ANALYTIC FAMILIES OF HOLOMORPHIC IFS

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Abstract. This paper deals with analytic families of holomorphic iterated function systems. More specifically, we prove a classification theorem of analytic families which depend continuously on a parameter when the space of holomorphic IFSs is endowed with the λ-topology. This classification theorem helps us generalize some results from [7] and give us a better understanding of the space of conformal IFSs when this latter is equipped with the λ-topology.

1. **Introduction**

The last 15 years have been a period of extensive study of single conformal iterated function systems (abbreviated to CIFSs). Recently, interest in families of such systems has emerged (see [1], [2], [6], [7] and [8], among others). In [6], Roy and Urbański studied the space $\mathrm{CIFS}(X, I)$ of all CIFSs sharing the same seed space $X \subset \mathbb{R}^d$ and the same alphabet I. When I is finite, they endowed $\mathrm{CIFS}(X, I)$ with a natural metric of pointwise convergence (pointwise meaning that corresponding generators are $C^1(X)$-close to one another). They showed that the topological pressure and the Hausdorff dimension functions are then continuous as functions of the underlying CIFSs (see Lemma 4.2 and Theorem 4.3 in [6]). When I is infinite, they discovered that these functions are generally not continuous when $\mathrm{CIFS}(X, I)$ is equipped with a "generalized" metric of pointwise convergence (see Theorem 5.2 and Lemma 5.3 in [6], as well as the example following these results). They thereafter introduced a new, weaker topology called λ-topology (see (5.1) in [6]). In that topology, they proved that the topological pressure and the Hausdorff dimension functions are both continuous (see Theorems 5.7 and 5.10 in [6]).

More recently, the authors of the present article have studied more thoroughly the pointwise and λ- topologies (see [7]).

The aim of this paper is to investigate some families in a subspace of $\mathrm{CIFS}(X, I)$ when $d = 2$ (or, equivalently, when $X \subset \mathbb{C}$) and when $\mathrm{CIFS}(X, I)$ is endowed with the λ-topology. The abovementioned subspace, denoted by $\mathrm{HIFS}(X, I)$, is the space of holomorphic iterated function systems. We shall study analytic families of HIFSs whose generators depend analytically on a complex parameter.

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After some preliminaries on single iterated function systems in section 2 and on families of CIFSs in section 3, we study in section 4 the properties of the pressure function seen as a function of two variables, that is, as a function of not only the usual parameter t but also of the underlying IFS Φ. In section 5, we enunciate a classification theorem of analytic families which depend continuously on a parameter when HIFS(X, I) is endowed with the λ-topology. Finally, in section 6, we use our classification theorem to generalize some results that we obtained in [7] and hence gain a better understanding of CIFS(X, I) when this latter is equipped with the λ-topology.

2. Preliminaries on Iterated Function Systems

Let us first describe the setting of conformal iterated function systems introduced in [4]. Let I be a countable (finite or infinite) index set (often called alphabet) with at least two elements, and let $\Phi = \{\varphi_i : X \to X \mid i \in I\}$ be a collection of injective contractions of a compact metric space (X, d_X) (sometimes coined seed space) for which there exists a constant $0 < s < 1$ such that $d_X(\varphi_i(x), \varphi_i(y)) \leq s\, d_X(x, y)$ for every $x, y \in X$ and for every $i \in I$. Any such collection Φ is called an iterated function system (abbr. IFS). We define the limit set J_Φ of this system as the image of the coding space I^∞ under a coding map π_Φ as follows. Let I^n denote the space of words of length n with letters in I, let $I^* := \bigcup_{n \in I\!N} I^n$ be the space of finite words, and I^∞ the space of one-sided infinite words with letters in I. For every $\omega \in I^* \cup I^\infty$, we write $|\omega|$ for the length of ω, that is, the unique $n \in I\!N \cup \{\infty\}$ such that $\omega \in I^n$. For $\omega \in I^n$ with $n \in I\!N$, we set $\varphi_\omega := \varphi_{\omega_1} \circ \varphi_{\omega_2} \circ \cdots \circ \varphi_{\omega_n}$. If $\omega \in I^* \cup I^\infty$ and $n \in I\!N$ does not exceed the length of ω, we denote by $\omega|_n$ the word $\omega_1\omega_2 \ldots \omega_n$. Since, given $\omega \in I^\infty$, the diameters of the compact sets $\varphi_{\omega|_n}(X)$, $n \in I\!N$, converge to zero and since these sets form a decreasing family, their intersection

$$\bigcap_{n=1}^{\infty} \varphi_{\omega|_n}(X)$$

is a singleton, and we denote its element by $\pi_\Phi(\omega)$. This defines the coding map $\pi_\Phi : I^\infty \to X$. Clearly, π_Φ is a continuous function when I^∞ is equipped with the topology generated by the cylinders $[i]_n = \{\omega \in I^\infty : \omega_n = i\}$, $i \in I$, $n \in I\!N$. The main object of our interest will be the limit set

$$J_\Phi = \pi_\Phi(I^\infty) = \bigcup_{\omega \in I^\infty} \bigcap_{n=1}^{\infty} \varphi_{\omega|_n}(X).$$

Observe that J_Φ satisfies the natural invariance equality, $J_\Phi = \bigcup_{i \in I} \varphi_i(J_\Phi)$. Note also that if I is finite, then J_Φ is compact, which is usually not the case when I is infinite.

An IFS $\Phi = \{\varphi_i\}_{i \in I}$ is said to satisfy the Open Set Condition (OSC) if there exists a nonempty open set $U \subset X$ such that $\varphi_i(U) \subset U$ for every $i \in I$ and $\varphi_i(U) \cap \varphi_j(U) = \emptyset$ for every pair $i, j \in I$, $i \neq j$.

An IFS Φ is called conformal (and thereafter a CIFS) if X is connected, $X = \overline{\mathrm{Int}_{I\!R^d}(X)}$ for some $d \in I\!N$, and if the following conditions are satisfied:

(i) Φ satisfies the OSC with $U = \mathrm{Int}_{I\!R^d}(X)$;
(ii) There exists an open connected set V, with $X \subset V \subset I\!R^d$, such that all the generators φ_i, $i \in I$, extend to C^1 conformal diffeomorphisms of V into V;
(iii) There exist $\gamma, l > 0$ such that for every $x \in X$ there is an open cone $\mathrm{Con}(x, \gamma, l) \subset \mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure γ, and altitude l;
(iv) There are two constants $L \geq 1$ and $\alpha > 0$ such that
$$\Big| |\varphi_i'(y)| - |\varphi_i'(x)| \Big| \leq L \|(\varphi_i')^{-1}\|_V^{-1} \|y - x\|^\alpha$$
for all $x, y \in V$ and all $i \in I$, where $\|\cdot\|_V$ is the supremum norm taken over V.

Remark 2.1. *It has been proved in Proposition 4.2.1 of* [5] *that if* $d \geq 2$*, then condition* (*iv*) *is satisfied with* $\alpha = 1$*. This condition is also fulfilled if* $d = 1$ *and the alphabet* I *is finite.*

The following useful fact has been also proved in Lemma 4.2.2 of [5].

Lemma 2.2. *For all* $\omega \in I^*$ *and all* $x, y \in V$ *we have that*
$$\Big| \log |\varphi_\omega'(y)| - \log |\varphi_\omega'(x)| \Big| \leq L(1 - s)^{-1} \|y - x\|^\alpha.$$

As an immediate consequence of this lemma we get the following.

(iv') Bounded Distortion Property (BDP): There exists a constant $K \geq 1$ such that
$$|\varphi_\omega'(y)| \leq K |\varphi_\omega'(x)|$$
for every $x, y \in V$ and every $\omega \in I^*$, where $|\varphi_\omega'(x)|$ denotes the norm of the derivative.

