

REAL ANALYTICITY OF HAUSDORFF DIMENSION FOR HIGHER DIMENSIONAL HYPERBOLIC GRAPH DIRECTED MARKOV SYSTEMS

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ABSTRACT. In this paper we prove that the Hausdorff dimension function of the limit sets of strongly regular, hyperbolic, conformal graph directed Markov systems living in higher dimensional Euclidean spaces \mathbb{R}^d , $d \geq 3$, and with an underlying finitely irreducible incidence matrix is real-analytic.

1. INTRODUCTION

In this paper we deal with the problem of regularity of the Hausdorff dimension function of the limit sets of strongly regular, hyperbolic, conformal graph directed Markov systems living in higher dimensional Euclidean spaces \mathbb{R}^d , $d \geq 3$, and whose underlying incidence matrix is finitely irreducible. The foundations of the theory of hyperbolic, conformal graph directed Markov systems were laid in [4]. This theory extends that of hyperbolic, conformal iterated function systems developed in [6]. Extensive research in this field has been done before. As we are mainly interested in this article in regularity properties of the Hausdorff dimension function, we would like to bring to the attention of our readers article [7], in which we explored the continuity of the Hausdorff dimension in the general case ($d \geq 1$), as well as papers [7] (sections 6 and 8), [1] and [3], where the problem of regularity of the Hausdorff dimension was treated in the complex plane (case $d = 2$), under the assumption that the generators of the systems are holomorphic. In [3] the real-analyticity was established under weaker assumptions (to our knowledge). In the present paper we go beyond the plane by investigating the case $d \geq 3$. In this case, the conformal generators of the systems are not analytic (unless they are affine), but in view of Liouville's Theorem (see [2], for instance) they all take on the same canonical form. This enables us to parametrize the space M_d of such maps by $q = 2 + \frac{1}{2}d(d+3)$ real parameters, and thereafter define natural notions of differentiability and real-analyticity on the space $(\mathbb{R}^q)^I$, where I is a countable set. In our framework I is the alphabet of the graph directed Markov systems. Our main result, Theorem 7.2, asserts that the Hausdorff dimension function is real-analytic when this function is restricted to an open subset W of $(\mathbb{R}^q)^I$ corresponding to a family of strongly regular, hyperbolic, conformal graph directed Markov systems with a common alphabet I , a common finitely irreducible incidence matrix and which satisfy the open set condition.

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The main idea of the proof is to ascribe canonically to each graph directed Markov system Φ its complexification $\hat{\Phi}$ acting on $\hat{\mathbb{C}}^d$. This complexification is no longer conformal (with respect to the Euclidean metric on $\hat{\mathbb{C}}^d$) but it permits us to define a holomorphic extension of the normalized derivatives of the generators of Φ (see (6.2)). With the help of this extension we define the family of Perron-Frobenius operators associated to the family of complexified systems. These operators constitute a holomorphic family (see Lemma 6.4). This fact and the quasi-compactness property of these operators allow us to apply Kato-Rellich perturbation theorem to conclude that the corresponding pressure function is real-analytic. It then follows from the implicit function theorem that the only zero of this pressure function, which we call Bowen's parameter, is real-analytic throughout W (see Theorem 7.1). It is important to note that in order to come to this conclusion we do not require that the open set condition be fulfilled. Whenever this latter is satisfied, a variant of Bowen's formula (see [4]), which identifies the Hausdorff dimension of the limit set of a conformal graph directed Markov system as the zero of the pressure function, permits us to conclude that the Hausdorff dimension function coincides with Bowen's parameter and thus is real-analytic (see Theorem 7.2). It is also worth pointing out that this fact was unknown until now even when the underlying alphabet I of the systems is finite.

2. PRELIMINARIES FROM THERMODYNAMIC FORMALISM ON SYMBOL SPACES

In this section we collect the fundamental ergodic (thermodynamic formalism) results in one-sided symbolic dynamics. All of them can be found with proofs in [4], [5] or [9]. Let I be a countable set and let $A : I \times I \rightarrow \{0, 1\}$ be a matrix, often called *incidence matrix*. A finite or countably infinite tuple (commonly called *word*) ω of elements of I is called A -admissible if $A_{ab} = 1$ for any two consecutive elements a, b of ω . The matrix A is said to be finitely irreducible provided that there exists a finite set F of finite A -admissible words such that for any two elements a, b of I there is a word γ in F such that the word $a\gamma b$ is A -admissible. Throughout the entire section the matrix A is assumed to be finitely irreducible. We denote the set of all countably infinite A -admissible words by

$$I_A^\infty = \{(\omega_n)_{n=1}^\infty \in I^\infty : \forall(n \in \mathbb{N}), A_{\omega_n \omega_{n+1}} = 1\}.$$

Obviously I_A^∞ is a closed subset of I^∞ when this latter is endowed with the product topology, and we equip the former with the topology inherited from the latter. We further denote by I_A^n the set of all A -admissible words of length $n \geq 1$, and by I_A^* the set of all finite A -admissible words. Recall that the left shift map $\sigma : I^\infty \rightarrow I^\infty$, which by definition drops the first entry of ω , is defined by the formula

$$\sigma((\omega_n)_{n=1}^\infty) = (\omega_{n+1})_{n=1}^\infty.$$

This shift is clearly continuous, and of course $\sigma(I_A^\infty) \subset I_A^\infty$. Given $\omega \in I_A^*$ we denote by $|\omega|$ the length of ω , that is, the unique $n \geq 1$ such that $\omega \in I_A^n$. If $\omega \in I_A^\infty$ and $n \geq 1$, then we will use the notation

$$\omega|_n := \omega_1 \dots \omega_n.$$

We will also denote the empty word by ε and declare that its length is 0. For $\omega, \tau \in I_A^\infty$, we define $\omega \wedge \tau \in I_A^\infty \cup I_A^* \cup \{\varepsilon\}$ as the longest initial subword common to ω and τ . Finally,

for every finite A -admissible word $\omega = \omega_1 \dots \omega_n$, set

$$[\omega] = \{\tau \in I_A^\infty : \tau|_n = \omega\}.$$

The set $[\omega]$ is called the cylinder generated by ω .

Let us now talk about functions, also called potentials, on I_A^∞ . Given $s \in (0, 1)$, recall that a function $\psi : I_A^\infty \rightarrow \mathbb{C}$ is s -Hölder continuous if

$$v_s(\psi) := \sup_{n \geq 1} v_{s,n}(\psi) < \infty,$$

where

$$v_{s,n}(\psi) = \sup \left\{ \frac{|\psi(\omega) - \psi(\tau)|}{s^n} : \omega, \tau \in I_A^\infty \text{ and } |\omega \wedge \tau| \geq n \right\}.$$

We will denote by \mathcal{H}_s the set of all s -Hölder continuous functions on I_A^∞ . Recall also that a function $\psi : I_A^\infty \rightarrow \mathbb{C}$ is said to be summable if

$$\sum_{i \in I} \exp(\sup(\operatorname{Re} \psi|_{[i]})) < \infty.$$

We henceforth denote by \mathcal{K}_s the set of all s -Hölder continuous, summable functions.

Now, let $\psi : I_A^\infty \rightarrow \mathbb{C}$ be a continuous function. Let also M_σ denote the set of all shift-invariant Borel probability measures on I_A^∞ . A measure $\mu \in M_\sigma$ is said to be a Gibbs state for ψ provided that there are constants $C \geq 1$ and $P \in \mathbb{R}$ such that

$$(2.1) \quad C^{-1} \leq \frac{\mu([\omega])}{\exp(\operatorname{Re} S_n \psi(\tau) - Pn)} \leq C$$

for all $n \geq 1$, all $\omega \in I_A^n$ and all $\tau \in [\omega]$. Note that if ψ has a Gibbs state, then ψ is summable. The topological pressure $P(\psi)$ is defined as follows:

$$P(\psi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\omega \in I_A^n} \exp(\sup(\operatorname{Re} S_n \psi|_{[\omega]})).$$

It follows from (2.1) that if ψ admits a Gibbs state, then $P = P(\psi)$. The proofs of the following two fundamental results can be found in [4], [5] or [9].

Theorem 2.1. (*Existence and Uniqueness of Gibbs States*) *For every Hölder continuous, summable potential $\psi : I_A^\infty \rightarrow \mathbb{C}$ there exists a unique Gibbs state μ_ψ on I_A^∞ . Furthermore, this state is ergodic.*

Theorem 2.2. (*Variational Principle*) *For every Hölder continuous, summable potential $\psi : I_A^\infty \rightarrow \mathbb{C}$ we have*

$$\sup \left\{ h_\mu(\sigma) + \int_{I_A^\infty} \operatorname{Re} \psi d\mu : \mu \in M_\sigma \text{ and } \int \operatorname{Re} \psi d\mu > -\infty \right\} = P(\psi) = h_{\mu_\psi}(\sigma) + \int_{I_A^\infty} \operatorname{Re} \psi d\mu_\psi$$

where μ_ψ is the unique Gibbs state for ψ , and is the only measure at which the supremum is taken on.

Any measure that realizes the supremum value in the above variational principle is called an equilibrium state for ψ . With this terminology, Theorem 2.2 can be reformulated as follows.

Theorem 2.3. *If $\psi : I_A^\infty \rightarrow \mathbb{C}$ is a Hölder continuous, summable potential, then the unique Gibbs state μ_ψ for ψ is the unique equilibrium state for ψ .*

We will in a moment define the Perron-Frobenius operator $\mathcal{L}_\psi : C_b(I_A^\infty) \rightarrow C_b(I_A^\infty)$, where $C_b(I_A^\infty)$ represents the space of all bounded, continuous complex-valued functions on I_A^∞ . But first, given $i \in I$ and $g : I_A^\infty \rightarrow \mathbb{C}$, define the mapping $g \circ i : I_A^\infty \rightarrow \mathbb{C} \cup \{-\infty\}$ by the formula

$$g \circ i(\omega) = \begin{cases} g(i\omega) & \text{if } A_{i\omega_1} = 1 \\ -\infty & \text{if } A_{i\omega_1} = 0 \end{cases}$$

if $g \in \mathcal{K}_s$, and

$$g \circ i(\omega) = \begin{cases} g(i\omega) & \text{if } A_{i\omega_1} = 1 \\ 0 & \text{if } A_{i\omega_1} = 0 \end{cases}$$

otherwise. Now fix $\psi \in \mathcal{K}_s$ and notice that for every $g \in C_b(I_A^\infty)$, the operator

$$\mathcal{L}_\psi(g) = \sum_{i \in I} e^{\psi \circ i} g \circ i$$

is well defined (with the convention that $e^{-\infty} = 0$, $0 \cdot (-\infty) = 0$ and $-\infty - (-\infty) = 0$), belongs to $C_b(I_A^\infty)$ and $\|\mathcal{L}_\psi(g)\|_\infty \leq \sum_{i \in I} \exp(\sup(\operatorname{Re} \psi|_{[i]})) \|g\|_\infty$. We have thus defined the operator \mathcal{L}_ψ acting continuously on $C_b(I_A^\infty)$ with

$$\|\mathcal{L}_\psi\|_\infty \leq \sum_{i \in I} \exp(\operatorname{Re} \sup(\psi|_{[i]})) < \infty.$$

In fact, as proved in [4], \mathcal{L}_ψ preserves the Banach space $\mathcal{H}_s^b := \mathcal{H}_s \cap C_b(I_A^\infty)$ when this latter is endowed with the s -norm, namely $\|g\|_s := \|g\|_\infty + v_s(g)$, where $g \in \mathcal{H}_s^b$. The operator \mathcal{L}_ψ is called Perron-Frobenius operator. As an immediate consequence of Theorem 2.4.6 from [4], we get the following:

