5), the phenomena we discover differentiate promptly our parabolic system from hyperbolic ones. The main results are stated below in Subsection 1.2. In order to perform our analysis of a parabolic horseshoe, we develop an appropriate form of thermodynamic formalism of σ-finite measures, we borrow from the theory of parabolic iterated function systems and develop the "parabolic" approach to study generalized physical measures and measures of maximal dimension, existing up to our knowledge, so far only in hyperbolic contexts.

1.2. **Statement of the main results.**

Let $f : S \to \mathbb{R}^2$ be a parabolic horseshoe map of smooth type and let $\omega \in S$ be its parabolic fixed point (see Section 5 for the definition and details). We call the set $\Lambda = \{x \in S : f^n(x) \in S \text{ for all } n \in \mathbb{Z}\}$ the parabolic horseshoe of f. Consider the potential $\phi_u : \Lambda \to \mathbb{R}$ defined by $\phi_u(x) = \log |Df(x)|E^u_x|$. We define the unstable pressure function by $P^u(t) = P(f|\Lambda, -t\phi_u) : \mathbb{R} \to \mathbb{R}$, where $P(f|\Lambda, .)$ denotes the topological pressure with respect to the dynamical system $f|\Lambda$. Our first result is a Bowen-Ruelle-Manning-McCluskey type of formula for the unstable dimension of Λ. It compiles results from Theorems 7.3 and 7.4.

Theorem 1.1. *Let $f : S \to \mathbb{R}^2$ be a parabolic horseshoe map of smooth type. Then the unstable dimension $t^u = \dim_H W^u(x) \cap \Lambda$ is independent of $x \in \Lambda$. Moreover, t^u is the smallest zero of the unstable pressure function $t \mapsto P^u(t)$ and $0 < t^u < 1$.*

Let $H_t(A)$ and $P_t(A)$ denote the t-dimensional Hausdorff respectively packing measure of a set A. In the case of hyperbolic horseshoes and more generally for uniformly hyperbolic sets on surfaces it is well-known that $W^u_{\text{loc}}(x) \cap \Lambda$ has positive and finite t^u-dimensional Hausdorff measure. The next result (see Theorem 7.5 in the text) shows that alone the occurrence of one parabolic fixed point can cause a drastic change on this phenomenon.

Theorem 1.2. *Let $f : S \to \mathbb{R}^2$ be a parabolic horseshoe map of smooth type and let $x \in \Lambda$. Then $H_{t^u}(W^u(x) \cap \Lambda) = 0$ and $0 < P_{t^u}(W^u(x) \cap \Lambda) < \infty$.*

Next, we discuss results concerning the equilibrium states of the potential $-t^u\phi_u$. In particular, we consider finite as well as σ-finite equilibrium states. Let μ_ω denote the Dirac δ-measure supported on the parabolic fixed point ω which is clearly an equilibrium state of the potential $-t^u\phi_u$. We show in Theorem 8.3 that there exists a unique (up to a multiplicative constant) ergodic conservative σ-finite equilibrium state μ_{t^u} of the potential $-t^u\phi_u$ which is distinct from μ_ω. It turns out that the question whether μ_{t^u} is finite is closely related to the behavior of f near ω. Let β be defined as in equation (5.1). Roughly speaking, the exponent β determines the rate at which orbits starting close to ω escape from ω. The following theorem compiles results from Theorems 8.1 and 8.3 in the text.

Theorem 1.3. *Let $f : S \to \mathbb{R}^2$ be a parabolic horseshoe map of smooth type. Then the following are equivalent:*

(i) *μ_{t^u} is finite, in which case P^u is not differentiable at t^u;*
(ii) *$t^u > 2\beta/(\beta+1)$.*

Since $t^u < 1$, the measure μ_{t^u} being finite implies that $\beta < 1$. On the other hand, by equation (5.1), $1+\beta$ is an upper bound for the maximal possible regularity of f, i.e., f is at most of class $C^{1+\beta}$. Therefore, if f is a C^2-diffeomorphism then μ_{t^u} is always an infinite measure. In order to reasonably speak about σ-finite equilibrium states, we develop in Section 3 an appropriate form of a thermodynamic formalism: variational principle (Theorem 3.2) and equilibrium states (Definition 3.3) for σ-finite measures on subshifts of finite type (keep in mind that our parabolic horseshoe is topologically conjugate to the full shift on two elements).

We now discuss two applications of our results concerning the existence of certain natural invariant measures of f. Recall that an ergodic invariant probability measure μ is called a generalized physical measure if its basin $\mathcal{B}(\mu)$ has the same Hausdorff dimension as the stable set of Λ (see Section 11 and [Wo] for more details). Applying Theorem 1.3 we are able to prove that the finiteness of the equilibrium state μ_{t^u} is equivalent to the existence of a generalized physical measure (see Theorem 11.2). In particular, there are parabolic horseshoes having no generalized physical measure (see Corollary 11.3). This contrasts the case of hyperbolic surface diffeomorphisms which always have a unique generalized physical measure (see [Wo]).

Another application concerns the existence of ergodic measures of maximal dimension. Given an invariant probability measure μ, we denote by $\dim_H \mu$ the Hausdorff dimension of μ (see (12.1) for the definition). Assume now that μ is ergodic. Following [BW1] we say that μ is an ergodic measure of maximal dimension if

$$\dim_H \mu = \sup_\nu \dim_H \nu,$$

where the supremum is taken over all ergodic invariant probability measures ν. These measures have recently been intensively studied in the context of hyperbolic diffeomorphism in [BW1]. It turns out that for hyperbolic sets on surfaces there always exists an ergodic measure of maximal dimension. Moreover, this measure is, in general, not unique (see the example in [Ra]). In contrast to these results, we show in Theorem 12.4 that for certain parabolic horseshoe maps there exists no ergodic measure of maximal dimension.

2. Symbol space and the shift map

In this section we recall some notions from symbolic dynamics. We will discuss simultaneously one-sided and two-sided shift maps. We denote by $\mathbb{Z}$ be the set of all integers and by $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ be the set of all non-negative integers. Given a countable, either finite or infinite set E and a function $A : E \times E \to \{0, 1\}$, called and incidence matrix, we define

$$E_A^+ = \left\{(\omega_n)_0^{+\infty} : A_{\omega_n\omega_{n+1}} = 1 \ \text{ for all } n \in \mathbb{N}\right\} \quad \text{and} \quad E_A^{+-} = \left\{(\omega_n)_{-\infty}^{+\infty} : A_{\omega_n\omega_{n+1}} = 1 \text{ for all } n \in \mathbb{Z}\right\}.$$

We refer to either of these sets as a shift or a symbol space. In the case when we do not want to specify the shift space or also when it is clear from the context which shift space is meant, we write E_A instead of E_A^+ or E_A^{+-}. Given any $s \in (0, 1)$, the space E_A can be endowed with the metric $\rho = \rho_s$ defined by

$$\rho\big((\omega_n)_n, (\tau_n)_n\big) = s^{\min\{n \geq 0\colon\, \omega_n \neq \tau_n \text{ or } \omega_{-n} \neq \tau_{-n}\}}.$$

Here we use the common convention that $s^{+\infty} = 0$. All the metrics ρ_s, $s \in (0, 1)$ are Hölder continuously equivalent and induce (the same) Tychonoff topology on E_A. It is well-known that E_A endowed with the Tychonoff topology is compact in the case when E is a finite set. The (left) shift map $\sigma : E_A \to E_A$ is defined by the formula

$$\sigma\big((x_n)_n\big) = (x_{n+1})_n.$$

Note that in the case of E_A^{+-}, the shift map is injective and in the case of E_A^+, the shift map performs cutting off the zero-th coordinate. Let $m \leq n$ and let $\omega = (\omega_m, \omega_{m+1}, \cdots, \omega_n) \in E^{n-m+1}$. We call

$$[\omega] = \{\tau \in E_A : \tau_j = \omega_j \text{ for all } m \leq j \leq n\},$$

the cylinder generated by ω. If $\omega \in E_A$ and $m \leq n$, we define

$$\omega|_m^n = (\omega_m, \omega_{m+1}, \cdots, \omega_n),$$

and, if $m = 0$, we frequently write $\omega|^n$ instead of $\omega|_0^n$. For every $n \geq 1$ a n-tuple τ of elements of in E is said to be A-admissible provided that $A_{ab} = 1$ for all pairs of consecutive elements ab in τ. The number n is then called the length of τ and is denoted by $|\tau|$. We denote by E_A^n the set of A-admissible tuples of length n. We also put $E_A^* = \bigcup_{n=1}^\infty E_A^n$. Denote by $\Pi : E_A^{+-} \to E_A^+$ the projection from E_A^{+-} to E_A^+ defined by

$$\Pi\big((\omega_n)_{-\infty}^{+\infty}\big) = (\omega_n)_0^{+\infty}.$$

If $D \subset E$ then we write D_A for $D_{A|_{D\times D}}$. Let $g : E_A \to \mathbb{R}$ be a function. Given an integer $n \in \mathbb{N}$ we define the n-th partition function $Z_n(D, g)$ by

$$Z_n(D, g) = \sum_{\omega \in D_A^n} \exp\left(\sup_{\tau \in [\omega] \cap D_A} \big(S_n g(\tau)\big)\right),$$

where

$$S_n g := \sum_{j=0}^{n-1} g \circ \sigma^j$$

In the case when $D = E$, then we simply write $Z_n(g)$ instead of $Z_n(E, g)$. A straightforward argument shows that the sequence $(\log Z_n(D, f))_{n\in\mathbb{N}}$ is subadditive. Therefore, we can define the topological pressure of g with respect to the shift map $\sigma : D_A \to D_A$ by

$$P_D(g) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(D,g) = \inf\left\{\frac{1}{n}\log Z_n(D,g) : n \in \mathbb{N}\right\}. \tag{2.1}$$

If $D = E$, then we simply write $P(g)$ instead of $P_E(g)$. Recall from [SU] that a function $g : E_A \to \mathbb{R}$ is said to be finitely acceptable if it is uniformly continuous and

$$\sup\big(g|_{[e]}\big) - \inf\big(g|_{[e]}\big) < \infty$$

for all $e \in E$. The following fact, which will be needed in the next section, was proven in [SU].

Theorem 2.1. *If $g : E_A \to \mathbb{R}$ is finitely acceptable, then*

$$P(g) = \sup\{P_D(g)\},$$

where the supremum is taken over all finite subsets D of E.

In the case when $E = \{0, 1, 2, \dots, d-1\}$, where $d \geq 2$, and the incidence matrix A consists of 1s only, we rather use the notation

$$\Sigma_d^+ := \{0,1,2,\dots,d-1\}_A^+ = \{0,1,2,\dots,d-1\}^{\mathbb{N}}$$

and

$$\Sigma_d^{+-} := \{0,1,2,\dots,d-1\}_A^{+-} = \{0,1,2,\dots,d-1\}^{\mathbb{Z}}.$$

Similarly as above, in the case when we do not want to specify the shift space or also when is clear from the context which shift space is meant, we write Σ_d instead of Σ_d^+ or Σ_d^{+-}.

3. Variational principle and σ-finite equilibrium states

Let $(X, \mathcal{A}, \mu)$ be a measure space, where μ is a σ-finite measure. Moreover, let $T : X \to X$ be a measurable map which is μ-invariant (that is, $\mu \circ T^{-1} = \mu$), ergodic and conservative (see [A] for the definitions). Consider a fixed set $F \in \mathcal{A}$ with $\mu(F) > 0$. Then the first return time $\tau := \tau_F : F \to \{1, 2, \dots\} \cup \{\infty\}$ given by the formula

$$\tau(x) = \min\{k \geq 1 : T^k(x) \in F\}$$

is finite in the complement (in F) of a set of measure zero. Therefore, the first return map $T_F : F \to F$ given by the formula

$$T_F(x) = T^{\tau_F(x)}(x)$$

is well-defined μ-a.e. on F. If $\phi : X \to \mathbb{R}$ is a measurable function, we set

$$\phi_F(x) = \sum_{j=0}^{\tau(x)-1} \phi \circ T^j(x),\ x \in F.$$

The function ϕ_F is $\mathcal{A}_F$-measurable, where $\mathcal{A}_F$ is the σ-algebra of all subsets of F that belong to $\mathcal{A}$. If in addition $\mu(F) < \infty$, then it is well-known (see [A] for example) that the conditional measure

μ_F on F defined by $\mu_F(B) = \mu(B)/\mu(F)$, $B \in \mathcal{A}_F$, is T_F-invariant. Also if ϕ is μ-integrable, that is $\int |\phi| d\mu < +\infty$, then ϕ_F is μ_F-integrable and

$$\int_F \phi_F d\mu_F = \frac{\int \phi d\mu}{\mu(F)}. \tag{3.1}$$

We now shall provide a short proof of the following, essentially immediate, consequence of Abramov's formula.

Theorem 3.1. *Let μ be a σ-finite measure on X, and let $T : X \to X$ be an ergodic, conservative and μ-invariant map. If E and F are measurable sets with $0 < \mu(E), \mu(F) < \infty$, then*

$$\mu(E) h_{\mu_E}(T_E) = \mu(F) h_{\mu_F}(T_F).$$

Proof. Put $D = E \cup F$. Obviously $0 < \mu(D) < +\infty$. Let T_E^D and T_F^D be the first return maps respectively to E and F induced by the map $T_D : D \to D$. It follows from Abramov's formula that

$$\mu_D(E) \mathrm{h}_{(\mu_D)_E}(T_E^D) = \mathrm{h}_{\mu_D}(T_D) = \mu_D(F) \mathrm{h}_{(\mu_D)_F}(T_F^D). \tag{3.2}$$

Since $\mu_D(E) = \mu(E)/\mu(D)$, $\mu_D(F) = \mu(F)/\mu(D)$, $T_E = T_E^D$, $T_F = T_F^D$ (as $E, F \subset D$), and $(\mu_D)_E = \mu_E$, $(\mu_D)_F = \mu_F$, we may conclude that $\mu(E)\mathrm{h}_{\mu_E}(T_E) = \mu(F)\mathrm{h}_{\mu_F}(T_F)$. $\square$

We denote this common value $\mu(F)\mathrm{h}_{\mu_F}(T_F)$ by $\mathrm{h}_\mu(T)$ and call it the entropy of the transformation T with respect to the invariant measure μ. Assume now that $T : X \to X$ is a continuous self-map of a compact metric space X and $\phi : X \to \mathbb{R}$ is a continuous function. Then it is a consequence of the well-known variational principle that

$$\sup \left\{ \mathrm{h}_\mu(T) + \int (\phi - P(\phi)) d\mu \right\} = 0,$$

where $P(\phi)$ denotes the topological pressure of the potential ϕ, and the supremum is taken over all ergodic T-invariant Borel probability measures on X. It therefore follows from (3.1) and Theorem 3.1 that

$$\sup \left\{ \mathrm{h}_{\mu_F}(T_F) + \int_F (\phi - P(\phi))_F d\mu_F \right\} = 0, \tag{3.3}$$

where the supremum is taken over all ergodic T-invariant Borel probability measures on X and all Borel sets $F \subset X$ with $\mu(F) > 0$. Denote by $\mathcal{M}_\phi^\sigma$ the family of all ergodic, conservative, σ-finite and T-invariant Borel measures on X for which

$$\int |\phi - P(\phi)| d\mu < \infty.$$

In the context of subshifts of finite type we obtain the following stronger version of the variational principle (3.3) which now also takes σ-finite measures into account.

Theorem 3.2. *Suppose that $E \neq \emptyset$ is a finite set and let $A : E \times E \to \{0, 1\}$ be an irreducible incidence matrix. If $\phi : E_A \to \mathbb{R}$ is a continuous function, then*

$$\sup \left\{ h_{\mu_F}(\sigma_F) + \int_F (\phi - P(\phi))_F d\mu_F : \mu \in \mathcal{M}_\phi^\sigma,\ 0 < \mu(F) < \infty \right\} = 0.$$

