Geometry of Markov Systems and Codimension-One Foliations

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Abstract

We show that the theory of graph directed systems can be used to study exceptional minimal sets of some foliated manifolds. A C^1-smooth embedding of a contractible parabolically Markov system into a holonomy pseudogroup of codimension one foliation allows us to describe in details the h-dimensional Haussdorff and packing measures of the intersection of the complete transversal with the exceptional minimal sets.

1 Introduction

Cantwell and Conlon [3] observed that there exists a special class of pseudogroups, called Markov pseudogroups, which are semiconjugated to subshifts of finite type. Markov pseudogroups appeared in natural way in the theory of foliations as holonomy pseudogroups of some closed, transversaly oriented, C^2-foliated manifolds of codimension one. A detailed introduction to the foliation theory the reader can find in [4]. However, for a convenience of the reader we shall recall few definitions.

Given a topological space X denote by $Homeo(X)$ the family of all homeomorphisms between open subset of X. If $g \in Homeo(X)$, then D_g is its domain and $R_g = g(D_g)$ is its range.

Definition 1. *Let M be a Riemannian manifold. A C^r pseudogroup Γ on M is a collection of C^r diffeomorphisms $h : D_h \to R_h$ between open subsets D_h and R_h of M such that*

1. *If $g, h \in \Gamma$ then $g \circ f : f^{-1}(R_f \cap D_g) \to g(R_f \cap D_g)$ is an element of Γ.*

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2. *If $h \in \Gamma$, then $h^{-1} \in \Gamma$.*

3. *$id_M \in \Gamma$*

4. *If $h \in \Gamma$ and $W \subset D_h$ is an open subset, then $h|_W \in \Gamma$.*

5. *If $h : D_h \to R_h$ is a C^r diffeomorphism betwen open subsets of M and if, for each point $p \in D_h$, there exists a neighborhood N of p in D_h such that $h|_N \in \Gamma$, then $h \in \Gamma$.*

For any set $G \subset Homeo(M)$ which satisfies the condition

$$\bigcup_{g \in G} \{D_g \cup R_g : g \in G\} = M,$$

there exists a unique smallest (in the sense of inclusion) pseudogroup $\Gamma(G)$ which contains G. Notice that $\gamma \in \Gamma(G)$ if and only if $\gamma \in Homeo(M)$ and for any $x \in D_\gamma$ there exist maps $g_1, g_2, ..., g_k \in G$, exponents $e_1, e_2, ..., e_k \in \{-1, 1\}$ and an open neighbourhood U of x in M such that

$$U \subset D_\gamma \textit{ and } \gamma|_U = g_1^{e_1} \circ ... \circ g_k^{e_k}|_U$$

The pseudogroup $\Gamma(G)$ is said to be *generated by* G. If the set G is finite, we say that the pseudogroup $\Gamma(G)$ is *finitely generated.*

Following [13] we write:

Definition 2. *A finite system $S = \{h_1, ..., h_m\}$ of $Homeo(M)$, $h_j : D_j \to R_j$, together with nonempty compact sets $K_j \subset R_j$ is called a Markov system if*

1. *$R_i \cap R_j = \emptyset$ when $i \neq j$.*

2. *either $K_i \subset Q_j$ or $K_i \cap D_j = \emptyset$,*

where $Q_j = h_j^{-1}(K_j)$.
If S is a Markov system and $\bigcup_{i=1}^m (D_{h_i} \cup R_{h_i}) = M$, then the pseudogroup $\Gamma(S)$ generated by the finite set S is called a Markov pseudogroup.

Notice that Markov pseudogroups are generated by maps $h_i, h_j \in S$ such that either $D_{h_i \circ h_j} = D_{h_i}$ or $D_{h_i \circ h_j} = \emptyset$. Therefore, the following definition is very useful:

Definition 3. *For any Markov system $S = \{h_1, ..., h_m\}$ one defines its transition matrix $P = [p_{ij}]_{i,j=1,...m}$, as follows*

$$p_{ij} \in \{0, 1\} \quad \textit{and} \quad p_{ij} = 1 \; iff \; K_j \subset Q_i.$$

The Markov invariant set Z_0 is defined as

$$Z_0 = Z \setminus int(Z), \textit{ where } Z = \bigcap_{n=1}^{\infty} \bigcup_{g \in S_n} K_g, \; S_n = \{h_{i_1} \circ ... \circ h_{i_n} : i_1, ..., i_n \leq m\},$$

$$\textit{and } K_g = g(Q_{i_n}) \textit{ when } g = h_{i_1} \circ ... \circ h_{i_n}.$$