Theorem 2.4. *If $\psi \in \mathcal{K}_s$, then $e^{\mathcal{P}(\psi)}$ is a simple isolated eigenvalue of the Perron-Frobenius operator $\mathcal{L}_\psi : \mathcal{H}_s^b \rightarrow \mathcal{H}_s^b$.*

One of the key facts needed to prove the real analyticity of Hausdorff dimension is Theorem 3.8 from [3]:

Theorem 2.5. *Let $\Gamma \subset \mathbb{C}$ be an open subset of \mathbb{C} , and suppose that the function $\gamma \mapsto \psi_\gamma \in \mathcal{K}_s$, $\gamma \in \Gamma$, is continuous. If the function $\gamma \mapsto \psi_\gamma(\omega) \in \mathbb{C}$, $\gamma \in \Gamma$, is holomorphic for every $\omega \in I_A^\infty$, then the function $\gamma \mapsto \mathcal{L}_{\psi_\gamma} \in L(\mathcal{H}_s^b)$ is also holomorphic.*

3. SELECTED FACTS ABOUT GRAPH DIRECTED MARKOV SYSTEMS

In this section we begin our study of graph directed Markov systems. Let us recall the definition of these systems (see also [4]). Graph directed Markov systems are based upon a directed multigraph and an associated incidence matrix, (V, I, in, to, A) . The multigraph consists of a finite set V of vertices and a countable (either finite or infinite) set of directed edges I and two functions $in, to : I \rightarrow V$. For each edge $i \in I$, $in(i)$ is the initial vertex of that edge and $to(i)$ its terminal vertex. The edge goes from $in(i)$ to $to(i)$. Also, an incidence matrix $A : I \times I \rightarrow \{0, 1\}$ is given. That matrix is an edge incidence matrix: it determines which edges may follow a given edge. And it does so while respecting the multigraph, that is it has the property that if $A_{uv} = 1$, then $to(u) = in(v)$. We will consider finite and

infinite walks through the vertex set which are consistent with the incidence matrix. That is, adopting the notation from section 2, we denote the set of infinite A -admissible words I_A^∞ on the alphabet I by

$$I_A^\infty = \{\omega \in I^\infty : A_{\omega_n \omega_{n+1}} = 1 \text{ for all } n \geq 1\}.$$

A hyperbolic Graph Directed Markov System (GDMS) consists of a directed multigraph and an associated incidence matrix, (V, I, in, to, A) , together with a set of non-empty compact metric spaces $\{X_v\}_{v \in V}$, a number s , $0 < s < 1$, and for every $i \in I$, a one-to-one contraction $\varphi_i : X_{to(i)} \rightarrow X_{in(i)}$ with a Lipschitz constant less than or equal to s . Briefly, the set

$$\Phi = \{\varphi_i : X_{to(i)} \rightarrow X_{in(i)}\}_{i \in I}$$

is called a hyperbolic GDMS, or simply GDMS for short. We now describe its limit set. For each $\omega \in I_A^*$, say $\omega \in I_A^n$, we consider the map coded by ω ,

$$\varphi_\omega := \varphi_{\omega_1} \circ \cdots \circ \varphi_{\omega_n} : X_{to(\omega_n)} \rightarrow X_{in(\omega_1)}.$$

For each $\omega \in I_A^\infty$, the sets $(\varphi_{\omega|_n}(X_{to(\omega_n)}))_{n=1}^\infty$ form a descending sequence of non-empty compact sets and therefore $\bigcap_{n=1}^\infty \varphi_{\omega|_n}(X_{to(\omega_n)}) \neq \emptyset$. Since for every $n \geq 1$ we have $\text{diam}(\varphi_{\omega|_n}(X_{to(\omega_n)})) \leq s^n \text{diam}(X_{to(\omega_n)}) \leq s^n \max\{\text{diam}(X_v) : v \in V\}$, we conclude that the intersection

$$\bigcap_{n=1}^\infty \varphi_{\omega|_n}(X_{to(\omega_n)})$$

is a singleton and we denote its element by $\pi_\Phi(\omega)$. In this way we have defined the coding map π_Φ from I_A^∞ to $X := \bigoplus_{v \in V} X_v$, the disjoint union of the compact sets X_v . The set

$$J_\Phi = \pi_\Phi(I_A^\infty)$$

is the limit set of the GDMS Φ .

A (hyperbolic) GDMS is called conformal (and thus is a (hyperbolic) CGDMS) if the following conditions are satisfied.

- (a) For every vertex $v \in V$, X_v is a compact connected subset of a Euclidean space \mathbb{R}^d (the dimension $d \geq 1$ being common to all $v \in V$) and $X_v = \overline{\text{Int}(X_v)}$.
- (b) *Open Set Condition (OSC)*: For all $a, b \in I$, $a \neq b$,

$$\varphi_a(\text{Int}(X_{to(a)})) \cap \varphi_b(\text{Int}(X_{to(b)})) = \emptyset.$$

- (c) For every vertex $v \in V$ there exists an open connected set $W_v \supset X_v$ such that for every $i \in I$ with $to(i) = v$, the map φ_i extends to a C^1 conformal diffeomorphism of W_v into $W_{in(i)}$.
- (d) *Cone Property*: There exist $\gamma, l > 0$ such that for every $x \in X \subset \mathbb{R}^d$ there exists an open cone $\text{Con}(x, \gamma, l) \subset \text{Int}(X)$ with vertex x , central angle of measure γ , and altitude l .
- (e) There are two constants $L \geq 1$ and $\alpha > 0$ such that

$$\left| \|\varphi'_i(y)\| - \|\varphi'_i(x)\| \right| \leq L \|(\varphi'_i)^{-1}\|^{-1} \|y - x\|^\alpha$$

for every $i \in I$ and every pair of points $x, y \in X_{to(i)}$, where $\|\varphi'_i(x)\|$ means the norm of the derivative of φ at x .

A GDMS Φ which satisfies (a) and (c) — (e) without necessarily satisfying (b) will be called a WPO CGDMS, where WPO stands for “With Potential Overlaps”. Obviously, a CGDMS is a WPO CGDMS without overlaps.

The following remarkable result was proved in [4].

Proposition 3.1. *If $d \geq 2$ and a GDMS $\Phi = \{\varphi_i\}_{i \in I}$ satisfies conditions (a) and (c), then it also satisfies condition (e) with $\alpha = 1$.*

As shown in [4], the following result is a rather straightforward consequence of (e).

Lemma 3.2. *If $d \geq 2$ and a GDMS $\Phi = \{\varphi_i\}_{i \in I}$ satisfies conditions (a) and (c) (and thus (e)), then for all $\omega \in I_A^*$ and all $x, y \in W_{to(\omega)}$, we have*

$$|\log \|\varphi'_\omega(y)\| - \log \|\varphi'_\omega(x)\|| \leq \frac{L}{1 - \alpha} \|y - x\|^\alpha.$$

As an immediate consequence of the previous lemma, observe the following.

(f) *Bounded Distortion Property (BDP):* There exists $K \geq 1$ such that for all $\omega \in I_A^*$ and all $x, y \in X_{to(\omega)}$

$$\|\varphi'_\omega(y)\| \leq K \|\varphi'_\omega(x)\|.$$

In a different vein, it was proved in [4] that for each $t \geq 0$ the following limit exists (and can be equal to ∞ but not $-\infty$):

$$P_\Phi(t) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\omega \in I_A^n} \|\varphi'_\omega\|^t.$$

This number is called the topological pressure of the parameter t . It is important to notice that pressure is well defined whenever (a) and (c) are satisfied.

From the graph of the pressure function, two fundamental parameters were identified in [4]. First, Bowen’s parameter $BP(\Phi)$ of a WPO CGDMS Φ is defined as

$$BP(\Phi) := \inf\{t \geq 0 : P_\Phi(t) < 0\}.$$

A second useful parameter is the finiteness parameter

$$\theta(\Phi) = \inf\{t \geq 0 : P_\Phi(t) < \infty\} = \sup\{t \geq 0 : P_\Phi(t) = \infty\}.$$

It turns out that the pressure function is non-increasing on $[0, \infty)$, strictly decreasing on $[\theta(\Phi), \infty)$, and convex and continuous on $(\theta(\Phi), \infty)$. It is also right-continuous at $\theta(\Phi)$.

We now recall Mauldin and Urbanski’s classification of systems in terms of the pressure function.

Definition 3.3. *A WPO CGDMS Φ is said to be regular if there exists $t \geq 0$ such that $P_\Phi(t) = 0$. Otherwise, Φ is said to be irregular.*

Definition 3.4. *A WPO CGDMS is said to be strongly regular if there exists $t \geq 0$ such that $0 < P_\Phi(t) < \infty$.*

Definition 3.5. *A WPO CGDMS is said to be cofinitely regular if all of its cofinite subsystems are regular. (A family $\Phi|_F := \{\varphi_i\}_{i \in F}$ is said to be a cofinite subsystem of a system $\Phi = \{\varphi_i\}_{i \in I}$ if $F \subset I$ and the difference $I \setminus F$ is finite.)*

The following fact, which relates the diverse kinds of regularity, can be found in [4].

Proposition 3.6. *Each cofinitely regular system is strongly regular, and each strongly regular system is regular.*

Finally, we state the central result of the theory of CGDMS. Let $\mathcal{F}in(I)$ denote the family of all finite subsets of I . The following characterization of $\text{HD}(J_\Phi)$, the Hausdorff dimension of the limit set J_Φ , is a variant of Bowen's formula. It was proved in [4] as Theorem 4.2.13 for finitely primitive CGDMS, but the proof can easily be improved to finitely irreducible CGDMS.

Theorem 3.7. *If a CGDMS Φ is finitely irreducible, that is, has an associated matrix A which is finitely irreducible, then*

$$\text{HD}(J_\Phi) = BP(\Phi) = \sup\{\text{HD}(J_{\Phi|_F}) : F \in \mathcal{F}in(I)\} \geq \theta(\Phi).$$

It follows immediately from this theorem that if Φ is regular, that is if $P_\Phi(t) = 0$ for some t , then that t is the only zero of the pressure function $P_\Phi(t)$ and $t = \text{HD}(J_\Phi)$. Moreover, a system Φ is strongly regular if and only if $\text{HD}(J_\Phi) > \theta(\Phi)$.

Note also that the theorem is generally not true when there are overlaps, that is when the OSC is not satisfied.

4. PRELIMINARIES ON DIFFERENTIABILITY

Throughout this section \mathcal{F} denotes either \mathbb{R} or \mathbb{C} . Let also $d \geq 1$ be an integer and I a countable (either finite or infinite) set. For every $F \subset I$ we denote by $P_F : (\mathcal{F}^d)^I \rightarrow (\mathcal{F}^d)^F$ the canonical projection from $(\mathcal{F}^d)^I$ onto $(\mathcal{F}^d)^F$. For every point $x \in (\mathcal{F}^d)^I$ and every set $F \subset I$ set

$$\mathcal{F}_{(x,F)} = P_{I \setminus F}^{-1}(P_{I \setminus F}(x)).$$

If the set F is finite, then $\mathcal{F}_{(x,F)}$ is canonically isomorphic to $\mathcal{F}^{d(\#F)}$ and is called the space box of $(\mathcal{F}^d)^I$ generated by x and F . Recall that the box topology on $(\mathcal{F}^d)^I$ has for a base the set of all Cartesian products $\prod_{i \in I} U_i$, where each U_i , $i \in I$, is an open subset of \mathcal{F}^d . A function $G : U \rightarrow B$ from U , an open subset of $(\mathcal{F}^d)^I$ in the box topology, to a Banach space B is said to be differentiable at a point $x \in U$ if its restriction to every space box $\mathcal{F}_{(x,F)}$ intersected with U , that is $G|_{\mathcal{F}_{(x,F)} \cap U}$, is differentiable at the point x . As usual, G is said to be differentiable on a set $V \subset U$ if it is differentiable at every point of V . Analogous definitions of C^r differentiability ($1 \leq r \leq \infty$), real analyticity, and complex analyticity (that is, holomorphicity) when $\mathcal{F} = \mathbb{C}$, can be readily made. Note that due to Hartogs Theorem, it suffices to consider singleton sets F in the definition of complex analyticity.

