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# Transversal families of hyperbolic skew-products

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Abstract. We study families of hyperbolic skew products with the transversality condition and in particular, the Hausdorff dimension of their fibers, by using thermodynamical formalism. The maps we consider can be non-invertible, and the study of their dynamics is influenced greatly by this fact.

We introduce and employ probability measures (constructed from equilibrium measures on the natural extension), which are supported on the fibers of the skew product. A stronger condition, that of Uniform Transversality is then considered in order to obtain a general formula for Hausdorff dimension of fibers for all base points and almost all parameters.

In the end we study a large class of examples of transversal hyperbolic families which locally depend linearly on the parameters, and also another class of examples related to complex dynamics.

1. **Introduction.** In this paper we consider skew product maps which are conformal in fibers, and fiberwise contracting, with mild assumptions about the base map. The motivation to deal with them comes from three directions, namely: general Axiom A endomorphisms (see [6] for example), smooth hyperbolic skew-products ([8] and [9]) and conformal iterated function systems with overlaps ([11] for example). Our goal, similar to the one in the two latter mentioned groups of papers, is to put light on what is the value of Hausdorff dimension of fibers, and more precisely a version of Bowen's formula.

Let us also note that non-invertibility of the maps we consider prevents one from using the same kind of approach as in the diffeomorphism case (see [3] for example).

It is known, and easy to see, that in general Bowen's formula fails for conformal iterated function systems with overlaps. In order to remedy this situation, the concept of transversality was introduced (see [10] and [5] for example). This is a measure theoretic assumption which permits to establish Bowen's formula for Lebesgue almost all iterated systems from a given family.

We asked ourselves whether one could define an appropriate concept of transversality for skew-products, and then, by using thermodynamic formalism to obtain generic (i.e. for almost all parameters) Bowen's formula. Indeed, we came up in this paper with the transversality condition which is formulated in (cf). Working with Rokhlin's natural extensions and canonical conditional measures, it allowed us to prove a generic Bowen's formula in Theorem 2.8 and Corollary 2.9.

Imposing a stronger condition, namely uniform transversality, we proved (see Theorem 2.10) a more precise Bowen's formula, which holds for almost all parameters and **all** points $x$ in the base space.

One can notice also that we can interpret iterated function systems with overlaps as skew-products, with their base map being a one-sided shift map, thus being investigated by our work. Moreover in the last section we consider maps which are more general than iterated function systems (although resembling them), namely maps of the form $F_\lambda(x,y) = (f(x), \lambda_i + \Phi_i(x,y,\lambda))$, for $x \in X_i$, where $f: I_1 \cup \ldots \cup I_d \to I$, and $X_i = I_* \cap I_i, i = 1, \ldots, d$. We show in Theorem 3.3 that this family $\{F_\lambda\}_{\lambda \in B_d(0,\eta)}$ is uniformly transversal.

In the last section, we complete our theoretical work with a rather large selection of elaborated examples; some of them (see Theorem 3.3 and Corollary 3.4) were motivated by conformal iterated function

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systems with overlaps, and the others were resulting from higher-dimensional holomorphic dynamics of skew products (see Theorem 3.5). All of these examples satisfy the transversality condition.

Some of these last families of maps $F_\lambda(z,w)$ are obtained by perturbations of skew products which have a hyperbolic map of one variable $f(z)$ in their first coordinate, and depend linearly or quadratically in $w$ in the second coordinate.

2. **Transversal Families of Hyperbolic Skew-Products.** Recall from [7] that a continuous self-map $f : X \to X$ of a compact metric space $(X,\rho)$ is called open distance expanding, provided that $f$ is open, Lipschitz continuous, and there are three constants $\eta > 0$, $\gamma > 1$ and an integer $k \geq 1$, such that $\rho(f^k(x), f^k(z)) \geq \gamma\rho(x,z)$ whenever $\rho(x,z) \leq \eta$. It is fairly easy to see that changing the metric $\rho$ in a bi-Lipschitz manner, we may assume without loss of generality that $k = 1$. There is an abundance of open distance expanding maps. We want to bring reader's attention now to one particular class of them, called expanding repellers. Let $U$ be a bounded open subset of a Euclidean space $\mathbb{R}^p$ with some $p \geq 1$.

A map $g : U \to \mathbb{R}^p$ is called an *expanding repeller* if and only if the following conditions are satisfied:

i) $g : U \to \mathbb{R}^p$ is a $C^{1+\gamma}$ endomorphism.
ii) $X = \bigcap_{n=0}^\infty g^{-n}(U)$ is a compact $g$-invariant ($g(X) = X$) subset of $U$. The map $g : X \to X$ is transitive.
iii) The map $g : X \to X$ is infinitesimally expanding, i.e. there exists $k \geq 1$ such that for all $x \in X$ and for all $v \in \mathbb{R}^p$, we have $||D_x g^k(v)|| \geq 2||v||$.

Clearly, $g : X \to X$ is an open distance (with respect to the Euclidean metric) expanding map. Frequently, perhaps even more appropriately, the word repeller is refered also to the set $X$.

Let us then take $f : X \to X$ an open distance expanding map and suppose it is transitive. Let $V$ be a bounded quasi-convex open subset of $\mathbb{R}^q$, $q \geq 1$. Being $D$-quasiconvex (with some $D \geq 1$) means that the internal distances are not bigger than Euclidean distances multiplied by $D$. In what follows quasi-convexity will be used only when the Mean Value Inequality is to be applied. So, in order to simplify notation, we will assume in the sequel that $V$ is convex.

**Definition 2.1.** *Suppose now that for all $x \in X$ there exists a $C^{1+\gamma}$ conformal endomorphism $\phi_x : V \to V$ conformally extendable to a neighborhood of $\overline{V}$ with the following properties.*

(a) $\kappa := \sup\{|(\phi_x)'(y)| : (x,y) \in X \times \overline{V}\} < 1$.
(b) $\underline{\kappa} := \inf\{|(\phi_x)'(y)| : (x,y) \in X \times \overline{V}\} > 0$.