Proof. In view of (3.3) it sufficient to demonstrate that

$$h_{\mu_F}(\sigma_F) + \int_F (\phi - P(\phi))_F d\mu_F \le 0 \tag{3.4}$$

for every $\mu \in \mathcal{M}_\phi^\sigma$ and every Borel set $F \subset E_A$ with $0 < \mu(F) < \infty$. In view of (3.1) and Theorem 3.1, given $\mu \in \mathcal{M}_\phi^\sigma$ it suffices to find at least one set F (with $\mu(F) \in (0,\infty)$) for which (3.4) holds. Since the measure μ is σ-finite (and non-zero) there exists a Borel set $Y \subset E_A$ with $\mu(Y) \in (0,\infty)$. Again, since μ is σ-finite, it is upper regular, and in consequence, there is an open set $G \subset E_A$ such that $Y \subset G$ and $\mu(G) < \infty$. Since G can be represented as a countable union of cylinder sets, there exists a cylinder $[\omega]$ with $\mu([\omega]) \in (0,\infty)$. Since the measure μ is shift-invariant, we may assume without loss of generality that $[\omega]$ is an initial cylinder, that is, $\omega\omega_0\omega_1\ldots\omega_q$ for some $q \ge 0$. Let F be the set of all those $\rho \in E_A$ that return to $[\omega]$ infinitely often under the forward iteration of the shift map σ in the one-sided case, and under both, forward and backward, iteration in the two-sided case. Obviously $\mu(F) \in (0,\infty)$. Clearly, for every $k \ge 1$ there exists a finite initial word $\rho^{(k)} = \rho_0^{(k)}\rho_1^{(k)}\ldots\rho_{u_k}^{(k)} \in E_A^\infty$ such that $\tau_F^{-1}(k) = [\rho^{(k)}]$ and all the words $\rho^{(k)}$, $k \ge 1$, are mutually incomparable. In addition, $\rho^{(k)}|_0^q = \omega$ and $\rho^{(k)}|_{u_k-q}^{u_k} = \omega$. Let now

$$E_\omega = \{\rho^{(k)}|_0^{u_k-q-1} : k \ge 1\}.$$

Notice that F coincides with the set E_ω^∞ of all infinite concatenations (from 0 to $+\infty$ in the one-sided case and from $-\infty$ to $+\infty$ in the two-sided case) of elements from E_ω. Furthermore, the map $I : F \to E_\omega^\infty$ which ascribes to each element $\rho \in F$ its unique representation as an infinite word over the alphabet E_ω, is a homeomorphism, and I establishes conjugacy between $\sigma_F : F \to F$ and the full shift map $\sigma : E_\omega^\infty \to E_\omega^\infty$. That is, the following diagram commutes,

$$\begin{tikzcd} F \arrow[r, "\sigma_F"] \arrow[d, "I"'] & F \arrow[d, "I"] \\ E_\omega^\infty \arrow[r, "\sigma"] & E_\omega^\infty \end{tikzcd}$$

or equivalently $I\circ\sigma_F = \sigma\circ I$. In what follows we may assume without loss of generality that $P(\phi) = 0$ and we may treat, via the conjugacy I the function $\psi = \phi_F$ as defined on the full shift space E_ω^∞. Since the function $\phi : E_A \to \mathbb{R}$ is continuous, the function $\psi : E_\omega^\infty \to \mathbb{R}$ is finitely acceptable. Hence, in view of Theorem 2.1,

$$P(\psi) = \sup\{P_D(\psi)\}, \tag{3.5}$$

where the supremum is taken over all finite subsets D of E_ω. For every $n \ge 1$ let D_n be the (finite) set of all those elements in E_ω whose length (when treated as elements in E_A^∞) is bounded above by n. Then $I^{-1}(D_n^\infty)$ is a compact subset of E_A^∞, the time of the first return from $I^{-1}(D_n^\infty)$ to $I^{-1}(D_n^\infty)$ of every element $x \in I^{-1}(D_n^\infty)$ is equal to $\tau_F(x) \le n$, and the first return map $\sigma_n : I^{-1}(D_n^\infty) \to I^{-1}(D_n^\infty)$ is continuous. Let ν be an arbitrary ergodic Borel probability σ_n-invariant measure on $I^{-1}(D_n^\infty)$. Then the formula

$$\hat\nu(B) = \left(\sum_{k=0}^n \nu\big(I^{-1}(D_n^\infty) \setminus \bigcup_{j=1}^k \sigma^{-j}\big(I^{-1}(D_n^\infty)\big)\right)^{-1} \sum_{k=0}^n \nu\big(I^{-1}(D_n^\infty)\sigma^{-k}(B) \setminus \bigcup_{j=1}^k \sigma^{-j}\big(I^{-1}(D_n^\infty)\big)\big)$$

defines a Borel probability σ-invariant measure on E_A such that $\hat{\nu}_{I^{-1}(D_n^\infty)} = \nu$. Now apllying the classical version of the variational principle and Abramov's formula along with (3.1), we obtain

$$\hat{\nu}\big(I^{-1}(D_n^\infty)\big)\left(\mathrm{h}_\nu(\sigma_n)+\int_{I^{-1}(D_n^\infty)}\psi\right)d\nu = \mathrm{h}_{\hat{\nu}}(\sigma)+\int_{E_A}\phi d\hat{\nu} \le P(\phi)=0.$$

Consequently, $\mathrm{h}_\nu(\sigma_n)+\int\psi d\nu \le 0$, and applying the variational principle (to the continuous map $\sigma_n : I^{-1}(D_n^\infty) \to I^{-1}(D_n^\infty))$ again, we conclude that $P(\sigma_n,\psi)\le 0$. It thus follows from (3.5) that $P(\psi)\le 0$. Since $\psi : E_\omega^\infty \to \mathbb{R}$ is continuous and $\int|\psi|d\mu < +\infty$, it now follows from Theorem 2.1.7 in [MU2] that $\mathrm{h}_{\mu_F}(\sigma_F)+\int_F\psi d\mu_F \le P(\psi)\le 0$. We are done. □

Passing to σ-finite equilibrium states, we start with the following:

Definition 3.3. *A σ-finite measure $\mu \in \mathcal{M}_\phi^\sigma$ is said to be an equilibrium state for $\phi : E_A \to \mathbb{R}$ provided that for every (or equivalently at least one) Borel set $F \subset E_A$ with $\mu(F)\in(0,\infty)$, we have*

$$h_{\mu_F}(\sigma_F)+\int_F(\phi - P(\phi))d\mu_F = 0.$$

We now prove the following:

Theorem 3.4. *Suppose that E is a finite set, $A : E\times E \to \{0,1\}$ is an irreducible incidence matrix and $\phi : E_A^+ \to \mathbb{R}$ is a continuous function. Then $\mu \in \mathcal{M}_E^\sigma(E_A^{+-})$ is an equilibrium state for $\phi\circ\Pi : E_A^{+-} \to \mathbb{R}$ if and only if $\mu\circ\Pi^{-1}$ is an equilibrium state for ϕ.*

Proof. Using shift invariance of the measure μ, we conclude that there exists a cylinder $F \subset E_A^+$ such that $0 < \mu(F) < \infty$. By a direct straightforward inspection, we verify the following three formulas:

$$(\mu\circ\Pi^{-1})_F = \mu_{\Pi^{-1}(F)}\circ\Pi^{-1}, \tag{3.6}$$

$$\tau_{\Pi^{-1}(F)} = \tau_F\circ\Pi, \tag{3.7}$$

and

$$\phi\circ\Pi_{\Pi^{-1}(F)} = \phi_F\circ\Pi. \tag{3.8}$$

Using (3.1) and (3.8), we obtain

$$\begin{aligned}\int\phi_F d(\mu\circ\Pi^{-1})_F &= \int\phi_F d\mu_{\Pi^{-1}(F)}\circ\Pi^{-1} = \int\phi_F\circ\Pi d\mu_{\Pi^{-1}(F)}\\ &= \int\phi\circ\Pi_{\Pi^{-1}(F)}d\mu_{\Pi^{-1}(F)}.\end{aligned} \tag{3.9}$$

Denote by α the partition of F induced by the first return time. Then α is an (even topological) generator of σ_F, and therefore, using (3.1), we obtain

$$\mathrm{h}_{(\mu\circ\Pi^{-1})_F}(\sigma_F) = \mathrm{h}_{\mu_{\Pi^{-1}(F)}\circ\Pi^{-1}}(\sigma_F) = \mathrm{h}_{\mu_{\Pi^{-1}(F)}\circ\Pi^{-1}}(\sigma_F,\alpha) = \mathrm{h}_{\mu_{\Pi^{-1}(F)}}\big(\sigma_{\Pi^{-1}(F)},\Pi^{-1}(\alpha)\big).$$

Since $\Pi^{-1}(\alpha)$ is a generator for $\sigma_{\Pi^{-1}(F)}$, we may conclude that

$$\mathrm{h}_{(\mu\circ\Pi^{-1})_F}(\sigma_F) = \mathrm{h}_{\mu_{\Pi^{-1}(F)}}\big(\sigma_{\Pi^{-1}(F)}\big).$$

Adding side by side this equality and (3.9), we obtain

$$\mathrm{h}_{(\mu\circ\Pi^{-1})_F}(\sigma_F) + \int \phi_F d(\mu\circ\Pi^{-1})_F = \mathrm{h}_{\mu_{\Pi^{-1}(F)}}\big(\sigma_{\Pi^{-1}(F)}\big) + \int \phi\circ\Pi_{\Pi^{-1}(F)} d\mu_{\Pi^{-1}(F)}.$$

This inequality immediately implies that $\mu \in \mathcal{M}_E^\sigma(E_A^{+-})$ is an equilibrium state for $\phi\circ\Pi : E_A^{+-} \to \mathbb{R}$ if and only if $\mu\circ\Pi^{-1}$ is an equilibrium state for ϕ. □

Since it is evident that two potentials cohomologous up to a constant (in the class of continuous functions) have the same equilibrium states, the following result is an immediate consequence of Theorem 3.4.

Theorem 3.5. *If $\phi : E_A^+ \to \mathbb{R}$ and $\psi : E_A^{+-} \to \mathbb{R}$ are two continuous functions such that $\phi\circ\Pi$ and ψ are cohomologous up to a constant in the class of continuous functions on E_A^{+-}, then $\mu \in \mathcal{M}_E^\sigma(E_A^{+-})$ is an equilibrium state for ψ if and only if $\mu\circ\Pi^{-1}$ is an an equilibrium state for ϕ.*

4. One-dimensional parabolic iterated function systems

The concept of a parabolic Cantor set was introduced in [U2]. The concept of a parabolic iterated function system was formally introduced in [MU1]. Both concepts largely overlap and the object dealt with in this and subsequent sections belongs to this overlap. We briefly summarize here the definition of a parabolic iterated function following [MU1] and partially adopting it to the much more concrete setting we will need in the sequel for our applications. Let Δ be a compact line segment. Suppose that we have at least two and at most finitely many $C^{1+\varepsilon}$ maps $\phi_i : \Delta \to \Delta$, $i \in I$, satisfying the following conditions:

(1) (Open Set Condition) $\phi_i(\mathrm{int}(\Delta)) \cap \phi_j(\mathrm{int}(\Delta)) = \emptyset$ for all $i \neq j$.
(2) $|\phi_i'(x)| < 1$ everywhere except for finitely many pairs (i, x_i), $i \in I$, for which x_i is the unique fixed point of ϕ_i and $|\phi_i'(x_i)| = 1$. Such pairs and indices i will be called parabolic and the set of parabolic indices will be denoted by Ω. All other indices will be called hyperbolic.
(3) $\forall n \geq 1$ $\forall \omega = (\omega_1, \omega_2, \cdots, \omega_n) \in I^n$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then

$$\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \ldots \phi_{\omega_n}$$

extends in a $C^{1+\varepsilon}$ manner to an open line segment V and maps V into itself.
(4) If i is a parabolic index, then $\bigcap_{n\geq 0} \phi_{i^n}(\Delta) = \{x_i\}$ and the diameters of the sets $\phi_{i^n}(\Delta)$ converge to 0.
(5) $\exists s < 1$ $\forall n \geq 1$ $\forall \omega \in I^n$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then $||\phi_\omega'|| \leq s$.
(6) (Bounded Distortion Property) $\exists K \geq 1$ $\forall n \geq 1$ $\forall \omega = (\omega_1, \cdots, \omega_n) \in I^n$ $\forall x, y \in V$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then

$$\frac{|\phi_\omega'(y)|}{|\phi_\omega'(x)|} \leq K.$$

We call such a system of maps $\mathcal{F} = \{\phi_i : i \in I\}$ a 1-dimensional subparabolic iterated function system. If $\Omega \neq \emptyset$, we call the system $\mathcal{F} = \{\phi_i : i \in I\}$ a 1-dimensional parabolic iterated function system. From now on throughout the entire section we assume the system $\mathcal{F}$ to be parabolic.
The Bounded Distortion Property (6) is not obvious in the 1-dimensional case. But there is a natural easily verifiable sufficient condition for this property to hold. Indeed, it was proved in [U1] and [U2] that condition (6) (Bounded Distortion Property) follows from all conditions (1)-(5) enlarged by the requirement that if i is a parabolic element, then the map ϕ_i has the following representation in a neighborhood of the the parabolic point x_i:

$$\phi_i(x) = x \pm a|x - x_i|^{\beta_i+1} + o(|x - x_i|^{\beta_i+1}) \tag{4.1}$$

with some $\beta_i > 0$ and with the sign "$-$" if $x - x_i \geq 0$ and the sign "$+$" if $x - x_i \leq 0$. It was also proven in [U1] and [U2] that for every parabolic element i and every $n \geq 0$,

$$|\phi'_{i^n}(x)| \asymp (n+1)^{-\frac{\beta_i+1}{\beta_i}} \quad \text{and} \quad |\phi'_{i^n}(x) - x| \asymp (n+1)^{-\frac{1}{\beta_i}}$$

outside every fixed open neighborhood of x_i. In particular

$$|\phi'_{i^n j}(x)| \asymp (n+1)^{-\frac{\beta_i+1}{\beta_i}} \quad \text{and} \quad |\phi'_{i^n j}(x) - x| \asymp (n+1)^{-\frac{1}{\beta_i}}$$

for every parabolic element i and every $j \in I \setminus \{i\}$ and all $n \geq 0$. Recall that the elements of the set $I \setminus \Omega$ are called hyperbolic (see condition 2). We extend this name to all the words appearing in conditions (5) and (6). By I^* we denote the set of all finite words with alphabet I and by I^∞ all infinite sequences with elements in I. It follows from (3) that for every hyperbolic word ω, $\phi_\omega(V) \subset V$. Note that our conditions insure that $\phi'_i(x) \neq 0$, for all i and $x \in V$. It has been proven in [MU1] that for all $\omega = (\omega_n)_{n\geq 0} \in I^\infty$ the intersection $\bigcap_{n\geq 0} \phi_{\omega_n}(\Delta)$ is a singleton. Furthermore,

$$\lim_{n\to\infty} \sup\{\operatorname{diam}(\phi_\omega(\Delta)) : \omega \in I^*, |\omega| = n)\} = 0.$$

Thus we can define the coding map $\pi : I^\infty \to \Delta$, defining $\pi(\omega)$ to be the only element of the intersection $\bigcap_{n\geq 0} \phi_{\omega_n}(\Delta)$, and this map is uniformly continuous. The limit set $J = J_{\mathcal{F}}$ of the system $\mathcal{F} = \{\phi_i\}_{i\in I}$ is defined to be $\pi(I^\infty)$. It turns out that $J_{\mathcal{F}}$ is compact and satisfies the following invariance property:

$$J = \cup_{i\in I}\phi_i(J).$$

Consider now the system $\mathcal{F}^*$ generated by I^* defined by

$$\mathcal{F}^* = \{\phi_{i^n j} : n \geq 1,\ i \in \Omega,\ i \neq j\} \cup \{\phi_k :\ k \in I \setminus \Omega\}.$$

It immediately follows from our assumptions that the following holds.

Theorem 4.1. *The system $\mathcal{F}^*$ is a hyperbolic (though with the infinite alphabet I^*) conformal iterated function system, i.e. $\mathcal{F}^*$ has no parabolic elements.*

$\mathcal{F}^*$ is called the hyperbolic iterated function system associated to the parabolic system $\mathcal{F}$. The limit set generated by the system $\mathcal{F}^*$ is denoted by J^*. A proof of the following lemma can be found in [MU1].

Lemma 4.2. *The limit sets J and J^* of the systems $\mathcal{F}$ and $\mathcal{F}^*$ respectively differ only by a countable set. More precisely, $J^* \subset J$ and $J \setminus J^*$ is countable.*

In this paper we will only be interested in the special case when $I = \{0, 1\}$ and $\phi_0(\Delta)\cap\phi_1(\Delta) = \emptyset$. Then the projection $\pi : \Sigma_2^+ \to J_S$ is a homeomorphism, there is a well-defined map $F : J_S \to J_S$, where $F(x) = \phi_i^{-1}(x)$ if $x \in \phi_i(\Delta)$, which has a $C^{1+\varepsilon}$ extension ϕ_i^{-1} to each interval $\phi_i(V)$, and the projection $\pi : \Sigma_2^+ \to J_S$ establishes a canonical conjugacy between the shift map $\sigma : \Sigma_2^+ \to \Sigma_2^+$ and the map $F : J_{\mathcal{F}} \to J_{\mathcal{F}}$. We will frequently invoke the results proven in [MU1] and [U2] concerning the dynamics of the map F and the geometry of the limit set $J_{\mathcal{F}}$ called a parabolic Cantor set in [U2].

5. Parabolic Smale's horseshoes

In this section we describe the main object of interest in this paper, a parabolic horseshoe of smooth type. Let $S \subset \mathbb{R}^2$ be a closed topological disk whose boundary is smooth except a finitely many (possibly none) points. We consider a $C^{1+\varepsilon}$-diffeomorphism $f : S \to \mathbb{R}^2$ onto its image having the following properties:

(a) The intersection $f(S) \cap S$ consists of two disjoint closed topological disks R_0 and R_1.
(b) The closure $\overline{f(S) \setminus S}$ consists of three mutually disjoint topological disks R_2, R_3, R_4.

Furthermore, it is required that there exist two 1-dimensional mutually transversal foliations W^u and W^s of $S \cup f(S)$ consisting of smooth connected leaves with the following properties:

(c) If $x \in R_i$, $i = 0, 1, 2, 3, 4$, then the sets $W^u(x) \cap R_i$ and $W^s(x) \cap R_i$ are connected.
(d) For all points $x, y \in R_i$, $i = 0, 1, 2, 3, 4$, $W^u(x) \cap W^s(y) \cap R_i$ is a singleton denoted by $[x, y]$.
(e) For every point $x \in R_i$, $i = 0, 1, 2, 3, 4$, the map $[\cdot, \cdot] : W^s(x) \cap R_i \times W^u(x) \cap R_i \to R_i$, $(y, z) \mapsto [y, z]$, is a homeomorphism.
(f) For every point $x \in R_i$, $i = 0, 1$, $f\big(W^u(x) \cap R_i\big) = W^u(f(x))$. If $f(x) \in R_j$, $j = 0, 1, 2, 3, 4$, then $f^{-1}\big(W^s(f(x)) \cap R_j\big) = W^s(x)$.

For every point $x \in S$ we denote by $E_x^{u/s}$ the tangent space of $W^{u/s}(x)$ at x. We use the notation $D_{u/s}f(x)$ for the derivative of the map $f : W^{u/s}(x)\cap S \to W^{u/s}(f(x))$; hence $D_{u/s}f(x) = Df(x)|E_x^{u/s}$ for all $x \in S$. Let ω be the fixed point of f lying in R_0. We require that the following conditions hold:

(g) There exists $0 < \gamma < 1$ such that $|D_s f(x)| \le \gamma$ for all $x \in S$.
(h) If $x \in S$ then $|D_u f(x)| \ge 1$. Moreover, if $x \in S \setminus W^s(\omega)$, then $|D_u f(x)| > 1$.
(i) $D_u f(\omega) = 1$.
(j) In the oriented arc-length parametrization of $W^u(\omega)$ starting at ω, we have

$$f^{-1}(x) = x \pm a|x|^{\beta+1} + o(|x|^{\beta+1}) \tag{5.1}$$

in a sufficiently small neighborhood of 0 with some positive constants a and β and with the sign "$-$" if $x \ge 0$ and the sign "$+$" if $x \le 0$.

We call a map $f : S \to \mathbb{R}^2$ satisfying properties (a) - (j) a parabolic horseshoe map.

Remark. We note that property (j) restricts the regularity of f, namely by (5.1) the map f is at most of class $C^{1+\beta}$. In particular, if $\beta < 1$, then f is not twice differentiable.