5. REFINED PROPERTIES OF CONFORMAL MAPPINGS

We now turn our attention to conformal mappings in higher-dimensional spaces. Let us fix an integer $d \geq 3$. For every $z = (z_1, z_2, \dots, z_d) \in \mathbb{C}^d$ we define

$$(5.1) \quad |||z|||^2 := \sum_{j=1}^d z_j^2 \quad \text{and} \quad \|z\|^2 := \sum_{j=1}^d |z_j|^2.$$

Of course, in general $|||z|||^2$ is a complex number and is not the square of a norm. However, $|||\cdot|||^2|_{\mathbb{R}^d} = \|\cdot\|^2|_{\mathbb{R}^d}$ and $\|\cdot\|^2$ are the squares of the standard Euclidean norms in \mathbb{R}^d

and \mathbb{C}^d , respectively. It is also worth noticing that $\left| \left| \left| z \right| \right|^2 \right| \leq \|z\|^2$ for every $z \in \mathbb{C}^d$. We will often implicitly consider \mathbb{R}^d as naturally embedded in \mathbb{C}^d . Open balls in \mathbb{R}^d will be denoted by $B(z, r)$, while open balls in \mathbb{C}^d will be denoted by $B_{\mathbb{C}^d}(z, r)$. We denote by $\hat{\mathbb{R}}^d$ and $\hat{\mathbb{C}}^d$ the one-point (Alexandrov) compactifications of \mathbb{R}^d and \mathbb{C}^d , respectively, obtained by adding the point ∞ . Recall that, given $a \in \mathbb{R}^d$ and $r > 0$, the inversion $i_{a,r} : \hat{\mathbb{R}}^d \rightarrow \hat{\mathbb{R}}^d$ in the sphere centred at a and of radius r is given by the formula

$$(5.2) \quad i_{a,r}(x) = r^2 \frac{x - a}{\|x - a\|^2} + a.$$

We further set $i_{\infty, \infty} = Id$. Obviously, $i_{a,r}$ is an involution (that is, $i_{a,r}^2 = Id$) which leaves the sphere $\partial B(a, r)$ invariant, and $a = i_{a,r}^{-1}(\infty)$. Liouville's Theorem (see [2, 10]) says that each C^1 conformal homeomorphism φ defined on a connected, open subset of \mathbb{R}^d extends to the entire space $\hat{\mathbb{R}}^d$ and takes on the form

$$(5.3) \quad \varphi(x) = \lambda A \circ i_{a,r}(x) + b,$$

where $\lambda > 0$, $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a (linear) isometry, $a \in \hat{\mathbb{R}}^d$, $r \in (0, \infty]$ and $b \in \mathbb{R}^d$. The number λ is often called the scalar factor of φ , and $a = \varphi^{-1}(\infty)$ its center of inversion. Note that for every $x \in \mathbb{R}^d$, the scaling factor of the derivative $\varphi'(x)$ is given by the formula

$$(5.4) \quad \|\varphi'(x)\| = \frac{\lambda r^2}{\|x - a\|^2} > 0.$$

For every $a \in \mathbb{R}^d$, let

$$Z_a = \{z \in \mathbb{C}^d : \left| \left| \left| z - a \right| \right|^2 = 0\}.$$

Clearly, $Z_a = a + Z_0$. Notice that the inversion $i_{a,r}$ extends to the open set $\hat{\mathbb{C}}^d \setminus Z_a$. Indeed, for every $z \in \hat{\mathbb{C}}^d \setminus Z_a$, set

$$(5.5) \quad i_{a,r}(z) = r^2 \frac{z - a}{\left| \left| \left| z - a \right| \right|^2} + a.$$

The C^1 conformal homeomorphism $\varphi : \hat{\mathbb{R}}^d \rightarrow \hat{\mathbb{R}}^d$ therefore extends canonically to a map from $\hat{\mathbb{C}}^d \setminus Z_a$ to $\hat{\mathbb{C}}^d$ by the formula

$$(5.6) \quad \varphi(z) = \lambda A \circ i_{a,r}(z) + b.$$

The “scaling factor of the derivative” of this extension is defined as

$$(5.7) \quad \left| \left| \left| \varphi'(z) \right| \right| = \frac{\lambda r^2}{\left| \left| \left| z - a \right| \right|^2}.$$

Observe that this factor is generally a nonzero complex number.

We will later need the following result:

Lemma 5.1. *If X is a compact subset of \mathbb{R}^d and $V \supset X$ is an open subset of \mathbb{R}^d , then*

$$\text{dist}_{\mathbb{C}^d} \left(X, \bigcup_{a \in \mathbb{R}^d \setminus V} Z_a \right) > 0.$$

Proof. Suppose on the contrary that $\text{dist}_{\mathbb{C}^d}(X, \cup_{a \in \mathbb{R}^d \setminus V} Z_a) = 0$. Then there exist sequences $(a^{(n)})_{n=1}^\infty \subset \mathbb{R}^d \setminus V$ and $(z^{(n)})_{n=1}^\infty \subset \mathbb{C}^d$ such that $z^{(n)} \in Z_{a^{(n)}}$ for every $n \geq 1$ and $\lim_{n \rightarrow \infty} \text{dist}_{\mathbb{C}^d}(X, z^{(n)}) = 0$. Since X is compact, by passing to a subsequence if necessary, we may assume without loss of generality that the sequence $(z^{(n)})_{n=1}^\infty$ converges to a point $x \in X$. Now, for every $n \geq 1$, we have

$$(5.8) \quad \begin{aligned} \left| \|z^{(n)} - a^{(n)}\| \right|^2 &= \sum_{j=1}^d \left((z_j^{(n)} - x_j) + (x_j - a_j^{(n)}) \right)^2 \\ &= \left| \|z^{(n)} - x\| \right|^2 + 2 \sum_{j=1}^d (z_j^{(n)} - x_j)(x_j - a_j^{(n)}) + \|x - a^{(n)}\|^2. \end{aligned}$$

To shorten the notation, set $\Delta := \text{dist}(X, \partial V)$. Since $a^{(n)} \in \mathbb{R}^d \setminus V$ and $x \in X$, we have

$$(5.9) \quad \|x - a^{(n)}\|^2 \geq \Delta^2 > 0.$$

Moreover, since $\lim_{n \rightarrow \infty} z^{(n)} = x$, there exists $N \geq 1$ such that for all $n \geq N$ we have

$$(5.10) \quad \|z^{(n)} - x\| \leq \frac{1}{8d} \min\{1, \Delta^2\}.$$

So, for all $n \geq N$ and those $1 \leq j \leq d$ such that $|x_j - a_j^{(n)}| \leq 1$, we have

$$(5.11) \quad 2|z_j^{(n)} - x_j| \cdot |x_j - a_j^{(n)}| \leq 2\|z^{(n)} - x\| \leq \frac{1}{4d} \Delta^2,$$

whereas for those j for which $|x_j - a_j^{(n)}| \geq 1$, we get

$$(5.12) \quad 2|z_j^{(n)} - x_j| \cdot |x_j - a_j^{(n)}| \leq 2\|z^{(n)} - x\| \cdot |x_j - a_j^{(n)}|^2 \leq \frac{1}{4d} \|x - a^{(n)}\|^2.$$

Thus, using (5.8) — (5.12), we obtain for every $n \geq N$

$$\begin{aligned} \left| \left| \|z^{(n)} - a^{(n)}\| \right|^2 \right| &\geq \|x - a^{(n)}\|^2 - \|z^{(n)} - x\|^2 - 2 \sum_{j=1}^d |z_j^{(n)} - x_j| \cdot |x_j - a_j^{(n)}| \\ &\geq \|x - a^{(n)}\|^2 - \frac{1}{4} \Delta^2 - \frac{1}{4d} \sum_{j=1}^d \max\{\Delta^2, \|x - a^{(n)}\|^2\} \\ &\geq \frac{1}{2} \|x - a^{(n)}\|^2 \geq \frac{1}{2} \Delta^2 > 0. \end{aligned}$$

Hence $z^{(n)} \notin Z_{a^{(n)}}$ for each $n \geq N$, and this contradiction completes the proof. \square

We will also need the following estimates:

Lemma 5.2. *Let X be a compact subset of \mathbb{R}^d and $V \supset X$ an open subset of \mathbb{R}^d . Let $\Delta := \text{dist}(X, \partial V) > 0$ and $\tilde{\Delta} = \Delta/(32d)$. Then for all $(w, \alpha) \in B_{\mathbb{C}^d}(X, \tilde{\Delta}) \times B_{\mathbb{C}^d}(\mathbb{R}^d \setminus V, \tilde{\Delta})$ we have*

$$\left| \left| \|w - \alpha\| \right|^2 \right| \geq \frac{3}{16} \|w - \alpha\|^2 > 0.$$

Proof. Let $(w, \alpha) \in B_{\mathbb{C}^d}(X, \tilde{\Delta}) \times B_{\mathbb{C}^d}(\mathbb{R}^d \setminus V, \tilde{\Delta})$ and take $(x, a) \in X \times (\mathbb{R}^d \setminus V)$ such that $(w, \alpha) \in B_{\mathbb{C}^d}(x, \tilde{\Delta}) \times B_{\mathbb{C}^d}(a, \tilde{\Delta})$. Then

$$\begin{aligned}
\left| \|w - \alpha\|^2 \right| &= \left| \|w - x + x - a + a - \alpha\|^2 \right| \\
&= \left| \|w - x\|^2 + \|x - a\|^2 + \|a - \alpha\|^2 \right. \\
&\quad \left. + 2 \sum_{j=1}^d [(w_j - x_j)(x_j - a_j) + (x_j - a_j)(a_j - \alpha_j) + (w_j - x_j)(a_j - \alpha_j)] \right| \\
&\geq \|x - a\|^2 - \left| \|w - x\|^2 \right| - \left| \|a - \alpha\|^2 \right| \\
&\quad - 2 \sum_{j=1}^d [|w_j - x_j| \cdot |x_j - a_j| + |x_j - a_j| \cdot |a_j - \alpha_j| + |w_j - x_j| \cdot |a_j - \alpha_j|] \\
&\geq \|x - a\|^2 - \|w - x\|^2 - \|a - \alpha\|^2 \\
&\quad - 2d \left[\|w - x\| \cdot \|x - a\| + \|x - a\| \cdot \|a - \alpha\| + \|w - x\| \cdot \|a - \alpha\| \right] \\
&\geq \|x - a\|^2 - 2\tilde{\Delta}^2 - 2d[2\tilde{\Delta}\|x - a\| + \tilde{\Delta}^2] \\
&\geq \|x - a\|^2 \left(1 - \frac{4\tilde{\Delta}d}{\Delta} \right) - 2(d+1)\tilde{\Delta}^2 \\
&\geq \frac{7}{8}\|x - a\|^2 - \frac{2(d+1)\|x - a\|^2}{1024d^2} \\
&\geq \frac{3}{4}\|x - a\|^2.
\end{aligned}$$