Examples of Markov pseudogroups and its minimal sets abound in the literature on foliation theory, let us list only few papers [12], [6], [7], [8]. From our point of view, the importance of Markov pseudogroups for foliation theory can be derived from the result of Cantwell and Conlon [5] which states that any one dimensional Markov pseudogroup can be realized as a holonomy pseudogroup of some foliated manifold. More precise formulation of this result and detailed proof is due to Walczak [13] :

Theorem 1. *(Thm 1.4.8 in [13]) If Γ is a Markov pseudogroup on a circle such that its Markov invariant set Z_0 contains a $\Gamma-$invariant minimal set C, then there exists a closed foliated 3-manifold (M, F), $dimF = 2$, an exceptional minimal set $E \subset M$, a complete transversal T and a homeomorphism $h : E \cap T \to C$ which conjugates $\Gamma|C$ to $H|E \cap T$, H being the holonomy pseudogroup of F acting on T.*

More results and a list of still open problems on Markov pseudogroups the reader can find in [1]. Another realization of a Markov pseudogroup, obtained by a hyperbolic Markov system, as a holonomy pseudogroup of codimension one foliation on a compact three manifold was provided by Biś, Hurder and Shive [2] in their construction of generalized Hirsch foliations.

2 Contracting and Parabolic Markov Systems

Let $\mathcal{S} = \{h_j : j \in I\}$, where I is a countable set, be a Markov system in the sense of Definition 2. Suppose X is a, not necessarily connected, 1-dimensional smooth manifold and all $D_j s$ and $R_j s$ are its proper subarcs. Suppose further that all homeomorphisms $h_j s$ have $C^{1+\varepsilon}$ extensions to $\bar{D}_j$, the closures of their domains. We call the Markov system $\mathcal{S}$ contractible provided that

$$s = \sup\{||h'_{ij}||_\infty : F_{ij} = 1\} < 1, \tag{1}$$

where $F_{ij} = 1$ if and only if $K_i \subseteq Q_j = h_j^{-1}(K_j)$, is equal to zero otherwise, and $h_{ij} = h_{j|K_i}$. The associated Markov pseudogroup is also called contractible. We want to associate to $\mathcal{S}$ a (conformal) graph directed Markov system $\hat{S}$ in the sense of [9]. Indeed, take $V = \{1, 2, \ldots, m\}$ to be the set of vertices, and $E = \{(i, j) : F_{ij} = 1\}$ to be the set of edges. Define the incidence matrics $A : E \times E \to \{0, 1\}$ by the formula

$$A_{(i,j)(k,l)} = \begin{cases} 1 & \text{if } i = l, \\ 0 & \text{if } i \neq l. \end{cases}$$

Put further

$$\varphi_{(i,j)} = h_{ij}$$

for all $(i, j) \in E$. $\hat{S} = \{\varphi_e\}_{e \in E}$ is our graph directed Markov system. In order to fulfill all the formal conditions from [10] we extend all the maps φ_e, $e \in E$, in a $C^{1+\varepsilon}$ fashion to some open intervals $\Delta_j \supset K_j$ such that all the components of $\Delta_j \setminus K_j$, $j = 1, \ldots, m$, have the same lengths and $|\varphi'_{(i,j)}(x)| \leq s$ for all $(i, j) \in E$ and all $x \in \Delta_i$. It is easy to notice that the limit set of the graph directed Markov system $\hat{S}$ is equal to $Z_0 = Z$ (this equality being a consequence of (1)), the Markov invariant set of $\mathcal{S}$ introduced in Definition 3.

Assume that the incidence matrix A is finitely primitive, meaning that there exists a finite set Λ of A-admissible words of the same length such that any for any two elements a and b of E there exists $\gamma \in \Lambda$ such that the word $a\gamma b$ is A-admissible. Let $h = HD(Z)$ be the Hausdorff dimension of the invariant set Z. Invoking appropriate theorems from [10], we can list the following:

Theorem 2. *If $\mathcal{S}$ is a contracting Markov system, then $0 < h < 1$, $H_h(Z) < \infty$ and $P_h(Z) > 0$, where H_h denotes the h-dimensional Hausdorff measure and P_h denotes h-dimensional packing measure. If S is finite, then in addition $H_h(Z) > 0$ and $P_h(Z) < \infty$. Furthermore, the measures $H_h|_Z$ and $P_h|_Z$ are equal up to a multiplicative constant.*

From now we assume that our contractible Markov system is finite, $I = \{1, ..., m\}$.