It is easy to see that for every $\tau \in \Sigma_2^{+-}$, the intersection

$$\bigcap_{n=-\infty}^{+\infty} f^{-n}(R_{\tau_n})$$

is a singleton, which we denote by $\Pi(\tau)$. The map $\Pi : \Sigma_2^{+-} \to S$ is a homeomorphism on its image $\Pi(\Sigma_2^{+-})$, which we denote by Λ. Hence,

$$\Lambda = \{x \in S : f^{\pm n}(x) \in S \text{ for all } n \in \mathbb{N}\}. \tag{5.2}$$

Obviously, Λ is a compact f-invariant set, i.e. $f(\Lambda) = \Lambda = f^{-1}(\Lambda)$. Moreover, the map

$$\Pi : \Sigma_2^{+-} \to \Lambda$$

establishes a topological conjugacy between the shift map $\sigma : \Sigma_2^{+-} \to \Sigma_2^{+-}$ and $f : \Lambda \to \Lambda$, i.e., the following diagram commutes:

$$\begin{tikzcd} \Sigma_2^{+-} \arrow[r, "\sigma"] \arrow[d, "\Pi"'] & \Sigma_2^{+-} \arrow[d, "\Pi"] \\ \Lambda \arrow[r, "f"] & \Lambda \end{tikzcd}$$

In the sequel we will frequently identify the cylinders on the symbol space Σ_2^{+-} and their images under the homeomorphism Π. Given a point $x \in R_i$, $i = 0, 1, 2, 3, 4$, we put

$$W^s_{\mathrm{loc}}(x) = W^s(x) \cap R_i.$$

These sets are called local stable manifolds. Given a point $x \in R_i$, $i = 0, 1, 2, 3, 4$, we define

$$W^u_i(x) = W^u(x) \cap R_i.$$

For $i \in \{0, 1\}$ we define

$$W^u_{i\oplus 1}(x) = W^u(x) \cap R_{i\oplus 1}(x),$$

where the symbol "$\oplus$" denotes the addition mod(2) in the group $\{0, 1\}$. Given two points $x, y \in f(S)$ we denote by

$$H^u_{x,y} : W^u(x) \to W^u(y)$$

the holonomy map along local stable manifolds. Moreover, we denote by

$$H^u : f(S) \to W^u(\omega)$$

the holonomy map from $f(S)$ to $W^u(\omega)$ along local stable manifolds, that is,

$$H^u(x) = H^u_{x,\omega}.$$

We define $\omega_0 = \omega$ and pick $\omega_1 \in \Lambda$ arbitrary having the property that $f(W^u_1(\omega)) = W^u(\omega_1)$. We say that the parabolic horseshoe $f : S \to \mathbb{R}^2$ is of smooth type if the holonomy map $H^u : f(S) \to W^u(\omega)$ and all the holonomy maps $H^u_{x,y} : W^u(x) \to W^u(y)$ are $C^{1+\varepsilon}$, and in addition, the derivative along unstable manifolds of the map $f^{-1} \circ H^u_{\omega,\omega_1} : W^u(\omega) \to W^u(\omega)$ has norm less than 1 at every poiny of $W^u(\omega)$ except possibly at ω. Given two points $x, y \in f(S)$ we denote by

$$H^s_{x,y} : W^s_{\mathrm{loc}}(x) \to W^s_{\mathrm{loc}}(y)$$

the holonomy map along unstable manifolds. We say that the parabolic horseshoe $f : S \to \mathbb{R}^2$ has a Lipschitz continuous unstable foliation if all the holonomy maps $H^s_{x,y} : W^s_{\mathrm{loc}}(x) \to W^s_{\mathrm{loc}}(y)$ are Lipschitz continuous with a uniform Lipschitz constant.
The following example provides a parabolic horseshoe map $f : S \to \mathbb{R}^2$ of smooth type having a Lipschitz continuous unstable foliation. In particular, the unstable and stable manifolds in R_0 and R_1 are vertical and horizontal lines respectively.

Example 1. An Almost Linear Parabolic Horseshoe Map.
Let $S \subset \mathbb{R}^2$ be a unit square and let $h : S \to \mathbb{R}^2$ be a linear horseshoe map with constant contraction rate $\lambda_s < \frac{1}{2}$ and constant expansion rate $\lambda_u > 2$, see Figure 1. Let $\omega = (\omega_2, \omega_2)$ be the fixed point of h contained in R_0. Let $\delta > 0$ such that $(\omega_1, \omega_2 + \delta), (\omega_1, \omega_2 - \delta) \in \mathrm{int} S$. Fix constants $\beta > 0, a > 0$ and $\eta > 1$. Let $\varphi : [-\delta, \delta] \to [-\delta, \delta]$ be a $C^{1+\epsilon}$-diffeomorphism satisfying the following properties:

(i) $\varphi^{-1}(t) = \lambda_u t \pm \lambda_u a|t|^{\beta+1}$ for t sufficiently close to 0,
and with the sign "$-$" if $t \geq 0$ and the sign "$+$" if $t \leq 0$;
(ii) $\lambda_u^{-1} < \varphi'(t) \leq \eta$ for all $t \in [-\delta, \delta] \setminus \{0\}$;
(iii) $\varphi'(-\delta) = \varphi'(\delta) = 1$.

Let $A_\delta \subset h(S)$ be defined as in Figure 1, that is, $A_\delta = [a_1, a_2] \times [b_1, b_2]$, where $b_1 = \omega_2 - \delta, b_2 = \omega_2 + \delta$ and $a_2 - a_1 = \lambda_s$. We define the map $g : h(S) \to h(S)$ by

$$g(x) = \begin{cases} (x_1, \omega_2 + \varphi(x_2 - \omega_2)) & \text{if} \quad x \in A_\delta \\ \\ x & \text{if} \quad x \in h(S) \setminus A_\delta \end{cases} \tag{5.3}$$

for all $x = (x_1, x_2) \in h(S)$. Clearly, g is a $C^{1+\epsilon}$-diffeomorphism which preserves vertical and horizontal lines in $R_0 \cup R_1 = S \cap h(S)$. We now define the map $f = g \circ h$. It follows immediately from the construction that f is a parabolic horseshoe map. In particular, property (5.1) holds. Indeed (5.1) is a consequence of the facts that the foliations $W^u|R_0 \cup R_1$ and $W^s|R_0 \cup R_1$ of f are given by vertical and horizontal lines respectively and that $\mathrm{pr}_2(h^{-1}(\omega_1, x_2)) = \omega_2 + \lambda_u^{-1}(x_2 - \omega_2)$ and $\mathrm{pr}_2(g^{-1}(\omega_1, x_2)) = \omega_2 + \varphi^{-1}(x_2 - \omega_2)$ if x_2 is sufficiently close to ω_2. Here pr_2 denotes the projection in $\mathbb{R}^2$ on the 2-th coordinate. Moreover, it is easy to see that f is of smooth type and has a Lipschitz continuous unstable foliation. A simple calculation shows that

$$|D_s f(x)| = \lambda_s \quad \text{and} \qquad 1 \leq |D_u f(x)| \leq \eta \lambda_u \quad \text{for all } x \in \Lambda. \tag{5.4}$$

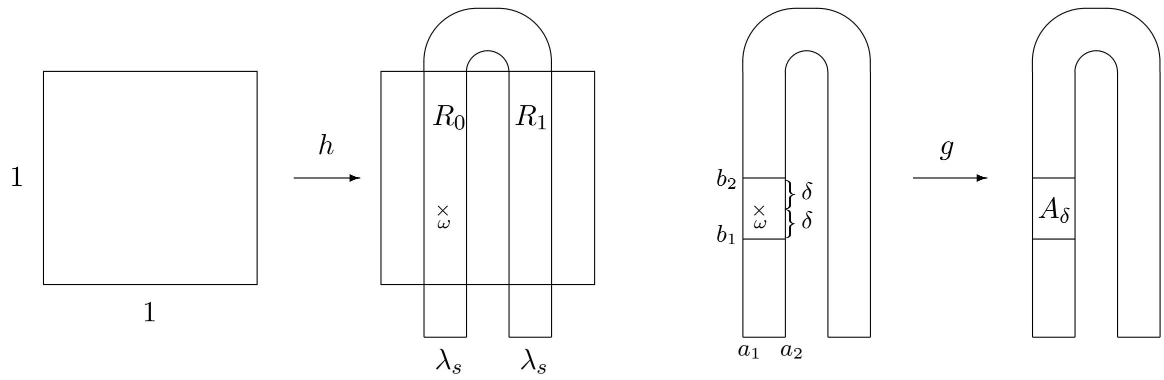

Figure 1. An Almost Linear Parabolic Horseshoe Map $f = g \circ h$

Remarks.
(i) We note that the diffeomorphism f is uniquely determined on $S \cap h(S)$ once we have chosen the constants $a, \beta, \eta, \delta, \lambda_u, \lambda_s$, the map φ and h. In particular, the contraction rate $\lambda_s \in (0, 1/2)$ can be chosen independently of $a, \beta, \eta, \delta, \lambda_u$, and φ.
(ii) The method of construction of an almost linear parabolic horseshoe map in Example 1 can be generalized. Namely, let h be a hyperbolic (not necessarily linear) $C^{1+\epsilon}$-horseshoe map which has the property that h as well as h^{-1} is of smooth type. Then we can construct analogously to Example 1 a $C^{1+\epsilon}$-diffeomorphism $g : h(S) \to h(S)$ such that $f = g \circ h$ is a parabolic horseshoe map of smooth type with a Lipschitz (even $C^{1+\alpha}$) unstable foliation.

6. Horseshoe and the associated parabolic iterated function system

In this section we introduce for a parabolic horseshoe map an associated parabolic iterated function system on the unstable manifold of the parabolic fixed point. The goal is to do this in such a way that the parabolic horseshoe and the parabolic iterated function system share many ergodic-theoretical features. Let f be a parabolic horseshoe map of smooth type. We define $\omega_0 = \omega$ and pick $\omega_1 \in \Lambda$ arbitrary having the property that $f(W_1^u(\omega)) = W^u(\omega_1)$. We introduce the 1-dimensional iterated function system Φ on $W^u(\omega)$ defined by the following two maps:

$$\phi_i = f^{-1} \circ H^u_{\omega,\omega_i} : W^u(\omega) \to W_i^u(\omega) \subset W^u(\omega), \ i = 0, 1. \tag{6.1}$$

It is easy to check that the iterated function system $\Phi = \{\phi_0, \phi_1\}$ satisfies all the requirements (in particular (4.1)) of a 1-dimensional parabolic iterated function system introduced in Section 4 except possibly item (2) (it may happen that $|\phi_1'(\omega)| = 1$). We shall now demonstrate that after a C^∞ change of the Riemannian metric on $W^u(\omega)$, condition (2) is also satisfied and $\Phi = \{\phi_0, \phi_1\}$

becomes a parabolic iterated function system. The following formula immediately follows from the definition of a smooth parabolic horseshoe.

$$|\phi_i'(x)| < 1 \text{ for all } x \in W^u(\omega) \setminus \{\omega\} \ \text{ and } \ |\phi_i'(\omega)| \le 1, \ i = 0, 1. \tag{6.2}$$

Since $\phi_1(\phi_1(\omega)), \phi_0(\phi_1(\omega)) \neq \phi_1(\omega)$, there exists a closed segment $T \subset W^u(\omega)$ with the following properties:

(a) $T \subset W_1^u(\omega)$.
(b) $\phi_1(\omega) \in \mathrm{int}_{W_1^u(\omega)} T$
(c) $\phi_1(T) \cap T = \emptyset$.

Define

$$M = \sup\{\max\{|\phi_0'(x)|, |\phi_1'(x)| : x \in T\}. \tag{6.3}$$

In view of (6.2), item (a) above, and the continuity of the functions $x \mapsto |\phi_0'(x)|, |\phi_1'(x)|$, we conclude that

$$M < 1. \tag{6.4}$$

Therefore there exists a C^∞ function $\rho : W^u(\omega) \to (0, 1]$ such that

$$M < \rho(x) < 1 \ \text{ for all } \ x \in \mathrm{int}T \tag{6.5}$$

and

$$\rho(x) = 1 \ \text{ for all } \ x \in W^u(\omega) \setminus T. \tag{6.6}$$

Consider the Riemannian metric $\rho(x)dx$ on $W^u(\omega)$ and let $|g'(x)|_\rho = \rho(g(x))|g'(x)|/\rho(x)$ be the norm of the derivative of a differentiable function $g : W^u(\omega) \to W^u(\omega)$ calculated with respect to the Riemannian metric $\rho(x)dx$. We shall prove the following.

Lemma 6.1. *We have $|\phi_i'(x)|_\rho < 1$ for all $x \in W^u(\omega) \setminus \{\omega\}$, $i = 0, 1$, $|\phi_0'(\omega)|_\rho = 1$ and $|\phi_0'(\omega)|_\rho < 1$.*

Proof. The equality $|\phi_0'(\omega)|_\rho = 1$ is immediate. Consider an arbitrary point $x \in \mathrm{int}T$. Then both $\phi_0(x), \phi_1(x) \notin T$ by (a) and (c) respectively. Thus, using (6.2)-(6.3), we obtain for $i = 0, 1$ that

$$|\phi_i'(x)|_\rho = \rho(\phi_i(x))|\phi_i'(x)|/\rho(x) = |\phi_i'(x)|/\rho(x) < M^{-1}|\phi_i'(x)| \le 1.$$

Next we assume that $x \in W^u(\omega) \setminus \mathrm{int}T$. Then

$$|\phi_i'(x)|_\rho = \rho(\phi_i(x))|\phi_i'(x)| \le |\phi_i'(x)|. \tag{6.7}$$

So, if $x \neq \omega$, then it follows from (6.2) that $|\phi_i'(x)|_\rho \le |\phi_i'(x)| < 1$. Hence, we are left to consider the case when $x = \omega$ and $i = 1$. It then follows from (b), (6.5), (6.2) and (6.2) that $|\phi_1'(\omega)|_\rho \rho(\phi_1(\omega))|\phi_1'(\omega)| \le \rho(\phi_1(\omega)) < 1$. ☐

Therefore, as long as we are dealing with the iterated function system Φ itself, we may assume without loss of generality that Φ is a parabolic iterated function system and, in particular, all the considerations from Section 4 apply.

Similarly as in the case of the horseshoe Λ we will frequently identify in the sequel the cylinders on the symbol space Σ_2^+ and their images under the homeomorphism $\pi : \Sigma_2^+ \to J_\Phi$. Note that

$$\phi_i^{-1} = (H^u_{\omega,\omega_i})^{-1} \circ f = H^u_{\omega_i,\omega} \circ f : W_i^u(\omega) \to W^u(\omega), \ i = 0, 1 \tag{6.8}$$

and

$$H^u_{\omega_i,\omega} = H^u|_{W^u(\omega_i)}. \tag{6.9}$$

We first prove a preliminary result.

Lemma 6.2. *For all $x \in S$ and all $n \geq 0$ we have that $(H^u \circ f)^k(x) \in S$ for all $k = 0, 1, \ldots, n$ if and only if $f^k(x) \in S$ for all $k = 0, 1, \ldots, n$. In this case we have*

$$W^s_{\text{loc}}\big((H^u \circ f)^n(x)\big) = W^s_{\text{loc}}(f^n(x)).$$

Proof. We shall prove this lemma by induction with respect to $n \geq 0$, For $n = 0$ there is nothing to prove. So, take $n \geq 1$ and suppose that our lemma is true for all non-negative integers less than n. Assume that $f^k(x) \in S$ for all $k = 0, 1, \ldots, n$. Then

$$(H^u \circ f)^n(x) = H^u \circ f\big((H^u \circ f)^{n-1}(x)\big) \in H^u \circ f\big(W^s_{\text{loc}}(f^{n-1}(x))\big) \subset H^u(W^s_{\text{loc}}(f^n(x))\big) \subset W^s_{\text{loc}}(f^n(x)),$$

and, in particular, $(H^u \circ f)^n(x) \in S$. We now assume that $(H^u \circ f)^k(x) \in S$ for all $k = 0, 1, \ldots, n$. Thus,

$$\begin{aligned} f^n(x) &= f(f^{n-1}(x)) \in f\big(W^s_{\text{loc}}\big((H^u \circ f)^{n-1}(x)\big)\big) \\ &\subset W^s_{\text{loc}}\big(f \circ (H^u \circ f)^{n-1}(x)\big) = W^s_{\text{loc}}\big(H^u \circ f) \circ (H^u \circ f)^{n-1}(x)\big) \\ &= W^s_{\text{loc}}((H^u \circ f)^n(x)). \end{aligned}$$

This implies that $f^n(x) \in S$. The desired result now follows from the induction assumption. □

For all integers $n \geq 0$ and also for $n = +\infty$ we define

$$S_n = \bigcap_{k=0}^{n} f^{-k}(S)$$

and

$$S_{-n} = \bigcap_{k=0}^{n} f^{k}(S).$$

Next, we prove the following.

Lemma 6.3. *If $n \geq 0$ and $x \in S_n$, then $W^s_{\text{loc}}(x) \subset S_n$.*

Proof. For $n = 0$ this is obvious because $W^s_{\text{loc}}(z) \subset S = S_0$ for all $z \in S$. So, suppose that our lemma is true for some $n \geq 0$ and fix a point $x \in S_{n+1}$. Then $x \in S_n$, and in view of our inductive hypothesis, $W^s_{\text{loc}}(x) \subset S_n$. Hence, $f^{n+1}\big(W^s_{\text{loc}}(x)\big)$ is well-defined and, as $f^{n+1}(x) \in S$, we get that $f^{n+1}\big(W^s_{\text{loc}}(x)\big) \subset W^s_{\text{loc}}\big(f^{n+1}(x)\big) \subset S$. This completes the proof. □

We recall that J_Φ is the limit set of the iterated function system Φ. The relation between this limit set and the horseshoe Λ is given by the following.

Lemma 6.4. $J_\Phi = \Lambda \cap W^u(\omega)$.