*If the conditions (a) and (b) are satisfied, then the map $F : U \times V \to \mathbb{R}^p \times V$ given by the formula*

$$F(x,y) = (f(x), \phi_x(y))$$

*will be called a* ***hyperbolic fiberwise conformal skew-product*** *provided that it is Lipschitz continuous (with respect to the sum metric on $X \times \mathbb{R}^q$) and the map $(x,y) \mapsto (f(x), \phi'_x(y))$ is also Lipschitz continuous; denote the common Lipschitz constant by $L_F$.*

Set

$$\Lambda = \bigcup_{x \in X} \bigcap_{n=0}^{\infty} \bigcup_{z \in f^{-n}(x)} \phi_z^n(\overline{V}),$$

where $\phi_z^n = \phi_{F^{n-1}(z)} \circ \phi_{f^{n-1}(z)} \circ \ldots \circ \phi_z : \overline{V} \to \overline{V}$ and $F^n(x,y) = (f^n(x), \phi_x^n(y))$; $\Lambda$ is called the basic set of the endomorphism $F$. Obviously

$$F(\Lambda) \subset \Lambda \;\text{ and }\; F(Y_x) \subset Y_{f(x)},$$

where

$$Y_x = \bigcap_{n=0}^{\infty} \bigcup_{z \in f^{-n}(x)} \phi_z^n(\overline{V}).$$

Let $\tilde{f} : \tilde{X} \to \tilde{X}$ be the Rokhlin's natural extension (inverse limit) of the endomorphism $f : X \to X$. For every $n \geq 0$ let $p_n : \tilde{X} \to X$ be the projection onto $n$th coordinate of $\tilde{X}$. Put

$$\hat{\Lambda} = \bigcup_{x \in X} p_0^{-1}(x) \times Y_x$$

and define the map $\hat{F} : \hat{\Lambda} \to \hat{\Lambda}$ by the formula

$$\hat{F}(\tilde{x}, y) = (\tilde{f}(\tilde{x}), \phi_{x_1}(y)).$$

Notice that the map $\hat{F} : \hat{\Lambda} \to \hat{\Lambda}$ is a homeomorphism and the mapping

$$((x_n, y_n)_0^\infty) \mapsto ((x_n, y_0)_0^\infty)$$

is a homeomorphism from $\tilde{\Lambda}$, the Rokhlin's natural extension of $F|_\Lambda$, to $\hat{\Lambda}$ which establishes a canonical topological conjugacy between the map $\tilde{F} : \tilde{\Lambda} \to \tilde{\Lambda}$ and the map $\hat{F} : \hat{\Lambda} \to \hat{\Lambda}$. Note that for every $\hat{x} \in \hat{X}$, $\{\phi_{x_n}^n(\overline{V})\}_{n=0}^\infty$ is descending (as $\phi_{x_{n+1}}^{n+1} = \phi_{x_n}^n \circ \phi_{x_{n+1}}$) sequence of compact sets whose diameters, by condition (e) converge to 0. Hence, the intersection

$$\bigcap_{n=0}^\infty \phi_{x_n}^n(\overline{V})$$

is a singleton, and denote its only element by $\pi(\tilde{x})$. So, we have defined a map

$$\pi : \tilde{X} \to \overline{V}.$$

It is easy to see that for every $x \in X$,

$$\pi(p_0^{-1}(x)) = Y_x.$$

Endow $\tilde{X}$ with a metric $\tilde{\rho}$ defined as follows.

$$\tilde{\rho}(\tilde{x}, \tilde{z}) = \sum_{n=0}^\infty \kappa^n \rho(x_n, z_n).$$

We shall prove the following.

**Proposition 2.2.** *The map $\pi : \tilde{X} \to \overline{V}$ is Lipschitz continuous.*

*Proof.* We shall first prove the following formula by induction

$$\left\|\phi_{x_n}^n(w) - \phi_{z_n}^n(w)\right\| \leq \sum_{j=0}^{n-1} \kappa^j \left|\left|\phi_{x_{j+1}}\left(\phi_{z_n}^{n-j-1}(w)\right) - \phi_{z_{j+1}}\left(\phi_{z_n}^{n-j-1}(w)\right)\right|\right| \tag{2.1}$$

for all $n \geq 1$, all $w \in \overline{V}$ and all $\tilde{x}, \tilde{z} \in \tilde{X}$. Indeed, for $n = 1$ we even have equality. Suppose the formula is true for some $n \geq 1$. Using the Mean Value Inequality we then get

$$\begin{aligned}
&\left\|\phi_{x_{n+1}}^{n+1}(w) - \phi_{z_{n+1}}^{n+1}(w)\right\| = \\
&= \left\|\phi_{x_n}^n\left(\phi_{x_{n+1}}(w)\right) - \phi_{x_n}^n\left(\phi_{z_{n+1}}(w)\right) + \phi_{x_n}^n\left(\phi_{z_{n+1}}(w)\right) - \phi_{z_n}^n\left(\phi_{z_{n+1}}(w)\right)\right\| \\
&\leq \left\|\phi_{x_n}^n\left(\phi_{x_{n+1}}(w)\right) - \phi_{x_n}^n\left(\phi_{z_{n+1}}(w)\right)\right\| + \left\|\phi_{x_n}^n\left(\phi_{z_{n+1}}(w)\right) - \phi_{z_n}^n\left(\phi_{z_{n+1}}(w)\right)\right\| \\
&\leq \kappa^n ||\phi_{x_{n+1}}(w) - \phi_{z_{n+1}}(w)|| + \sum_{j=0}^{n-1} \kappa^j \left\|\phi_{x_{j+1}}\left(\phi_{z_n}^{n-j-1}\left(\phi_{z_{n+1}}(w)\right)\right) - \phi_{z_{j+1}}\left(\phi_{z_n}^{n-j-1}\left(\phi_{z_{n+1}}(w)\right)\right)\right\| \\
&= \kappa^n \left\|\phi_{x_{n+1}}(w) - \phi_{z_{n+1}}(w)\right\| + \sum_{j=0}^{n-1} \kappa^j \left\|\phi_{x_{j+1}}\left(\phi_{z_{n+1}}^{n-j}(w)\right) - \phi_{z_{j+1}}\left(\phi_{z_{n+1}}^{n-j}(w)\right)\right\| \\
&= \sum_{j=0}^{n} \kappa^j \left\|\phi_{x_{j+1}}\left(\phi_{z_{n+1}}^{n-j}(w)\right) - \phi_{z_{j+1}}\left(\phi_{z_{n+1}}^{n-j}(w)\right)\right\|.
\end{aligned}$$

The inductive proof of formula (2.1) is complete. Continuing the estimates in this formula, we obtain

$$\left\|\phi^n_{x_n}(w)-\phi^n_{z_n}(w)\right\| \le L_F\sum_{j=0}^{n-1}\kappa^j\rho(x_{j+1},z_{j+1}) \le L_F\sum_{j=0}^{\infty}\kappa^j\rho(x_{j+1},z_{j+1}).$$