Theorem 3. *If $\mathcal{S}$ is a contracting Markov system, then there exists a constant $c \geq 1$ such that for all $r \in (0, 1]$ and all $z \in Z$,*

$$c^{-1} \leq \frac{H_h(B(z,r))}{r^h} \leq c.$$

Theorem 4. *$BD(Z) = PD(Z) = HD(Z)$, where $BD(Z)$ and $PD(Z)$ are respectively the box counting and packing dimensions of Z.*

Now, replace in the above considerations, condition (1) by the following. For all $i, j \in \{1, \ldots, m\}$ with $A_{ij} = 1$ and all $x \in K_i$,

$$|h'_{ij}(x)| \leq 1,$$

and if $|h'_{ij}(x)| = 1$, then $h_{ij}(x) = x$. Such point x is called parabolic. The set Ω of parabolic points is asumed to be nonempty and $K_i \cap \Omega$ contains at most one point, for all $i \in I$. Asumme also that the maps h_{ij} are C^2. Call any such system $\mathcal{S}$ parabolic Markov. Then, Theorem 2, Theorem 3 and Theorem 4 take on the following form:

Theorem 5. *If $\mathcal{S}$ is a parabolic Markov system, then the h-dimensional Hausdorff measure of Z vanishes whereas the h-dimensional packing measure is finite and positive.*

Theorem 6. *Suppose that $\mathcal{S}$ is a parabolic Markov system. Then for any $z \in Z$, we have*

$$\liminf_{r\to 0} \frac{P_h(B(z,r))}{r^h} \in (0, +\infty] \quad \textit{and} \quad \limsup\nolimits_{r\to 0} \tfrac{P_h(B(z,r))}{r^h} = +\infty,$$

where P_h denotes the h-dimensional packing measure on Z.

Theorem 7. *If $\mathcal{S}$ is a parabolic Markov system, then $BD(Z) = PD(Z) = HD(Z)$.*

3 Contracting and Parabolic Markov systems versus codimension-one foliations

Denote the unit disc, the unit circle, a circle and an open ball in the complex plane respectively by:

$$D = \{w \in \mathbb{C} : |w| \leq 1\}$$

$$S^1 = \{w \in \mathbb{C} : |w| = 1\}$$
$$S(z,r) = \{w \in \mathbb{C} : |w - z| = r\}$$
$$B(z,r) = \{w \in \mathbb{C} : |w - z| \leq r\}$$

Choose an integer $n > 1$ and an analytic embedding $\varphi : S^1 \to S^1 \times D$ so that its homotopy class is equal to ng, where g is a generator of the fundamental group of the solid torus.

Now we recall the construction of a generalized Hirsch foliation in codimension one, which was presented in details in the Section 2 of [2], in the following way. Choose a non-zero interior point $z_0 \in D$ (such that $0 < |z_0| < 1$) and positive $\varepsilon > 0$ such that $0 < 2\varepsilon < \min\{|z_0|, 1-|z_0|\}$. Now define n-punctured disc

$$P = D \setminus (B(z_0, \varepsilon) \cup B(z_1, \varepsilon) \cup ... \cup B(z_{n-1}, \varepsilon))$$

where for any $0 \leq m < n$ the complex number $z_m = \rho^m z_0$ and $\rho = e^{\frac{2\pi}{n}i}$.

The analityc 3-manifold N_1 with boundary is defined as the quotient of $\mathbb{R} \times P$ by the equivalence relation $\sim$ that identifies the points (x, z) and $(x + 1, \rho z)$. Notice that N_1 is diffeomorphic to the solid torus $S^1 \times D$ from which an open tubular neighborhood of $\varphi(S^1)$ was removed. Remember that the embedding $\varphi : S^1 \to S^1 \times D$ winds n times around the core. The boundary of N_1 consists of two disjoint tori, $\partial N_1 = \partial^+ N_1 \cup \partial^- N_1$, where

$$\partial^+ N_1 = (\mathbb{R} \times S^1)/\sim$$

and

$$\partial^- N_1 = (\mathbb{R} \times ((S(z_0, \varepsilon) \cup S(z_1, \varepsilon) \cup ... \cup S(z_{n-1}, \varepsilon)))/\sim$$

The manifold N_1 admits a foliation $\mathcal{F}_{N_1} = \{\{x\} \times P : x \in [0, 1)\}$ by compact 2-manifolds with boundary. Notice that the intersection of the leaves of $\mathcal{F}_{N_1}$ with the boundary tori consists of circles, therefore each boundary torus is foliated by circles. Gluing the boundary $\partial^+ N_1$ with the boundary $\partial^- N_1$ by a properly chosen diffeomorphism $f : \partial^+ N_1 \to \partial^- N_1$, which maps the foliations of the boundary tori each to the other, we get a foliated c manifold N with foliation $\mathcal{F}$. To construct such a diffeomorphism f choose an immersion $H : S^1 \to S^1$ of degree n. Notice that the choice of H is equivalent to the choice of a diffeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that $h(x + 1) = h(x) + 1$. So, $H = h\ mod(1)$.