Proof. We shall show first by induction that $J_\Phi \subset S_n$ for all $n \geq 0$. Indeed, if $z \in J_\Phi$, then $z \in \phi_0(W^u(\omega)) \cup \phi_1(W^u(\omega)) = W^u_0(\omega) \cup W^u_1(\omega) \subset S = S_0$ and our inclusion is proved for $n = 0$. So, suppose that $n \geq 1$ and that $J_\Phi \subset S_k$ for all $k = 0, 1, \ldots, n-1$. Let us consider an arbitrary point $z \in J_\Phi$. We write $z = \pi(\tau)$, where $\tau \in I^\infty$. Then $z \in J_\Phi \subset S_{n-1}$ and $z = \phi_{\tau_0}(\pi(\sigma(\tau)))$. Hence,

$$f^n(z) = f^n \circ \phi_{\tau_0}(\pi(\sigma(\tau))) = f^n \circ f^{-1} \circ H^u_{\omega,\omega_{\tau_0}}(\pi(\sigma(\tau))) = f^{n-1}\big(H^u_{\omega,\omega_{\tau_0}}(\pi(\sigma(\tau)))\big).$$

Since $H^u_{\omega,\omega_{\tau_0}}(\pi(\sigma(\tau))) \in W^s_{\rm loc}(\pi(\sigma(\tau)))$ and since $\pi(\sigma(\tau)) \in J_\Phi \subset S_{n-1}$, we may conclude from Lemma 6.3 that $H^u_{\omega,\omega_{\tau_0}}(\pi(\sigma(\tau))) \in S_{n-1}$. Thus, $f^n(z) \in S$, which implies that $z \in S_n$. Therefore, the inductive proof is complete, and we obtain that $J_\Phi \subset S_{+\infty}$. Since also $J_\Phi \subset W^u_0(\omega) \cup W^u_1(\omega) \subset S_{-\infty}$, we therefore obtain that

$$J_\Phi \subset W^u(\omega) \cap S_{+\infty} \cap S_{-\infty} = W^u(\omega) \cap \Lambda. \tag{6.10}$$

In order to prove the opposite inclusion we consider an arbitrary point $z \in \Lambda \cap W^u(\omega)$. We shall prove by induction that there exists an infinite word $\tau = \emptyset\tau_1\tau_2\ldots \in \{\emptyset\} \times I^\infty$, such that for every $n \geq 0$, $z = \phi_{\tau|_n}(x)$ for some $x \in W^u(\omega)$. Indeed, for $n = 0$ we have $z = \phi_\emptyset(z)$ and $z \in W^u(\omega)$. So, suppose that for some $n \geq 1$, the word $\tau|_n = \emptyset\tau_1\tau_2\ldots\tau_n$ has been constructed. This means that $z = \phi_{\tau|_n}(x)$ with some $x \in W^u(\omega)$. As $z \in \Lambda$, we have $f^n(z) \in S$, and, in view of Lemma 6.2, $(H \circ f)^n(z) \in S$. Using (6.8) and (6.9), we therefore obtain that $x = \phi^{-1}_{\tau|_n}(z) = (H^u \circ f)^n(z) \in S$. And applying the last part of Lemma 6.2 along with the fact that $f^{n+1}(z) \in S$, we get that

$$f(x) = f \circ (H^u \circ f)^n(z) \in f\big(W^s_{\rm loc}(f^n(z))\big) \subset W^s_{\rm loc}(f^{n+1}(z)) \subset S.$$

Moreover,

$$f(x) \in f(S \cap W^u(\omega)) = f\big(W^u_0(\omega) \cup W^u_1(\omega)\big) = f\big(W^u_0(\omega)\big) \cup f\big(W^u_1(\omega)\big) = W^u(\omega) \cup W^u(\omega_1),$$

and thus $f(x) = H^u_{\omega,\omega_i}(y)$ with some $i \in \{0,1\}$ and $y \in W^u(\omega)$. Consequently, $x = f^{-1} \circ H^u_{\omega,\omega_i}(y) = \phi_i(y)$. Thus $z = \phi_\tau(\phi_i(y)) = \phi_{\tau i}(y)$, and the inductive proof is complete by putting $\tau_{n+1} = i$. Note that $z = \pi(\tau) \in J_\Phi$. This gives the inclusion $W^u(\omega) \cap \Lambda \subset J_\Phi$. Combining this with (6.10) completes the proof. $\square$

7. Topological pressures, unstable dimension, Hausdorff and packing measures

Let f be a parabolic horseshoe map of smooth type. Since the holonomy maps along local stable manifolds are smooth, it follows that

$$t^u \stackrel{\rm def}{=} \dim_H W^u(\omega) \cap \Lambda = \dim_H W^u(x) \cap \Lambda$$

is independent of $x \in \Lambda$. We call the quantity t^u the unstable dimension of the set Λ. The main goal of this section is to establish a Bowen-Ruelle-Manning-McCluskey type of formula for t^u. In order to do this we will make use of Lemma 6.4, will introduce the topological pressure of several dynamical systems and potentials, and will apply results from the thermodynamic formalism of parabolic iterated function systems derived in [MU1] and [U2]. First we consider different pressure functions. For all $t \geq 0$ let $\hat{P}(t)$ denote the pressure function associated with the iterated function system Φ as defined in Section 4 (see [MU1], [U2] for details). Moreover, we define $P^u(t) = P(f, -t\log|D_u f|)$, where $P(f, -t\log|D_u f|)$ is the ordinary topological pressure of the potential $-t\log|D_u f| : \Lambda \to \mathbb{R}$

with respect to the dynamical system $f|\Lambda$. We claim that the following diagram commutes:

$$\begin{array}{ccc} \Lambda & \xrightarrow{f} & \Lambda \\ H^u\big\downarrow & & \big\downarrow H^u \\ J_\Phi & \xrightarrow{H^u\circ f} & J_\Phi \end{array} \tag{7.1}$$

that is,

$$H^u \circ f = (H^u \circ f) \circ H^u. \tag{7.2}$$

Indeed, if $x \in \Lambda$, then $W^s_{\rm loc}(x) = W^s_{\rm loc}(H^u(x))$. This implies that $W^s_{\rm loc}(f(x)) = W^s_{\rm loc}(f(H^u(x)))$, and therefore $H^u(f(x)) = H(f(H^u(x)))$, which proves the claim. We define

$$\tilde{P}(t) = P\big(H^u \circ f, -t\log|D(H^u\circ f)|\big) \quad\text{and}\quad \overline{P}(t) = P\big(f, -t\log|D(H^u\circ f)|\circ H^u\big).$$

Since $H^u \circ f : J_\Phi \to J_\Phi$ is the dynamical system generated by the inverses ϕ_0^{-1} and ϕ_1^{-1}, we immediately obtain that

$$\tilde{P}(t) = \hat{P}(t),\ t \geq 0. \tag{7.3}$$

We now prove the following.

Lemma 7.1. *For every $t \geq 0$, we have that $P^u(t) = \hat{P}(t)$.*

Proof. Since the diagram (7.1) commutes, we have that

$$\overline{P}(t) \geq \tilde{P}(t). \tag{7.4}$$

for all $t \geq 0$. Differentiating both sides of equation (7.2) along the unstable manifolds, we get $(D_uH^u \circ f)\cdot D_uf = D(H^u\circ f)\circ H^u \cdot D_uH^u$. This implies that

$$\log|D(H^u\circ f)\circ H^u| - \log|D_uf| = \log|D_uH^u\circ f| - \log|D_uH^u|,$$

and consequently the potentials $-t\log|D(H^u\circ f)\circ H^u|$ and $-t\log|D_uf|$ are cohomologous for all $t \geq 0$. Hence,

$$\overline{P}(t) = P^u(t), \tag{7.5}$$

and combining this along with (7.4) and (7.3), we conclude that

$$P^u(t) \geq \hat{P}(t). \tag{7.6}$$

Now we shall prove that

$$\tilde{P}(t) \geq \overline{P}(t)$$

for all $t \geq 0$. Indeed, fix $\varepsilon > 0$. Since all the holonomy maps between all unstable manifolds along local stable manifolds are continuous, it is easy to see that there is $\delta \in (0,\varepsilon)$ such that if $x, y \in W^u(\omega)$ and $d_{W^u(\omega)}(x,y) \leq \delta$, then

$$d\big(z, H^u_{\omega,z}(y)\big) \leq \varepsilon \tag{7.7}$$

for all $z \in W^s_{\rm loc}(x)$. Since the diffeomorphism $f : \Lambda \to \Lambda$ contracts on stable manifolds uniformly and since all these manifolds are "uniformly" smooth, there exists a universal constant $c > 0$ such that for every $y \in \Lambda$ there exists a set $I_y \subset W^s_{\rm loc}(y)$, which is (n,ε)-spanning set for $f|_{W^u(y)}$ and whose cardinality does not exceed $c\varepsilon^{-1}$. Now fix $\tilde{E}_n(\varepsilon)$, a minimal (n,δ)-spanning set for the dynamical system $H^u \circ f : J_\Phi \to J_\Phi$. We claim that the set

$$E_n(\varepsilon) = \bigcup_{y\in\tilde{E}_n(\varepsilon)} I_y$$

is $(n, 2\varepsilon)$-spanning set for the dynamical system $f : \Lambda \to \Lambda$. Indeed, consider an arbitrary point $z \in \Lambda$. Since $\tilde{E}_n(\varepsilon)$ is (n,δ)-spanning for $H^u \circ f : J_\Phi \to J_\Phi$, there is $y \in \tilde{E}_n(\varepsilon)$ such that $d^n_{H^u\circ f}(H^u(z), y) \le \delta$. Let $x = [z, y]$. Equation (7.7) yields,

$$d^n_f(z,x) \le \varepsilon.$$

Also, there exists a point $\rho \in I_y$ such that $d^n_f(x,\rho) \le \varepsilon$. Hence, $d^n_f(z,\rho) \le 2\varepsilon$ and our claim is proved. Let $\psi = -t\log|(H^u \circ f)'|$ where $\psi : J_\Phi \to \mathbb{R}$. Define

$$S_n(\psi\circ H^u) = \sum_{j=0}^{n-1} (\psi\circ H^u)\circ f^j$$

and let

$$\tilde{S}_n\psi = \sum_{j=0}^{n-1} \psi\circ (H^u\circ f)^j.$$

Since for every $x \in J_\Phi \subset W^u(\omega)$, $H^u(I_y) = \{y\}$ and since the diagram (7.1) commutes, we obtain

$$\begin{aligned}\sum_{x\in E_n(\varepsilon)} \exp\big(S_n(\psi\circ H^u)(x)\big) &= \sum_{y\in\tilde{E}_n(\varepsilon)}\sum_{z\in I_y} \exp\big(S_n(\psi\circ H^u)(z)\big) = \sum_{y\in\tilde{E}_n(\varepsilon)}\sum_{z\in I_y} \exp\big(\tilde{S}_n\psi(H^u(z))\big)\\ &= \sum_{y\in\tilde{E}_n(\varepsilon)}\sum_{z\in I_y} \exp\big(\tilde{S}_n\psi(y)\big) \le \sum_{y\in\tilde{E}_n(\varepsilon)} \#I_y \exp\big(\tilde{S}_n\psi(y)\big)\\ &\le c\varepsilon^{-1} \sum_{y\in\tilde{E}_n(\varepsilon)} exp\big(\tilde{S}_n\psi(y)\big).\end{aligned}$$

Therefore,

$$\liminf_{n\to\infty}\frac{1}{n}\log\sum_{x\in E_n(\varepsilon)}\exp\big(S_n(\psi\circ H^u)(x)\big) \le \liminf_{n\to\infty}\frac{1}{n}\log\sum_{y\in\tilde{E}_n(\varepsilon)} exp\big(\tilde{S}_n\psi(y)\big).$$

Since the lower limit on the right-hand side of this inequality converges to $\tilde{P}(t)$ if $\varepsilon \searrow 0$ (as the sets $E_n(\varepsilon)$ were chosen to be minimal) and since the lower limit as $\varepsilon \searrow 0$ on the left-hand side is larger than or equal to $\overline{P}(t)$, we get that $\tilde{P}(t) \ge \overline{P}(t)$. Finally, by using (7.3), (7.5), and (7.6), we may conclude that $\hat{P}(t) = P^u(t)$. □

Combining Lemma 7.1 and Bowen's formula proven in [U2] (compare [MU1]) provides the following.

Proposition 7.2. *The unstable dimension t^u of Λ is the first zero of the pressure function $t \mapsto \hat{P}(t)$, $t \ge 0$.*

As an immediate consequence of Proposition 7.2 and Lemma 7.1 we obtain the following.

Theorem 7.3. *The unstable dimension t^u of Λ is the smallest zero of the unstable pressure function $t \mapsto P^u(t)$, $t \geq 0$.*

Now, let us take more fruits of the results proven in [U2]. First, Theorem 6.5 from [U2] and Lemma 6.4 give the following.

Theorem 7.4. *The unstable dimension $t^u \in (0,1)$.*

Remark. It follows from work in [DV] that if f is a C^2-diffeomorphism (which in particular implies that $\beta \geq 1$), then $t^u \geq \frac{1}{2}$.

Let us denote by $H_t(A)$ respectively $P_t(A)$ the t-dimensional Hausdorff respectively packing measure of the set A. Combining Theorem 7.4, Lemma 6.4, and Theorem 6.4 of [U2] provides the following.

Theorem 7.5. $\mathrm{H}_{t^u}(W^u(z) \cap \Lambda) = 0$ *and* $0 < P_{t^u}(W^u(z) \cap \Lambda) < \infty$ *for all* $z \in \Lambda$.

We end this section with the following result.

Theorem 7.6. *The unstable pressure function $t \mapsto P^u(t)$ is real-analytic on $(0, t^u)$.*

Proof. Consider the hyperbolic iterated function system $\Phi^* = \{\phi_{0^n1}\}_{n=0}^{\infty}$ associated to the system Φ as in Section 4. Consider the two-parameter family $G_{t,s}$ of the functions

$$g_{t,s}^{0^n1}(z) = t \log |\phi'_{o^n1}(z)| - s(n+1) : t, s \in \mathbb{R},\ n \geq 0,\ z \in W^u(\omega).$$

With the terminology of Section 3 of [MU2] (see also [U3] and [HMU], where these were introduced) we shall prove the following.

Lemma 7.7. *For all $t, s \in \mathbb{R}$ the family $G_{t,s}$ is Hölder continuous. For all $(t,s) \in \mathbb{R} \times (0, +\infty)$ the family $G_{t,s}$ is summable.*

Proof. The fact that the family $G_{t,s}$ is Hölder follows immediately from the sentence located just beneath the proof of Theorem 8.4.2 in [MU2]. Since $||\phi'_{0^n1}|| \asymp (n+1)^{-\frac{\beta+1}{\beta}}$ (see Lemma 2.3 in [U2]), we see that if $t \in \mathbb{R}$ and $s > 0$, then

$$\begin{aligned}
\sum_{n \geq 0} \exp\bigl(\sup\bigl(g_{t,s}^{0^n1}\bigr)\bigr) &= \sum_{n \geq 0} \exp\bigl(\sup\bigl(t \log |\phi'_{o^n1}(z)| - s(n+1) : z \in W^u(\omega)\bigr)\bigr) \\
&= \sum_{n \geq 0} e^{-s(n+1)} ||\phi'_{0^n1}||^t \asymp \sum_{n \geq 0} (n+1)^{-\frac{\beta+1}{\beta} t} e^{-s(n+1)} \\
&< +\infty.
\end{aligned}$$

This precisely means that our family $G_{t,s}$ is summable, and we are done. □

Let $g_{t,s} : \{0^n1 : n \geq 0\}^{\mathbb{N}} \to \mathbb{R}$ be the amalgamated function (see [MU2], comp. [U3] and [HMU]) of the family $\{g_{t,s}^{0^n1}\}_{n=0}^{\infty}$. This function is given by the formula

$$g_{t,s}(\tau) = g_{t,s}^{\tau_0}(\pi_*(\sigma_*(\tau))) = t \log |\phi'_{\tau_0}| - s|\tau_0|, \tag{7.8}$$

where $\sigma_* : \{0^n 1 : n \geq 0\}^{\mathbb{N}} \to \{0^n 1 : n \geq 0\}^{\mathbb{N}}$ is the shift map associated with the iterated function system Φ^* and $\pi_* : \{0^n 1 : n \geq 0\}^{\mathbb{N}} \to J_{\Phi_*}$ is the corresponding canonical projection. Summability of the family $G_{t,s}$ proven in Lemma 7.7 precisely means summability of amalgamated function $g_{t,s}$. Hölder continuity of this amalgamated function follows from Lemma 7.7 and Lemma 3.1.3 from [MU2]. The following lemma is now an immediate consequence of Theorem 2.6.12 and Proposition 3.1.4, both from [MU2].

Lemma 7.8. *The function $(t,s) \mapsto P(G_{t,s})$, $(t,s) \in \mathbb{R} \times (0,+\infty)$ is real-analytic in both variables t and s, where the topological pressure $P(G_{t,s})$ is defined in Section 3.1 of* [MU2].

We now prove the conclusion of Theorem 7.6. Since the dynamical system $H^u \circ f : J_\Phi \to J_\Phi$ is expansive, it follows from Theorem 3.12 in [DU] that for every $t \geq 0$ there exists a Borel probability measure m_t supported on J_Φ and such that

$$m_t(H^u \circ f(A)) = \int_A e^{\tilde{P}(t)} |(H^u \circ f)'|^t dm_t \tag{7.9}$$

for all Borel sets $A \subset J_\Phi$ having the property $H^u \circ f|_A$ is one-to-one. Hence,

$$m_t(\phi_i(E)) = \int_E e^{-\tilde{P}(t)} |\phi_i'|^t dm_t$$

for $i = 0, 1$ and E, any Borel subset of J_Φ. In addition, $m_t\big(\phi_0(W^u(\omega)) \cap \phi_1(W^u(\omega))\big) = 0$ as these sets $\phi_0(W^u(\omega))$ and $\phi_1(W^u(\omega))$ are disjoint. We therefore get by a straightforward induction that

$$m_t\big(\phi_{0^n 1}(E)\big) = \int_E \exp\big(g_{t,P(t)}^{0^n 1}\big) dm_t$$

and

$$m_t\big(\phi_{0^n 1}(W^u(\omega)) \cap \phi_{0^k 1}(W^u(\omega))\big) = 0$$

for all $t \geq 0$ and all $n, k \geq 0$ with $n \neq k$. We have replaced here $\tilde{P}(t)$ by $P^u(t)$ due to Lemma 7.1 and (7.3). If now $t \in (0, t^u)$, then $P^u(t) > 0$ the family $G_{tP(t)}$ is Hölder and summable due to Lemma 7.7. So, looking at the definition of $G_{tP(t)}$-conformal measures, i.e. formulas (3.5) and (3.6) from [MU2], we see that m_t is a unique $G_{tP(t)}$-conformal measure and that

$$P\big(g_{t,s}\big) = P\big(G_{tP(t)}\big) = 0, \ t \in (0, t^u). \tag{7.10}$$

Looking at Theorem 3.2.3, Corollary 2.7.5 and Proposition 2.6.13 from [MU2] and at the formula (7.8), we conclude that for all $t \in (0, t^u)$,

$$\frac{\partial P\big(g_{t,s}\big)}{\partial s}|_{(t,P(t)} = \int -|\tau_0| d\tilde{\mu}_t(\tau) \neq 0, \tag{7.11}$$

where $\tilde{\mu}_t = \tilde{\mu}_{g_{t,P(t)}}$ is the σ_*-invariant Gibbs state proved to exist by Corollary 2.7.5 of [MU2]. Hence, applying formula (7.11) (also using (7.10)) the proof follows by applying Lemma 7.8 and the Implicit Function Theorem. □

8. Equilibrium states

In this section we provide a complete description of all ergodic σ-finite equilibrium states of the potential $-t^u\phi_u$, where

$$\phi^u = \log|D_u f| : \Lambda \to \mathbb{R},$$

with respect to the dynamical system $f : \Lambda \to \Lambda$. We start our analysis with the potential $-t^u\tilde{\phi}_u$, where

$$\tilde{\phi}_u := \log|D(H^u \circ f)| : J_\Phi \to \mathbb{R}$$

with respect to the dynamical system $H^u \circ f : J_\Phi \to J_\Phi$. Let $\tilde{\mu}_\omega$ be the Dirac δ-measure supported on ω. Let $\tilde{m}_{t^u}$ be the t^u-conformal measure established in (7.9) with $t = t^u$. We note that $\tilde{P}(t^u) = 0$. It follows from Theorem 7.2 in [U2] that there exists a unique (up to a multiplicative constant) ergodic σ-finite $H^u \circ f$-invariant measure $\tilde{\mu}_{t^u}$ on J_Φ equivalent to $\tilde{m}_{t^u}$. The measure $\tilde{\mu}_{t^u}$ is ergodic and conservative. The following necessary and sufficient condition for the measure $\tilde{\mu}_{t^u}$ to be finite was established in [U2].