As demonstrated in [4], infinite CIFSs, unlike finite ones, can be splitted into two main classes: irregular and regular systems. This dichotomy can be expressed in terms of the absence or existence of a zero for the topological pressure function. Recall that the topological pressure $\mathrm{P}_\Phi(t)$, $t \geq 0$, is defined as follows. For every $n \in I\!N$, set
$$\mathrm{P}_\Phi^{(n)}(t) = \sum_{\omega \in I^n} \|\varphi_\omega'\|^t,$$
where $\|\cdot\| := \|\cdot\|_X$ is the supremum norm over X. Then
$$\mathrm{P}_\Phi(t) = \lim_{n \to \infty} \frac{1}{n} \log \mathrm{P}_\Phi^{(n)}(t) = \inf_{n \in I\!N} \frac{1}{n} \log \mathrm{P}_\Phi^{(n)}(t).$$
Recall also that the shift map $\sigma : I^* \cup I^\infty \to I^* \cup I^\infty$ is defined for each $\omega \in I^* \cup I^\infty$ as
$$\sigma\Big(\{\omega_n\}_{n=1}^{|\omega|}\Big) = \{\omega_{n+1}\}_{n=1}^{|\omega|-1}.$$

If the function $\zeta_\Phi : I^\infty \to I\!\!R$ is given by the formula

$$\zeta_\Phi(\omega) = \log|\varphi'_{\omega_1}(\pi(\sigma(\omega)))|,$$

then $\mathrm{P}_\Phi(t) = \mathrm{P}(t\zeta_\Phi)$, where $\mathrm{P}(t\zeta_\Phi)$ is the classical topological pressure of the function $t\zeta_\Phi$ when I is finite (so the space I^∞ is compact), and is understood in the sense of [3] and [5] when I is infinite. The finiteness parameter θ_Φ of a system is defined by $\inf\{t \geq 0 : \mathrm{P}_\Phi^{(1)}(t) < \infty\} = \inf\{t \geq 0 : \mathrm{P}_\Phi(t) < \infty\}$. In [4], it was shown that the topological pressure function P_Φ is non-increasing on $[0,\infty)$, (strictly) decreasing, continuous and convex on $[\theta_\Phi,\infty)$, and $\mathrm{P}_\Phi(d) \leq 0$. Of course, $\mathrm{P}_\Phi(0) = \infty$ if and only if I is infinite. The following characterization of the Hausdorff dimension h_Φ of the limit set J_Φ was stated as Theorem 3.15 in [4]. For every $F \subset I$, we write $\Phi|_F$ for the subsystem $\{\varphi_i\}_{i\in F}$ of Φ.

Theorem 2.3.

$$h_\Phi = \sup\{h_{\Phi|_F} : F \subset I \text{ is finite }\} = \inf\{t \geq 0 : \mathrm{P}_\Phi(t) \leq 0\}.$$

In particular, if $\mathrm{P}_\Phi(t) = 0$*, then* $t = h_\Phi$*.*

Subsequently, a system Φ was called regular provided there is some $t \geq 0$ such that $\mathrm{P}_\Phi(t) = 0$. It follows from the strict decrease of P_Φ on $[\theta_\Phi,\infty)$ that such a t is unique.

Regular systems can also be naturally divided into subclasses. Following [4] still, a system Φ was said to be strongly regular if $0 < \mathrm{P}_\Phi(t) < \infty$ for some $t \geq 0$. As an immediate application of Theorem 2.3, a system Φ is strongly regular if and only if $h_\Phi > \theta_\Phi$. On the other hand, a system Φ was called cofinitely regular provided that every nonempty cofinite subsystem $\Phi' = \{\varphi_i\}_{i\in I'}$ (i.e. I' is a cofinite subset of I) is regular. A finite system is clearly cofinitely regular, and it was shown in [4] that an infinite system is cofinitely regular exactly when the pressure is infinite at the finiteness parameter, that is, $\mathrm{P}_\Phi(\theta_\Phi) = \infty$. Note that every cofinitely regular system is strongly regular, and every strongly regular system is regular. Finally, recall that critically regular systems are those regular systems for which $P_\Phi(\theta_\Phi) = 0$.

3. Preliminaries on Families of CIFSs

When dealing with families of CIFSs, we denote the set of all conformal iterated function systems with phase space X and alphabet I by $\mathrm{CIFS}(X,I)$. Obviously, $\mathrm{CIFS}(X,I)$ can be endowed with different topologies.

When I is finite, $\mathrm{CIFS}(X,I)$ is naturally endowed with the metric of pointwise convergence. This metric asserts that the distance between CIFSs $\Phi = \{\varphi_i\}_{i\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$ is

$$\rho(\Phi,\Psi) = \sum_{i\in I}\Big(\|\varphi_i - \psi_i\| + \|\varphi'_i - \psi'_i\|\Big).$$

It was proved in [6] that, given $t \geq 0$, the pressure function $P(t) : \mathrm{CIFS}(X,I) \to I\!\!R$, $\Phi \mapsto P_\Phi(t)$, is continuous (see Lemma 4.2) and that so is the Hausdorff dimension function $h : \mathrm{CIFS}(X,I) \to [0,d]$, $\Phi \mapsto h_\Phi$ (see Theorem 4.3).

When I is infinite, the situation is more intricate. First, recall that we may assume that $I = I\!\!N$ without loss of generality. Henceforth, we accordingly abbreviate CIFS$(X, I\!\!N)$ to CIFS(X). The set CIFS(X) can easily be equipped with a metric of pointwise convergence. In [6], such a metric was introduced by defining the distance between two CIFSs Φ and Ψ as

$$\rho_\infty(\Phi,\Psi) = \sum_{i=1}^{\infty} 2^{-i} \min\Big\{1, \|\varphi_i - \psi_i\| + \|\varphi_i' - \psi_i'\|\Big\}.$$

Recall also that $\|\cdot\| := \|\cdot\|_X$ is the supremum norm over X. In particular, this implies that each term in a sequence $\{\Phi^n\}$ admits a neighbourhood V_{Φ^n} of X (cf. definition of CIFS) and the intersection of these neighbourhoods may not be a neighbourhood of X. A potential consequence of this is that each $\{\Phi^n\}$ has a minimal constant of bounded distortion K_{Φ^n} but these constants may not be uniformly bounded.

Roy and Urbański observed that the pressure and Hausdorff dimension functions are generally not continuous when CIFS(X) is endowed with the metric ρ_∞ (see the example following Lemma 5.3 in [6]). This triggered the introduction of a topology called the λ-topology (see section 5 in [6]). In that topology, a sequence $\{\Phi^n\}$ converges to Φ provided that $\{\Phi^n\}$ converges to Φ in the pointwise topology and that there exist constants $C > 0$ and $N \in I\!\!N$ such that

$$\Big|\log\|\varphi_i'\| - \log\|(\varphi_i^n)'\|\Big| \leq C \tag{3.1}$$

for all $i \in I\!\!N$ and all $n \geq N$. A set $F \subset$ CIFS(X) is declared to be closed if the λ-limit of every λ-converging sequence of CIFSs in F belongs to F. Several topological properties of CIFS(X) were given in [6] and [7]. Among others, let us mention that this topology is not metrizable, for it does not even satisfy the first axiom of countability (see Proposition 5.7 in [7]). Nonetheless, this topology proves to be useful, for it is easy to determine whether a sequence converges or not in that topology and, according to Roy and Urbański, the Hausdorff dimension function is then continuous everywhere on CIFS(X) (see Theorem 5.10 in [6]). In fact, the combination of Theorem 5.7 in [6] with Lemma 5.22 in [7] shows that the pressure function is continuous wherever it possibly can be.

Whichever topology we choose to endow CIFS(X) with, there are some subspaces of particular interest. Following the notation in [7], let SIFS(X) represent the subset of CIFS(X) comprising all similarity iterated function systems, that is, iterated function systems whose generators are similarities. Let also IR$(X) \subset$ CIFS(X) be the subset of irregular systems, while R$(X) \subset$ CIFS(X) will represent the subset of regular systems. Denote further by CR$(X) \subset$ R(X) the subset of critically regular systems, by SR$(X) \subset$ R(X) the subset of strongly regular systems, and by CFR$(X) \subset$ SR(X) the subset of cofinitely regular systems. Finally, we will denote by FSR(X) the set SR$(X) \setminus$ CFR(X).

In this paper, we will concentrate on the case $d = 2$ or, equivalently, $X \subset \mathbb{C}$. In this case, a natural subset of CIFS(X, I) is:

Definition 3.1. *The set* $\mathrm{HIFS}(X,I)$ *consists of those systems* $\Phi = \{\varphi_i\}_{i\in I} \in \mathrm{CIFS}(X,I)$ *which admit an open connected neighbourhood* V *of* $X \subset \mathbb{C}$ *such that for each* $i \in I$, *the map* $x \mapsto \varphi_i(x)$ *is holomorphic on* V.

More specifically, we will be interested in analytic families of HIFSs.