Since

$$\|x - a\| \geq \|w - \alpha\| - \|x - w\| - \|\alpha - a\| \geq \|w - \alpha\| - 2\tilde{\Delta} \geq \frac{1}{2}\|w - \alpha\| \geq \frac{1}{2}(\Delta - 2\tilde{\Delta}) > 0,$$

we conclude that

$$\left| \|w - \alpha\|^2 \right| \geq \frac{3}{4} \left(\frac{1}{2} \|w - \alpha\| \right)^2 = \frac{3}{16} \|w - \alpha\|^2 \geq \frac{3}{16} (\Delta - 2\tilde{\Delta})^2 > 0.$$

□

Lemma 5.3. *If X is a compact subset of \mathbb{R}^d and $V \supset X$ is an open subset of \mathbb{R}^d , then there exists a constant $C(X, V) > 0$, depending only on $\Delta := \text{dist}(X, \partial V)$ and d , such that for all $(z, \alpha), (w, \beta) \in B_{\mathbb{C}^d}(X, \tilde{\Delta}) \times B_{\mathbb{C}^d}(\mathbb{R}^d \setminus V, \tilde{\Delta})$, where $\tilde{\Delta} := \Delta/(32d)$, we have*

$$\begin{aligned}
&\max \left\{ \left| \frac{\|z - \alpha\|^2}{\|w - \beta\|^2} - 1 \right|, \left| \frac{\|z - \alpha\|^2}{\|w - \beta\|^2} - 1 \right| \right\} \\
&\leq C(X, V) \left(\|z - w\|^2 + \|\alpha - \beta\|^2 + \|z - w\| + \|z - w\| \cdot \|\alpha - \beta\| + \|\alpha - \beta\| \right).
\end{aligned}$$

Proof. Let $(z, \alpha), (w, \beta) \in B_{\mathbb{C}^d}(X, \tilde{\Delta}) \times B_{\mathbb{C}^d}(\mathbb{R}^d \setminus V, \tilde{\Delta})$. Then

$$\begin{aligned} \left| \frac{\|z - \alpha\|^2}{\|w - \beta\|^2} - 1 \right| &= \frac{\left| \|z - w\|^2 + \|\alpha - \beta\|^2 + 2 \sum_{j=1}^d \operatorname{Re} \left[(z_j - w_j) \overline{(w_j - \beta_j)} + (z_j - w_j) \overline{(\beta_j - \alpha_j)} + (w_j - \beta_j) \overline{(\beta_j - \alpha_j)} \right] \right|}{\|w - \beta\|^2} \\ &\leq \frac{\left\{ \begin{aligned} &\|z - w\|^2 + \|\alpha - \beta\|^2 \\ &+ 2d \left[\|z - w\| \cdot \|w - \beta\| + \|z - w\| \cdot \|\beta - \alpha\| + \|w - \beta\| \cdot \|\beta - \alpha\| \right] \end{aligned} \right\}}{\|w - \beta\|^2} \\ &\leq (\Delta')^{-2} \left(\|z - w\|^2 + \|\alpha - \beta\|^2 \right) \\ &\quad + 2d(\Delta')^{-1} \left(\|z - w\| + (\Delta')^{-1} \|z - w\| \cdot \|\alpha - \beta\| + \|\alpha - \beta\| \right), \end{aligned}$$

where $\Delta' = \Delta - 2\tilde{\Delta}$.

Similarly,

$$\begin{aligned} \left| \frac{|||z - \alpha|||^2}{|||w - \beta|||^2} - 1 \right| &= \frac{\left| |||z - w|||^2 + |||\alpha - \beta|||^2 + 2 \sum_{j=1}^d \left[(z_j - w_j)(w_j - \beta_j) + (z_j - w_j)(\beta_j - \alpha_j) + (w_j - \beta_j)(\beta_j - \alpha_j) \right] \right|}{|||w - \beta|||^2} \\ &\leq \frac{\left\{ \begin{aligned} &\|z - w\|^2 + \|\alpha - \beta\|^2 \\ &+ 2d \left[\|z - w\| \cdot \|w - \beta\| + \|z - w\| \cdot \|\beta - \alpha\| + \|w - \beta\| \cdot \|\beta - \alpha\| \right] \end{aligned} \right\}}{(3/16)\|w - \beta\|^2} \\ &\leq \frac{16}{3} \left\{ \begin{aligned} &(\Delta')^{-2} \left(\|z - w\|^2 + \|\alpha - \beta\|^2 \right) \\ &+ 2d(\Delta')^{-1} \left(\|z - w\| + (\Delta')^{-1} \|z - w\| \cdot \|\alpha - \beta\| + \|\alpha - \beta\| \right) \end{aligned} \right\}, \end{aligned}$$

where we have relied upon Lemma 5.2 to bound from below the denominator in the second-to-last inequality. Simply take $C(X, V) = (32/3) \max\{(\Delta')^{-2}, 6d(\Delta')^{-1} \max\{1, (\Delta')^{-1}\}\}$.

□

We will also need a slightly different type of estimates:

Lemma 5.4. *Let X be a compact subset of \mathbb{R}^d and $V \supset X$ an open subset of \mathbb{R}^d . Let $\Delta := \operatorname{dist}(X, \partial V) > 0$ and $\tilde{\Delta} := \Delta/(32d)$. Then for every $0 < r < \tilde{\Delta}$ and every couple $(z, \alpha) \in B_{\mathbb{C}^d}(X, r) \times B_{\mathbb{C}^d}(\mathbb{R}^d \setminus V, r)$ we have*

$$\left| \frac{\|z - \alpha\|^2}{|||z - \alpha|||^2} - 1 \right| \leq \frac{64dr}{3\|z - \alpha\|}.$$

Proof. Let $(z, \alpha) \in B_{\mathbb{C}^d}(X, r) \times B_{\mathbb{C}^d}(\mathbb{R}^d \setminus V, r)$. Then

$$\begin{aligned}
\left| \frac{\|z - \alpha\|^2}{\|\|z - \alpha\|\|^2} - 1 \right| &= \frac{\left| \|z - \alpha\|^2 - \|\|z - \alpha\|\|^2 \right|}{\|\|z - \alpha\|\|^2} \\
&\leq \frac{16}{3} \frac{\left| \sum_{j=1}^d (z_j - \alpha_j)(\bar{z}_j - \bar{\alpha}_j) - \sum_{j=1}^d (z_j - \alpha_j)^2 \right|}{\|z - \alpha\|^2} \\
&\leq \frac{16}{3\|z - \alpha\|^2} \sum_{j=1}^d |z_j - \alpha_j| \left| (\bar{z}_j - z_j) + (\alpha_j - \bar{\alpha}_j) \right| \\
&\leq \frac{32}{3\|z - \alpha\|} \sum_{j=1}^d \left(|\operatorname{Im}(\bar{z}_j)| + |\operatorname{Im}(\alpha_j)| \right) \\
&= \frac{64dr}{3\|z - \alpha\|}.
\end{aligned}$$

□

Now, let

$$M_d := (0, \infty) \times (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \times \operatorname{SO}(d) \subset \mathbb{R}^q,$$

where $q = 2 + \frac{1}{2}d(d+3)$. Note that M_d is an open subset of \mathbb{R}^q . Given $e = (\lambda_e, r_e, a_e, b_e, A_e) \in M_d$, set (cf. formulas (5.3) and (5.4))

$$\varphi_e = \lambda_e A_e \circ i_{a_e, r_e} + b_e.$$

More generally, put

$$\mathbb{C}M_d := \mathbb{C} \setminus \{0\} \times \mathbb{C} \setminus \{0\} \times \mathbb{C}^d \times \mathbb{C}^d \times \operatorname{SO}_{\mathbb{C}}(d) \subset \mathbb{C}^q.$$

Observe that $\mathbb{C}M_d$ is an open subset of \mathbb{C}^q , and for every $e \in \mathbb{C}M_d$ define φ_e again as above (cf. formulas (5.5) and (5.6)).

Now, given X a compact subset of \mathbb{R}^d and $V \supset X$ an open subset of \mathbb{R}^d , as well as $s > 0$, let

$$M_d(X, V)_s = \left\{ e \in M_d : a_e \in \mathbb{R}^d \setminus V \text{ and } \sup_{x \in X} \|\varphi'_e(x)\| \leq s \right\}.$$

Our last goal in this section is to prove Lemma 5.7, which will later be used to “complexify” the space of graph directed Markov systems. In order to prove it, we first establish two preliminary lemmas.

Lemma 5.5. *Let X be a compact subset of \mathbb{R}^d and $V \supset X$ an open subset of \mathbb{R}^d . Then for every $\varepsilon > 0$ and every $e \in M_d(X, V)_1$ there exists $\delta = \delta_e(\varepsilon) > 0$ such that if $f \in B_{\mathbb{C}^q}(e, \delta) \cap \mathbb{C}M_d$, then $\|\varphi_f - \varphi_e\|_X \leq \varepsilon$, where $\|\varphi_f - \varphi_e\|_X := \sup_{x \in X} \|\varphi_f(x) - \varphi_e(x)\|$.*

Proof. Fix $e \in M_d(X, V)_1$ and $0 < \varepsilon < \min\{1, 1/(4C(X, V))\}$, where $C(X, V)$ is the constant arising from Lemma 5.3. Set $\Delta := \operatorname{dist}(X, \partial V)$ and $\tilde{\Delta} := \Delta/(32d)$. Take

$$\delta = \varepsilon \min \left\{ (\sqrt[6]{2} - 1) \min\{\lambda_e, r_e, 1\}, \frac{\min\{\Delta, 1\}}{48 \max\{\lambda_e, \lambda_e r_e^2, r_e^2, \|a_e\| + 1, \lambda_e(\|a_e\| + 1)\}} \right\}.$$

If $f \in B_{\mathbb{C}^q}(e, \delta) \cap \mathbb{C}M_d$, then one can show that the following conditions are satisfied:

$$(5.13) \quad \|a_f - a_e\| \leq \min\left\{\frac{\varepsilon}{16(1+3d)}, \tilde{\Delta}\right\},$$

$$(5.14) \quad \|b_f - b_e\| \leq \frac{\varepsilon}{4},$$

$$(5.15) \quad \left\|\lambda_f r_f^2 A_f - \lambda_e r_e^2 A_e\right\| \leq \Delta \frac{\varepsilon}{4},$$

$$(5.16) \quad \left\|\lambda_f A_f a_f - \lambda_e A_e a_e\right\| \leq \frac{\varepsilon}{4}$$

and

$$(5.17) \quad |\lambda_f| |r_f|^2 \|A_f\| \leq 2\lambda_e r_e^2 \|A_e\| = 2\lambda_e r_e^2.$$