Theorem 8.1. *The measure $\tilde{\mu}_{t^u}$ is finite if and only if*

$$t^u > 2\frac{\beta}{\beta+1}.$$

We shall prove the following:

Theorem 8.2. *The measures $\tilde{\mu}_{t^u}$ and $\tilde{\mu}_\omega$ are the only ergodic equilibrium states of the potential $-t^u\tilde{\phi}_u$ with respect to the dynamical system $H^u \circ f : J_\Phi \to J_\Phi$.*

Proof. Since by Lemma 3.3 and Theorem 4.3 in [U2], $\tilde{P}(-t^u\tilde{\phi}_u) = 0$ and since $\int -t^u\tilde{\phi}_u d\tilde{\mu}_\omega = 0$, it follows that the Dirac δ-measure $\tilde{\mu}_\omega$ is an equilibrium state of the potential $-t^u\tilde{\phi}_u$. Next, we demonstrate that $\tilde{\mu}_{t^u}$ is also an equilibrium state. We define

$$\rho = \frac{d\tilde{\mu}_{t^u}}{d\tilde{m}_{t^u}}. \tag{8.1}$$

It follows from Theorem 7.2 in [U2] that $\rho|_{[0^n1|^n_0]} \asymp n+1$. Since in addition $\tilde{m}_{t^u}\big([0^n1|^n_0]\big) \asymp (n+1)^{-\frac{\beta+1}{\beta}t^u}$ and since

$$|(H^u \circ f)'(z) - 1| \asymp (\beta+1)z^\beta \asymp (n+1)^{-1}$$

for all $z \in [0^n1|^n_0]$ (so $\log|(H^u \circ f)'(z)| \asymp (n+1)^{-1}$), we may conclude that

$$0 \le \int -t^u\tilde{\phi}_u d\tilde{\mu}_{t^u} < \infty. \tag{8.2}$$

Now notice that $0 < \tilde{\mu}_{t^u}([1|^0_0]) < \infty$. In order to see that $\tilde{\mu}_{t^u}$ is an equilibrium state for $-t^u\tilde{\phi}_u$ let us induce the map $H^u \circ f$ on the cylinder $[1|^0_0]$. Denote $\tilde{\mu}_{t^u[1|^0_0]}$ by $\tilde{\mu}_1$. Consider the partition α of $[1|^0_0]$ generated by the first return time. Obviously α is a generating partition for the measure $\tilde{\mu}_1$ and the first return map on $[1|^0_0]$. Since for $\tilde{\mu}_1$-a.e. point $\tau \in [1|^0_0]$ there is an infinite sequence $\{k_n\}_{n=1}^\infty$ of positive integers such that $\tau_{k_n} = 1$, and since $\tilde{\mu}_1$ is equivalent to $\tilde{m}_{t^u}|_{[1|^0_0]}$ with the Radon-Nikodym derivative bounded away from zero and infinity, we get that

$$\tilde{\mu}_1\big([\tau|_0^{k_n}]\big) \asymp \tilde{m}_{t^u}\big([\tau|_0^{k_n}]\big) \asymp \exp\big(-t^u S_{k_n}\tilde{\phi}_u(\tau)\big).$$

But $[\tau|_0^{k_n}] = \alpha^n(\tau)$ and $S_{k_n}\big(-t^u\tilde{\phi}_u\big)(\tau)S_n^1\big(-t^u\tilde{\phi}_u^1\big)(\tau)$, where α^n is the n-th refined partition with respect to the first return map on $[1|^0_0]$, S_j^1 is the j-th ergodic sum with respect to the first return map on $[1|^0_0]$, and $-t^u\tilde{\phi}_u^1 = \big(-t^u\tilde{\phi}_u\big)|_{[1|^0_0]}$. So, $\tilde{\mu}_1(\beta^n(\tau)) = \exp\big(-t^u S_n^1\tilde{\phi}_u^1(\tau)\big)$. Applying now Shannon-McMillan-Breiman Theorem and Birkhoff's Ergodic Theorem (along with the observation that $\tilde{\mu}_1$ is ergodic because $\tilde{m}_{t^u}$ is), we get that $\mathrm{h}_{\tilde{\mu}_1} + \int(-t^u\tilde{\phi}_u^1)d\tilde{\mu}_1 = 0$, the equality that demonstrates that $\tilde{\mu}_{t^u}$ is an equilibrium state for $-t^u\tilde{\phi}_u$.

In order to prove that $\tilde{\mu}_\omega$ and $\tilde{\mu}_{t^u}$ are the only ergodic conservative equilibrium states for $-t^u\tilde{\phi}_u$, suppose that $\tilde{\mu}$ is an ergodic conservative equilibrium state for $-t^u\tilde{\phi}_u$ different from $\tilde{\mu}_\omega$. Suppose that $\tilde{\mu}$ has an atom. Because of ergodicity and conservativity of $\tilde{\mu}$, this measure must be supported on a periodic orbit of σ containing this atom. But then $\tilde{\mu}$ is (up to a multiplicative constant) a probability measure, $\mathrm{h}_{\tilde{\mu}} = 0$ and $\int\tilde{\phi}_u d\tilde{\mu} > 0$ since $\tilde{\mu} \neq \tilde{\mu}_\omega$. Consequently, $\mathrm{h}_{\tilde{\mu}} + \int(-t^u\tilde{\phi}_u)d\tilde{\mu} < 0$ contrary to the fact that $\tilde{\mu}$ is an ergodic conservative equilibrium state for $-t^u\tilde{\phi}_u$. So, $\tilde{\mu}$ is atomless, and similarly as in the proof of Theorem 3.2, there exists an initial cylinder F, with the first coordinate equal to 1, such that $\tilde{\mu}(F) \in (0,\infty)$. Let α be the countable partition of F induced by the first return time. We shall prove the following.

Claim 1. $\mathrm{h}_{\tilde{\mu}_F}(\alpha) < \infty$.

Proof. Suppose on the contrary that $\mathrm{h}_{\tilde{\mu}_F}(\alpha) = \infty$. It then immediately follows from Shannon-McMillan-Breiman Theorem that

$$\lim_{n\to\infty}\frac{-\log\big(\tilde{\mu}_F(\alpha^n(\omega))\big)}{n} = \infty \tag{8.3}$$

for $\tilde{\mu}_F$-a.e. $\omega \in F$, say $\omega \in F_1$. Since $\tilde{\mu} \in \mathcal{M}_{-t^u\tilde{\phi}_u}$, the function $|\tilde{\phi}_u| = -\tilde{\phi}_u$ is $\tilde{\mu}$-integrable, and therefore $0 < \chi := \int|\tilde{\phi}_u|d\tilde{\mu}_F < \infty$. Thus, by Birkhoff's Ergodic Theorem,

$$\lim_{n\to\infty}\frac{1}{n}S_n^F\tilde{\phi}_u^F(\omega) = \chi \tag{8.4}$$

for $\tilde{\mu}$-a.e. $\omega \in F$, say $\omega \in F_2$. Put $F_3 = F_1 \cap F_2$. Then $\tilde{\mu}_F(F_3) = 1$. Since ϕ_1 is a hyperbolic element of our parabolic iterated function system Φ, it follows from item (b) of Section 4 and the definition of measure $\tilde{\mu}_{t^u}$ that for every $\omega \in F$ and every $n \geq 1$, $\tilde{\mu}_{t^uF}(\alpha^n(\omega)) \asymp \exp\big(-t^u S_n^F\tilde{\phi}_u(\omega)\big)$. Therefore, using (8.4), we get for all $\omega \in F_3$ and all $n \geq 1$ large enough, that

$$\tilde{\mu}_{t^uF}(\alpha^n(\omega)) \geq \exp\big(-t^u(\chi+1)n\big).$$

Combining this and (8.3), we see that for all $\omega \in F_3$,

$$\lim_{n\to\infty}\frac{\tilde{\mu}_F(\alpha^n(\omega))}{\tilde{\mu}_{t^uF}(\alpha^n(\omega))} = 0.$$

Thus $\tilde{\mu}_F(F_3) = 0$ and this contradiction finishes the proof of Claim 1. □

Let $J_{\tilde{\mu}_F}^{-1} : F \to [0,1]$ be the inverse of the (weak) Jacobian of the measure $\tilde{\mu}_F$ with respect to the first return map $\sigma_F : F \to F$, i.e

$$\tilde{\mu}_F(B) = \int_{\sigma_F(B)} J_{\tilde{\mu}_F}^{-1} \circ (\sigma_F|_B)^{-1} d\tilde{\mu}_F$$

for every Borel set $B \subset F$ such that $T|_B$ is injective. Let $\mathcal{L}_{\tilde{\mu}_F} : L^1(\tilde{\mu}_F) \to L^1(\tilde{\mu}_F)$ be the Perron-Frobenius operator associated to the measure $\tilde{\mu}_F$. The operator $\mathcal{L}_{\tilde{\mu}_F}$ is given by the formula

$$\mathcal{L}_{\tilde{\mu}_F} g(z) = \sum_{x \in \sigma_F^{-1}(z)} J_{\tilde{\mu}_F}^{-1}(x) g(x).$$

Notice that for the measure $\tilde{\mu}_{t^u}$ we have

$$J_{\tilde{\mu}_{t^u F}}^{-1}(z) = \frac{\rho(z)}{\rho(\sigma_F(z))} |(H^u \circ f)'_F(z)|^{-t^u} \tag{8.5}$$

and this Jacobian is a continuous function. Furthermore,

$$\mathcal{L}_{\tilde{\mu}_{t^u F}} g(z) = \sum_{x \in \sigma_F^{-1}(z)} \frac{\rho(x)}{\rho(z)} |(H^u \circ f)'_F(z)|^{-t^u} g(x) \tag{8.6}$$

and $\mathcal{L}_{\tilde{\mu}_{t^u F}} g : F \to \mathbb{R}$ is continuous for every continuous function $g : F \to \mathbb{R}$. In particular,

$$\mathcal{L}_{\tilde{\mu}_{t^u F}} \rho = \rho,$$

and this equality holds throughout the whole set F. We now shall prove the following:

Claim 2. *$J_{\tilde{\mu}_F}^{-1}(z) = J_{\tilde{\mu}_{t^u F}}^{-1}(z)$ for $\tilde{\mu}_F$-a.e. $z \in F$.*

Proof. Since $\mathcal{L}_{\tilde{\mu}_F}(\mathbb{1}) = \mathbb{1}$ and since $\mathcal{L}_{\tilde{\mu}_{t^u F}}(\mathbb{1}) = \mathbb{1}$, applying (8.6), we get

$$\begin{aligned}
1 = \int \mathbb{1} d\tilde{\mu}_F &= \int \mathcal{L}_{\tilde{\mu}_{t^u F}}(\mathbb{1}) d\tilde{\mu}_F \sum_{x \in \sigma_F^{-1}(z)} \big(J_{\tilde{\mu}_F}^{-1}(x)\big)^{-1} J_{\tilde{\mu}_{t^u F}}^{-1}(x) J_{\tilde{\mu}_F}^{-1}(x) d\tilde{\mu}_F(z) \\
&= \int \mathcal{L}_{\tilde{\mu}_F}\big(\big(J_{\tilde{\mu}_F}^{-1}(x)\big)^{-1} J_{\tilde{\mu}_{t^u F}}^{-1}\big) d\tilde{\mu}_F = \int \mathcal{L}_{\tilde{\mu}_F}\big(J_{\tilde{\mu}_F}^{-1}(x)\big)^{-1} J_{\tilde{\mu}_{t^u F}}^{-1} d\tilde{\mu}_F \\
&\geq 1 + \int \log\big(\big(J_{\tilde{\mu}_F}^{-1}(x)\big)^{-1} J_{\tilde{\mu}_{t^u F}}^{-1}\big) d\tilde{\mu}_F \geq 1 + \int \log\big(J_{\tilde{\mu}_F} \rho \cdot (\rho \circ \sigma_F)^{-1} |(H^u \circ f)'_F|^{-t^u}\big) d\tilde{\mu}_F \qquad (8.7) \\
&= 1 + \int \log(J_{\tilde{\mu}_F}) d\tilde{\mu}_F + \int \log \rho d\tilde{\mu}_F - t^u \int \log(\rho \circ \sigma_F) - t^u \int \log |(H^u \circ f)'_F| d\tilde{\mu}_F \\
&= 1 + \int \log(J_{\tilde{\mu}_F}) d\tilde{\mu}_F - t^u \int \tilde{\phi}_F^u d\tilde{\mu}_F.
\end{aligned}$$

Applying Claim 1 yields, $\int \log(J_{\tilde{\mu}_F}) d\tilde{\mu}_F = \mathrm{h}_{\tilde{\mu}_F}(\sigma_F)$. Therefore, since $P(-t^u \tilde{\phi}^u) = 0$ and since $\tilde{\mu}$ is an equilibrium state of $-t^u \tilde{\phi}^u$, we conclude that the most right-hand sided formula in (8.7) is equal to 1. Hence, the signs "$\geq$" appearing in (8.7) must be all equal to the "$=$" sign s. As a consequence, $J_{\tilde{\mu}_F}^{-1} = J_{\tilde{\mu}_{t^u F}}^{-1}$-$\tilde{\mu}_F$ a.e. The proof of Claim 2 is complete. □

Since $\sigma_F^n(A) = F$ for all $n \geq 1$ and all $A \in \alpha^n$, it immediately follows from Claim 2, (8.5), (8.1), and the Bounded Distortion Property (item (6) in Section 4), that if $\tilde{\mu}_F(A) > 0$, then $\tilde{\mu}_F(A) \asymp \tilde{\mu}_{t^u F}(A)$.

Thus, the measure $\tilde{\mu}_F$ is absolutely continuous with respect to the measure $\tilde{\mu}_{t^uF}(A)$, and since the latter is ergodic, $\tilde{\mu}_F = \tilde{\mu}_{t^uF}$. Hence, $\tilde{\mu} = \tilde{\mu}_{t^u}$, and we are done. $\square$

Let μ_ω be the Dirac δ-measure on Λ supported on ω. The following, main result of this section, provides a complete description of the structure of equilibrium states of the potential $-t^u\phi_u : \Lambda \to \mathbb{R}$, where ϕ_u is given by the formula

$$\phi_u(x) = \log |D_u f(x)|, \tag{8.8}$$

with respect to the dynamical system $f : \Lambda \to \Lambda$.