Definition 3.2. *A family* $\{\Phi^\gamma\}_{\gamma\in\Gamma} = \{\{\varphi_i^\gamma\}_{i\in I}\}_{\gamma\in\Gamma}$ *in* $\mathrm{HIFS}(X,I)$ *is called analytic if ...*

- Γ *is a connected, finite-dimensional complex manifold; and*
- *for each* $\gamma_0 \in \Gamma$, *there exists a neighbourhood* Γ_0 *of* γ_0 *and a bounded neighbourhood* V_0 *of* $X \subset \mathbb{C}$ *with respect to which each system* $\Phi^\gamma \in \mathrm{HIFS}(X,I)$, $\gamma \in \Gamma_0$, *and such that for each* $i \in I$, *the map*
$$(\gamma, x) \mapsto \varphi_i^\gamma(x), \quad (\gamma, x) \in \Gamma_0 \times V_0,$$
is holomorphic.

4. The Pressure as a Function of Two Variables

In this section, we make some observations about the pressure function seen as a function of two variables, that is, as a function of not only the parameter t but also of the underlying CIFS Φ. These observations mostly follow from earlier results presented in [6] and [7].

When the alphabet I is finite, the pressure behaves well.

Lemma 4.1. *If* I *is a finite alphabet, then the pressure function* $(\Phi, t) \mapsto P(\Phi, t) = P(t\zeta_\Phi)$, $\Phi \in \mathrm{CIFS}(X,I)$, $t \geq 0$, *is continuous.*

Proof. In the proof of Lemma 4.2 in [6], it was shown that the function $\Phi \mapsto \zeta_\Phi \in C(I^\infty)$ is continuous. Since the pressure function $P : C(I^\infty) \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant 1, we deduce that
$$\begin{aligned}
\left|P(\Phi,t) - P(\Phi_0,t_0)\right| &= \left|P(t\zeta_\Phi) - P(t_0\zeta_{\Phi_0})\right| \\
&\leq \|t\zeta_\Phi - t_0\zeta_{\Phi_0}\| \\
&\leq \|t\zeta_\Phi - t\zeta_{\Phi_0}\| + \|t\zeta_{\Phi_0} - t_0\zeta_{\Phi_0}\| \\
&= |t| \cdot \|\zeta_\Phi - \zeta_{\Phi_0}\| + |t - t_0| \cdot \|\zeta_{\Phi_0}\|,
\end{aligned}$$
for every $\Phi, \Phi_0 \in \mathrm{CIFS}(X,I)$ and $t, t_0 \geq 0$. Thus, the pressure function $(\Phi, t) \mapsto P(\Phi, t)$ is continuous. ■

For an infinite alphabet, we obtain immediately the following.

Lemma 4.2. *The pressure function* $(\Phi, t) \mapsto P(\Phi, t)$, $\Phi \in \mathrm{CIFS}(X)$, $t \geq 0$, *is lower semi-continuous when* $\mathrm{CIFS}(X)$ *is endowed with the metric* ρ_∞ *of pointwise convergence. Moreover, it is continuous on the set* $\cup_{\Phi\in\mathrm{CIFS}(X)}\{\Phi\} \times Fin(\Phi)^c$, *and discontinuous on* $\cup_{\Phi\in\mathrm{CIFS}(X)}\{\Phi\} \times (Fin(\Phi) \cap [0,d))$. *In fact, in this latter case, given any* $\Phi \in \mathrm{CIFS}(X)$ *and* $t \in Fin(\Phi) \cap [0,d)$, *the function* $\Psi \mapsto P(\Psi, t)$ *is discontinuous at* Φ.

Proof. Using Theorem 2.1.5 in [5] as well as Lemma 4.1, for any finite subset F of $I\!N$ we have that

$$\liminf_{(\Phi,t)\to(\Phi_0,t_0)} P(\Phi,t) \geq \liminf_{(\Phi,t)\to(\Phi_0,t_0)} P_F(\Phi,t) = \lim_{(\Phi,t)\to(\Phi_0,t_0)} P_F(\Phi,t) = P_F(\Phi_0,t_0).$$

Since this is true for every finite subset F of $I\!N$, we deduce from Theorem 2.1.5 in [5] that

$$\liminf_{(\Phi,t)\to(\Phi_0,t_0)} P(\Phi,t) \geq \sup_{F\subset I\!N, F\text{finite}} P_F(\Phi_0,t_0) = P(\Phi_0,t_0).$$

Observe also that $P(\Phi,t)=\infty$ for any $t\notin Fin(\Phi)$ and every $\Phi\in \mathrm{CIFS}(X)$. Thus, the pressure function is upper semi-continuous at any point in the set $\cup_{\Phi\in\mathrm{CIFS}(X)}\{\Phi\}\times Fin(\Phi)^c$.

Finally, given $\Phi\in\mathrm{CIFS}(X)$ and $t\in Fin(\Phi)\cap[0,d)$, the function $\Psi\mapsto P(\Psi,t)$ is discontinuous at Φ according to the last paragraph in the proof of Lemma 4.4 in [7]. ■

In terms of the λ-topology, we have the following result:

Lemma 4.3. *The pressure function* $(\Phi,t)\mapsto P(\Phi,t)$ *is continuous on the set* $\cup_{\Phi\in\mathrm{CIFS}(X)}\{\Phi\}\times Fin(\Phi)^c$ *and sequentially continuous on* $\cup_{\Phi\in\mathrm{CIFS}(X)}\{\Phi\}\times Fin(\Phi)$ *when* $\mathrm{CIFS}(X)$ *is endowed with the* λ*-topology.*

Proof. The continuity of the pressure on the set $\cup_{\Phi\in\mathrm{CIFS}(X)}\{\Phi\}\times Fin(\Phi)^c$ follows from Lemma 4.2 above. The sequential continuity on $\cup_{\Phi\in\mathrm{CIFS}(X)}\{\Phi\}\times Fin(\Phi)$ follows from Lemma 5.22 in [7] and the fact that if $\{\Phi^n\}$ converges to Φ in the λ-topology, then $Fin(\Phi^n)=Fin(\Phi)$ for all n large enough, as observed before Theorem 5.20 in [7]. ■

We now investigate the properties of the pressure function $(\gamma,t)\mapsto P(\gamma,t)=P(\Phi^\gamma,t)$ as a function of two variables among analytic families in $\mathrm{HIFS}(X,I)$.

We first prove that every analytic family in $\mathrm{HIFS}(X,I)$ constitutes a continuous family in $\mathrm{CIFS}(X,I)$ when this latter is equipped with the pointwise topology.

Theorem 4.4. *An analytic family* $\{\Phi^\gamma\}_{\gamma\in\Gamma}$ *regarded as the function*

$$\begin{array}{cccc}\Phi & : & \Gamma & \to & \mathrm{CIFS}(X,I)\\ & & \gamma & \mapsto & \Phi^\gamma\end{array}$$

is continuous when $\mathrm{CIFS}(X)$ *is endowed with the metric* ρ_∞ *of pointwise convergence.*

Proof. We give the proof in the case of an infinite alphabet I. A similar proof can be derived in the finite case.

If Φ were discontinuous at some $\gamma_0\in\Gamma$, there would exist $\varepsilon>0$ and a sequence $\{\gamma_n\}$ such that $\gamma_n\to\gamma_0$ but $\rho_\infty(\Phi^{\gamma_n},\Phi^{\gamma_0})\geq\varepsilon$ for every $n\in I\!N$. Choose $J\in I\!N$ such that $\sum_{j>J}2^{-j}<\varepsilon/2$. Then

$$\sum_{j\leq J} 2^{-j}\min\Big\{1, \|\varphi_j^{\gamma_n}-\varphi_j^{\gamma_0}\| + \|(\varphi_j^{\gamma_n})'-(\varphi_j^{\gamma_0})'\|\Big\} > \varepsilon/2$$

since, otherwise, we would have $\rho_\infty(\Phi^{\gamma_n}, \Phi^{\gamma_0}) < \varepsilon$. Replacing the sequence $\{\gamma_n\}$ by one of its subsequences if necessary, it follows from this that there is an $i \le J$ such that

$$\|\varphi_i^{\gamma_n} - \varphi_i^{\gamma_0}\| + \|(\varphi_i^{\gamma_n})' - (\varphi_i^{\gamma_0})'\| > \frac{\varepsilon}{2J}$$

for every $n \in I\!N$. Consequently, each subsequence of $\{\varphi_i^{\gamma_n}\}_{n\in I\!N}$ either does not converge uniformly on X to $\varphi_i^{\gamma_0}$ or the corresponding subsequence of derivatives does not converge uniformly on X to $(\varphi_i^{\gamma_0})'$.