Since $\|a_f - a_e\| \leq \tilde{\Delta}$, it follows from Lemma 5.2 that

$$\varphi_f(x) = \lambda_f A_f \left(r_f^2 \frac{x - a_f}{\|x - a_f\|^2} + a_f \right) + b_f$$

is well defined for all $x \in X$. Then for all $x \in X$ we have

$$\begin{aligned} & \|\varphi_f(x) - \varphi_e(x)\| \\ = & \left\| \left(\lambda_f r_f^2 A_f - \lambda_e r_e^2 A_e \right) \frac{x - a_e}{\|x - a_e\|^2} + \lambda_f r_f^2 A_f \left(\frac{x - a_f}{\|x - a_f\|^2} - \frac{x - a_e}{\|x - a_e\|^2} \right) \right\| \\ & + \left\| (\lambda_f A_f a_f - \lambda_e A_e a_e) + (b_f - b_e) \right\| \\ \leq & \|x - a_e\|^{-1} \|\lambda_f r_f^2 A_f - \lambda_e r_e^2 A_e\| + |\lambda_f| |r_f|^2 \|A_f\| \left\| \frac{x - a_f}{\|x - a_f\|^2} - \frac{x - a_e}{\|x - a_e\|^2} \right\| \\ & + \|\lambda_f A_f a_f - \lambda_e A_e a_e\| + \|b_f - b_e\| \\ \leq & \Delta^{-1} \Delta \frac{\varepsilon}{4} + |\lambda_f| |r_f|^2 \|A_f\| \frac{\left\| \|x - a_e\|^2(x - a_f) - \|x - a_f\|^2(x - a_e) \right\|}{\|x - a_f\|^2 \|x - a_e\|^2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ \leq & \frac{3\varepsilon}{4} + |\lambda_f| |r_f|^2 \|A_f\| \frac{\left\| \|x - a_e\|^2(x - a_f) - \|x - a_f\|^2(x - a_e) \right\|}{\|x - a_f\|^2 \|x - a_e\|^2}, \end{aligned}$$

where we used (5.14), (5.15) and (5.16). Now

$$(5.18) \quad \begin{aligned} & \left\| \|x - a_e\|^2(x - a_f) - \|x - a_f\|^2(x - a_e) \right\| \\ = & \left\| \|x - a_e\|^2(a_e - a_f) + (\|x - a_e\|^2 - \|x - a_f\|^2)(x - a_e) \right\| \\ \leq & \|a_e - a_f\| \cdot \|x - a_e\|^2 + \|x - a_e\| \left| \|x - a_e\|^2 - \|x - a_f\|^2 \right|. \end{aligned}$$

Furthermore,

$$(5.19) \quad \begin{aligned} \left| \|x - a_e\|^2 - \|x - a_f\|^2 \right| &= \left| \sum_{j=1}^d (a_{f_j} - a_{e_j}) \left((x_j - a_{f_j}) + (x_j - a_{e_j}) \right) \right| \\ &\leq d \|a_f - a_e\| \left(\|x - a_f\| + \|x - a_e\| \right). \end{aligned}$$

In virtue of (5.13) and Lemma 5.3 (recall that $0 < \varepsilon < \min\{1, 1/(4C(X, V))\}$), we have

$$\frac{1}{2} \leq \frac{\left| \|x - a_f\|^2 \right|, \|x - a_f\|^2}{\|x - a_e\|^2} \leq \frac{3}{2}.$$

Combining this with (5.18) and (5.19), we get

$$\begin{aligned} \frac{\left| \|x - a_e\|^2 (x - a_f) - \|x - a_f\|^2 (x - a_e) \right|}{\left| \|x - a_f\|^2 \|x - a_e\|^2 \right|} &\leq \frac{\|a_f - a_e\|}{\left| \|x - a_f\|^2 \right|} + \frac{d \|a_f - a_e\| \left(\|x - a_f\| + \|x - a_e\| \right)}{\left| \|x - a_f\|^2 \|x - a_e\| \right|} \\ &\leq \frac{\|a_f - a_e\|}{(1/2) \|x - a_e\|^2} + d \|a_f - a_e\| \frac{3 \|x - a_e\|}{(1/2) \|x - a_e\|^3} \\ &= 2(1 + 3d) \frac{\|a_f - a_e\|}{\|x - a_e\|^2}. \end{aligned}$$

Since $\|\varphi'_e(x)\| \leq 1$, we get from (5.4) that $\lambda_e r_e^2 \leq \|x - a_e\|^2$. Using this, (5.17), the above inequality and (5.13), we deduce that

$$\begin{aligned} |\lambda_f \|r_f\|^2 \|A_f\| \frac{\left| \|x - a_e\|^2 (x - a_f) - \|x - a_f\|^2 (x - a_e) \right|}{\left| \|x - a_f\|^2 \|x - a_e\|^2 \right|} &\leq 2\lambda_e r_e^2 \cdot 2(1 + 3d) \frac{\|a_f - a_e\|}{\|x - a_e\|^2} \\ &= 4(1 + 3d) \|a_f - a_e\| \leq \varepsilon/4. \end{aligned}$$

We thus conclude that $\|\varphi_f(x) - \varphi_e(x)\| < \varepsilon$. Since this is true for every $x \in X$, we deduce that $\|\varphi_f - \varphi_e\|_X \leq \varepsilon$. Finally, because f was chosen arbitrarily in $B_{\mathbb{C}^q}(e, \delta) \cap \mathbb{C}M_d$, the proof is complete. \square

Here is the second preliminary lemma.

Lemma 5.6. *Let X be a compact subset of \mathbb{R}^d and $V \supset X$ an open subset of \mathbb{R}^d . Fix $s \in (0, 1]$. Then for every $\varepsilon > 0$ there exists $r > 0$ such that for all $e \in M_d(X, V)_s$ there is $\delta = \delta_\varepsilon > 0$ such that if $f \in B_{\mathbb{C}^q}(e, \delta) \cap \mathbb{C}M_d$, then $\|\varphi'_f(z)\| \leq (1 + \varepsilon)s$ for all $z \in B_{\mathbb{C}^d}(X, r)$.*

Proof. For every $g \in \mathbb{C}M_d$ and every $z \in \mathbb{C}^d \setminus Z_{a_g}$ define the matrix $(\hat{g}_{ij}(z))_{1 \leq i, j \leq d}$ by the formula

$$\hat{g}_{ij}(z) = \frac{(z_i - a_{g_i})(z_j - a_{g_j})}{\|z - a_g\|^2},$$

where, to allege notation, $a_{g_i} := (a_g)_i$ denotes the i -th component of a_g . Let $\varepsilon \in (0, 1/2)$. Let $\tilde{\varepsilon} \in (0, \varepsilon]$ be such that $(1 + \tilde{\varepsilon})^2(1 - \tilde{\varepsilon})^{-1}(1 + 12d\tilde{\varepsilon}) \leq 1 + \varepsilon$. Let $\Delta := \text{dist}(X, \partial V)$ and

take $0 < r \leq 3\Delta\tilde{\varepsilon}/(128d)$ such that, according to Lemma 5.3,

$$(5.20) \quad \max \left\{ \left| \frac{\|z - \alpha\|^2}{\|w - \beta\|^2} - 1 \right|, \left| \frac{\| \|z - \alpha\|^2}{\| \|w - \beta\|^2} - 1 \right| \right\} \leq \tilde{\varepsilon} < 1/2$$

for every $(z, \alpha), (w, \beta) \in B_{\mathbb{C}^d}(X, r) \times B_{\mathbb{C}^d}(\mathbb{R}^d \setminus V, r)$. Since $0 < r < \tilde{\Delta} := \Delta/(32d)$, Lemma 5.4 gives

$$(5.21) \quad \left| \frac{\|z - \alpha\|^2}{\| \|z - \alpha\|^2} - 1 \right| \leq \frac{64dr}{3\|z - \alpha\|} \leq \frac{64dr}{3(\Delta - 2r)} \leq \frac{128dr}{3\Delta} \leq \tilde{\varepsilon}.$$

for every couple $(z, \alpha) \in B_{\mathbb{C}^d}(X, r) \times B_{\mathbb{C}^d}(\mathbb{R}^d \setminus V, r)$.

Now, let $e \in M_d(X, V)_s$. Take $0 < \delta \leq r$ such that if $f \in B_{\mathbb{C}^q}(e, \delta) \cap \mathbb{C}M_d$, then $|\lambda_f| |r_f|^2 \leq (1 + \tilde{\varepsilon})\lambda_e r_e^2$ and $\|A_f\| \leq (1 + \tilde{\varepsilon})\|A_e\| = 1 + \tilde{\varepsilon}$. Let $f \in B_{\mathbb{C}^q}(e, \delta) \cap \mathbb{C}M_d$. Fix $1 \leq i, j \leq d$ and $z \in B_{\mathbb{C}^d}(X, r)$. Then there exists $x \in X$ such that $\|z - x\| < r$. We thereafter have

$$\begin{aligned} |\hat{f}_{ij}(z) - \hat{e}_{ij}(x)| &= \left| \left((x_i - a_{e_i})(x_j - a_{e_j}) - (z_i - a_{f_i})(z_j - a_{f_j}) \right) \|x - a_e\|^{-2} \right. \\ &\quad \left. + \left(\|x - a_e\|^{-2} - \| \|z - a_f\|^2 \right) (z_i - a_{f_i})(z_j - a_{f_j}) \right| \\ &\leq \|x - a_e\|^{-2} \left| (x_i - a_{e_i})(x_j - a_{e_j}) - (z_i - a_{f_i})(z_j - a_{f_j}) \right| \\ &\quad + \|z - a_f\|^2 \left| \|x - a_e\|^{-2} - \| \|z - a_f\|^2 \right| \\ &\leq \|x - a_e\|^{-2} \left| (x_i - a_{e_i}) \left((x_j - z_j) + (a_{f_j} - a_{e_j}) \right) + (z_j - a_{f_j}) \left((x_i - z_i) + (a_{f_i} - a_{e_i}) \right) \right| \\ &\quad + \left| \frac{\|z - a_f\|^2}{\|x - a_e\|^2} - \frac{\|z - a_f\|^2}{\| \|z - a_f\|^2} \right| \\ &\leq \|x - a_e\|^{-2} \left[\|x - a_e\| \left(\|x - z\| + \|a_f - a_e\| \right) + \|z - a_f\| \left(\|x - z\| + \|a_f - a_e\| \right) \right] \\ &\quad + \left| \frac{\|z - a_f\|^2}{\|x - a_e\|^2} - 1 \right| + \left| \frac{\|z - a_f\|^2}{\| \|z - a_f\|^2} - 1 \right| \\ &\leq \|x - a_e\|^{-2} \left(\|x - a_e\| + \|z - a_f\| \right) \left(r + \|f - e\| \right) + 2\tilde{\varepsilon}, \end{aligned}$$

where we successively used (5.20) with $\alpha = a_f$, $w = x$ and $\beta = a_e$, and (5.21) with $\alpha = a_f$, keeping in mind that $a_e \in \mathbb{R}^d \setminus V$ since $e \in M_d(X, V)_s$. Since $\delta \leq r$ and $f \in B_{\mathbb{C}^q}(e, \delta) \cap \mathbb{C}M_d$, we obtain

$$|\hat{f}_{ij}(z) - \hat{e}_{ij}(x)| \leq \|x - a_e\|^{-2} \left(\|x - a_e\| + \|z - a_f\| \right) \cdot 2r + 2\tilde{\varepsilon}.$$

Using (5.20) again, we deduce that

$$\begin{aligned} |\hat{f}_{ij}(z) - \hat{e}_{ij}(x)| &\leq \|x - a_e\|^{-2} \left(\|x - a_e\| + 2\|x - a_e\| \right) \cdot 2r + 2\tilde{\varepsilon} \\ &= 6r\|x - a_e\|^{-1} + 2\tilde{\varepsilon} \leq 6\Delta^{-1}r + 2\tilde{\varepsilon} < 3\tilde{\varepsilon}. \end{aligned}$$

Since the above argument is in particular valid with f replaced by e , we have

$$|\hat{f}_{ij}(z) - \hat{e}_{ij}(z)| \leq |\hat{f}_{ij}(z) - \hat{e}_{ij}(x)| + |\hat{e}_{ij}(x) - \hat{e}_{ij}(z)| < 6\tilde{\varepsilon}.$$