Theorem 8.3. *There are exactly two (up to a multiplicative constant) ergodic equilibrium states for the potential $-t^u\phi_u$. Namely, μ_ω and μ_{t^u}. Moreover, we have $\mu_{t^u} \circ (H^u)^{-1} = \tilde{\mu}_{t^u}$.*

Proof. For the purposes of this proof we denote cylinders on J_Φ by $[\tau|_j^k]^+$, $\tau \in I^* \cup \Sigma_2^{+-} \cup \Sigma_2^+$. Since the functions $-t^u\tilde{\phi}_u \circ H^u$ and $-t^u\phi_u$ are cohomologous with respect to the dynamical system $f : \Lambda \to \Lambda$, it follows from Corollary 3.5 that μ is an equilibrium state for $-t^u\phi_u$ if and only if $\mu \circ (H^u)^{-1}$ is an equilibrium state for $-t^u\tilde{\phi}_u$. Let us assume that

$$\mu \circ (H^u)^{-1} = \nu \circ (H^u)^{-1},$$

where μ and ν are some Borel σ-finite f-invariant measures on Λ. Then, for every cylinder $[\omega|_k^n]$, $k \le n$, we have

$$\begin{aligned}\nu([\omega|_k^n]) &= \nu\big(\sigma^k([\omega|_k^n])\big) = \nu([\omega|_0^{n-k}]) = \nu \circ (H^u)^{-1}([\omega|_0^{n-k}]^+) = \mu \circ (H^u)^{-1}([\omega|_0^{n-k}]^+) \\ &= \mu([\omega|_0^{n-k}]) = \mu([\omega|_k^n]).\end{aligned}$$

Hence $\nu = \mu$. Therefore, by using that $\mu_\omega \circ (H^u)^{-1} = \tilde{\mu}_\omega$, we obtain that in order to complete the proof it suffices to show that there exists a σ-finite ergodic and conservative f-invariant measure μ_{t^u} on Λ such that $\mu_{t^u} \circ (H^u)^{-1} = \tilde{\mu}_{t^u}$. Indeed, since $\tilde{\mu}_{t^u}(\{\omega\}) = 0$, $\tilde{\mu}_{t^u}$ can be treated as a Borel σ-finite measure on $J_\Phi \setminus \{\omega\}$. Let V be the vector space consisting of all real-valued continuous functions with compact support defined on $\Lambda \setminus (H^u)^{-1}(\omega)$ and the vector subspace M of V consisting of all functions of the form $g \circ H^u$, where $g : J_\Phi \setminus \{\omega\} \to \mathbb{R}$ is a continuous function with compact support. Treating $\tilde{\mu}_{t^u}$ as a positive linear functional on M, that is $\tilde{\mu}_{t^u}(g \circ H^u) = \int_{J_\Phi \setminus \{\omega\}} g d\tilde{\mu}_{t^u}$, it follows from Theorem 2.6.2 in [E] that $\tilde{\mu}_{t^u}$ can be extended to a positive linear functional $\mu^*_{t^u}$ on V. Given any function $\psi \in V$ define $\psi^* : \Lambda \setminus (H^u)^{-1}(\omega) \to \mathbb{R}$ by the formula

$$\psi^*(x) = \sup\{\psi(y) : H^u(y) = H^u(x)\}.$$

We notice that $\psi^* \ge \psi$ and $\psi^* \in M$. It follows from the proof of Theorem 2.6.2 in [E] that $\mu^*_{t^u}$ can be constructed in such a way that

$$\mu^*_{t^u}(\psi) \le \tilde{\mu}_{t^u}(\psi^*). \tag{8.9}$$

Now suppose that $\{\psi_n : \Lambda \setminus (H^u)^{-1}(\omega) \to \mathbb{R}\}_{n=1}^\infty$ is a sequence of continuous functions uniformly converging to the function identically equal to zero and such that there is a compact set $T \subset \Lambda \setminus (H^u)^{-1}(\omega)$ containing the supports of all continuous functions ψ_n, $n \ge 1$. Denoting by $\psi_n^{**} : J_\Phi \setminus \{\omega\} \to \mathbb{R}$ the unique continuous function with the property that $\psi_n^* = \psi_n^{**} \circ H^u$, we see that the sequence $\{\psi_n^{**}\}_{n=1}^\infty$ converges uniformly to zero and that for every $n \ge 1$ the topological support of the function ψ_n^{**} is contained in $H^u(T)$, which is a compact subset of $J_\Phi \setminus \{\omega\}$. It therefore follows from (8.9) that

$$\lim_{n\to\infty} \mu^*_{t^u}(\psi_n) \le \lim_{n\to\infty} \tilde{\mu}^*_{t^u}(\psi_n^*) = \lim_{n\to\infty} \tilde{\mu}^*_{t^u}(\psi_n^{**}) \le 0.$$

Replacing ψ_n by $-\psi_n$ and using that $-\lim_{n\to\infty}\mu^*_{t^u}(\psi_n)=\lim_{n\to\infty}\mu^*_{t^u}(-\psi_n)$, we may conclude that $\lim_{n\to\infty}\mu^*_{t^u}(\psi_n)\le 0$. Hence, $\lim_{n\to\infty}\mu^*_{t^u}(\psi_n)=0$, and it follows from Riesz's representation theorem that the functional $\mu^*_{t^u}$ can be identified with a Radon measure on $\Lambda\setminus(H^u)^{-1}(\omega)$. Now let $\tau\in\{0,1\}^*\setminus\{0^n:n\ge 0\}$. Fix $j\le k$ with $k-j=|\tau|-1$. We shall prove that for every $n\ge|j|$,

$$\mu^*_{t^u}\circ\sigma^{-n}\big([\tau|_j^k]\big)=\tilde\mu_{t^u}\big([\tau|_0^{|\tau|-1}\big). \tag{8.10}$$

Indeed, since $\mu^*_{t^u}$ is an extension of $\tilde\mu_{t^u}$, the invariance of the measure $\tilde\mu_{t^u}$ implies that

$$\begin{aligned}\mu^*_{t^u}\circ\sigma^{-n}\big([\tau|_j^k]\big)&=\mu^*_{t^u}\big([\tau|_{j+n}^{k+n}]\big)=\mu^*_{t^u}\left(\bigcup_{|\gamma|=j+n}[\gamma\tau|_0^{k+n}]\right)=\sum_{|\gamma|=j+n}\mu^*_{t^u}\big([\gamma\tau|_0^{k+n}]\big)\\&=\sum_{|\gamma|=j+n}\tilde\mu_{t^u}\big([\gamma\tau|_0^{k+n}]^+\big)=\tilde\mu_{t^u}\left(\bigcup_{|\gamma|=j+n}[\gamma\tau|_0^{k+n}]^+\right)=\tilde\mu_{t^u}\big([\tau|_{j+n}^{k+n}]\big)\\&=\tilde\mu_{t^u}\circ\sigma^{-n}\big([\tau|_0^{k-j}]^+\big)=\tilde\mu_{t^u}\big([\tau|_0^{|\tau|-1}\big).\end{aligned}$$

This proves (8.10). Now let l_∞ be the Banach space consisting of all bounded real sequences and let $L:l_\infty\to\mathbb{R}$ be a Banach limit. It follows from (8.10) that if A is a Borel subset of one of the cylinders of the form $[0^q1|_0^q]$, $q\ge 0$, then $\{\mu^*_{t^u}\circ\sigma^{-n}(A)\}_{n=0}^\infty\in l_\infty$, and we can define

$$\mu_{t^u}(A)=L\big(\{\mu^*_{t^u}\circ\sigma^{-n}(A)\}_{n=0}^\infty\big). \tag{8.11}$$

We shall show first that for every $q\ge 0$, formula (8.11) defines a Borel finite measure on $[0^q1|_0^q]$. Obviously, $\mu_{t^u}(\emptyset)=0$ and μ_{t^u} is monotone since L and $\mu^*_{t^u}$ are. Now suppose that $\{B_k\}_{k=1}^\infty$ is a sequence of mutually disjoint Borel subsets of $[0^q1|_0^q]$. Since L is a bounded operator, we therefore obtain

$$\begin{aligned}\mu_{t^u}\left(\bigcup_{k=1}^\infty B_k\right)&=L\left(\left\{\mu^*_{t^u}\circ\sigma^{-n}\left(\bigcup_{k=1}^\infty B_k\right)\right\}_{n=0}^\infty\right)=L\left(\left\{\sum_{k=1}^\infty\mu^*_{t^u}\circ\sigma^{-n}\big(B_k\big)\right\}_{n=0}^\infty\right)\\&=L\left(\sum_{k=1}^\infty\left\{\mu^*_{t^u}\circ\sigma^{-n}\big(B_k\big)\right\}_{n=0}^\infty\right)=\sum_{k=1}^\infty L\left(\left\{\mu^*_{t^u}\circ\sigma^{-n}\big(B_k\big)\right\}_{n=0}^\infty\right)=\sum_{k=1}^\infty\mu_{t^u}\big(B_k\big).\end{aligned}$$

Thus, μ_{t^u} is a Borel measure on $[0^q1|_0^q]$, and its finiteness follows from (8.10). Since all the sets $[0^q1|_0^q]$, $q\ge 0$, are mutually disjoint, the formula

$$\mu_{t^u}(B)=\sum_{q=0}^\infty\mu_{t^u}(B\cap[0^q1|_0^q])$$

defines a Borel measure on $\bigcup_{q=0}^\infty[0^q1|_0^q]=\Lambda\setminus(H^u)^{-1}(\omega)$. Since all the measures $\mu_{t^u}([0^q1|_0^q])$, $q\ge 0$, are finite, μ_{t^u} is a σ-finite measure. We now may extend μ_{t^u} on the entire set Λ by defining $\mu_{t^u}((H^u)^{-1}(\omega))=0$. Now for every $\tau\in\{0,1\}^*\setminus\{0^n:n\ge 0\}$, by using (8.11) and (8.10), we obtain that

$$\mu_{t^u}\circ(H^u)^{-1}([\tau|_0^k]^+)=\mu_{t^u}([\tau|_0^k])=L\big(\{\mu^*_{t^u}\circ\sigma^{-n}([\tau|_0^k])\}_{n=0}^\infty\big)=\tilde\mu_{t^u}([\tau|_0^k]^+),$$

where $k=|\tau|-1$. Therefore, $\mu_{t^u}\circ(H^u)^{-1}=\tilde\mu_{t^u}$ on $J_\Phi\setminus\{\omega\}$. Since in addition $\mu_{t^u}\circ(H^u)^{-1}(\omega)=\tilde\mu_{t^u}(\omega)$, we see that

$$\mu_{t^u}\circ(H^u)^{-1}=\tilde\mu_{t^u} \tag{8.12}$$

on the entire set J_Φ. Since for every $q \geq 0$ and for every Borel set $A \subset [0^q 1|0^q]$ we have that $\sigma^{-1}(A) \subset [1|^0_0] \cup [0^{q+1}1|^{q+1}_0]$, we may conclude that

$$\begin{aligned}
\mu_{t^u}(\sigma^{-1}(A)) &= \mu_{t^u}\big(\sigma^{-1}(A) \cap [1|^0_0] \cup \sigma^{-1}(A) \cap [0^{q+1}1|^{q+1}_0]\big) \\
&= \mu_{t^u}\big(\sigma^{-1}(A) \cap [1|^0_0]\big) + \mu_{t^u}\big(\sigma^{-1}(A) \cap [0^{q+1}1|^{q+1}_0]\big) \\
&= L\big(\{\mu^*_{t^u} \circ \sigma^{-n}\big(\sigma^{-1}(A) \cap [1^0_0]\big)\}_{n=0}^\infty\big) + L\big(\{\mu^*_{t^u} \circ \sigma^{-n}\big(\sigma^{-1}(A) \cap [0^{q+1}1^{q+1}_0]\big)\}_{n=0}^\infty\big) \\
&= L\big(\{\mu^*_{t^u} \circ \sigma^{-n}\big(\sigma^{-1}(A) \cap [1^0_0]\big) + \mu^*_{t^u} \circ \sigma^{-n}\big(\sigma^{-1}(A) \cap [0^{q+1}1^{q+1}_0]\big)\}_{n=0}^\infty\big) \\
&= L\big(\{\mu^*_{t^u} \circ \sigma^{-(n+1)}(A)\}_{n=0}^\infty\big) = \mu_{t^u}(A).
\end{aligned}$$

Since in addition, by (8.12),

$$\begin{aligned}
\mu_{t^u}(\sigma^{-1}((H^u)^{-1}(\omega))) &= \mu_{t^u}\big((H^u \circ \sigma)^{-1}(\omega)\big) = \mu_{t^u}\big((\sigma \circ H^u)^{-1}(\omega)\big) \\
&= \mu_{t^u} \circ (H^u)^{-1}(\sigma^{-1}(\omega)) = \tilde{\mu}_{t^u}(\sigma^{-1}(\omega)) = 0 = \mu_{t^u}((H^u)^{-1}(\omega)),
\end{aligned}$$

we conclude that the measure μ_{t^u} is $\sigma(f)$-invariant. So, we are left to show that the measure μ_{t^u} is ergodic and conservative. To prove ergodicity assume that $E \subset \Lambda$ is a Borel set such that $\sigma^{-1}(E) = E$ and $\mu_{t^u}(E) > 0$. Let μ_E be the measure on Λ given by the formula $\mu_E(A) = \mu_{t^u}(A \cap E)$. We notice that μ_E is f-invariant (as μ_{t^u} is) and $f^{-1}(E) = E$. Thus, $\mu_E \circ (H^u)^{-1}$ is a shift-invariant measure on J_Φ. Since for every Borel set $B \subset J_\Phi$,

$$\mu_E \circ (H^u)^{-1}(B) = \mu_E((H^u)^{-1}(B) \cap E) \leq \mu_{t^u}((H^u)^{-1}(B)) = \tilde{\mu}_{t^u}(B),$$

we see that the measure $\mu_E \circ (H^u)^{-1}$ is absolutely continuous with respect to $\tilde{\mu}_{t^u}$. Invoking now the fact that $\tilde{\mu}_{t^u}$ is ergodic and conservative, we conclude that $\mu_E \circ (H^u)^{-1}$ is a constant multiple of $\tilde{\mu}_{t^u}$. Normalizing $\tilde{\mu}_{t^u}$ appropriately we may assume without loss of generality that $\mu_E \circ (H^u)^{-1} = \tilde{\mu}_{t^u}$. Thus, $\mu_E \circ (H^u)^{-1} = \mu_{t^u} \circ (H^u)^{-1}$, and therefore (see the first part of the proof) $\mu_E = \mu_{t^u}$. In particuilar, $\mu_{t^u}(\Lambda \setminus E) = 0$. This establishes the ergodicity of the measure μ_{t^u}. Now, since $f : \Lambda \to \Lambda$ is invertible, Proposition 1.2.1 in [A] yields that μ_{t^u} is conservative. This completes the proof. □

9. Conditional Measures

Suppose that $(X, \mathcal{A}, \nu)$ is a σ-finite measure space. Suppose also that $\overline{\mathcal{A}}$ is a sub-σ-algebra of $\mathcal{A}$. It easily follows from the probabilistic case that for every ν-integrable function $g : X \to \mathbb{R}$ there exists $E(g|\overline{\mathcal{A}}) : X \to \mathbb{R}$, a unique expected value of g with respect to the σ-algebra $\overline{\mathcal{A}}$, i.e. an $\overline{\mathcal{A}}$-measurable function for which

$$\int_A E(g|\overline{\mathcal{A}}) d\nu = \int_A g d\nu$$

for every set $A \in \overline{\mathcal{A}}$. Let $\mathcal{P} = \mathcal{P}_{\overline{\mathcal{A}}}$ be the measurable partition generated by the σ-algebra $\overline{\mathcal{A}}$. The canonical system $\{\nu^x\}_{x \in X}$ of ν conditional measures with respect to the σ-algebra $\overline{\mathcal{A}}$ (and partition $\mathcal{P}$) is given by the following formula:

$$\nu^x(B \cap \mathcal{P}(x)) = E(\mathbb{1}_B|\overline{\mathcal{A}})(x) \tag{9.1}$$

for every set $B \in \mathcal{A}$. We note that for ν-a.e. $x \in X$ the value $\nu^x(B \cap \mathcal{P}(x))$ is independent of the choice of a set $B' \in \mathcal{A}$ with the property that $B' \cap \mathcal{P}(x) = B \cap \mathcal{P}(x)$. Since ν is σ-finite, it easily follows from Martingale's Convergence Theorem that if $\{\mathcal{A}_n\}_{n=0}^{\infty}$ is an ascending sequence of sub-σ-algebras of $\overline{\mathcal{A}}$, generating $\overline{\mathcal{A}}$, then

$$E(1\!\!1_B|\overline{\mathcal{A}})(x) = \lim_{n\to\infty} E\big(1\!\!1_B|\mathcal{A}_n\big)(x) \tag{9.2}$$

for ν-a.e. $x \in X$. We now consider the σ-finite measure space $(\Lambda, \mathcal{B}, \mu_{t^u})$, the partition $\mathcal{P}^u$ of Λ into unstable manifolds $W_i^u(x)$, $x \in \Lambda$, $i = 0, 1$, and the σ-algebra $\mathcal{B}^u$ generated by the partition $\mathcal{P}^u$. In the language of the symbol space Σ_2^{+-} the unstable manifolds take on the form $W_0^u(\gamma) = [\gamma|_{-\infty}^0]$. Without confusion we will frequently use either of the two languages: symbolic or "differentiable". For every $n \geq 0$ let B_n^u be the finite σ-algebra generated by the cylinders $[\gamma|_{-n}^0]$, $\gamma \in \Sigma_2^{+-}$. Obviously, $\{\mathcal{B}_n^u\}_{n=0}^{\infty}$ is an ascending sequence of sub-σ-algebras generating $\mathcal{B}^u$. Applying (9.1) and (9.2), we see that for every $\gamma \in \Sigma_2^{+-}$ and every $\tau \in \{0,1\}^q$, we have

$$\begin{aligned}
\mu_{t^u}^{\gamma}\big([\gamma|_{-\infty}^0\tau|_{-\infty}^q]\big) &= \mu_{t^u}^{\gamma}\big([\tau|_1^q]\cap[\gamma|_{-\infty}^0]\big) = E\left(1\!\!1|_{[\tau|_1^q]}|B^u\right)(\gamma) = \lim_{n\to\infty} E\left(1\!\!1|_{[\tau|_1^q]}|B_n^u\right)(\gamma)\\
&= \lim_{n\to\infty}\frac{\mu_{t^u}\big([\tau|_1^q]\cap[\gamma|_{-n}^0]\big)}{\mu_{t^u}\big([\gamma|_{-n}^0]\big)} = \lim_{n\to\infty}\frac{\mu_{t^u}\big([\gamma|_{-n}^0\tau|_{-n}^q]\big)}{\mu_{t^u}\big([\gamma|_{-n}^0]\big)}\\
&= \lim_{n\to\infty}\frac{\mu_{t^u}\big([\gamma|_{-n}^0\tau|_0^{n+q}]\big)}{\mu_{t^u}\big([\gamma|_{-n}^0|_0^n]\big)} = \lim_{n\to\infty}\frac{\tilde{\mu}_{t^u}\big([\gamma|_{-n}^0\tau|_0^{n+q}]^+\big)}{\tilde{\mu}_{t^u}\big([\gamma|_{-n}^0|_0^n]^+\big)},
\end{aligned}$$

where we could have written the second last equality sign since the measure μ_{t^u} is shift-invariant and we wrote the second equality sign because of Theorem 8.3. We recall that $\rho = \frac{d\tilde{\mu}_{t^u}}{d\tilde{m}_{t^u}}$. Using the fact that $\tilde{P}(t^u) = P(t^u) = 0$, we further can write by making use of (7.9) that

$$\mu_{t^u}^{\gamma}\big([\gamma|_{-\infty}^0\tau|_{-\infty}^q]\big) = \lim_{n\to\infty}\frac{\int_{J_\Phi}|\phi'_{\gamma|_{-n}^0\tau}|^{t^u}\rho\circ\phi_{\gamma|_{-n}^0\tau}d\tilde{\mu}_{t^u}}{\int_{J_\Phi}|\phi'_{\gamma|_{-n}^0}|^{t^u}\rho\circ\phi_{\gamma|_{-n}^0}d\tilde{\mu}_{t^u}} \asymp \lim_{n\to\infty}\frac{\int_{J_\Phi}|\phi'_{\gamma|_{-n}^0\tau}|^{t^u}d\tilde{\mu}_{t^u}}{\int_{J_\Phi}|\phi'_{\gamma|_{-n}^0}|^{t^u}d\tilde{\mu}_{t^u}},$$

where, and we wrote the comparability sign using (8.2). We now assume that $\tau \neq 0^q$ and $\gamma|_{-\infty}^0 \neq 0^\infty|_{-\infty}^0$. Let $i \geq 0$ be the least integer such that $\gamma_{-i} = 1$. Then the distortion property allows us to continue as follows:

$$\begin{aligned}
\mu_{t^u}^{\gamma}\big([\gamma|_{-\infty}^0\tau|_{-\infty}^q]\big) &\asymp K_{i,j}^{(3)}\frac{||\phi'_{\gamma|_{-n}^0\tau}||^{t^u}}{||\phi'_{\gamma|_{-n}^0}||^{t^u}} \asymp K_{i,j}^{(2)}\frac{||\phi'_{\gamma|_{-n}^0}||^{t^u}||\phi'_\tau||^{t^u}}{||\phi'_{\gamma|_{-n}^0}||^{t^u}} = K_{i,j}^{(2)}||\phi'_\tau||^{t^u}\\
&\asymp K'_{i,j}||\phi'_{\rho_0\tau}||^h \asymp K_{i,j}\tilde{m}_h\big([\rho_0\tau|_0^q]\big),
\end{aligned}$$

where the constants $K_{i,j}^{(3)}, K_{i,j}^{(2)}, K'_{i,j}$ and $K_{i,j}$ depend on i and $j = q - l$, where l last position of the letter 1 in the word $\tau = \tau_1\tau_2\ldots\tau_q$. Since the conformal measure $\tilde{\mu}_{t^u}$ is a constant multiple of packing measure $P_{t^u}|_{J_\Phi}$ (see Theorem 6.4 in [U2]), since the holonomy maps between unstable manifolds are uniformly Lipschitz, and since $H\big([\gamma|_{-\infty}^0\tau|_{-\infty}^q]\big) = [\gamma_0\tau|_0^q]$, we obtain that

$$\mu_{t^u}^{\gamma}\big([\gamma|_{-\infty}^0\tau|_{-\infty}^q]\big) \asymp K_{i,j}P_{t^u}\big([\gamma|_{-\infty}^0\tau|_{-\infty}^q]\big).$$