However, given the neighbourhood V from the definition of an analytic family, the generators $\{\varphi_j^\gamma\}_{j\in I\!N,\gamma\in\Gamma}$ form a family of holomorphic maps uniformly bounded on V, say by a constant M, due to the forward invariance of V. By Cauchy's Integral Formula, their derivatives are thereby uniformly bounded on any compact subset K of V by a constant that depends on M, K and V only (but neither on j nor on γ). So the generators constitute an equicontinuous family on V and thus are a normal family by Arzelà-Ascoli's Theorem. Therefore, the sequence $\{\varphi_i^{\gamma_n}\}_{n\in I\!N}$ admits a subsequence that converges uniformly on an open neighbourhood U of X relatively compact in V. By Hurwitz's Theorem, so will the associated derivatives subsequence. Moreover, in any analytic family, the function $\gamma \mapsto \varphi_j^\gamma(x)$, $\gamma \in \Gamma$, is holomorphic for each $x \in V$ and $j \in I\!N$. Thus, the sequence $\{\varphi_i^{\gamma_n}\}_{n\in I\!N}$ converges pointwise to $\varphi_i^{\gamma_0}$ on V. Hence $\{\varphi_i^{\gamma_n}\}_{n\in I\!N}$ has a subsequence that converges uniformly on U to $\varphi_i^{\gamma_0}$. The derivatives subsequence will then converge uniformly to $(\varphi_i^{\gamma_0})'$ on U, and in particular on X. This is a contradiction. ■

In light of Theorem 4.4, it is relevant to ask whether an analytic family is continuous when CIFS(X) is endowed with the λ-topology.

Remark 4.5. *An analytic family $\{\Phi^\gamma\}_{\gamma\in\Gamma}$ regarded as the function*

$$\begin{array}{cccc} \Phi & : & \Gamma & \to & \mathrm{CIFS}(X) \\ & & \gamma & \mapsto & \Phi^\gamma \end{array}$$

is generally not continuous when CIFS(X) *is endowed with the λ-topology.*

Indeed, if this were the case then the finiteness parameter function $\gamma \mapsto \theta_{\Phi^\gamma}$ would be locally constant according to Lemma 5.4 in [6]. Since Γ is connected, that function would in fact be constant. However, Example 6 in section 8 of [6] presents an analytic family whose finiteness parameter varies.

An immediate consequence of Theorem 4.4 is the following.

Corollary 4.6. *Within an analytic family $\{\Phi^\gamma\}_{\gamma\in\Gamma}$, the pressure function $(\gamma, t) \mapsto P(\gamma, t) := P(\Phi^\gamma, t)$, is lower semi-continuous when* CIFS(X) *is endowed with the pointwise topology. Moreover, it is continuous on the set $\cup_{\gamma\in\Gamma}\{\gamma\} \times Fin(\Phi^\gamma)^c$.*

Proof. It follows immediately from Theorem 4.4 that the map $(\gamma, t) \mapsto (\Phi^\gamma, t)$ is continuous, while Lemma 4.2 asserts that the map $(\Phi, t) \mapsto P(\Phi, t)$ is lower semi-continuous. The composition of these two functions, the pressure function $(\gamma,t)\mapsto P(\gamma,t):=P(\Phi^\gamma,t)$, is thus lower semi-continuous. The second assertion follows from a similar argument. ■

Remark 4.7. *Note that $(\gamma,t)\mapsto P(\gamma,t)$ may be discontinuous at some points in $\cup_{\gamma\in\Gamma}\{\gamma\}\times Fin(\Phi^\gamma)$. For instance, Examples 1, 2 and 3 in section 8 of* [6] *show that the function $\gamma\mapsto P(\gamma,t)$ may be discontinuous at $t=\theta_{\Phi^\gamma}$ if $\Phi^\gamma\notin \mathrm{CFR}(X)$. In all three examples, this function is discontinuous on the unit circle.*

We further have the following result when I is finite.

Theorem 4.8. *Let I be a finite alphabet. If $\{\Phi^\gamma\}_{\gamma\in\Gamma}$ is an analytic family in* $\mathrm{HIFS}(X,I)$, *then for every $t\geq 0$ the pressure function $\gamma\mapsto P(\gamma,t)$, is a continuous, plurisubharmonic function.*

Proof. Fix $t\geq 0$. Since I is finite, each Φ^γ is regular. According to the proof of Theorem 6.3 in [6], fixing any $\gamma_0\in\Gamma$, we have that

$$P(\gamma,t)=\sup_{\mu\in M(\gamma_0)}\left\{h_\mu(\sigma)+t\int_{I^\infty}\zeta_{\Phi^\gamma}(\omega)d\mu(\omega)\right\},$$

and that the function $\gamma\mapsto\int_{I^\infty}\zeta_{\Phi^\gamma}(\omega)d\mu(\omega)$ is finite, positive and pluriharmonic for every $\mu\in M(\gamma_0)$. Thus, $P(\gamma,t)$ is the supremum of a family of pluriharmonic functions that are uniformly bounded from above by $\log|I|$. Then $P(\gamma,t)$ is a plurisubharmonic function uniformly bounded from above by $\log|I|$. ■

In the infinite case, if we consider analytic families that are continuous with respect to the λ-topology on $\mathrm{HIFS}(X)$, then we have the following.

Theorem 4.9. *Let $t\geq 0$ and $\{\Phi^\gamma\}_{\gamma\in\Gamma}$ be an analytic family in* $\mathrm{HIFS}(X)$ *such that $\gamma\mapsto\Phi^\gamma\in$* $\mathrm{HIFS}(X)$ *is continuous with respect to the λ-topology. Suppose that there exists $\gamma_0\in\Gamma$ such that $P(\gamma_0,t)<\infty$. Then the function $\gamma\mapsto P(\gamma,t)$, $\gamma\in\Gamma$, is continuous and plurisubharmonic. In particular, this function satisfies the Maximum Principle.*

Proof. Using Theorem 5.7 in [6], $\gamma\mapsto P(\gamma,t)$ is continuous. Moreover, using Theorem 2.1.5 in [5], the Variational Principle and Fubini's Theorem, it follows that the function $\gamma\mapsto P(\gamma,t)$ is a supremum of a family of pluriharmonic functions. Hence it is plurisubharmonic. ■

All of the forthcoming results rely upon the following theorem.

Theorem 4.10. *Let $t\geq 0$ and $\{\Phi^\gamma\}_{\gamma\in\Gamma}$ be an analytic family in* $\mathrm{HIFS}(X)$ *such that*

(1) *for every $\gamma_0\in\Gamma$ there exists a neighbourhood Γ_0 of γ_0 and $\eta_0\in(0,1)$ such that $|\kappa_\omega(\gamma)-1|<\eta_0$ for every $\omega\in I^\infty$ and all $\gamma\in\Gamma_0$, where*

$$\kappa_\omega(\gamma):=\frac{(\varphi^\gamma_{\omega_1})'(\pi_\gamma(\sigma(\omega)))}{(\varphi^{\gamma_0}_{\omega_1})'(\pi_{\gamma_0}(\sigma(\omega)))}.$$

(2) *there exists* $\gamma_1 \in \Gamma$ *such that* $P(\gamma_1, t) < \infty$.

Then the function $\gamma \mapsto P(\gamma, t)$, $\gamma \in \Gamma$, *is real-analytic.*

Proof. This result follows from page 13 of [9]. Although condition (b) at the beginning of section 4 of that paper requires the existence of a γ_0 such that Φ^{γ_0} is strongly regular, this assumption is not needed to show that $\gamma \mapsto P(\gamma, t)$ is real-analytic. ■

Remark 4.11. *Condition (1) above is a local version of condition (c) in* [6].

Those analytic families that depend continuously on the parameter γ when HIFS(X) is equipped with the λ-topology enjoy the following property.

Lemma 4.12. *Let* $\{\Phi^\gamma\}_{\gamma\in\Gamma}$ *be an analytic family such that* $\gamma \mapsto \Phi^\gamma \in \mathrm{HIFS}(X)$ *is continuous with respect to the* λ*-topology. Then condition (1) in Theorem 4.10 holds.*

Proof. This lemma is a consequence of the claim below and the observation following Corollary 6.2 in [6].