So, letting $n\to\infty$, we get

$$||\pi(\tilde x)-\pi(\tilde z)|| \le L_F\sum_{j=0}^{\infty}\kappa^j\rho(x_{j+1},z_{j+1}) \le L_F\frac{\rho(\tilde x,\tilde z)}{\kappa}.$$

We are done. □

For every continuous potential $g:\tilde X\to\mathbb{R}$ let $\mathrm{P}(g)=\mathrm{P}(\tilde f,g)$ be the topological pressure of $g$ with respect to the dynamical system $\tilde f:\tilde X\to\tilde X$. For the topological pressure and its basic properties see for ex. [1] and [7]. Now consider the potential $\zeta=\zeta_F:\tilde X\to\mathbb{R}$ given by the formula

$$\zeta(\tilde x)=\log|\phi'_{x_0}(\pi(\tilde x))|$$

This potential is Hölder continuous because of Proposition 2.2. It is easy to see that the function $t\mapsto \mathrm{P}(\tilde f,t\zeta)$ is convex, Lipschitz continuous, strictly decreasing, and

$$\lim_{t\to-\infty}\mathrm{P}(\tilde f,t\zeta)=+\infty \text{ and } \lim_{t\to+\infty}\mathrm{P}(\tilde f,t\zeta)=-\infty.$$

Thus there exists exactly one $t\in\mathbb{R}$, denoted by $h$, such that $\mathrm{P}(\tilde f,h\zeta)=0$. Since $\mathrm{P}(\tilde f,0\zeta)=\mathrm{h_{top}}(\tilde f)>0$, we see that $h>0$. The number $h$ is called Bowen's stable zero of the basic set $\Lambda$. Our goal from now on throughout this section is to provide a geometric characterization of this zero $h$ of the above pressure function in the framework of smooth families of hyperbolic fiberwise conformal skew-products.

Endow the space $C^{1+\gamma}(\overline V)$ of all $C^{1+\gamma}$ differentiable endomorphisms from $\overline V$ into $\overline V$ with the norm $||\cdot||_\gamma$ given by the formula

$$||\phi||_\gamma=||\phi||_\infty+||\phi'||_\infty+v_\gamma(\phi'),$$

where

$$v_\gamma(\phi')=\inf\{L>0:|\phi'(y)-\phi'(x)|\le L|y-x|^\gamma \text{ for all } x,y\in\overline V\}.$$

Obviously $C^{1+\gamma}(\overline V)$ endowed with this norm becomes a Banach space. Denote the metric induced by the norm $||\cdot||_\gamma$ by $\rho_\gamma$.

**Definition 2.3.** *In the above setting, fix $d\ge 1$ and an open set $W\subset\mathbb{R}^d$ and consider a family $\Phi=\{\phi^\lambda_x:\overline V\to V\}_{(\lambda,x)\in W\times X}$ of maps from $C^{1+\gamma}(\overline V)$ satisfying the following conditions.*

- (af) *Conditions (a) and (b) with the same constants $\kappa,\underline{\kappa}\in(0,1)$.*
- (bf) *The map $(\lambda,x)\mapsto\phi^\lambda_x\in C^{1+\gamma}(\overline V)$ defined on $W\times X$ is continuous.*
- (cf) *(Transversality Condition)*

$$\forall(x\in X)\,\forall(\lambda_0\in W)\,\exists(\delta(x,\lambda_0)>0)\,\exists(C_1>0)\,\forall(\tilde x,\tilde y\in p_0^{-1}(x))\,\forall(r>0)$$
$$x_1\ne y_1\Rightarrow l_d\big(\{\lambda\in B(\lambda_0,\delta(x,\lambda_0)):||\pi_\lambda(\tilde x)-\pi_\lambda(\tilde y)||\le r\}\big)\le C_1r^q,$$

*where $l_d$ denotes the $d$-dimensional Lebesgue measure on $\mathbb{R}^d$ and $\pi_\lambda:\tilde X\to\overline V$ is the canonical projection induced by the skew-product $F_\lambda:U\times\overline V\to\mathbb{R}^p\times\overline V$ given by the formula*

$$F_\lambda(x,y)=(f(x),\phi^\lambda_x(y)).$$

*Any such family $\Phi$ is said to be* ***transversal*** *and the canonically induced family $\overline\Phi=\{F_\lambda\}_{\lambda\in W}$ is also called* ***transversal***.

For all $\lambda,\lambda'\in W$ put

$$||F_\lambda||_\gamma=\sup\{||\phi^\lambda_x||_\gamma:x\in X\} \text{ and } \overline\rho_\gamma(F_\lambda,F_{\lambda'})=\sup\{\rho_\gamma(\phi^\lambda_x,\phi^{\lambda'}_x):x\in X\}.$$

Condition (bf) can be now rephrased as follows.

(b'f) The function $\lambda \mapsto F_\lambda$, $\lambda \in W$, is continuous.

In order to prove Bowen's formula for the family $\overline{\Phi}$, we need some auxiliary facts.

**Lemma 2.4.**
$$\forall(\eta>0)\exists(\delta>0)\forall(\lambda_0\in W)\forall(\lambda\in B(\lambda_0,\delta)\cap W)\forall(\tilde{x}\in\tilde{X})\forall(n\geq 0)$$
$$e^{-\eta n}\leq \frac{||\big(\phi_{x_n}^{\lambda,n}\big)'||}{||\big(\phi_{x_n}^{\lambda_0,n}\big)'||}\leq e^{\eta n}.$$