Lemma 1. *([2], p. 76-77) Given diffeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that $h(x+1) = h(x)+n$, the map $\tilde{f} : \mathbb{R} \times D \to \mathbb{R} \times D$ described by the formula*

$$\tilde{f}(x, z) = (h(x), z_1 + \varepsilon z e^{\frac{2\pi x i}{n}})$$

induces a map $f : \partial^+ N_1 \to \partial^- N_1$.

Finally, define

$$N = N_1/\sim_f,$$

where $\sim_f$ identifies the points (x, z) anf $f(x, z)$. Then, the leaves of $\mathcal{F}_{N_1} \cap \partial^+ N_1$ are mapped to leaves $\mathcal{F}_{N_1} \cap \partial^- N_1$, which yields that N has a foliation $\mathcal{F}_N$ whose leaves are the union of n-punctured discs.

The foliation $\mathcal{F}_N$ on N is called a *generalized Hirsch foliation.*

The foliation $\mathcal{F}_N$ on N admits a complete transversal $T : S^1 \to N$. Observe that the foliation $\mathcal{F}_{N_1}$ on N_1 is defined by fibration, therefore $\mathcal{F}_{N_1}$ has no holonomy. So, all the holonomy of $\mathcal{F}_N$ is introduced by the identyfication of the outer boundary $\partial^+ N_1$ with the inner boundary $\partial^- N_1$ via the diffeomorphism f.

The immersion $H : S^1 \to S^1$ of degree n induces an equivalence relation on S^1: two points $x, y \in S^1$ are said to be in the same "grand orbit" of H if there exist positive integers k and l such that $H^k(x) = H^l(y)$ (cf. Milnor [11]). The grand orbit of a point x is denited by $O(x)$.

Recall that a subset $K \subset S^1$ is said to be H-invariant if for all $x \in K$ the grand orbit $O(x)$ is contained in K.

Definition 4. *An invariant set K is called minimal if it is closed and for all $x \in K$ the H-orbit $O(x)$ is dense in K. A minimal set K is exceptional if it is nowhere dense and is not finite.*

Our first, obvious application to the theory of foliations is the following.

Theorem 8. *Suppose $\mathcal{F}$ is a smooth codimension 1 foliation on a Riemannian manifold M and T is a complete transversal for $\mathcal{F}$. If the holonomy pseudogroup of $\mathcal{F}$ acting on T is generated by a contracting (parabolicalic) Markov system then there is an expectional minimal set E for $\mathcal{F}$ such that the theorems Theorem 2, Theorem 3 and Theorem 4 (Theorem 5, Theorem 6 and Theorem 7) are true with $Z = E \cap T$.*

Theorem 9. *If $\mathcal{S} = \{h_1, ..., h_m\}$ is either a contracting or parabolic Markov C^r-system on the circle S^1, $r \geq 1$, such that all maps h_i are defined on on the closed interval $I_0 \subset S^1$, then there exists a generalized Hirsch foliation $(N, \mathcal{F})$, codim$\mathcal{F} = 1$, an exceptional minimal set $E \subseteq N$, a complete transversal T and a C^r-diffeomorphism, $f : E \cap T \to J_{\mathcal{G}}$ (the Markov invariant set of the pseudogroup $\mathcal{G}$ generated by a Markov system $\mathcal{S}$).*

Proof. Take a contracting or parabolic Markov C^r-system $\mathcal{S} = \{h_1, ..., h_m\}$ and a closed intervals $I_j = [a_j, b_j], j = 0, 1, ..., m$, such that

1. $h_i : I_0 \to I_i \subset I_0$,
2. $I_i \cap I_j = \emptyset \; for \; i \neq j$,
3. $|h_i^{'}(x)| \leqslant 1$ for any $x \in I_0$ and the equality $|h_i^{'}(x_0)| = 1$ holds at most for one point x_0 of I_i.