Claim. Let $\{\Phi^\gamma\}_{\gamma\in\Gamma}$ be an analytic family in HIFS(X). For a fixed $\omega \in I^\infty$, the function $\gamma \mapsto \pi_\gamma(\omega)$ is analytic, and furthermore (although we do not use the following to prove the lemma), $\{\gamma \mapsto \pi_\gamma(\omega)\}_{\omega\in I^\infty}$ is equicontinuous on Γ.

Proof of the claim. Let $x \in X$. Fix for a moment $\omega \in I^\infty$. Then $\pi_\gamma(\omega) = \lim_{n\to\infty} \varphi^\gamma_{\omega_1} \circ \cdots \circ \varphi^\gamma_{\omega_n}(x)$. Hence, $\gamma \mapsto \pi_\gamma(\omega)$ is a limit of a sequence of uniformly bounded holomorphic maps defined on Γ. Since a family of uniformly bounded holomorphic maps is normal, it follows that $\gamma \mapsto \pi_\gamma(\omega)$ is holomorphic. Furthermore, $\{\gamma \mapsto \pi_\gamma(\omega)\}_{\omega\in I^\infty}$ is uniformly bounded. Hence, it is normal and thereby equicontinuous. ■

Combining the fact that for any given $\omega \in I^\infty$, the function $\gamma \mapsto \pi_\gamma(\omega)$ is holomorphic and the observation following Corollary 6.2 in [6], the lemma follows immediately. ■

Remark 4.7, Theorems 4.9 and 4.10, and Lemma 4.12 justify the assumption of continuous dependence on γ of the families studied in the sequel.

5. A Classification Theorem

We now give a complete classification of analytic families which are continuous with respect to the λ-topology.

Definition 5.1. *Let* $\{\Phi^\gamma\}_{\gamma\in\Gamma}$ *be an analytic family in* HIFS(X). *We define the following subsets of parameters:*

- $\mathrm{R}_\Gamma(X) := \{\gamma \in \Gamma : \Phi^\gamma \in \mathrm{R}(X)\}$;
- $\mathrm{SR}_\Gamma(X) := \{\gamma \in \Gamma : \Phi^\gamma \in \mathrm{SR}(X)\}$;
- $\mathrm{CFR}_\Gamma(X) := \{\gamma \in \Gamma : \Phi^\gamma \in \mathrm{CFR}(X)\}$;
- $\mathrm{FSR}_\Gamma(X) := \{\gamma \in \Gamma : \Phi^\gamma \in \mathrm{SR}(X) \setminus \mathrm{CFR}(X)\}$;
- $\mathrm{CR}_\Gamma(X) := \{\gamma \in \Gamma : \Phi^\gamma \in \mathrm{CR}(X)\}$;
- $\mathrm{IR}_\Gamma(X) := \{\gamma \in \Gamma : \Phi^\gamma \in \mathrm{IR}(X)\}$;

- $\mathrm{RAH}(\Gamma) := \{\gamma_0 \in \Gamma : \gamma \mapsto h_{\Phi^\gamma}$ *is real-analytic in a neighbourhood of* $\gamma_0\}$*; and*
- $\mathrm{NPHH}(\Gamma) := \{\gamma_0 \in \Gamma : \gamma \mapsto h_{\Phi^\gamma}$ *is not pluriharmonic in any neighbourhood of* $\gamma_0\}$.

Theorem 5.2. *Let* $\{\Phi^\gamma\}_{\gamma\in\Gamma}$ *be an analytic family in* $\mathrm{HIFS}(X)$ *such that* $\gamma \mapsto \Phi^\gamma \in \mathrm{CIFS}(X)$ *is continuous with respect to the* λ*-topology. Let* θ *be the constant such that* $\theta_{\Phi^\gamma} \equiv \theta$ *on* Γ *(such a constant exists according to Lemma 5.4 in* [6]*). Then exactly one of the following statements holds.*

(I) $\Gamma = \mathrm{CFR}_\Gamma(X) = \mathrm{RAH}(\Gamma)$*, and the function* $\gamma \mapsto h_{\Phi^\gamma}$*,* $\gamma \in \Gamma$*, is identically equal to a constant larger than* θ*.*

(II) $\Gamma = \mathrm{CFR}_\Gamma(X) = \mathrm{RAH}(\Gamma) = \mathrm{NPHH}(\Gamma)$.

(III) $\Gamma = \mathrm{FSR}_\Gamma(X) = \mathrm{RAH}(\Gamma)$*, and the function* $\gamma \mapsto h_{\Phi^\gamma}$*,* $\gamma \in \Gamma$*, is identically equal to a constant larger than* θ*.*

(IV) $\Gamma = \mathrm{FSR}_\Gamma(X) = \mathrm{RAH}(\Gamma) = \mathrm{NPHH}(\Gamma)$.

(V) $\Gamma = \mathrm{R}_\Gamma(X) \setminus \mathrm{CFR}_\Gamma(X) = \mathrm{NPHH}(\Gamma)$*, with* $\mathrm{FSR}_\Gamma(X) \neq \emptyset$ *and* $\mathrm{CR}_\Gamma(X) \neq \emptyset$*.*
Furthermore, $\mathrm{CR}_\Gamma(X) = \partial(\mathrm{FSR}_\Gamma(X))$*.*
Also, $\mathrm{FSR}_\Gamma(X) \subset \mathrm{RAH}(\Gamma)$*.*

(VI) $\Gamma = \mathrm{CR}_\Gamma(X) = \mathrm{RAH}(\Gamma)$*, and the function* $\gamma \mapsto h_{\Phi^\gamma}$*,* $\gamma \in \Gamma$*, is identically equal to the constant* θ*.*

(VII) $\mathrm{FSR}_\Gamma(X) \neq \emptyset$*,* $\mathrm{CR}_\Gamma(X) \neq \emptyset$ *and* $\mathrm{IR}_\Gamma(X) \neq \emptyset$*, while* $\mathrm{CFR}_\Gamma(X) = \emptyset$*.*
Moreover, $\emptyset \neq \partial(\mathrm{IR}_\Gamma(X)) \subset \mathrm{CR}_\Gamma(X) = \partial(\mathrm{FSR}_\Gamma(X))$*.*
Moreover, $(\mathrm{FSR}_\Gamma(X)) \cup \mathrm{IR}_\Gamma(X) \subset \mathrm{RAH}(\Gamma)$*.*
Moreover, $\emptyset \neq \partial(\mathrm{IR}_\Gamma(X)) \subset \Gamma \setminus \mathrm{RAH}(\Gamma)$*.*
Furthermore, $\mathrm{NPHH}(\Gamma) = \mathrm{R}_\Gamma(X) \setminus \mathrm{CFR}_\Gamma(X)$*.*

(VIII) $\Gamma = \mathrm{IR}_\Gamma(X) = \mathrm{RAH}(\Gamma)$*, and the function* $\gamma \mapsto h_{\Phi^\gamma}$*,* $\gamma \in \Gamma$*, is identically equal to the constant* θ*.*

Proof. **Case (1):** Suppose that $\mathrm{CFR}_\Gamma(X) \neq \emptyset$. Since $\mathrm{CFR}_\Gamma(X)$ is clopen by Lemma 5.9 in [6] and Γ is connected, we deduce that $\Gamma = \mathrm{CFR}_\Gamma(X)$. Now, if the function $\gamma \mapsto h_{\Phi^\gamma}$, $\gamma \in \Gamma$, is not constant, then combining Theorem 6.1 and Corollary 6.4 in [6] we obtain $\Gamma = \mathrm{RAH}(\Gamma) = \mathrm{NPHH}(\Gamma)$. This corresponds to statement (II).

On the other hand, if the function $\gamma \mapsto h_{\Phi^\gamma}$, $\gamma \in \Gamma$, is identically equal to a constant τ, then by Theorem 2.4 in [6] we have $\tau > \theta$. This is statement (I).

Case (2): Suppose next that $\Gamma = \mathrm{FSR}_\Gamma(X)$. If $\gamma \mapsto h_{\Phi^\gamma}$, $\gamma \in \Gamma$, is not constant, then combining Theorem 6.1 and Corollary 6.4 in [6] we get $\Gamma = \mathrm{RAH}(\Gamma) = \mathrm{NPHH}(\Gamma)$. This is statement (IV).