*Proof.* Fix $y\in\overline{V}$. Using the Mean Value Inequality and condition (a), we get
$$\begin{aligned}
&||\phi_{x_{n+1}}^{\lambda,n+1}(y)-\phi_{x_{n+1}}^{\lambda_0,n+1}(y)||\leq\\
&\qquad\leq ||\phi_{x_1}^{\lambda}\big(\phi_{x_{n+1}}^{\lambda,n}(y)\big)-\phi_{x_1}^{\lambda_0}\big(\phi_{x_{n+1}}^{\lambda,n}(y)\big)||+||\phi_{x_1}^{\lambda_0}\big(\phi_{x_{n+1}}^{\lambda,n}(y)\big)-\phi_{x_1}^{\lambda_0}\big(\phi_{x_{n+1}}^{\lambda_0,n}(y)\big)||\\
&\qquad\leq ||\phi_{x_1}^{\lambda}-\phi_{x_1}^{\lambda_0}||_\infty+||\big(\phi_{x_1}^{\lambda_0}\big)'||_\infty||\phi_{x_{n+1}}^{\lambda,n}(y)-\phi_{x_{n+1}}^{\lambda_0,n}(y)||\\
&\qquad\leq ||\phi_{x_1}^{\lambda}-\phi_{x_1}^{\lambda_0}||_\infty+\kappa||\phi_{x_{n+1}}^{\lambda,n}(y)-\phi_{x_{n+1}}^{\lambda_0,n}(y)||.
\end{aligned}$$
Thus, by induction
$$||\phi_{x_n}^{\lambda,n}(y)-\phi_{x_n}^{\lambda_0,n}(y)||\leq(1-\kappa)^{-1}||\phi_{x_1}^{\lambda}-\phi_{x_1}^{\lambda_0}||_\infty\leq(1-\kappa)^{-1}\overline{\rho}_\gamma(F_\lambda,F_{\lambda_0}).$$
Hence, for every $0\leq k\leq n$, we get that
$$\begin{aligned}
&||\big(\phi_{x_k}^{\lambda}\big)'\big(\phi_{x_n}^{\lambda,n-k}(y)\big)-\big(\phi_{x_k}^{\lambda_0}\big)'\big(\phi_{x_n}^{\lambda_0,n-k}(y)\big)||\leq\\
&\qquad\leq ||\big(\phi_{x_k}^{\lambda}\big)'\big(\phi_{x_n}^{\lambda,n-k}(y)\big)-\big(\phi_{x_k}^{\lambda_0}\big)'\big(\phi_{x_n}^{\lambda,n-k}(y)\big)||+\\
&\qquad\quad+||\big(\phi_{x_k}^{\lambda_0}\big)'\big(\phi_{x_n}^{\lambda,n-k}(y)\big)-\big(\phi_{x_k}^{\lambda_0}\big)'\big(\phi_{x_n}^{\lambda_0,n-k}(y)\big)||\\
&\qquad\leq ||\big(\phi_{x_k}^{\lambda}\big)'-\big(\phi_{x_k}^{\lambda_0}\big)'||_\infty+v_\gamma\big(\big(\phi_{x_k}^{\lambda_0}\big)'\big)||\phi_{x_n}^{\lambda,n-k}(y)-\phi_{x_n}^{\lambda_0,n-k}(y)||^\gamma\\
&\qquad\leq \overline{\rho}_\gamma(F_\lambda,F_{\lambda_0})+||F_{\lambda_0}||_\gamma(1-\kappa)^{-\gamma}\overline{\rho}_\gamma^\gamma(F_\lambda,F_{\lambda_0})\\
&\qquad\leq \big(1+(1-\kappa)^{-\gamma}||F_{\lambda_0}||_\gamma\big)\overline{\rho}_\gamma^\gamma(F_\lambda,F_{\lambda_0}),
\end{aligned}$$
where the last inequality was written assuming that $\overline{\rho}_\gamma(F_\lambda,F_{\lambda_0})\leq 1$. Since $\log|b/a|\leq|b-a|/|b|$, we further get using (af) that
$$\log\frac{|\big(\phi_{x_k}^{\lambda}\big)'\big(\phi_{x_n}^{\lambda,n-k}(y)\big)|}{|\big(\phi_{x_k}^{\lambda_0}\big)'\big(\phi_{x_n}^{\lambda_0,n-k}(y)\big)|}\leq\underline{\kappa}^{-1}\big(1+(1-\kappa)^{-\gamma}||F_{\lambda_0}||_\gamma\big)\overline{\rho}_\gamma^\gamma(F_\lambda,F_{\lambda_0}).$$
Using the Chain Rule, we therefore get
$$\frac{1}{n}\log\frac{|\big(\phi_{x_n}^{\lambda,n}\big)'(y)|}{|\big(\phi_{x_n}^{\lambda_0,n}\big)'(y)|}=\frac{1}{n}\sum_{k=1}^{n}\log\frac{|\big(\phi_{x_k}^{\lambda}\big)'\big(\phi_{x_n}^{\lambda,n-k}(y)\big)|}{|\big(\phi_{x_k}^{\lambda_0}\big)'\big(\phi_{x_n}^{\lambda_0,n-k}(y)\big)|}\leq\underline{\kappa}^{-1}\big(1+(1-\kappa)^{-\gamma}||F_{\lambda_0}||_\gamma\big)\overline{\rho}_\gamma^\gamma(F_\lambda,F_{\lambda_0}).$$
So, the lemma follows by invoking (b'f), the uniform (decreasing $W$ if necessary) continuity of the function $\lambda\mapsto F_\lambda$ and the distortion property of $\phi_{x_n}^\lambda$ on $V$. □

Our next auxiliary result is this.

**Lemma 2.5.** *If $\Phi=\{F_\lambda\}_{\lambda\in W}$ is a transversal family of hyperbolic fiberwise conformal skew-products, then for every $\beta\in(0,q)$ and for all $x\in X$ there exists a constant $C>0$ such that for all $\tilde{z},\tilde{w}\in p_0^{-1}(x)$ with $z_1\neq w_1$, we have*
$$\int_{B(\lambda_0,\delta(x,\lambda_0))}\frac{d\lambda}{||\pi_\lambda(\tilde{w})-\pi_\lambda(\tilde{z})||^\beta}\leq C.$$

*Proof.* Applying the transversality condition (cf), we estimate as follows.