Hence $\|\hat{f} - \hat{e}\| < 6d\tilde{\varepsilon}$. Now, the derivative matrix $\varphi'_f(z)$ of φ_f , written as the matrix of partial derivatives in complex coordinates at a point $z \in \mathbb{C}^d$, is equal to $\lambda_f r_f^2 \|z - a_f\|^{-2} A_f (Id - 2\hat{f})(z)$. By our choice of δ we have $|\lambda_f| |r_f|^2 \leq (1 + \tilde{\varepsilon}) \lambda_e r_e^2$ and $\|A_f\| \leq (1 + \tilde{\varepsilon}) \|A_e\| = 1 + \tilde{\varepsilon}$. Moreover, $\|(Id - 2\hat{f}) - (Id - 2\hat{e})\| = 2\|\hat{f} - \hat{e}\| < 12d\tilde{\varepsilon}$. Using once again (5.20) as well as the above estimates, we get

$$\begin{aligned} \|\varphi'_f(z)\| &\leq (1 + \tilde{\varepsilon}) \lambda_e r_e^2 \cdot (1 - \tilde{\varepsilon})^{-1} \|x - a_e\|^{-2} \cdot (1 + \tilde{\varepsilon}) \cdot (1 + 12d\tilde{\varepsilon}) \\ &\leq (1 + \varepsilon) \lambda_e r_e^2 \|x - a_e\|^{-2} \leq (1 + \varepsilon) s. \end{aligned}$$

Since this is true for every $z \in B_{\mathbb{C}^d}(x, r)$, the result follows. \square

Now, given two compact subsets X_1, X_2 of \mathbb{R}^d , two open subsets V_1, V_2 of \mathbb{R}^d containing X_1 and X_2 respectively, and $s \in (0, 1)$, we denote by $\mathbb{R}_s((X_1, V_1); (X_2, V_2))$ the subset of M_d consisting of all e such that

$$(5.22) \quad \varphi_e(X_1) \subset X_2, \quad \varphi_e(V_1) \subset V_2$$

and

$$(5.23) \quad \sup\{\|\varphi'_e(x)\| : x \in X_1\} \leq s.$$

Note that $\mathbb{R}_s((X_1, V_1); (X_2, V_2)) \subset M_d(X_1, V_1)_s$. We define analogously the complex subset $\mathbb{C}_s((X_1, V_1); (X_2, V_2))$ of $\mathbb{C}M_d$, requiring this time that the X_i 's be compact subsets of \mathbb{C}^d and that the V_i 's be open subsets of \mathbb{C}^d , each containing the corresponding X_i .

Lemma 5.7. *Suppose that X_1 and X_2 are compact subsets of \mathbb{R}^d , and that V_1 and V_2 are open subsets of \mathbb{R}^d containing X_1 and X_2 , respectively. Fix $s \in (0, 1)$. Then there exists a constant $R := R_s((X_1, V_1); (X_2, V_2)) > 0$ such that for every $0 < \tilde{R} \leq R$, each $e \in \mathbb{R}_s((X_1, V_1); (X_2, V_2))$ admits $\delta_e(\tilde{R}) > 0$ such that*

$$B_{\mathbb{C}^q}(e, \delta_e(\tilde{R})) \cap \mathbb{C}M_d \subset \mathbb{C}_{\frac{s+1}{2}} \left((\overline{B_{\mathbb{C}^d}(X_1, \tilde{R})}, B_{\mathbb{C}^d}(X_1, 2\tilde{R})); (\overline{B_{\mathbb{C}^d}(X_2, \tilde{R})}, B_{\mathbb{C}^d}(X_2, 2\tilde{R})) \right).$$

Proof. Apply Lemma 5.6 with $X = X_1$, $V = V_1$ and $\varepsilon > 0$ so small that

$$(1 + \varepsilon)s \leq \frac{s+1}{2}.$$

Let $r > 0$ be the corresponding number mentioned in that lemma. Fix an element $e \in \mathbb{R}_s((X_1, V_1); (X_2, V_2)) \subset M_d(X_1, V_1)_s$ and denote by $\delta_{e,1} > 0$ the corresponding number δ produced in Lemma 5.6. The lemma then asserts that

$$(5.24) \quad \|\varphi'_f(z)\| \leq \frac{s+1}{2}$$

for all $z \in B_{\mathbb{C}^d}(X_1, r)$ and all $f \in B_{\mathbb{C}^q}(e, \delta_{e,1}) \cap \mathbb{C}M_d$. Now, fix $0 < \tilde{r} \leq r$ and apply Lemma 5.5 with

$$\varepsilon = \frac{1-s}{4} \tilde{r}.$$

Denote the resulting $\delta > 0$ by $\delta_{e,2}(\tilde{r})$. Lemma 5.5 then states that

$$(5.25) \quad \|\varphi_f - \varphi_e\|_{X_1} \leq \frac{1-s}{4} \tilde{r}$$

for all $f \in B_{\mathbb{C}^q}(e, \delta_{e,2}(\tilde{r})) \cap \mathbb{C}M_d$. Now, set $\delta_e(\tilde{r}) := \min\{\delta_{e,1}, \delta_{e,2}(\tilde{r})\}$. Consider an arbitrary radius $\eta \in [\tilde{r}/2, \tilde{r}]$ and take any point $z \in B_{\mathbb{C}^d}(X_1, \eta)$. Then there exists $x \in X_1$ such that $\|z - x\| < \eta$. Using (5.24) and (5.25), we get that for all $f \in B_{\mathbb{C}^q}(e, \delta_e(\tilde{r})) \cap \mathbb{C}M_d$

$$\begin{aligned} \|\varphi_f(z) - \varphi_e(x)\| &\leq \|\varphi_f(z) - \varphi_f(x)\| + \|\varphi_f(x) - \varphi_e(x)\| \\ &\leq \frac{s+1}{2}\|z - x\| + \frac{1-s}{4}\tilde{r} < \frac{s+1}{2}\eta + \frac{1-s}{4}\tilde{r} \leq \eta. \end{aligned}$$

Since $\varphi_e(x) \in X_2$, we deduce that $\varphi_f(z) \in B_{\mathbb{C}^d}(X_2, \eta)$. Consequently, $\varphi_f(B_{\mathbb{C}^d}(X_1, \eta)) \subset B_{\mathbb{C}^d}(X_2, \eta)$ and $\varphi_f(\overline{B_{\mathbb{C}^d}(X_1, \eta)}) \subset \overline{B_{\mathbb{C}^d}(X_2, \eta)}$. Looking at the first of these inclusions with $\eta = \tilde{r}/2$, at the latter one with $\eta = \tilde{r}$, and considering (5.24), we conclude that

$$B_{\mathbb{C}^q}(e, \delta_e(\tilde{r}/2)) \cap \mathbb{C}M_d \subset \mathbb{C}_{\frac{s+1}{2}} \left((\overline{B_{\mathbb{C}^d}(X_1, \tilde{r}/2)}, B_{\mathbb{C}^d}(X_1, \tilde{r})) ; (\overline{B_{\mathbb{C}^d}(X_2, \tilde{r}/2)}, B_{\mathbb{C}^d}(X_2, \tilde{r})) \right).$$

Setting $R = \tilde{r}/2$ and $\tilde{R} = \tilde{r}/2$ completes the proof. \square

6. ANALYTICITY OF PERRON-FROBENIUS OPERATORS

Throughout this section $d \geq 3$. Fix a countable (finite or infinite) set I and an incidence matrix $A : I \times I \rightarrow \{0, 1\}$. Fix also $E \in M_d^I \subset (\mathbb{R}^q)^I$ such that $\Phi_E = (\varphi_{E_i})_{i \in I}$ is a WPO CGDMS with incidence matrix $A_{E_{i_1} E_{i_2}} := A_{i_1 i_2}$, a finite vertex set V_E , and corresponding pairs (X_v, W_v) , $v \in V_E$. Denote the common contraction rate by $s = s_E \in (0, 1)$. Put

$$R_E := \inf \left\{ R_s((X_{to(E_i)}, W_{to(E_i)}) ; (X_{in(E_i)}, W_{in(E_i)})) : i \in I \right\} > 0,$$

where the number $R_s((X_{to(E_i)}, W_{to(E_i)}) ; (X_{in(E_i)}, W_{in(E_i)})) > 0$ results from the application of Lemma 5.7 with $X = X_{to(E_i)}$, $V = W_{to(E_i)}$ and $s = s_E$. Note that the number R_E is positive since it is the infimum of a finite set of positive numbers. Set also

$$\Gamma_{E_i} := B_{\mathbb{C}^q}(E_i, \delta_{E_i}(R_E)) \cap \mathbb{C}M_d$$

for each $i \in I$, where each number $\delta_{E_i}(R_E) > 0$ results from the application of Lemma 5.7 with $X = X_{to(E_i)}$, $V = W_{to(E_i)}$, $s = s_E$, and $e = E_i$. In view of Lemma 5.7, for each $D \in \Pi_{i \in I} \Gamma_{E_i}$ the collection $\Phi_D = (\varphi_{D_i})_{i \in I} \in (\mathbb{C}M_d)^I \subset (\mathbb{C}^q)^I$ forms a hyperbolic GDMS with contraction rate less than or equal to $(s+1)/2$, the same set of vertices V_E , corresponding pairs $(\overline{B_{\mathbb{C}^d}(X_v, R_E)}, B_{\mathbb{C}^d}(X_v, 2R_E))$, $v \in V_E$, and the ‘‘same’’ incidence matrix generated by $A : I \times I \rightarrow \{0, 1\}$, that is, $A_{D_{i_1} D_{i_2}} := A_{i_1 i_2}$.

However, the Φ_D 's are not WPO CGDMSs because they are not conformal in general. Nevertheless, since $\mathbb{C}M_d$ is open in \mathbb{C}^q , the set $\Pi_{i \in I} \Gamma_{E_i} \subset (\mathbb{C}^q)^I$ is open with respect to the box topology. Observe also that $\Pi_{i \in I} (\Gamma_{E_i} \cap M_d) \subset (\mathbb{R}^q)^I$ is open because M_d is open in \mathbb{R}^q , and that if $D \in \Pi_{i \in I} (\Gamma_{E_i} \cap M_d)$, then Φ_D restricted to the generating pairs $(\overline{B}(X_v, R_E), B(X_v, 2R_E))$, $v \in V_E$, is a WPO CGDMS. For every $D \in \Pi_{i \in I} \Gamma_{E_i}$, the coordinate projections $i \mapsto D_i$ induce the mapping $\omega \mapsto \omega_D = D_{\omega_1} D_{\omega_2} D_{\omega_3} \dots \in \{D_i : i \in I\}_A^\infty$ defined on I_A^∞ . Note also that we will sometimes slightly abuse notation. For instance, we will shorten the notation for the coding maps from π_{Φ_D} to π_D . Our first result in this section is the following:

Proposition 6.1. *For every $\omega \in I_A^\infty$ the projection map $D \mapsto \pi_D(\omega_D) \in \mathbb{C}^d$, $D \in \Pi_{i \in I} \Gamma_{E_i}$, is holomorphic.*

Proof. For every $v \in V_E$ fix a point $x_v \in X_v$. Then for every $D \in \Pi_{i \in I} \Gamma_{E_i}$, we have

$$(6.1) \quad \pi_D(\omega_D) = \lim_{n \rightarrow \infty} \varphi_{\omega_D|_n}(x_{to(\omega_n)}).$$

Now, let U be the intersection of $\Pi_{i \in I} \Gamma_{E_i}$ with some space box. Because of (5.6) and the first part of (5.1), for every $n \geq 1$ the mapping $D \mapsto \varphi_{\omega_D|_n}(x_{to(\omega_n)})$, $D \in U$, is holomorphic. Since the ranges of these maps are bounded (in fact, contained in $\cup_{v \in V_E} \overline{B_{\mathbb{C}^d}}(X_v, R_E)$), we conclude that the sequence of maps $D \mapsto \varphi_{\omega_D|_n}(x_{to(\omega_n)})$, $n \geq 1$, is normal. Consequently, all its limit points are holomorphic functions. Thus, $D \mapsto \pi_D(\omega_D) = \lim_{n \rightarrow \infty} \varphi_{\omega_D|_n}(x_{to(\omega_n)})$ is holomorphic. \square

Furthermore, we have the following property.