If, however, the function $\gamma \mapsto h_{\Phi^\gamma}$, $\gamma \in \Gamma$, is identically equal to a constant τ, then by Theorem 2.4 in [6] we have $\tau > \theta$. This is statement (III).

Case (3): Suppose that $\Gamma = \mathrm{R}_\Gamma(X) \setminus \mathrm{CFR}_\Gamma(X)$, that $\mathrm{FSR}_\Gamma(X) \neq \emptyset$, and $\mathrm{CR}_\Gamma(X) \neq \emptyset$. Then, by Lemma 5.8 in [6] we know that $\mathrm{FSR}_\Gamma(X)$ is a proper non-empty open subset of Γ. Moreover, by Theorem 6.1 in [6] we have $\mathrm{FSR}_\Gamma(X) \subset \mathrm{RAH}(\Gamma)$. Now, if there exists a parameter $\gamma_0 \in \mathrm{FSR}_\Gamma(X)$ such that $\gamma \mapsto h_{\Phi^\gamma}$ is pluriharmonic around γ_0, then by Corollary 6.4

and Theorem 2.4 in [6], we deduce that the function $\gamma \mapsto h_{\Phi^\gamma}$ is identically equal to a constant $\tau > \theta$ in the connected component U_0 of $\mathrm{FSR}_\Gamma(X)$ containing γ_0. Since $\mathrm{CR}_\Gamma(X) \neq \emptyset$ by assumption and since Γ is connected, the boundary ∂U_0 of U_0 in Γ is not empty. Moreover, $\partial U_0 \subset \mathrm{CR}_\Gamma(X)$. Then, according to Theorem 5.10 in [6], the function $\gamma \mapsto h_{\Phi^\gamma}$ is continuous and therefore $h_{\Phi^\gamma} = \tau$ for every $\gamma \in \partial U_0$. However, using Theorem 2.3 in [6], we know that $h_{\Phi^\gamma} = \theta$ for every $\gamma \in \mathrm{CR}_\Gamma(X)$. This is a contradiction. Consequently, $\overline{\mathrm{FSR}_\Gamma(X)} \subset \mathrm{NPHH}(\Gamma)$. Furthermore, Theorem 4.10 above and Theorems 2.3 and 2.4 in [6] imply that $\mathrm{CR}_\Gamma(X) = \partial(\mathrm{FSR}_\Gamma(X))$. Thus, we obtain $\Gamma = \overline{\mathrm{FSR}_\Gamma(X)} = \mathrm{NPHH}(\Gamma)$. This means that, in this case, statement (V) holds.

Case (4): Suppose next that $\Gamma = \mathrm{CR}_\Gamma(X)$. Then, by Theorem 2.3 in [6], statement (VI) holds.

Case (5): Suppose now that $\Gamma = \mathrm{IR}_\Gamma(X)$. Then, by Theorem 2.3 in [6], statement (VIII) holds.

Case (6): Suppose next that $\mathrm{CR}_\Gamma(X) \neq \emptyset$ and $\mathrm{IR}_\Gamma(X) \neq \emptyset$. Then $\gamma \mapsto P(\gamma, \theta)$ is not constant on Γ. Theorems 4.10 and 4.9 imply that the function $\gamma \mapsto P(\gamma, \theta)$ is a non-constant real-analytic function that satisfies the Maximum Principle. It follows that for any parameter $\gamma_0 \in \mathrm{CR}_\Gamma(X)$, there exists a parameter γ, arbitrarily close to γ_0, such that $P(\gamma, \theta) > 0$. Notice that $P(\gamma, \theta) > 0$ implies that $\gamma \in \mathrm{SR}_\Gamma(X)$. Hence, $\mathrm{CR}_\Gamma(X) \subset \partial(\mathrm{FSR}_\Gamma(X))$. On the other hand, the continuity of the function $\gamma \mapsto P(\gamma, \theta)$ implies that $\partial(\mathrm{FSR}_\Gamma(X)) \subset \mathrm{CR}_\Gamma(X)$. Thus, $\partial(\mathrm{FSR}_\Gamma(X)) = \mathrm{CR}_\Gamma(X)$. The continuity of the function $\gamma \mapsto P(\gamma, \theta)$ also implies that $\partial(\mathrm{IR}_\Gamma(X)) \subset \mathrm{CR}_\Gamma(X)$, while the connectedness of Γ ensures that $\partial(\mathrm{IR}_\Gamma(X)) \neq \emptyset$.

Moreover, using Theorems 5.10 and 6.1 in [6], we deduce that $(\mathrm{FSR}_\Gamma(X)) \cup \mathrm{IR}_\Gamma(X) \subset \mathrm{RAH}(\Gamma)$.

In order to show that $\partial(\mathrm{IR}_\Gamma(X)) \subset \Gamma \backslash \mathrm{RAH}(\Gamma)$, let $\gamma_0 \in \partial(\mathrm{IR}_\Gamma(X))$. Then, $\gamma_0 \in \mathrm{CR}_\Gamma(X) = \partial(\mathrm{FSR}_\Gamma(X))$. If $\gamma_0 \in \mathrm{RAH}(\Gamma)$, then the function $\gamma \mapsto h_{\Phi^\gamma}$ is identically equal to the constant θ in a neighborhood of γ_0. However, this contradicts Theorem 2.4 in [6]. Hence, we obtain that $\partial(\mathrm{IR}_\Gamma(X)) \subset \Gamma \setminus \mathrm{RAH}(\Gamma)$.

By the same argument as in case (3), we obtain $\mathrm{NPHH}(\Gamma) = (\mathrm{FSR}_\Gamma(X)) \cup \mathrm{CR}_\Gamma(X)$.

Thus, in this case, statement (VII) holds.

Case (7): Finally, suppose that $\mathrm{FSR}_\Gamma(X) \neq \emptyset$ and $\mathrm{IR}_\Gamma(X) \neq \emptyset$. Then, by Theorem 5.7 in [6], we have $\mathrm{CR}_\Gamma(X) \neq \emptyset$. Therefore, by case (6), statement (VII) holds.

We are done. ■

This classification theorem further highlights some differences between analytic families and the entire space $\mathrm{CIFS}(X)$. Whereas we know that $\partial(\mathrm{IR}(X)) = \mathrm{CR}(X)$ while $\partial(\mathrm{SR}(X)) \subset \mathrm{CR}(X)$ in $\mathrm{CIFS}(X)$, there exist analytic families of types (V) and (VII).

Corollary 5.3. *Under the assumptions of Theorem 5.2, the set* $\Gamma \setminus \mathrm{RAH}(\Gamma)$ *is included in a proper real-analytic subvariety of* Γ. *In particular,* $\mathrm{RAH}(\Gamma)$ *is open and dense in* Γ.

Proof. If the family $\{\Phi^\gamma\}_{\gamma \in \Gamma}$ is of type (V) or (VII), then $\Gamma \setminus \mathrm{RAH}(\Gamma)$ is included in the set $\mathrm{CR}_\Gamma(X) = \{\gamma \in \Gamma \mid P(\gamma, \theta) = 0\}$, which is a proper real-analytic subset of Γ due to Theorem 4.10. ■

Note that every type of family described in Theorem 5.2 exists. It is indeed possible to construct families of every type and in the following example we present a family of type (V).

Example 5.4. *Let $z_1, z_2, z_3 \in \mathbb{C}$ be complex numbers of modulus $1/2$ forming an equilateral triangle. Let $\{c_i\}_{i\geq 3}$ be a sequence of positive numbers such that*

$$\sum_{i=3}^{\infty} c_i^t = \begin{cases} \infty & \text{if } t < 1 \\ 1 & \text{if } t = 1 \\ < \infty & \text{if } t > 1. \end{cases}$$

Let $\Gamma = D(0,r) \subset \mathbb{C}$ be a small disk around 0 and $X = \overline{\mathbb{D}} = \overline{D(0,1)} \subset \mathbb{C}$ be the closed unit disk. For each $\gamma \in \Gamma$, let

$$\Phi^\gamma := \begin{cases} \varphi_1^\gamma(z) := \frac{4}{3}(\frac{1}{2}+\gamma)^2(z-z_1)+z_1, \\ \varphi_2^\gamma(z) := \frac{4}{3}(\frac{1}{2}-\gamma)^2(z-z_2)+z_2, \\ \varphi_i^\gamma(z) := \frac{1}{3}(\psi_i(z)-z_3)+z_3, \qquad i \geq 3, \end{cases}$$

where $\{\psi_i\}_{i\geq 3}$ is a family of similitudes such that $\psi_i(\mathbb{D}) \subset \mathbb{D}$, $\psi_i(\mathbb{D}) \cap \psi_j(\mathbb{D}) = \emptyset$ whenever $i \neq j$, and $\psi_i'(z) \equiv c_i$. We claim that the family $\{\Phi^\gamma\}_{\gamma\in\Gamma}$ is of type (V).