$$\begin{aligned}
\int_{B(\lambda_0,\delta(x,\lambda_0))} \frac{d\lambda}{||\pi_\lambda(\tilde w)-\pi_\lambda(\tilde z)||^\beta} = & \\
&= \int_0^\infty l_d\left(\left\{\lambda\in B(\lambda_0,\delta(x,\lambda_0)) : \frac{1}{||\pi_\lambda(\tilde w)-\pi_\lambda(\tilde z)||^\beta}\ge t\right\}\right)dt \\
&= \beta\int_0^\infty l_d\big(\{\lambda\in B(\lambda_0,\delta(x,\lambda_0)) : ||\pi_\lambda(\tilde w)-\pi_\lambda(\tilde z)||\le r\}\big)r^{-\beta-1} \\
&= \beta\int_0^{\delta(x,\lambda_0)} l_d\big(\{\lambda\in B(\lambda_0,\delta(x,\lambda_0)) : ||\pi_\lambda(\tilde w)-\pi_\lambda(\tilde z)||\le r\}\big)r^{-\beta-1}+ \\
&\quad +\beta\int_{\delta(x,\lambda_0)}^\infty l_d\big(\{\lambda\in B(\lambda_0,\delta(x,\lambda_0)) : ||\pi_\lambda(\tilde w)-\pi_\lambda(\tilde z)||\le r\}\big)r^{-\beta-1} \\
&\le C_1\beta\int_0^{\delta(x,\lambda_0)} r^{q-\beta-1}dr+\beta l_d(B(\lambda_0,\delta(x,\lambda_0)))\int_{\delta(x,\lambda_0)}^\infty r^{-\beta-1}dr \\
&\le C_1\beta(q-\beta)^{-1}(2\delta(x,\lambda_0))^{q-\beta}+\beta l_d(B(\lambda_0,\delta(x,\lambda_0)))\mathrm{diam}(V)^{-\beta}<+\infty.
\end{aligned}$$

□

**Lemma 2.6.** *Given $\varepsilon, a>0$ put $\eta=\frac{-\varepsilon\log\kappa}{2a+\varepsilon}$ and take $\delta=\delta(\eta)$ coming from Lemma 2.4 ascribed to $\eta$. Then for all $\tilde x\in\tilde X$ and all $n\ge 0$,*

$$||\lambda-\lambda_0||<\delta \ \Rightarrow\ ||(\phi_{x_n}^{\lambda_0,n})'||_\infty^{a+\frac{\varepsilon}{2}}\le||(\phi_{x_n}^{\lambda,n})'||_\infty^a.$$

*Proof.* Applying Lemma 2.4, we get

$$\begin{aligned}
||(\phi_{x_n}^{\lambda_0,n})'||_\infty^{a+\frac{\varepsilon}{2}} &\le \exp\big(\eta n\big(a+\frac{\varepsilon}{2}\big)\big)||(\phi_{x_n}^{\lambda,n})'||_\infty^{a+\frac{\varepsilon}{2}} \\
&\le \exp\big(\eta n\big(a+\frac{\varepsilon}{2}\big)\big)\kappa^{\frac{\varepsilon}{2}n}||(\phi_{x_n}^{\lambda,n})'||_\infty^a \\
&= \exp\big(-\frac{\varepsilon}{2}\log\kappa n\big)\kappa^{\frac{\varepsilon}{2}n}||(\phi_{x_n}^{\lambda,n})'||_\infty^a = ||(\phi_{x_n}^{\lambda,n})'||_\infty^a.
\end{aligned}$$

□

For every $\lambda\in W$ denote by $h_\lambda$ the Bowen's stable zero of the basic set $\Lambda_\lambda$. We now shall prove a technical fact, which will easily imply our main result.

**Lemma 2.7.** *Suppose that $\Phi=\{F_\lambda\}_{\lambda\in W}$ is a transversal family of hyperbolic fiberwise conformal skew-products. Then for all $x\in X$ we have*

(a)
$$\forall(\lambda_0\in W)\,\forall(\varepsilon>0)\,\exists(\delta>0)$$
$$\mathrm{HD}(Y_{\lambda,x})\ge\min\{h_{\lambda_0},q\}-\varepsilon$$

*for $l_d$-a.e. $\lambda\in B(\lambda_0,\delta)$ and*

(b) *If $h_{\lambda_0}>q$, then there exists $\delta>0$ such that*
$$l_q(Y_{\lambda,x})>0$$
*for $l_d$-a.e. $\lambda\in B(\lambda_0,\delta)$.*

*Proof.* Put $h=\min\{h_{\lambda_0},q\}$. Since the potential $h_{\lambda_0}\zeta_{F_{\lambda_0}}$ is Hölder continuous, there exists a unique equilibrium (Gibbs) state $\mu$ for this potential and the dynamical system $\tilde f:\tilde X\to\tilde X$. Since $f:X\to X$ is a distance expanding map, for every $r>0$ sufficiently small, say $r\in(0,R]$, every $z\in X$ and every $n\ge 0$ there exists a unique continuous inverse branch $f_z^{-n}:B(f^n(z),r)\to X$ of $f^n$ sending $f^n(z)$ to $z$. We now want to look at the Gibbs measure $\mu$ in greater detail. A straightforward adaptation of the proof of Lemma 1.6, p.11 in [1] results in the existence of a Hölder continuous function $\zeta_+$ that is cohomologous to $h_{\lambda_0}\zeta_{F_{\lambda_0}}$ and depends only on the 0th coordinate, in particular $\zeta_+$ can be regarded as a Hölder continuous function defined on $X$. Then $\mu = \tilde{\mu}_+$, where $\mu_+$ is the Gibbs (equilibrium) state for the potential $\zeta_+ : \tilde{X} \to \mathbb{R}$. Also $\mu_\circ p_n^{-1} = \mu_+$ for all $n \geq 0$, and $\mathrm{P}(\zeta_+) = \mathrm{P}\big(h_{\lambda_0}\zeta_{F_{\lambda_0}}\big) = 0$. Let $\mathcal{L}_+ : C(X) \to C(X)$ be the Perron-Frobenius operator determined by the potential $\zeta_+ : X \to \mathbb{R}$. It is then well-known (see [7], Ch. 4 for ex.) that there exists $m_+$, a Borel probability measure on $X$ being a fixed point of the dual operator $\mathcal{L}_+^* : C^*(X) \to C^*(X)$. This means that

$$m_+(f(A)) = \int_A e^{-\zeta_+} dm_+$$

whenever $A$ is a Borel subset of $X$ such that $f|_A : A \to f(A)$ is one-to-one. In particular, for every $x \in X$, every $r \in (0, R]$ and every Borel set $A \subset B(f^n(x), r)$

$$m_+\big(f_x^{-n}(A)\big) = \int_A \exp\big(S_n\zeta_+ \circ f_x^{-n}\big) dm_+ \asymp \exp\big(S_n\zeta_+(x)\big) m_+(A), \tag{2.2}$$

where we say that two positive quantities $A_n, B_n$ are **comparable**, written $A_n \asymp B_n$ if there exists a positive constant $C$ (called a *comparability constant*) such that $C^{-1} \leq \frac{A_n}{B_n} \leq C$; in our case the comparability constant is independent of $r$, $x$ and $n$.