We may assume that the interval $I_0 = [0, c]$, where $c < 1$. Denote the unique fixed point of h_i by x_i^*. Let $a_0 = \min\{x_i^* : 1 \leqslant i \leqslant m\}$, $b_0 = \max\{x_i^* : 1 \leqslant i \leqslant m\}$.

Without loosing the generality of considerations, we can assume that the interval $I_i = [a_i, b_i]$, $1 \leq i \leq m$ and

$$0 = a_0 = a_1 < b_1 < a_2 < ... < b_{m-1} < a_m < b_m = b_0$$

Following the Example 6.1 in [2] we define a C^r-diffeomorphism $h : [0, 1] \to [0, m]$ in the following way:

1. $h|_{[a_i,b_i]} = (i-1) + h_i$ for any $1 \leq i \leq m-1$.
2. $h|_{[b_i,a_{i+1}]} = f_i$ where $f_i : [b_i, a_{i+1}] \to [i-1+c, i]$ is a C^r-diffeomorphism chosen so that h is C^r at the points b_i and a_{i+1}, $1 \leq i \leq m-1$.
3. $h|_{[c,1]} = f_m$ where the C^r-diffeomorphism $f_m : [c, 1] \to [c, 1]$ satisfies the conditions:
   (a) f_m has a unique attracting fixed point at $x_0 = \frac{1+c}{2}$,
   (b) $f_m(c) = c$ and $f_m(1) = 1$,
   (c) $f_m|_{(c,1)}$ is a contraction of the open interval $(c, 1)$ to the attracting fixed point x_0.
   (d) h is C^r at the points $b_m = c$ and 1.

Let $H : S^1 \to S^1$ be the immersion of degree m, defined by $H = h(mod 1)$, and define the open set $U \subset S^1$ to be the union of the $H-$orbits of the open interval $(c, 1)$. Putting $K = S^1 \setminus U$ we get that $K \subset I_1 \cup .... \cup I_m$. Modifying slightly the proof of Lemma 2.1 in [2] we get

Lemma 2. *Let $H : S^1 \to S^1$ be the immersion of degree m, defined by $H = h(mod 1)$. Then there exists a unique minimal set $J_{\mathcal{G}} \subset S^1$ with respect to H. Moreover, $J_{\mathcal{G}} = K$.*

Gluing the outer boundary $\partial^+ N_1$ to the inner boundary $\partial^- N_1$ via the diffeomorphism h we obtain a three dimensional manifold N. The foliation $\mathcal{F}_N$ on N admits a complete transversal. It can be constructed by the embedding

$$\hat{t} : \mathbb{R} \to \mathbb{R} \times P$$

where $\hat{t}(x) = (x, 0)$. Notice that

$$\hat{t}(x+1) = (x+1, 0) \sim (x, \rho 0) = (x, 0) = \hat{t}(x)$$

Passing to quotient manifold we get $t : S^1 \to N_1$. By construction we obtain that for any leaf $L \in \mathcal{F}_{N_1}$ the intersection $L \cap t(S^1) \neq \emptyset$. Therefore, after the process of gluing outer and inner boundary we get a complete transversal $T : S^1 \to N$. The construction of the foliation $\mathcal{F}_N$ on N yields that for the exceptional minimal set E of $\mathcal{F}_N$ we have

$$E \cap T(S^1) = K = J_{\mathcal{G}},$$

which completes the proof.

■

Corollary 1. *Consequently all the above theorems (Theorem 2)– (Theorem 7) apply with the set Z replaced by $E \cap T$.*

Similarly, making the construction in the proof of Theorem 1.4.8 [13] C^1-smooth, we get the following

Theorem 10. *If $\mathcal{G}$ is a contractible (parabolically) Markov pseudogroup on a circle, then there exists a closed foliated* 3*-manifold* $(M, \mathcal{F})$*,*$\dim \mathcal{F} = 2$*, an expectional minimal set* $E \subseteq M$*, a complete transversal* T *and a* C^1*-diffeomorphism* $M : E \cap T \to J_{\mathcal{G}}$ *(the Markov invariant set of* $\mathcal{G}$*). Consequently all the above theorems (Theorem 2)– (Theorem 7) apply with the set* Z *replaced by* $E \cap T$.

Now, if E is an expectional minimal set for a codemension 1 foliation $\mathcal{F}$ and T is a complete transversal for $\mathcal{F}$, then E is locally diffeomorphic to the Cartesian product of $E \cap T$ and an interval. Consequently, Theorems 8 and Theorem 9 remain true with $E \cap T$ replaced by "sufficiently small" open subsets of E. The dimension h then stands for $HD(E) = HD(E \cap T) + 1$.

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