Indeed, $\{\Phi^\gamma\}_{\gamma\in\Gamma}$ is an analytic family in HIFS(X). Moreover, $\gamma \mapsto \Phi^\gamma$ is continuous with respect to the λ-topology.

Furthermore,

$$P(\gamma,t) = \log\left(\left(\frac{4}{3}\right)^t \left|\frac{1}{2}+\gamma\right|^{2t} + \left(\frac{4}{3}\right)^t \left|\frac{1}{2}-\gamma\right|^{2t} + \left(\frac{1}{3}\right)^t \sum_{i=3}^{\infty} c_i^t\right).$$

Therefore $\theta_{\Phi^\gamma} = 1$ for every γ and

$$\begin{aligned} P(\gamma,\theta_{\Phi^\gamma}) = P(\gamma,1) &= \log\left(\frac{4}{3}\left|\frac{1}{2}+\gamma\right|^2 + \frac{4}{3}\left|\frac{1}{2}-\gamma\right|^2 + \frac{1}{3}\right) \\ &= \log\left(\frac{4}{3}\left(\frac{1}{2}+2|\gamma|^2\right) + \frac{1}{3}\right) = \log\Big(1+\frac{8}{3}|\gamma|^2\Big). \end{aligned}$$

This latter function of γ takes its minimum value 0 at $\gamma = 0$ and is positive for each small $\gamma \neq 0$. Thus, $\{\Phi^\gamma\}_{\gamma\in\Gamma}$ is of type (V). ■

6. Consequences of the Classification Theorem

Propositions 5.13–5.16 in [7] assert that $\partial(\mathrm{SR}(X)) = \partial(\mathrm{FSR}(X)) \subset \mathrm{CR}(X)$, despite that $\mathrm{CR}(X) \neq \partial(\mathrm{SR}(X))$. Moreover, Proposition 5.17 of that same paper provides a condition under which a critically regular SIFS is in the boundary of the subset of strongly regular CIFSs. Using our classification theorem, we partially generalize this result.

Theorem 6.1. *Suppose that $X \subset \mathbb{C}$ is star shaped with $x_0 \in \mathrm{Int}(X)$ for center. Let $\Phi = \{\varphi_i\} \in \mathrm{CR}(X) \cap \mathrm{HIFS}(X)$ be a system such that there exists a finite set $F \subset \mathbb{N}$ for which*

$$\bigcup_{i \in F} \varphi_i(X) \subset \mathrm{Int}(X) \setminus \overline{\bigcup_{j \notin F} \varphi_j(X)} = \emptyset.$$

Then $\Phi \in \partial(\mathrm{SR}(X)) = \partial(\mathrm{FSR}(X))$. More precisely, there exists an open, connected neighbourhood Γ of $\{s \in \mathbb{R} : 0 < s \leq 1\}$ in $\mathbb{C}$ such that, setting

$$\Phi^\gamma = \left\{ \begin{array}{ll} \varphi_i, & i \notin F, \\ \varphi_i \circ \psi_\gamma, & i \in F \end{array} \right\}$$

for each $\gamma \in \Gamma$, where $\psi_\gamma(x) = \gamma(x - x_0) + x_0$, the analytic family $\{\Phi^\gamma\}_{\gamma \in \Gamma}$ in $\mathrm{HIFS}(X)$ satisfies all of the following:

(1) $\Phi^1 = \Phi$;
(2) $\gamma \in \Gamma \mapsto \Phi^\gamma \in \mathrm{HIFS}(X)$ *is continuous with respect to the λ-topology;*
(3) $\{\Phi^\gamma\}_{\gamma \in \Gamma}$ *is of type (VII);*
(4) $1 \in \mathrm{CR}_\Gamma(X) = \partial(\mathrm{FSR}_\Gamma(X))$;
(5) $\{s \in \mathbb{R} : 0 < s < 1\} \cap \mathrm{IR}_\Gamma(X) \neq \emptyset$ *and* $\{s \in \mathbb{R} : 0 < s \leq 1\} \cap \partial(\mathrm{IR}_\Gamma(X)) \neq \emptyset$; *and*
(6) *for any* $\gamma_0 \in \partial(\mathrm{IR}_\Gamma(X))$, *we have* $\gamma_0 \in \partial(\mathrm{IR}_\Gamma(X)) \subset \mathrm{CR}_\Gamma(X) = \partial(\mathrm{SR}_\Gamma(X) \cap \mathrm{CFR}_\Gamma(X))$ *and* $\gamma \mapsto h_{\Phi^\gamma}$ *is not real-analytic in any neighbourhood of* γ_0 *in* Γ.

Proof. Using the local bounded distortion property (see Corollary 4.1.4 in [5]), the fact that all generators φ_i, $i \in \mathbb{N}$, are contractions with a common ratio, and the Mean Value Inequality, we can construct a decreasing family $\mathcal{F}$ of open, connected neighbourhoods of X whose intersection is X and with respect to each of which Φ is in $\mathrm{HIFS}(X)$. (In fact, $X_\varepsilon := B(X, \varepsilon)$ is such a neighbourhood for every $\varepsilon > 0$ small enough). This family $\mathcal{F}$ therefore has a member $U \supset X$ such that

$$\bigcup_{i \in F} \varphi_i(\overline{U}) \subset \mathrm{Int}(X) \setminus \overline{\bigcup_{j \notin F} \varphi_j(X)} = \emptyset.$$

Choose $\delta > 0$ such that $\overline{X_\delta} \subset U$. Thereafter choose an open neighbourhood Γ of $\{s \in \mathbb{R} : 0 < s \leq 1\}$ in $\mathbb{C}$ such that

$$\bigcup_{\gamma \in \Gamma} \psi_\gamma(\overline{X_\delta}) \subset U.$$

Finally, take for V any neighbourhood in the family $\mathcal{F}$ such that $V \subset X_\delta$. Then $\varphi_i(V) \subset V$ for all $i \in \mathbb{N}$ and

$$\varphi_k \circ \psi_\gamma(V) \subset \varphi_k \circ \psi_\gamma(\overline{X_\delta}) \subset \varphi_k(U) \subset \mathrm{Int}(X) \setminus \overline{\cup_{j \notin F} \varphi_j(X)}$$

for every $k \in F$ and every $\gamma \in \Gamma$. With this neighbourhood V, we have that $\{\Phi^\gamma\}_{\gamma \in \Gamma}$ is an analytic family in $\mathrm{HIFS}(X)$. Moreover, $\gamma \mapsto \Phi^\gamma \in \mathrm{HIFS}(X)$ is clearly continuous with respect to the λ-topology. Furthermore, if $s > 0$ is small enough, then by the argument given in the proof of Proposition 5.11 in [7], we obtain that $s \in \mathrm{IR}_\Gamma(X)$. Thus, Theorem 5.2 implies that the analytic family $\{\Phi^\gamma\}_{\gamma \in \Gamma}$ is of type (VII). In particular, $1 \in \mathrm{CR}_\Gamma(X) = \partial(\mathrm{FSR}_\Gamma(X))$.

Now, let

$$S := \sup\Big\{ s \in I\!R : 0 < s < 1,\ s \in \mathrm{IR}_\Gamma(X) \Big\}.$$

Then $S \in \partial(\mathrm{IR}_\Gamma(X))$. Thus, statement (5) holds.

Finally, by Theorem 5.2, statement (6) holds. ■

By a similar argument, we get the following, more general result.