We now may conclude that

$$\mu_{t^u}^{\gamma}|_{[\gamma|_{-\infty}^0]\cap[1|_n^n]} \asymp K_{i,0}P_{t^u}|_{[\gamma|_{-\infty}^0]\cap[1|_n^n]}$$

for all $n \geq 0$. Since in addition $\mu_{t^u}^{\gamma}\big([\gamma|_{-\infty}^0 0^\infty]\big) = P_{t^u}\big([\gamma|_{-\infty}^0 0^\infty]\big) = 0$ (also using that μ_{t^u} is atomless), and since $\big([\gamma|_{-\infty}^0] = [\gamma|_{-\infty}^0 0^\infty] \cup \bigcup_{n=0}^{\infty}[1|_n^n]$, we therefore have proven the following main result of this section.

Theorem 9.1. *For every $i = 0, 1$ and all $x \in \Lambda \cap R_i$ the conditional measure $\mu^x_{t^u}$ on the unstable manifold $W^u_i(x)$ is equivalent to the packing measure P_{t^u} restricted to the manifold $W^u_i(x)$.*

10. Dimension of the horseshoe

In this section we establish formulas for the stable dimension and the dimension of the parabolic horseshoe. In this and in the subsequent sections we consider probability measures rather than σ-finite measures. In particular, the notion of equilibrium states will be from this point on exclusively used in the context of probability measures. Let f be a parabolic horseshoe map. We denote by $\mathcal{M}$ the space of all Borel f-invariant probability measures on Λ endowed with weak* topology. This makes $\mathcal{M}$ to a compact convex space. Moreover, we denote by $\mathcal{M}_E \subset \mathcal{M}$ the subset of ergodic measures. Let $\mu \in \mathcal{M}$. We define

$$\lambda_u(\mu) = \int_\Lambda \log |D_u f| \, d\mu \quad \text{and} \quad \lambda_s(\mu) = \int_\Lambda \log |D_s f| \, d\mu. \tag{10.1}$$

Note that $\lambda_u(\mu)$ and $\lambda_s(\mu)$ coincide with the μ-average of the pointwise Lyapunov exponents of f. It follows from properties (g), (h) and (i) of the parabolic horseshoe (see section 5) that

$$\lambda_u(\mu) \geq 0 \quad \text{and} \quad \lambda_s(\mu) < \log \gamma < 0 \tag{10.2}$$

for all $\mu \in \mathcal{M}$, where $\gamma < 1$ is the constant in property (g) of the parabolic horseshoe. We say that $\mu \in \mathcal{M}$ is a hyperbolic measure if $\lambda_u(\mu) > 0$. In this case we refer to $\lambda_{u/s}(\mu)$ as the positive/negative Lyapunov exponent of μ. Recall that μ_ω denotes the Dirac-δ measure supported on the parabolic fixed point ω. We begin with a preliminary result.

Lemma 10.1. *Let $\mu \in \mathcal{M}$. Then μ is a hyperbolic measure if and only if $\mu \neq \mu_\omega$.*

Proof. We first consider the case when μ is ergodic. Obviously, if $\mu = \mu_\omega$ then $\lambda_u(\mu) = 0$, and μ is not hyperbolic. Assume now that $\mu \neq \mu_\omega$. Therefore, Birkhoff's Ergodic Theorem implies that $\mu(W^s(\omega)) < 1$, and since μ is ergodic we obtain $\mu(W^s(\omega)) = 0$. Note that $|D_u f(x)| > 1$ for all $x \in \Lambda \setminus W^s(\omega)$. Therefore, it follows from the definition of the Lyapunov exponent and the fact that $x \mapsto D_u f(x)$ is continuous that $\lambda_u(\mu) > 0$. By using that $\lambda_s(\nu) < 0$ for all $\nu \in \mathcal{M}$ we obtain that μ is hyperbolic. Finally, the case when μ is not ergodic follows from the ergodic case by using an ergodic decomposition of μ. $\square$

We define the stable pressure function $P^s : \mathbb{R} \to \mathbb{R}$ by $P^s(t) = P(f|\Lambda, t\phi_s)$, where $P^s(f|\Lambda, .)$ is the ordinary topological pressure of the dynamical system $f|\Lambda$ and $\phi_s = \log |D_s f| : \Lambda \to \mathbb{R}$. Next we establish a Bowen-Ruelle-Manning-McCluskey type of formula for the stable dimension of Λ.

Theorem 10.2. *Let f a parabolic horseshoe map having a Lipschitz continuous unstable foliation. Then $t^s \stackrel{\text{def}}{=} \dim_H W^s_{\text{loc}}(x) \cap \Lambda$ does not depend on $x \in \Lambda$. Moreover, t^s is given by the unique solution of*

$$P^s(t) = 0, \tag{10.3}$$

and $0 < t^s < 1$.

Proof. The proof is analogous to the case uniformly hyperbolic sets on surfaces, see [MM]. Therefore, we provide only a sketch. First, note that since the unstable foliation is Lipschitz continuous, it follows immediately that $\dim_H W^s_{\text{loc}}(x) \cap \Lambda$ is independent of $x \in \Lambda$. Observe that $t \mapsto t\phi_s$ is strictly decreasing. Therefore, P^s is also strictly decreasing. Hence t^s is well-defined. Note that $h_{\mu_\omega}(f)=0$ and $\lambda_s(\mu_\omega)<0$. In particular, μ_ω is not an equilibrium state of $t^s\phi^s$. On the other hand, $f|\Lambda$ is expansive, which implies that the entropy map $\nu\mapsto h_\nu(f)$ is upper semi-continuous on $\mathcal{M}$. Hence, for every $\varphi\in C(\Lambda,\mathbb{R})$ there exists at least one equilibrium state. Let μ be an ergodic equilibrium state of $t^s\phi^s$. It now follows from the definition of t^s that $t^s=\mathrm{h}_\mu(f)/\lambda_s(\mu)$. Applying a result of Mendosa (see [Me, Theorem 1]) we deduce that $t^s\leq \dim_H W^s_{\mathrm{loc}}(x)\cap\Lambda$. Finally, the proofs of the inequalities $\dim_H W^s_{\mathrm{loc}}(x)\cap\Lambda\leq t^s<1$ and $t^s>0$ are analogous to the case of uniformly hyperbolic sets on surfaces (see [MM]). □

Remarks.
(i) Similar to the case of uniformly hyperbolic surface diffeomorphisms one can show that the potential $t\phi_s$ has a unique equilibrium state for all $t\geq 0$.
(ii) We note that Theorem 10.2 in particular applies to almost linear parabolic horseshoe maps (see Example 1) as well as to the more general case of small perturbations of hyperbolic horseshoes of smooth type which were discussed in part (ii) of the remark after Example 1.

Theorem 10.3. *Let f be a parabolic horseshoe map of smooth type having a Lipschitz continuous unstable foliation. Then* $\dim_H\Lambda=t^u+t^s$.

Proof. Let $i=0,1$. We consider the set $\Lambda_i=\Lambda\cap R_i$. Given $x\in\Lambda_i$ it follows from property (e) of the parabolic horseshoe (see section 5) that

$$[\cdot,\cdot]:W^u(x)\cap\Lambda_i\times W^s(x)\cap\Lambda_i\to\Lambda_i$$

is a homeomorphism. Moreover, since f is of smooth type and since the unstable foliation of f is Lipschitz continuous, it follows that $[\cdot,\cdot]$ as well as $[\cdot,\cdot]^{-1}$ are Lipschitz continuous and therefore preserve Hausdorff dimension. Similar, as in the case hyperbolic sets on surfaces one can show that $\dim_H W^s(x)\cap\Lambda_i=\overline{\dim}_B W^s(x)\cap\Lambda_i$. Using the formula for the Hausdorff dimension of products we obtain $\dim_H W^u(x)\cap\Lambda_i\times W^s(x)\cap\Lambda_i=\dim_H W^s(x)\cap\Lambda_i+\dim_H W^u(x)\cap\Lambda_i$. This completes the proof. □

We define the stable set of Λ by

$$W^s(\Lambda)=\left\{x\in S: f^n(x)\in S \text{ for all } n\in\mathbb{N},\ \lim_{n\to\infty} dist(f^n(x),\Lambda)=0\right\}. \tag{10.4}$$

Similarly we define the unstable set $W^u(\Lambda)$ of Λ as the stable set of Λ with respect to f^{-1}. It follows immediately from the properties of the parabolic horseshoe that

$$W^{s/u}(\Lambda)=\bigcup_{x\in\Lambda}W^{s/u}(x). \tag{10.5}$$

Note that (10.5) is also a consequence of the Shadowing Lemma.

Theorem 10.4. *Let f be a parabolic horseshoe map of smooth type having a Lipschitz continuous unstable foliation. Then* $\dim_H W^{s/u}(\Lambda)=t^{u/s}+1<2$.

Proof. We only proof the formula for the dimension of the stable set. The proof for the unstable set is analogous. Note that by Theorem 7.4, $t^u<1$ which gives the right-hand side inequality.

Combining that Λ has a local product structure with (10.5) implies that it suffices to prove that

$$\dim_H \left(\bigcup_{y \in W^u_{\mathrm{loc}}(x) \cap \Lambda} W^s_{\mathrm{loc}}(y) \right) = t^u + 1 \tag{10.6}$$

for all $x \in \Lambda$. Let $x \in \Lambda$. Set $A_x = W^u_{\mathrm{loc}}(p) \cap \Lambda$. Since Λ has a local product structure, there exists a homeomorphism

$$H : A_x \times (-1,1) \to \bigcup_{y \in A_x} W^s_{\mathrm{loc}}(y), \tag{10.7}$$

with the property that $H(y \times (-1,1)) = W^s_{\mathrm{loc}}(y)$ for all $y \in A_x$. Moreover, since f is of smooth type and since f has a Lipschitz continuous unstable foliation it follows that H as well as H^{-1} are Lipschitz continuous. Applying Theorem 10.2 completes the proof. □

11. Generalized physical measures

Let f be parabolic horseshoe map of smooth type having a Lipschitz continuous unstable foliation. In this section provide a classification for f having a generalized physical measure. Given $\mu \in \mathcal{M}$ we define the basin of μ by

$$\mathcal{B}(\mu) = \left\{ x \in S : f^n(x) \in S \text{ for all } n \in \mathbb{N}, \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} = \mu \right\}. \tag{11.1}$$

Here $\delta_{f^i(x)}$ denotes the Dirac-δ measure on $f^i(x)$. The basin of μ is sometimes also called the set of future generic points of μ, see [DGS] and [Ma]. A measure $\mu \in \mathcal{M}_E$ is called a *physical measure* if $B(\mu)$ has positive Lebesgue measure. Obviously,

$$\mathcal{B}(\mu) \subset W^s(\Lambda). \tag{11.2}$$

Therefore, by Theorem 10.4 the map f can not have a physical measure. Following [Wo] we say that $\mu \in \mathcal{M}_E$ a generalized physical measure if $\mathcal{B}(\mu)$ is as large as possible in the sense that

$$\dim_H \mathcal{B}(\mu) = \dim_H W^s(\Lambda). \tag{11.3}$$

We now prove a formula for the Hausdorff dimension of the basin of a hyperbolic measure.

Proposition 11.1. *Let $\mu \in \mathcal{M}_E \setminus \{\mu_\omega\}$. Then*

$$\dim_H \mathcal{B}(\mu) = \frac{h_\mu(f)}{\lambda_u(\mu)} + 1. \tag{11.4}$$

Proof. Note that by Lemma 10.1, $\lambda_u(\mu) > 0$; hence the right-hand side of (11.4) is well-defined. It is easy to see that if $y \in f(S)$ then

$$y \in \mathcal{B}(\mu) \text{ if and only if } \exists x \in \Lambda \text{ with } y \in W^s_{\mathrm{loc}}(x). \tag{11.5}$$

Combining (11.5) with property (e) of the parabolic horseshoe (see section 3) it follows that it is sufficient to show that for each $x \in \Lambda$,

$$\dim_H \left(\bigcup_{y \in A_x} W^s_{\mathrm{loc}}(y) \right) = \frac{h_\nu(f)}{\lambda_u(\nu)} + 1, \tag{11.6}$$

where $A_x = W^u_{\mathrm{loc}}(x) \cap \mathcal{B}(\mu)$. Pick $x \in \Lambda$. The same methods used by Manning [Ma] in the context of hyperbolic surface diffeomorphisms (see [Me] for the analogous result in the non-uniformly hyperbolic setting) can be used to show that

$$\dim_H A_x = \frac{h_\mu(f)}{\lambda_u(\mu)}. \tag{11.7}$$

Therefore, (11.6) can be shown analogously as Theorem 10.4 by using the bi-Lipschitz continuous homeomorphism H. □

Recall that $\phi_u = \log|D_u f| : \Lambda \to \mathbb{R}$. We now present our main result about generalized physical measures.

Theorem 11.2. *Let f be a parabolic horseshoe of smooth type having a Lipschitz continuous unstable foliation. Then the following are equivalent:*

(i) *f admits a generalized physical measure;*
(ii) *The potential $-t^u\phi_u$ has more than one (finite) equilibrium state;*
(iii) *The potential $-t^u\phi_u$ has precisely two ergodic (finite) equilibrium states;*
(iv) *P^u is not differentiable at t^u;*
(v) *$t^u > 2\frac{\beta}{\beta+1}$.*

Proof. (i)⇒(ii): Let μ be a generalized physical measure of f. It follows from Theorem 10.4 and Proposition 11.1 that $t^u = h_\mu(f)/\lambda_u(\mu)$. Thus, μ is an ergodic equilibrium state of the potential $-t^u\phi_u$. Using that $\mu \neq \mu_\omega$ implies (ii). The implication (ii)⇒(iii) is a consequence of Theorem 8.3. Moreover, (iii)⇒(iv) follows from [J, Corollary 1]. The implication (iv)⇒(v) follows from [J, Corollary 1] and Theorem 8.1. Finally, if (v) holds, then by Theorem 8.1 there exists an ergodic equilibrium state μ of the potential $-t^u\phi_u$ with $\mu \neq \mu_\omega$. Hence, $t^u = h_\mu(f)/\lambda_u(\mu)$, and we may conclude from Theorem 10.4 and Proposition 11.1 that μ is a generalized physical measure for f. □

As an application of Theorem 11.2 we construct parabolic horseshoe maps with as well as without generalized physical measures.

Corollary 11.3. *There exists a parabolic horseshoe map having a generalized physical measure as well as one having no generalized physical measure.*

Proof. We first construct an example of a parabolic horseshoe map having a generalized physical measure. Pick $0 < \beta < 1, \eta > 1$ and $\lambda_u > 2$ such that

$$\frac{\log 2}{\log \eta + \log \lambda_u} > 2\frac{\beta}{\beta + 1}. \tag{11.8}$$

Let f be a parabolic horseshoe map as defined in Example 1 with the corresponding constants β, η and λ_u. In particular, f is of smooth type and has a Lipschitz continuous unstable foliation. It follows from (5.4) that

$$\lambda_u(\nu) \leq \log \lambda_u + \log \eta \tag{11.9}$$

for all $\nu \in \mathcal{M}$. Let μ denote the measure of maximal entropy of f, i.e. the unique measure satisfying $h_\mu(f) = \log 2$. Therefore, (11.9) and Theorem 1 in [Me] imply that $t^u > 2\frac{\beta}{\beta+1}$. We now may conclude from Theorem 11.2 that f has a generalized physical measure. The existence of a parabolic horseshoe map having no generalized physical measure can be easily seen. Just pick any parabolic horseshoe map f of smooth type having a Lipschitz continuous unstable foliation (for instance a map as in Example 1) with $\beta \geq 1$. Recall that $t^u < 1$ (see Theorem 7.4); hence $t^u < 2\beta/(\beta+1)$, and therefore, Theorem 11.2 implies that f does not have a generalized physical measure. $\square$

Corollary 11.4. *The existence of a generalized physical measure is not a (topological) conjugacy invariant.*

Proof. By Corollary 11.3 there exist parabolic horseshoe maps f_k, $k = 1, 2$ such that f_1 has a generalized physical measure and f_2 does not have a generalized physical measure. Since $f_1|\Lambda_1$ and $f|\Lambda_2$ are both topological conjugate to the shift map $\sigma : \Sigma_2^{+-} \to \Sigma_2^{+-}$, the result follows. $\square$

12. Measures of maximal dimension

In this section we discuss the existence of ergodic measures of maximal dimension for a particular subclass of parabolic horseshoe maps. In particular, we provide a criteria which guarantees that no ergodic measure of maximal dimension exists. Recall that in this section the notion of equilibrium states is meant in the space of probability measures.