Since (see [7], Ch.4) the Radon-Nikodym derivative $\frac{d\mu_+}{dm_+}$ is a continuous function bounded away from zero and infinity, we get, using (2.2) and cohomology of $\zeta_+$ and $h_{\lambda_0}\zeta_{F_{\lambda_0}}$, for every $r \in (0, R]$, every $z \in X$ and all $n \geq 0$ that

$$\begin{aligned}
\mu\big(p_n^{-1} \circ f_z^{-n}(B(f^n(z), r))\big) &= \tilde{\mu}_+\big(p_n^{-1} \circ f_z^{-n}(B(f^n(z), r))\big) = \mu_+\big(f_z^{-n}(B(f^n(z), r))\big) \\
&\asymp m_+\big(f_z^{-n}(B(f^n(z), r))\big) \\
&\asymp \exp\big(S_n\zeta_+(x)\big) m_+(B(f^n(z), r)) \\
&\asymp \exp\big(h_{\lambda_0} S_n\zeta_{F_{\lambda_0}}(\tilde{z})\big) \mu_+(B(f^n(z), r)) \\
&= \left|\big(\phi_z^{\lambda_0, n}\big)'(\pi_{\lambda_0}(\tilde{z}))\right|^{h_{\lambda_0}} \tilde{\mu}_+ \circ p_0^{-1}(B(f^n(z), r)) \\
&\asymp \big|\big|(\phi_z^{\lambda_0, n})'\big|\big|^{h_{\lambda_0}} \mu\big(p_0^{-1}(B(f^n(z), r))\big),
\end{aligned} \tag{2.3}$$

where $\tilde{z}$ was an arbitrary auxiliary point in $p_0^{-1}(z)$ and all the comparability constants appearing in this calculation are independent of $r$, $z$ and $n$. Now, fix $x \in X$, $r \in (0, R]$, $n \geq 0$ and $\xi \in f^{-n}(x)$. Put

$$\mu_{x,n}(\xi) = \overline{\lim_{r \to 0}} \frac{\mu\big(p_n^{-1}\big(f_\xi^{-n}(B(x, r))\big)}{\mu\big(p_0^{-1}B(x, r))\big)}. \tag{2.4}$$

This formula defines a probability measure on the finite set $f^{-n}(x)$. Since for all $n \geq 1$ and all $z \in f^{-(n-1)}(x)$,

$$\begin{aligned}
\mu_{x,n} \circ f^{-1}(z) &= \sum_{w \in f^{-1}(z)} \mu_{x,n}(w) = \sum_{w \in f^{-1}(z)} \overline{\lim_{r \to 0}} \frac{\mu\big(p_n^{-1}\big(f_w^{-n}(B(x, r))\big)}{\mu\big(p_0^{-1}(x, r))\big)} \\
&= \overline{\lim_{r \to 0}} \big(\mu\big(p_0^{-1}(x, r)\big)\big)\big)^{-1} \sum_{w \in f^{-1}(z)} \mu\big(p_n^{-1}\big(f_w^{-n}(B(x, r))\big) \\
&= \overline{\lim_{r \to 0}} \big(\mu\big(p_0^{-1}(x, r)\big)\big)\big)^{-1} \mu\left(\bigcup_{w \in f^{-1}(z)} p_n^{-1}\big(f_w^{-n}(B(x, r))\big)\right) \\
&= \overline{\lim_{r \to 0}} \frac{\mu\big(p_{n-1}^{-1}\big(f_z^{-(n-1)}(B(x, r))\big)}{\mu\big(p_0^{-1}(x, r))\big)} \\
&= \mu_{x,n-1}(z),
\end{aligned}$$

the sequence $\big(\mu_{x,n}\big)_1^\infty$ is consistent with respect to the sequence of maps $\big(f : f^{-n}(x) \to f^{-(n-1)}(x)\big)_1^\infty$ in the sense of Definition 3.6.3 from [4]. It therefore follows from Daniel-Kolmogorov Consistency Theorem

(Proposition 3.6.4 in [4]) that there exists a measure $\mu_x$ on $p_0^{-1}(x)$ such that $\mu_x \circ p_n^{-1} = \mu_{x,n}$ for all $n \geq 0$. Hence it follows from (2.3) and (2.4) that for all $x \in X$, all $r > 0$, all $n \geq 0$ and all $\xi \in f^{-n}(x)$, we have

$$\mu_x(p_n^{-1}(\xi)) = \overline{\lim_{r\to 0}} \frac{\mu\big(p_n^{-1}\big(f_\xi^{-n}(B(x,r))\big)\big)}{\mu\big(p_0^{-1}B(x,r)\big)\big)} \asymp ||(\phi_\xi^{\lambda_0,n})'||^{h_{\lambda_0}} \tag{2.5}$$

and the universal comparability constant is independent of $r$, $x$, $n$ and $\xi$.

Given $\varepsilon > 0$, let $0 < \delta = \min\{\delta(\eta), \delta(x, \lambda_0)\}$, where $\eta = \frac{-\varepsilon \log \kappa}{2h-\varepsilon}$ comes from Lemma 2.6 with $a = h - \varepsilon$. By the potential-theoretic characterization of Hausdorff dimension (see [2]), it suffices to prove that

$$\begin{aligned} R_x(\lambda) &= \iint_{\overline{V}\times\overline{V}} \frac{d(\mu_x \circ \pi_\lambda^{-1} \times \mu_x \circ \pi_\lambda^{-1})(w,z)}{||w-z||^{h-\varepsilon}} \\ &= \iint_{p_0^{-1}(x)\times p_0^{-1}(x)} \frac{d\mu_2(\tilde{w},\tilde{z})}{||\pi_\lambda(\tilde{w}) - \pi_\lambda(\tilde{z})||^{h-\varepsilon}} < +\infty, \end{aligned} \tag{2.6}$$

where $\mu_2 = \mu_x \times \mu_x$ is the product measure on $p_0^{-1}(x) \times p_0^{-1}(x)$. And in turn, in order to prove (2.6), it is enough to show that

$$\int_{B(\lambda_0,\delta)} R_x(\lambda) d\lambda < +\infty.$$

For every $n \geq 1$ and every $\xi \in f^{-n}(x)$, let

$$A_\xi = \{(\tilde{w},\tilde{z}) \in p_0^{-1}(x) \times p_0^{-1}(x) : w_n = z_n = \xi \text{ and } w_{n+1} \neq z_{n+1}\}.$$