Theorem 6.2. *Let $X \subset \mathbb{C}$ and $\Phi = \{\varphi_i\} \in \mathrm{CR}(X) \cap \mathrm{HIFS}(X)$ be such that there exist a finite set $F \subset I\!N$ and a simply connected subdomain W of $\mathrm{Int}(X)$ such that*

$$\bigcup_{i \in F} \varphi_i(X) \subset W \ \text{ and } W \bigcap \overline{\bigcup_{j \notin F} \varphi_j(X)} = \emptyset.$$

Then $\Phi \in \partial(\mathrm{SR}(X)) = \partial(\mathrm{FSR}(X))$. More precisely, there exists an analytic family $\{\Phi^\gamma\}_{\gamma \in \Gamma}$ in $\mathrm{HIFS}(X)$ with an open, connected neighbourhood Γ of $\{s \in I\!R : 0 < s \leq 1\}$ in $\mathbb{C}$ such that

1. $\Phi^1 = \Phi$;
2. $\gamma \in \Gamma \mapsto \Phi^\gamma \in \mathrm{HIFS}(X)$ *is continuous with respect to the λ-topology;*
3. $\{\Phi^\gamma\}_{\gamma \in \Gamma}$ *is of type (VII);*
4. $1 \in \partial(\mathrm{FSR}_\Gamma(X))$;
5. $\{s \in I\!R : 0 < s < 1\} \cap \mathrm{IR}_\Gamma(X) \neq \emptyset$ *and* $\{s \in I\!R : 0 < s \leq 1\} \cap \partial(\mathrm{IR}_\Gamma(X)) \neq \emptyset$; *and*
6. *for any $\gamma_0 \in \partial(\mathrm{IR}_\Gamma(X))$, we have $\gamma_0 \in \partial(\mathrm{IR}_\Gamma(X)) \subset \mathrm{CR}_\Gamma(X) = \partial(\mathrm{SR}_\Gamma(X) \cap \mathrm{CFR}_\Gamma(X))$ and $\gamma \mapsto h_{\Phi^\gamma}$ is not real-analytic in any neighbourhood of γ_0 in Γ.*

Proof. Let A be a simply connected subdomain of W such that $\cup_{i \in F} \varphi_i(X) \subset A \subset \overline{A} \subset W$. Let also $a \in A$. By the Riemann Mapping Theorem, there exists an analytic family $\{\psi_\gamma\}_{\gamma \in \Gamma}$ of injective holomorphic maps from A to W with an open, connected neighbourhood Γ of $\{s \in I\!R : 0 < s \leq 1\}$ in $\mathbb{C}$ such that $\psi_1(z) \equiv z$ and $\psi_s(z) \to a$ as $|s| \to 0$ uniformly on A.

For each $\gamma \in \Gamma$, let

$$\Phi^\gamma = \left\{ \begin{array}{ll} \varphi_i, & i \notin F, \\ \psi_\gamma \circ \varphi_i, & i \in F \end{array} \right\}$$

Then $\{\Phi^\gamma\}_{\gamma \in \Gamma}$ is an analytic family in $\mathrm{HIFS}(X)$. Note that a neighbourhood common to all Φ^γ's can be found as follows. As explained in the proof of Theorem 6.1, we can construct a decreasing family $\mathcal{F}$ of open, connected neighbourhoods of X whose intersection is X and with respect to each of which Φ is in $\mathrm{HIFS}(X)$. This family therefore has a member $V \supset X$ such that $\cup_{i \in F} \varphi_i(V) \subset A$. Then $\varphi_i(V) \subset V$ for all $i \in I\!N$ and

$$\psi_\gamma \circ \varphi_i(V) \subset \psi_\gamma(A) \subset W \subset \mathrm{Int}(X) \subset V$$

for every $i \in F$. Moreover, $\gamma \in \Gamma \mapsto \Phi^\gamma \in \mathrm{HIFS}(X)$ is continuous with respect to the λ-topology. Furthermore, by the argument given in the proof of Proposition 5.11 in [7], every $\gamma \in \Gamma$ with small enough modulus is in $\mathrm{IR}_\Gamma(X)$. Hence, the family $\{\Phi^\gamma\}_{\gamma \in \Gamma}$ is of type (VII). In particular, $1 \in \mathrm{CR}_\Gamma(X) = \partial(\mathrm{FSR}_\Gamma(X))$. The remaining statements follow from the same arguments as those given in the proof of Theorem 6.1. ■

We now give a better description of the set $(\mathrm{FSR}(X)) \cap \mathrm{HIFS}(X)$.

Proposition 6.3. *Suppose that X is star shaped with $x_0 \in \mathrm{Int}(X)$ for center. Let $\Phi = \{\varphi_i\} \in (\mathrm{FSR}(X)) \cap \mathrm{HIFS}(X)$. Then there exists an open, connected neighbourhood Γ of $\{s \in \mathbb{R} : 0 < s \le 1\}$ in $\mathbb{C}$ such that, setting $\Phi^\gamma = \{\varphi_i \circ \psi_\gamma\}_{i\in I}$ for each $\gamma \in \Gamma$, where $\psi_\gamma(x) = \gamma(x - x_0) + x_0$, the analytic family $\{\Phi^\gamma\}_{\gamma\in\Gamma}$ in $\mathrm{HIFS}(X)$ satisfies*

(1) $\Phi^1 = \Phi$;
(2) $\Phi^\gamma \to \Phi$ *as* $\gamma \nearrow 1$ *in* $(0, 1)$ *with respect to the* λ*-topology;*
(3) $\gamma \in \Gamma \mapsto \Phi^\gamma \in \mathrm{HIFS}(X)$ *is continuous with respect to the* λ*-topology;*
(4) $\{\Phi^\gamma\}_{\gamma\in\Gamma}$ *is of type (VII);*
(5) $\{s \in \mathbb{R} : 0 < s < 1\} \cap \mathrm{IR}_\Gamma(X) \neq \emptyset$ *and* $\{s \in \mathbb{R} : 0 < s < 1\} \cap \partial(\mathrm{IR}_\Gamma(X)) \neq \emptyset$*; and*
(6) *for any* $\gamma_0 \in \partial(\mathrm{IR}_\Gamma(X))$, *we have* $\gamma_0 \in \partial(\mathrm{IR}_\Gamma(X)) \subset \mathrm{CR}_\Gamma(X) = \partial(\mathrm{SR}_\Gamma(X) \cap \mathrm{CFR}_\Gamma(X))$ *and* $\gamma \mapsto h_{\Phi^\gamma}$ *is not real-analytic in any neighbourhood of* γ_0 *in* Γ.

Proof. It is easy to see that the first three statements hold. Moreover, if $\gamma \in \Gamma$ and $|\gamma|$ is small enough, then $\gamma \in \mathrm{IR}_\Gamma(X)$. Since $\gamma \in \mathrm{SR}_\Gamma(X)$ if γ is close enough to 1, statement (4) follows from Theorem 5.2. By the same argument as that in the proof of Theorem 6.1, we can show that statements (5) and (6) hold. ■

Finally, we take a brief look at a subset of $\mathrm{CR}(X) \cap \mathrm{HIFS}(X)$.

Definition 6.4. *Let $X \subset \mathbb{C}$. Let $\mathrm{HSPT}(X)$ be the set of $\Phi \in \mathrm{CR}(X) \cap \mathrm{HIFS}(X)$ such that there exists an analytic family $\{\Phi^\gamma\}_{\gamma\in\Gamma}$ in $\mathrm{HIFS}(X)$ with $\Gamma = \{z \in \mathbb{C} : |z| < 1\}$ satisfying all of the following:*

- $\Phi^0 = \Phi$;
- $\gamma \mapsto \Phi^\gamma \in \mathrm{HIFS}(X)$ *is continuous with respect to the* λ*-topology;*
- $\{\Phi^\gamma\}_{\gamma\in\Gamma}$ *is of type (VII);*
- $0 \in \partial(\mathrm{IR}_\Gamma(X)) \subset \mathrm{CR}_\Gamma(X) = \partial(\mathrm{FSR}_\Gamma(X))$*; and*
- $\gamma \mapsto h_{\Phi^\gamma}$ *is not real-analytic in any neighbourhood of* 0.

"HSPT" stands for "**H**olomorphic **S**imultaneous **P**hase **T**ransition".

Theorem 6.5. *Let $X \subset \mathbb{C}$. Then $\mathrm{HSPT}(X)$ is a dense subset of $\mathrm{HIFS}(X)$ when this latter is endowed with the metric of pointwise convergence.*

Proof. This proposition follows immediately from combining the argument in the proof of Lemma 4.10 in [7] (taking the similitude S^n so that $S^n(X)$ is included in a small ball in $\varphi_n(X)$) with Theorem 5.2. ■

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