Let $f : S \to \mathbb{R}^2$ be a parabolic horseshoe map. We say that f has constant contraction rate if there is $0 < c < 1$ such that $|D_s f(x)| = c$ for all $x \in \Lambda$. For example the almost linear parabolic horseshoe maps in Example 1 have constant contraction rate $c < 1/2$. Given $\mu \in \mathcal{M}$ we define the Hausdorff dimension of μ by

$$\dim_H \mu = \inf\{\dim_H A : \mu(A) = 1\}. \tag{12.1}$$

Following [BW1] we say that $\mu \in \mathcal{M}_E$ is an ergodic measure of maximal dimension if

$$\dim_H \mu = \sup\{\dim_H \nu : \nu \in \mathcal{M}_E\} \stackrel{\text{def}}{=} \delta(f). \tag{12.2}$$

It follows from work in [BW2] that the definition of $\delta(f)$ in (12.2) is the same when the supremum is taken over all (not necessarily ergodic) measures in $\mathcal{M}$. Let $\mu \in \mathcal{M} \setminus \{\mu_\omega\}$. It follows from Young's formula [Y] (also using Lemma 10.1) that

$$\text{if } \mu \text{ is ergodic, then } \dim_H \mu = d(\mu), \tag{12.3}$$

where

$$d(\mu) \stackrel{\text{def}}{=} h_\mu(f)\left(\frac{1}{\lambda_u(\mu)} - \frac{1}{\lambda_s(\mu)}\right).$$

We now define a one-parameter family of measures $(\nu_t)_{t\in[0,t^u]}$ which will be crucial for the analysis of measures of maximal dimension. For $t \in [0, t^u)$ we define ν_t to be the unique equilibrium state of the potential $-t\phi_u$. Note that ν_t is well-defined. Indeed, P^u is differentiable in $[0, t^u)$. This is a consequence of Theorem 7.6 and the fact that $f|\Lambda$ has a unique measure of maximal entropy. Thus, by [J, Corollary 1] the potential $-t\phi_u$ has a unique equilibrium state. Next, we define ν_{t^u}. In the case when the potential $-t^u\phi_u$ has more than one equilibrium state we define ν_{t^u} to be the unique hyperbolic ergodic equilibrium state of $-t^u\phi_u$. Otherwise, we define $\nu_{t^u} = \mu_\omega$.

The results of this section will be based on a careful analysis of the dimension of the measures ν_t. For simplicity we write $h(t) = h_{\nu_t}(f)$ and $\lambda_u(t) = \lambda_u(\nu_t)$ for all $t \in [0, t^u]$. Thus,

$$P^u(t) = h(t) - t\lambda_u(t). \tag{12.4}$$

It follows from standard properties of the topological pressure (see for example [J]) that if P^u is differentiable at t_0 then

$$\frac{dP^u(t_0)}{dt} = -\lambda_u(t_0). \tag{12.5}$$

We first prove a preliminary result.

Lemma 12.1. *We have the following:*

(i) *If $\nu_{t^u} = \mu_\omega$ then $P^u \in C^1([0,t^u])$;*
(ii) *If $\nu_{t^u} \neq \mu_\omega$ then $P^u \in C^1([0,t^u))$.*

Proof. We already know that P^u is real-analytic on $(0,t^u)$ (see Theorem 7.6). Therefore, we only have to consider $t=0$ and $t=t^u$. Since ν_0 is the unique measure of maximal entropy it follows from [J] that P^u is differentiable at 0. Similarly, if $\nu_{t^u} = \mu_\omega$ then P^u is differentiable at t^u. We claim that if $\nu_{t^u} = \mu_\omega$ then P^u is C^1 in a left neighborhood of t^u. Let $t_n \leq t^u$ with $t_n \to t^u$ for $n \to \infty$. By (12.5) is suffices to show that $\lambda^u(t_n) \to \lambda_u(t^u)$ for $n \to \infty$. By convexity of P^u and (12.5), λ_u is decreasing. Thus, $a = \lim_{n\to\infty} \lambda_u(t_n)$ exists. By compactness of $\mathcal{M}$ there exists $\mu \in \mathcal{M}$ such that μ is a weak* cluster point of the measures ν_{t_n}. Since λ_u is continuous on $\mathcal{M}$ we may conclude that $\lambda_u(\mu) = a$. Using that the entropy map $\nu \mapsto h_\nu(f)$ is upper semi-continuous on $\mathcal{M}$ (also using (12.4)) we obtain that μ is an equilibrium state of the potential $-t^u\phi_u$; hence $\mu = \nu_{t^u}$ which proves the claim. The proof of the statement that P^u is C^1 in a right neighborhood of 0 is entirely analogous. □

Since P^u is real-analytic in $(0,t^u)$, (12.5) implies that λ_u is also real-analytic in $(0,t^u)$. Hence, by (12.4), h is also real-analytic in $(0,t^u)$. Moreover,

$$\frac{dh(t_0)}{dt} = \frac{t_0 d\lambda_u(t_0)}{dt} \tag{12.6}$$

for all $t_0 \in (0,t^u)$. We now prove another preliminary result.

Lemma 12.2. *The functions λ_u and h are continuous in $[0,t^u]$. Moreover,*

$$\{\lambda_u(t) : t \in [0,t^u]\} = [\lambda_u(0), \lambda_u(t^u)]. \tag{12.7}$$

Proof. We first consider the case $\nu_{t^u} = \mu_\omega$. In this case (12.4), (12.5) and Lemma 12.1 imply that the functions λ_u and h are continuous in $[0,t^u]$. Moreover, by (12.5), $\{\lambda_u(t) : t \in [0,t^u]\}$ is a compact interval; thus, (12.7) follows from the fact that $t \mapsto \lambda_u(t)$ is decreasing in $[0,t^u]$.

Next, we consider the case $\nu_{t^u} \neq \mu_\omega$. Similarly as above, we can show that λ_u is continuous in $[0,t^u)$. We now prove the continuity of λ_u at t^u. Let $ES(t^u)$ be the set of all equilibrium states of the potential $-t^u\phi_u$. It is well-known that $ES(t^u)$ is a compact convex set whose extreme points are the ergodic measures in $ES(t^u)$, see e.g. [J]. Therefore, Theorem 8.3 implies that

$$ES(t^u) = \{s\mu_\omega + (1-s)\nu_{t^u} : s \in [0,1]\}. \tag{12.8}$$

Since λ_u is decreasing in $[0,t^u)$ it follows that $\lim_{t\to t^u-} \lambda_u(t)$ exists. Since $\mathcal{M}$ is compact, there exist a sequence $(t_n)_{n\in\mathbb{N}}$ converging to t^u from the left and a measure $\mu \in \mathcal{M}$ such that μ is a weak* limit of the sequence of measures $(\nu_{t_n})_{n\in\mathbb{N}}$. Since the entropy map is upper semi-continuous, we may conclude from the continuity of λ_u that μ is an equilibrium state of the potential $-t^u\phi_u$. Thus, $\mu = s\mu_\omega + (1-s)\nu_{t^u}$ for some $s \in [0,1]$. We claim that $s=0$. Indeed, otherwise there would exist $\epsilon > 0$ and $\delta > 0$ such that $\lambda_u(t) < \lambda_u(t^u) - \epsilon$ for all $t < t^u$ with $t^u - t < \delta$. But this contradicts the convexity of P^u and the fact that ν_{t^u} is an equilibrium state of $-t^u\phi_u$. We conclude that λ_u is continuous at t_0. Finally, the continuity of h and identity (12.7) can be shown analogously as in the case $\nu_{t^u} = \mu_\omega$ (see above). □

The following result shows that each ergodic measure of maximal dimension must be contained in the family of measures $(\nu_t)_{t\in[0,t^u]}$.

Theorem 12.3. *Let f be a parabolic horseshoe map of smooth type with constant contraction rate. Then*

$$\delta(f) = \sup\{\dim_H \nu_t : t \in [0, t^u]\}. \tag{12.9}$$

Moreover, if $\mu_{\max}$ is an ergodic measure of maximal dimension for f then there is $t \in [0, t^u]$ such that $\mu = \nu_t$.

Proof. Since f has constant contraction rate, there exists $0 < c < 1$ such that

$$\lambda_s(\nu) = \log c \tag{12.10}$$

for all $\nu \in \mathcal{M}$. Let $\mu \in \mathcal{M}_E$. We claim that there exists $t \in [0, t^u]$ such that $\dim_H \mu \le \dim_H \nu_t$. Note that the claim clearly implies (12.9). To prove the claim we first consider the case $\lambda_u(\mu) > \lambda_u(\nu_0)$. Using that ν_0 is the unique measure of maximal entropy, we conclude from (12.3) and (12.10) that $d(\mu) < d(\nu_0)$.

Next we assume that $\lambda_u(\mu) \in [\lambda_u(\nu_0), \lambda_u(\nu_{t^u})]$. It follows from (12.7) that there exists $t \in [0, t^u]$ with $\lambda_u(\nu_t) = \lambda_u(\mu)$. We now may conclude from the definition of ν_t and (12.4) that $h_{\nu_t}(f) \ge h_\mu(f)$. As before, (12.3) and (12.10) gives $d(\mu) \le d(\nu_t)$.

Finally, we consider the case $\lambda_u(\mu) < \lambda_u(\nu_{t^u})$. Clearly, this is only possible if $\nu_{t^u} \ne \mu_\omega$. Hence, ν_{t_u} is the unique ergodic hyperbolic equilibrium state of the potential $-t^u \phi_u$. We now may conclude from (12.3), (12.10) and the definition of t^u that

$$\dim_H \nu_{t^u} = t^u - \frac{h_{\nu_{t^u}}(f)}{\log c}. \tag{12.11}$$

On the other hand, by (12.4), $h_{\nu_{t^u}}(f) \ge h_\mu(f)$. We conclude that $d(\nu_{t^u}) > d(\mu)$. This completes the proof of the claim.

Assume now that $\mu_{\max}$ is an ergodic measure of maximal dimension for f. Applying the same arguments as before to $\mu_{\max}$ instead of μ implies that there exists $t_{\max} \in [0, t^u]$ such that $\lambda_u(\mu_{\max}) = \lambda_u(\nu_{t_{\max}})$ and $h_{\nu_t}(f) \ge h_{\mu_{\max}}(f)$. On the other hand, the fact that $\mu_{\max}$ is an ergodic measure of maximal dimension and (12.3) imply $h_{\mu_{\max}}(f) \ge h_{\nu_t}(f)$; hence $h_{\mu_{\max}}(f) = h_{\nu_t}(f)$. We conclude that $\mu_{\max}$ is an ergodic equilibrium state of the potential $-t_{\max}\phi_u$. Using that $\mu_{\max} \ne \mu_\omega$ implies $\mu_{\max} = \nu_{t_{\max}}$. This completes the proof. ☐

We now present the main result of this section.

Theorem 12.4. *f be a parabolic horseshoe map of smooth type with constant contraction rate c. Suppose that μ_ω is the unique equilibrium state of the potential $-t^u\phi_u$ and that*

$$\eta = \inf\left\{\frac{P^u(t)}{\lambda_u(\nu_t)^2} : t \in [0, t^u)\right\} > 0 \tag{12.12}$$

Then there exists $c_0 = c_0(\eta) > 0$ such that if $0 < c < c_0$, then f has no ergodic measure of maximal dimension.

Proof. Note that in order to show that f has no ergodic measure of maximal dimension it suffices to prove that

$$t \mapsto d(\nu_t) \tag{12.13}$$

is strictly increasing on $[0, t^u)$. This follows from Theorem 12.3 and the fact that $\dim_H \nu_{t^u} = 0$. We claim that P^u is not affine on $[0, t^u)$. Indeed, because otherwise (12.5) would imply that the measure of maximal entropy ν_0 is an equilibrium state of the potential $-t^u\phi_u$, in contradiction to the hypothesis. Since P^u a real-analytic convex function on $(0,t^u)$, we may conclude that

$$\frac{d^2P^u(t_0)}{dt^2} \geq 0, \tag{12.14}$$

where all the zeros of d^2P^u/d^2t in $(0,t^u)$ are isolated. Moreover, $\nu_{t_0} \neq \mu_\omega$ for all $t_0 \in [0,t^u)$. Thus ν_t is a hyperbolic measure (see Lemma 10.1). Let now $t_0 \in (0,t^u)$. It follows from (12.3), (12.5), (12.6) and an elementary calculation that

$$\begin{aligned} \frac{d}{dt}d(\nu_{t_0}) &= \frac{d}{dt}\left(\frac{h(t_0)}{\lambda_u(t_0)} - \frac{h(t_0)}{\log c}\right) \\ &= -\lambda_u'(t_0)\frac{P^u(t_0)}{\lambda_u(t_0)^2} - \frac{t_0\lambda_u'(t_0)}{\log c} \\ &= \frac{d^2P^u(t_0)}{dt^2}\left(\frac{P^u(t_0)}{\lambda_u(t_0)^2} + \frac{t_0}{\log c}\right) \\ &\stackrel{\text{def}}{=} \frac{d^2P^u(t_0)}{dt^2}A(t_0). \end{aligned} \tag{12.15}$$

We define $c_0 = \exp(-t^u/\eta)$. It follows from an easy computation that if $0 < c < c_0$ then $A(t_0) > 0$ for all $t_0 \in (0,t^u)$. Therefore, (12.15) implies that

$$\frac{d}{dt}d(\nu_{t_0}) \geq 0, \tag{12.16}$$

where all the zeros of $\frac{d}{dt}d(\nu_{t_0})$ in $(0,t^u)$ are isolated. We conclude that $d(\nu_t)$ is strictly increasing in $[0,t^u)$. This completes the proof. □

Remarks.
(i) We note that by L'Hospital's rule the hypotheses (12.12) holds for instance in the case when P^u has uniformly bounded second derivatives on $[0,t^u)$.
(ii) A strategy to construct a parabolic horseshoe map having no ergodic measure of maximal dimension is as follows: Consider an almost linear parabolic horseshoe map f as given in Example 1 such that $-t^u\log|D_uf|$ has only one ergodic equilibrium state (for example one can chose $\beta \geq 1$) and that (12.12) holds. In particular, f is of smooth type and has constant contraction rate; hence Theorem 12.4 applies. As mentioned in the Remark (i) after Example 1, we may decrease the contraction rate c without changing D_uf on $W^u(\omega)$ insuring that $c < c_0$. Therefore, Lemma 7.1 guarantees that $P^u(t), t \in [0,t^u)$ is not affected by this procedure, and therefore, (12.12) still holds. Finally, applying Theorem 12.4 shows that f does not have an ergodic measure of maximal dimension.

References

[A] J. Aaronson, *An introduction to infinite ergodic theory*, Mathematical Surveys and Monographs, 50. American Mathematical Society, Providence, RI, 1997.

[BW1] L. Barreira and C. Wolf, *Measures of maximal dimension for hyperbolic diffeomorphisms*, Comm. Math. Phys. **239** (2003), 93-113.

[BW2] L. Barreira and C. Wolf, *Local dimension and ergodic decompositions*, preprint.

[DU] M. Denker and M. Urbański, *On the existence of conformal measures*, Trans. A.M.S. 328 (1991), 563-587.

[DV] L. Diaz and M. Viana, *Discontinuity of Hausdorff dimension and limit capacity on arcs of diffeomorphisms*, Ergodic Theory Dyn. Systems 9 (1989), no. 3, 403–425.

[DGS] M. Denker, C. Grillenberger and K. Sigmund, *Ergodic theory on compact spaces*, Lecture Notes in Math. **527**, Springer-Verlag, Berlin, 1976.

[E] Edwards, *Functional Analysis: Theory and Applications*, Dover 1995.

[J] O. Jenkinson, *Rotation, Entropy, and Equilibrium States*, Trans. Amer. Math. Soc. **353** (2001), 3713-3739.

[Ma] A. Manning, *A relation between Lyapunov exponents, Hausdorff dimension and entropy*, Ergodic Theory Dynamical Systems 1 (1981), no. 4, 451–459 (1982).

[Me] L. Mendoza, *The entropy of C^2 surface diffeomorphisms in terms of Hausdorff dimension and a Lyapunov exponent*, Ergodic Theory Dynam. Systems **5** (1985), no. 2, 273–283.

[MM] A. Manning and H. McCluskey, *Hausdorff dimension for Horseshoes*, Ergod. Theory and Dyn. Syst. **3** (1983), 251 - 260.

[HMU] P. Hanus, D. Mauldin, M. Urbański, *Thermodynamic formalism and multifractal analysis of conformal infinite iterated function systems*, Acta Math. Hungarica, 96 (2002), 27-98.

[MU1] D. Mauldin, M. Urbański, *Parabolic iterated function systems*, Ergod. Th. & Dynam. Sys. 20 (2000), 1423-1447

[MU2] D. Mauldin, M. Urbański, *Graph Directed Markov Systems: Geometry and Dynamics of Limit Sets*, Cambridge Univ. Press. 2003.

[PU] F. Przytycki, M. Urbański, *Fractals in the Plane - Ergodic Theory Methods*, to appear Cambridge Univ. Press, available on Urbański's webpage.

[Ra] M. Rams, *Measures of maximal dimension for linear horseshoes*, preprint.

[Ru] D. Ruelle, *Thermodynamic formalism*, Addison-Wesley, Reading, MA, 1978.

[SU] B. Stratmann, M. Urbański, *Pseudo-Markov systems and infinitely generated Schottky groups*, preprint 2005.

[U1] M. Urbański, *Parabolic Cantor sets*, Preprint 1995, available on Urbański's webpage.

[U2] M. Urbański, *Parabolic Cantor sets*, Fund. Math. 151 (1996), 241-277.

[U3] M. Urbański, *Hausdorff measures versus equilibrium states of conformal infinite iterated function systems*, Periodica Math. Hung., 37 (1998), 153-205.

[Wa] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics 79, Springer, 1981.

[Wo] C. Wolf, *Generalized physical and SRB measures for hyperbolic diffeomorphisms*, Journal of Statistical Physics, to appear.

[Y] L.-S. Young, *Dimension, entropy and Lyapunov exponents*, Ergod. Theory and Dyn. Syst **2** (1982), 109-124.

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