Lemma 6.2. *The family of projection maps $\{D \mapsto \pi_D(\omega_D)\}_{\omega \in I_A^\infty}$ is equicontinuous at the point E .*

Proof. Fix $\varepsilon > 0$, and for every $i \in I$ take $\xi_i > 0$ so small that $B_{\mathbb{C}^q}(E_i, \xi_i) \subset \Gamma_{E_i}$ and that according to Lemma 5.5 (applied with $X = X_{to(E_i)}$, $V = W_{to(E_i)}$ and $e = E_i$) we have

$$\sup\{\|\varphi_f - \varphi_{E_i}\|_{X_{to(E_i)}} : f \in B_{\mathbb{C}^q}(E_i, \xi_i)\} \leq \varepsilon.$$

For every $n \geq 1$, set

$$\Delta_n := \sup\{\|\varphi_{\omega_D|_n} - \varphi_{\omega_E|_n}\|_{X_{to(\omega_n)}} : D \in \Pi_{i \in I} B_{\mathbb{C}^q}(E_i, \xi_i), \omega \in I_A^\infty\},$$

and observe that $\Delta_n \leq \sqrt{d}(\max\{\text{diam}(X_v) : v \in V_E\} + 2R_E)$. For every $n \geq 1$, every $D \in \Pi_{i \in I} B_{\mathbb{C}^q}(E_i, \xi_i)$, every $\omega \in I_A^\infty$ and every $x \in X_{to(\omega_{n+1})}$, we have

$$\begin{aligned} \|\varphi_{\omega_D|_{n+1}}(x) - \varphi_{\omega_E|_{n+1}}(x)\| &= \left\| \begin{aligned} &\left(\varphi_{\omega_D|_n}(\varphi_{(\omega_D)_{n+1}}(x)) - \varphi_{\omega_D|_n}(\varphi_{(\omega_E)_{n+1}}(x)) \right) \\ &+ \left(\varphi_{\omega_D|_n}(\varphi_{(\omega_E)_{n+1}}(x)) - \varphi_{\omega_E|_n}(\varphi_{(\omega_E)_{n+1}}(x)) \right) \end{aligned} \right\| \\ &\leq \left(\frac{s+1}{2}\right)^n \|\varphi_{(\omega_D)_{n+1}}(x) - \varphi_{(\omega_E)_{n+1}}(x)\| + \Delta_n \\ &\leq \left(\frac{s+1}{2}\right)^n \|\varphi_{(\omega_D)_{n+1}} - \varphi_{(\omega_E)_{n+1}}\|_{X_{to(\omega_{n+1})}} + \Delta_n \\ &\leq \left(\frac{s+1}{2}\right)^n \varepsilon + \Delta_n. \end{aligned}$$

Therefore

$$\Delta_{n+1} \leq \Delta_n + \varepsilon \left(\frac{s+1}{2}\right)^n.$$

Thus, by induction,

$$\Delta_n \leq \varepsilon \sum_{j=1}^n \left(\frac{s+1}{2}\right)^j \leq \varepsilon \sum_{j=0}^{\infty} \left(\frac{s+1}{2}\right)^j = \frac{\varepsilon}{1 - \frac{s+1}{2}} = \frac{2\varepsilon}{1-s}.$$

Applying (6.1), we obtain that $\|\pi_D(\omega_D) - \pi_E(\omega_E)\| \leq 2\varepsilon/(1-s)$. Since $\varepsilon > 0$ was chosen arbitrarily, the family $\{D \mapsto \pi_D(\omega_D)\}_{\omega \in I_A^\infty}$ is equicontinuous at E . \square

Now, for every $\omega \in I_A^\infty$ define the function $\psi_\omega : \Pi_{i \in I} \Gamma_{E_i} \rightarrow \mathbb{C}$ by

$$(6.2) \quad \psi_\omega(D) := \frac{\|\|\varphi'_{(\omega_D)_1}(\pi_D(\sigma\omega_D))\|\|}{\|\|\varphi'_{(\omega_E)_1}(\pi_E(\sigma\omega_E))\|\|}.$$

where $\|\|\varphi'_f(z)\|\|$ was defined in (5.7), while $\|\|\varphi'_e(x)\|\|$ is the scaling factor of φ_e given in (5.4). Note that it follows immediately from Proposition 6.1 that each function ψ_ω , $\omega \in I_A^\infty$, is holomorphic. Moreover, we have the following.

Lemma 6.3. *For every $i \in I$ there exists $\rho_i > 0$ such that $B_{\mathbb{C}^q}(E_i, \rho_i) \subset \Gamma_{E_i}$ and*

$$|\psi_\omega(D) - 1| < \frac{1}{2}$$

for all $\omega \in I_A^\infty$ and all $D \in \Pi_{i \in I} B_{\mathbb{C}^q}(E_i, \rho_i)$.

Proof. This lemma is a direct consequence of (5.4), (5.7), Lemma 5.3 and Lemma 6.2. \square

Now, let $\log : B_{\mathbb{C}}(1, 1/2) \rightarrow \mathbb{C}$ be the branch of the logarithm determined by the requirement that $\log 1 = 0$. With ρ_i as specified above, for every $D \in \Pi_{i \in I} B_{\mathbb{C}^q}(E_i, \rho_i)$ and every $\omega \in I_A^\infty$, set

$$(6.3) \quad \zeta_\omega(D) := \log \psi_\omega(D) + \log \|\|\varphi'_{(\omega_E)_1}(\pi_E(\sigma\omega_E))\|\|.$$

Fix $t_0 > \theta(\Phi_E)$, where $\theta(\Phi_E)$ is the finiteness parameter of the WPO CGDMS Φ_E . Aiming to apply Theorem 2.5, we shall now prove the main result of this section:

Lemma 6.4. *For every $(t, D) \in \{\xi \in \mathbb{C} : \operatorname{Re}(\xi) > t_0\} \times \Pi_{i \in I} B_{\mathbb{C}^q}(E_i, \rho_i)$ we have that $t\zeta_{(\cdot)}(D) \in \mathcal{K}_{\frac{s+1}{2}}$. Moreover, the function $(t, D) \mapsto \mathcal{L}_{t\zeta_{(\cdot)}(D)} \in L(\mathcal{H}_{\frac{s+1}{2}}^b)$, $(t, D) \in \{\xi \in \mathbb{C} : \operatorname{Re}(\xi) > t_0\} \times \Pi_{i \in I} B_{\mathbb{C}^q}(E_i, \rho_i/2)$ is holomorphic.*

Proof. Fix $(t, D) \in \{\xi \in \mathbb{C} : \operatorname{Re}(\xi) > t_0\} \times \Pi_{i \in I} B_{\mathbb{C}^q}(E_i, \rho_i)$. Since, by Lemma 6.3, $|\log \psi_{(\cdot)}(D)|$ is uniformly bounded by $\log(3/2)$, summability of the potential $t\zeta_{(\cdot)}(D)$ follows immediately from the choice of t_0 .

We shall now check that the function $t\zeta_{(\cdot)}(D) \in \mathcal{H}_{\frac{s+1}{2}}$. In view of (5.7), for all $\omega, \tau \in I_A^\infty$ with $\omega_1 = \tau_1$, we have

$$|t\zeta_\tau(D) - t\zeta_\omega(D)| = |t| \left| \log \left(\frac{\|\|\pi_D(\sigma\omega_D) - a_D\|\|^2}{\|\|\pi_D(\sigma\tau_D) - a_D\|\|^2} \right) \right|,$$

where $a_D := a_{(\omega_D)_1} = a_{(\tau_D)_1}$. Thanks to our choice of the Γ_{E_i} 's and because $B_{\mathbb{C}^q}(E_i, \rho_i) \subset \Gamma_{E_i}$ for each $i \in I$, we can apply Lemma 5.3 with $X = X_{to(E_{\omega_1})} \ni \pi_E(\sigma\omega_E)$, $V = W_{to(E_{\omega_1})}$, $z = \pi_D(\sigma\omega_D)$, $w = \pi_D(\sigma\tau_D)$, and $\alpha = \beta = a_D$. With $C_1 := \max_{v \in V_E} C(X_v, W_v)$, $C_2 := \max\{\operatorname{diam}(X_v) : v \in V_E\} + 2R_E$ and $C_3 := \max\{\operatorname{diam}(X_v) : v \in V_E\}$, it follows that

$$\begin{aligned} \left| \frac{\|\|\pi_D(\sigma\omega_D) - a_D\|\|^2}{\|\|\pi_D(\sigma\tau_D) - a_D\|\|^2} - 1 \right| &\leq C_1 \max \left\{ \|\|\pi_D(\sigma\omega_D) - \pi_D(\sigma\tau_D)\|\|^2, \|\|\pi_D(\sigma\omega_D) - \pi_D(\sigma\tau_D)\|\| \right\} \\ &\leq C_1 C_2 \|\|\pi_D(\sigma\omega_D) - \pi_D(\sigma\tau_D)\|\| \\ &\leq C_1 C_2 C_3 \left(\frac{s+1}{2} \right)^{|\omega \wedge \tau| - 1}. \end{aligned}$$

Thus, there exists a constant $C_4 \geq 0$ such that

$$(6.4) \quad |t\zeta_\tau(D) - t\zeta_\omega(D)| \leq C_4 |t| \left(\frac{s+1}{2} \right)^{|\omega \wedge \tau|}.$$

So $t\zeta_{(\cdot)}(D) \in \mathcal{H}_{\frac{s+1}{2}}$, and in consequence $t\zeta_{(\cdot)}(D) \in \mathcal{K}_{\frac{s+1}{2}}$. On the other hand, it follows immediately from Proposition 6.1 that for every $\omega \in I_A^\infty$, the function $D \mapsto \psi_\omega(D)$ is holomorphic, and consequently $(t, D) \mapsto t\zeta_\omega(D) \in \mathbb{C}$, $(t, D) \in \{\xi \in \mathbb{C} : \operatorname{Re}(\xi) > t_0\} \times \prod_{i \in I} B_{\mathbb{C}^q}(E_i, \rho_i)$, is holomorphic. In view of Theorem 2.5 and Hartogs Theorem, in order to complete the proof, it suffices to demonstrate that the function $(t, D) \mapsto t\zeta_{(\cdot)}(D) \in \mathcal{K}_{\frac{s+1}{2}}$, with (t, D) in a one-dimensional space box neighbourhood of (t_0, E) , is continuous. Obviously, this function is continuous with respect to the variable t . So, we may assume that t is fixed and D is a one-dimensional space box neighbourhood of $\prod_{i \in I} B_{\mathbb{C}^q}(E_i, \rho_i)$, say $D \in B_{\mathbb{C}}(b, \rho_i)$ for some fixed $i \in I$ and some component b of E_i . Fix $\omega \in I_A^\infty$. Since the function $D \mapsto \log \psi_\omega(D)$ is holomorphic and bounded in modulus by $\log(3/2)$ on $B_{\mathbb{C}}(b, \rho_i)$, and satisfies $\log \psi_\omega(E) = 0$, its Taylor series expansion on $B_{\mathbb{C}}(b, \rho_i)$ takes on the form

$$(6.5) \quad \log \psi_\omega(D) = \sum_{n=1}^{\infty} a_n(\omega)(D - b)^n,$$

and from Cauchy's estimates

$$|a_n(\omega)| \leq \rho_i^{-n} \log(3/2).$$

Hence if $F \in B_{\mathbb{C}}(b, \rho_i)$, then

$$(6.6) \quad \begin{aligned} \log \psi_\omega(F) - \log \psi_\omega(D) &= \sum_{n=1}^{\infty} a_n(\omega) \left((F - b)^n - (D - b)^n \right) \\ &= \sum_{n=1}^{\infty} a_n(\omega) (F - D) \sum_{j=0}^{n-1} (F - b)^{n-1-j} (D - b)^j \end{aligned}$$