By the Mean Value Inequality, we get for all $(\tilde{w},\tilde{z}) \in A_\xi$ that

$$\begin{aligned} ||\pi_\lambda(\tilde{f}^{-n}(\tilde{w})) - \pi_\lambda(\tilde{f}^{-n}(\tilde{z}))|| &= ||(\phi_\xi^{\lambda,n})^{-1}\big(\pi_\lambda(\tilde{w})\big) - (\phi_\xi^{\lambda,n})^{-1}\big(\pi_\lambda(\tilde{z})\big)|| \\ &\leq ||(\phi_\xi^{\lambda,n})'||^{-1} ||\pi_\lambda(\tilde{w}) - \pi_\lambda(\tilde{z})||. \end{aligned} \tag{2.7}$$

By Lemma 2.6, we have

$$||(\phi_\xi^{\lambda,n})'||^{h-\varepsilon} \geq ||(\phi_\xi^{\lambda_0,n})'||^{h-\frac{\varepsilon}{2}} \geq ||(\phi_\xi^{\lambda_0,n})'||^{h_{\lambda_0}-\frac{\varepsilon}{2}}. \tag{2.8}$$

Hence, changing the order of integration, using (2.7), (2.8) and Lemma 2.5 ($(\tilde{f}^{-n}(\tilde{w}))_0 = \xi = (\tilde{f}^{-n}(\tilde{z}))_0$, $(\tilde{f}^{-n}(\tilde{w}))_1 = w_{n+1} \neq z_{n+1} = (\tilde{f}^{-n}(\tilde{z}))_1$), we get

$$\begin{aligned} &\int_{B(\lambda_0,\delta)} R_x(\lambda) d\lambda = \\ &= \iint_{p_0^{-1}(x)\times p_0^{-1}(x)} \int_{B(\lambda_0,\delta)} \frac{d\lambda}{||\pi_\lambda(\tilde{w}) - \pi_\lambda(\tilde{z})||^{h-\varepsilon}} d\mu_2(\tilde{w},\tilde{z}) \\ &= \sum_{n=0}^{\infty} \sum_{\xi\in f^{-n}(x)} \iint_{A_\xi} \int_{B(\lambda_0,\delta)} \frac{d\lambda}{||\pi_\lambda(\tilde{w}) - \pi_\lambda(\tilde{z})||^{h-\varepsilon}} d\mu_2(\tilde{w},\tilde{z}) \\ &\leq \sum_{n=0}^{\infty} \sum_{\xi\in f^{-n}(x)} \iint_{A_\xi} \int_{B(\lambda_0,\delta)} ||(\phi_\xi^{\lambda,n})'||^{\varepsilon-h} \frac{d\lambda}{||\pi_\lambda(\tilde{f}^{-n}(\tilde{w})) - \pi_\lambda(\tilde{f}^{-n}(\tilde{z}))||^{h-\varepsilon}} d\mu_2(\tilde{w},\tilde{z}) \\ &\leq \sum_{n=0}^{\infty} \sum_{\xi\in f^{-n}(x)} \iint_{A_\xi} ||(\phi_\xi^{\lambda_0,n})'||^{\frac{\varepsilon}{2}-h_{\lambda_0}} \int_{B(\lambda_0,\delta)} \frac{d\lambda}{||\pi_\lambda(\tilde{f}^{-n}(\tilde{w})) - \pi_\lambda(\tilde{f}^{-n}(\tilde{z}))||^{h-\varepsilon}} d\mu_2(\tilde{w},\tilde{z}) \\ &\leq C \sum_{n=0}^{\infty} \sum_{\xi\in f^{-n}(x)} \iint_{A_\xi} ||(\phi_\xi^{\lambda_0,n})'||^{\frac{\varepsilon}{2}-h_{\lambda_0}} d\mu_2(\tilde{w},\tilde{z}). \end{aligned} \tag{2.9}$$

Now, using (2.5), we can continue (2.9) as follows ($A_\xi \subset p_n^{-1}(\xi)$).

$$\begin{aligned}
\int_{B(\lambda_0,\delta)} R_x(\lambda)d\lambda &\preceq \sum_{n=0}^{\infty} \sum_{\xi\in f^{-n}(x)} \iint_{A_\xi} ||(\phi_\xi^{\lambda_0,n})'||^{\frac{\varepsilon}{2}} \mu_x^{-1}(p_n^{-1}(\xi))d\mu_2 \\
&= \sum_{n=0}^{\infty} \kappa^{\frac{n\varepsilon}{2}} \sum_{\xi\in f^{-n}(x)} \mu_x^{-1}(p_n^{-1}(\xi))\mu_2(A_\xi) \\
&\leq \sum_{n=0}^{\infty} \kappa^{\frac{n\varepsilon}{2}} \mu_x(p_0^{-1}(x)) \\
&= \sum_{n=0}^{\infty} \kappa^{\frac{n\varepsilon}{2}} < +\infty,
\end{aligned}$$

and we are done with part (a).

(b) Put $\eta = \frac{-\varepsilon \log \kappa}{2h_{\lambda_0}+\varepsilon}$ and determine $\delta = \delta(\eta)$ by Lemma 2.6 with $a = 1$ and $\varepsilon$ replaced by $\varepsilon/h_{\lambda_0}$. We use the same setup and notation as in the proof of part (a); in particular $\mu$ denotes the same Gibbs state. For every $\lambda \in B(\lambda_0, \delta)$, let

$$\nu_\lambda = \mu_x \circ \pi_\lambda^{-1}.$$

It suffices to show that $\nu_\lambda << l_q$. We shall prove that

$$R = \int_{B(\lambda_0,\delta)} \int_{\mathbb{R}} \underline{D}(\nu_\lambda, z)d\nu_\lambda(z)d\lambda = \int_{B(\lambda_0,\delta)} \int_{\overline{V}} \underline{D}(\nu_\lambda, z)d\nu_\lambda(z)d\lambda < \infty,$$

where

$$\underline{D}(\nu_\lambda, z) = \liminf_{r\searrow 0} \frac{\nu_\lambda(B(z, r))}{r^q}.$$

Having this, we will have $\underline{D}(\nu_\lambda, z) < +\infty$ for $\nu_\lambda$-a.e. $z \in \overline{V}$ and Theorem 2.12 in [2] will imply that $\nu_\lambda$ is absolutely continuous with respect to $l_q$. So, starting the proof that $R < \infty$, we apply Fatou's lemma to get

$$R \leq \liminf_{r\searrow 0} \int_{B(\lambda_0,\delta)} \int_{\overline{V}} \frac{\nu_\lambda(B(z, r))}{r^q} d\nu_\lambda(z)d\lambda. \tag{2.10}$$