Consequently, if $D, F \in B_{\mathbb{C}}(b, \rho_i/2)$, then

$$\begin{aligned} |\log \psi_\omega(F) - \log \psi_\omega(D)| &\leq \sum_{n=1}^{\infty} \rho_i^{-n} \log(3/2) |F - D| \sum_{j=0}^{n-1} |F - b|^{n-1-j} |D - b|^j \\ &\leq \log(3/2) |F - D| \sum_{n=1}^{\infty} \rho_i^{-n} \sum_{j=0}^{n-1} (\rho_i/2)^{n-1} \\ &= \log(3/2) \rho_i^{-1} \left(\sum_{n=1}^{\infty} n 2^{-(n-1)} \right) |F - D|. \end{aligned}$$

It then follows from (6.3) that

$$(6.7) \quad \|t\zeta_{(\cdot)}(F) - t\zeta_{(\cdot)}(D)\|_\infty \leq \log(3/2) |t| \rho_i^{-1} \left(\sum_{n=1}^{\infty} n 2^{-(n-1)} \right) |F - D|.$$

Now, fix $\omega, \tau \in I_A^\infty$ with $\omega_1 = \tau_1$. It follows from (6.3) and (6.4) that

$$|\log \psi_\tau(D) - \log \psi_\omega(D)| \leq C_4 \left(\frac{s+1}{2} \right)^{|\omega \wedge \tau|}.$$

Combining this with (6.5), Cauchy's estimates yield

$$|a_n(\tau) - a_n(\omega)| \leq C_4 \rho_i^{-n} \left(\frac{s+1}{2} \right)^{|\omega \wedge \tau|}.$$

So, if $D, F \in B_{\mathbb{C}}(b, \rho_i/2)$, we get using (6.6) that

$$\begin{aligned} & \left| \left(\log \psi_\tau(F) - \log \psi_\tau(D) \right) - \left(\log \psi_\omega(F) - \log \psi_\omega(D) \right) \right| \\ (6.8) \quad &= \left| \sum_{n=1}^{\infty} (a_n(\tau) - a_n(\omega))(F - D) \sum_{j=0}^{n-1} (F - b)^{n-1-j} (D - b)^j \right| \\ &\leq C_4 \rho_i^{-1} \left(\sum_{n=1}^{\infty} n 2^{-(n-1)} \right) |F - D| \left(\frac{s+1}{2} \right)^{|\omega \wedge \tau|}. \end{aligned}$$

Now, given a function $g : I_A^\infty \rightarrow \mathbb{C}$ we denote (as the reader will recall) by $v_s(g)$ the s -variation of g , that is, $v_s(g) := \sup\{|g(\omega) - g(\tau)|/s^n : \omega|_n = \tau|_n\}$, and by $\|g\|_s$ the s -norm of g , namely $\|g\|_s := \|g\|_\infty + v_s(g)$ (cf. section 2). Then it follows immediately from (6.8) that

$$v_{\frac{s+1}{2}} \left(\log \psi_{(\cdot)}(F) - \log \psi_{(\cdot)}(D) \right) \leq C_4 \rho_i^{-1} \left(\sum_{n=1}^{\infty} n 2^{-(n-1)} \right) |F - D|,$$

and consequently

$$v_{\frac{s+1}{2}} \left(t\zeta_{(\cdot)}(F) - t\zeta_{(\cdot)}(D) \right) \leq C_4 |t| \rho_i^{-1} \left(\sum_{n=1}^{\infty} n 2^{-(n-1)} \right) |F - D|.$$

Together with (6.7), this yields

$$\|t\zeta_{(\cdot)}(F) - t\zeta_{(\cdot)}(D)\|_{\frac{s+1}{2}} \leq \left(C_4 + \log(3/2) \right) |t| \rho_i^{-1} \left(\sum_{n=1}^{\infty} n 2^{-(n-1)} \right) |F - D|.$$

Thus the proof of continuity is complete and we are done. \square

7. REAL-ANALYTICITY OF HAUSDORFF DIMENSION

Sticking to the same framework as in the preceding section, $d \geq 3$, I is a countable (finite or infinite) alphabet and $A : I \times I \rightarrow \{0, 1\}$ an incidence matrix. Fix $E \in M_d^I \subset (\mathbb{R}^q)^I$ such that $\Phi_E = (\varphi_{E_i})_{i \in I}$ is a strongly regular WPO CGDMS with a finite set of vertices V_E and corresponding pairs (X_v, W_v) , $v \in V_E$. To each element $D \in \Pi_{i \in I} \Gamma_{E_i}$, ascribe uniquely a hyperbolic GDMS $\Phi_D = (\varphi_{D_i})_{i \in I} \in \mathbb{C}M_d^I \subset (\mathbb{C}^q)^I$ as in the previous section. Recall that if $D \in \Pi_{i \in I} (\Gamma_{E_i} \cap M_d)$, then Φ_D restricted to the generating pairs $(\overline{B}(X_v, R_E), B(X_v, 2R_E))_{v \in V_E}$ is a WPO CGDMS. Moreover, the limit sets of the restricted and unrestricted systems Φ_D are the same and contained in \mathbb{R}^d . The first result of this section is the following.

Theorem 7.1. *For every $i \in I$ there is $R_i > 0$ such that the function $D \mapsto BP(\Phi_D)$, $D \in \Pi_{i \in I} B(E_i, R_i)$, is real-analytic.*

Proof. For every $t \in \mathbb{C}$ with $\operatorname{Re}(t) > \theta(\Phi_E)$ and $D \in \Pi_{i \in I} \Gamma_{E_i}$ put $\mathcal{L}_{t,D} := \mathcal{L}_{t\zeta(\cdot)(D)} \in L(\mathcal{H}_{\frac{s+1}{2}}^b)$. Fix a space box $U \subset \mathbb{C} \times \Pi_{i \in I} B_{\mathbb{C}^q}(E_i, \rho_i/2)$ containing (t_0, E) (with ρ_i is as defined in Lemma 6.3). Apply Lemma 6.4 with $t_0 = t_E := BP(\Phi_E) > \theta(\Phi_E)$ (the strict inequality holds since Φ_E is strongly regular). In view of that lemma and Theorem 2.4, Kato-Rellich perturbation theorem applies and yields $R_i \in (0, \rho_i/2]$ (in fact, $R_i = \rho_i/2$ except for finitely many $i \in I$) and a holomorphic function $\lambda : U \cap (\mathbb{C} \times \Pi_{i \in I} B_{\mathbb{C}^q}(E_i, R_i)) \rightarrow \mathbb{C}$ such that $\lambda(t_E, E) = e^{P_E(t_E)}$ and such that $\lambda(t, D)$ is a simple isolated eigenvalue of the operator $\mathcal{L}_{t,D} : \mathcal{H}_{\frac{s+1}{2}}^b \rightarrow \mathcal{H}_{\frac{s+1}{2}}^b$ with the remainder of the spectrum separated uniformly from $\lambda(t, D)$. In particular, decreasing $R_i > 0$ appropriately, there is $\eta > 0$ such that

$$(7.1) \quad \sigma(\mathcal{L}_{t,D}) \cap B_{\mathbb{C}}(e^{P_E(t_E)}, \eta) = \{\lambda(t, D)\}$$

for all $(t, D) \in U$, where $\sigma(\mathcal{L}_{t,D})$ denotes the spectrum of $\mathcal{L}_{t,D}$. Since $e^{P_D(t)}$ is the spectral radius $r(\mathcal{L}_{t,D})$ of the operator $\mathcal{L}_{t,D}$ for all $(t, D) \in U \cap (\mathbb{R} \times M_d^I)$, in view of the semicontinuity of the spectral set function (see Theorem 10.20 on page 256 in [8]), by taking R_i 's sufficiently small we also have that $r(\mathcal{L}_{t,D}) \in [0, e^{P_E(t_E)} + \eta)$, and along with (7.1) this implies that $\lambda(t, D) = e^{P_D(t)}$ for all $(t, D) \in U \cap (\mathbb{R} \times M_d^I)$. Consequently, the function $(t, D) \mapsto P_D(t)$, $(t, D) \in U \cap (\mathbb{R} \times M_d^I)$, is real-analytic. Since $P_E(t_E) = 0$ and since

$$\left. \frac{\partial P}{\partial t} \right|_{(t_E, E)} = \int \log \|\varphi'_{(\omega_E)_1}(\pi_E(\sigma\omega_E))\| d\mu_E(\omega) < 0,$$

where μ_E is the Gibbs (equilibrium) state of the potential $\omega \mapsto t_E \log \|\varphi'_{(\omega_E)_1}(\pi_E(\sigma\omega_E))\|$, it follows from the implicit function theorem that, with appropriately smaller R_i 's, there exists a real-analytic function $t : \hat{U} \rightarrow \mathbb{R}$ defined on \hat{U} , the projection of U onto the set $\Pi_{i \in I} B(E_i, R_i)$, such that $t(E) = t_E$ and $P_D(t(D)) = 0$. Hence $t(D) = BP(\Phi_D)$ and we are done. \square

As an immediate consequence of this theorem and Bowen's formula, we obtain the following:

Theorem 7.2. *Suppose that the incidence matrix $A : I \times I \rightarrow \{0, 1\}$ is finitely irreducible and that Φ_E is a strongly regular CGDMS. If with R_i , $i \in I$, as in Theorem 7.1 or smaller, all the systems Φ_D , $D \in \Pi_{i \in I} B(E_i, R_i)$, satisfy the OSC, then the Hausdorff dimension function $D \mapsto \operatorname{HD}(J_{\Phi_D})$, $D \in \Pi_{i \in I} B(E_i, R_i)$, is real-analytic.*

Remark 7.3. *Suppose that the incidence matrix $A : I \times I \rightarrow \{0, 1\}$ is finitely irreducible and that $\Phi_E = (\varphi_{E_i})_{i \in I}$ is a strongly regular CGDMS. If Φ_E satisfies the super strong separation condition, that is, if for every $i \in I$*

$$\varphi_{E_i}(X_{t_0(E_i)}) \cap \overline{\bigcup_{j \in I \setminus \{i\}} \varphi_{E_j}(X_{t_0(E_j)})} = \emptyset,$$

then by proceeding inductively with respect to a good order on I , we can construct $R_i^ \in (0, \infty)$ such that for each $D \in \Pi_{i \in I} B(E_i, R_i^*)$, the system Φ_D also satisfies the super strong*

separation condition. In particular, each such Φ_D satisfies the OSC and Theorem 7.2 applies to that family of systems.

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