Now, use the definition of $\nu_\lambda$ to change the variable, write $\nu_\lambda(B(z, r))$ as an integral of the characteristic function, and change the variable once again to obtain

$$\begin{aligned}
\int_{\overline{V}} \nu_\lambda(B(z, r))d\nu_\lambda(z) &= \int_{p_0^{-1}(x)} \mu_x \circ \pi_\lambda^{-1}\big(B(\pi_\lambda(\tilde{z}), r)\ \big)d\mu_x \circ \pi_\lambda^{-1}(\tilde{z}) \\
&= \iint_{p_0^{-1}(x)\times p_0^{-1}(x)} 1\!\!1_{\pi_\lambda^{-1}\left(B(\pi_\lambda(\tilde{z}),r)\right)}(\tilde{w})d\mu_x \circ \pi_\lambda^{-1}(\tilde{w})d\mu_x \circ \pi_\lambda^{-1}(\tilde{z}) \\
&= \iint_{p_0^{-1}(x)\times p_0^{-1}(x)} 1\!\!1_{\{\tilde{w}\in\tilde{X}:||\pi_\lambda(\tilde{w})-\pi_\lambda(\tilde{z})||<r\}} d\mu_2(\tilde{w}, \tilde{z}).
\end{aligned}$$

Inserting this to (2.10) and changing the order of integration, gives

$$\begin{aligned}
R &\leq \liminf_{r\searrow 0} r^{-q} \iint_{p_0^{-1}(x)\times p_0^{-1}(x)} l_d\big(\{\lambda \in B(\lambda_0, \delta) : ||\pi_\lambda(\tilde{w}) - \pi_\lambda(\tilde{z})|| < r\}\big)d\mu_2(\tilde{w}, \tilde{z}) \\
&= \liminf_{r\searrow 0} r^{-q} \sum_{n=0}^{\infty} \sum_{\xi\in f^{-n}(x)} \iint_{A_\xi} l_d\big(\{\lambda \in B(\lambda_0, \delta) : ||\pi_\lambda(\tilde{w}) - \pi_\lambda(\tilde{z})|| < r\}\big)d\mu_2(\tilde{w}, \tilde{z}).
\end{aligned}$$

By (2.7), Lemma 2.6 with $a=1$ and $\varepsilon$ replaced by $\varepsilon/h_{\lambda_0}$, and (cf), we get for all $(\tilde{w},\tilde{z})\in A_\xi$ that

$$
\begin{aligned}
l_d\big(\{\lambda\in B(\lambda_0,\delta): ||\pi_\lambda(\tilde{w})-\pi_\lambda(\tilde{z})||<r\}\big) &\le \\
&\le l_d\big(\{\lambda\in B(\lambda_0,\delta): ||\pi_\lambda(\tilde{f}^{-n}(\tilde{w}))-\pi_\lambda(\tilde{f}^{-n}(\tilde{z}))||<r||(\phi_\xi^{\lambda,n})'||^{-1}\}\big)\\
&\le l_d\big(\{\lambda\in B(\lambda_0,\delta): ||\pi_\lambda(\tilde{f}^{-n}(\tilde{w}))-\pi_\lambda(\tilde{f}^{-n}(\tilde{z}))||<r||(\phi_\xi^{\lambda_0,n})'||^{-\left(1+\frac{\varepsilon}{2h_{\lambda_0}}\right)}\}\big)\\
&\le C_1 r^q||(\phi_\xi^{\lambda_0,n})'||^{-q\left(1+\frac{\varepsilon}{2h_{\lambda_0}}\right)}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
R &\preceq \sum_{n=0}^{\infty}\sum_{\xi\in f^{-n}(x)}\iint_{A_\xi}||(\phi_\xi^{\lambda_0,n})'||^{-q\left(1+\frac{\varepsilon}{2h_{\lambda_0}}\right)}d\mu_2\\
&\le \sum_{n=0}^{\infty}\sum_{\xi\in f^{-n}(x)}||(\phi_\xi^{\lambda_0,n})'||^{-q\left(1+\frac{\varepsilon}{2h_{\lambda_0}}\right)}\mu_2(A_\xi)\\
&\le \sum_{n=0}^{\infty}\sum_{\xi\in f^{-n}(x)}||(\phi_\xi^{\lambda_0,n})'||^{-q\left(1+\frac{\varepsilon}{2h_{\lambda_0}}\right)}\mu_x^2(p_n^{-1}(\xi)).
\end{aligned}
$$

But it follows from (2.5) that

$$
\begin{aligned}
||(\phi_\xi^{\lambda_0,n})'||^{-q\left(1+\frac{\varepsilon}{2h_{\lambda_0}}\right)} &\le ||(\phi_\xi^{\lambda_0,n})'||^{-(h_{\lambda_0}-\varepsilon)\left(1+\frac{\varepsilon}{2h_{\lambda_0}}\right)}\\
&= ||(\phi_\xi^{\lambda_0,n})'||^{-h_{\lambda_0}}||(\phi_\xi^{\lambda_0,n})'||^{\frac{\varepsilon}{2}+\frac{\varepsilon^2}{2h_{\lambda_0}}}\\
&\le \mu_x^{-1}(p_n^{-1}(\xi))||(\phi_\xi^{\lambda_0,n})'||^{\frac{\varepsilon}{2}}\\
&\le \kappa^{\frac{\varepsilon n}{2}}\mu_x^{-1}(p_n^{-1}(\xi)).
\end{aligned}
$$

Hence,

$$
R\preceq\sum_{n=0}^{\infty}\sum_{\xi\in f^{-n}(x)}\kappa^{\frac{\varepsilon n}{2}}\mu_x(p_n^{-1}(\xi))=\sum_{n=0}^{\infty}\kappa^{\frac{\varepsilon n}{2}}\mu_x(p_0^{-1}(x))=\sum_{n=0}^{\infty}\kappa^{\frac{\varepsilon n}{2}}<+\infty.
$$

We are done. □

We are now in a position to provide the proof of the following main result of this section.

**Theorem 2.8.** *Suppose that $\Phi=\{F_\lambda\}_{\lambda\in W}$ is a transversal family of hyperbolic fiberwise conformal skew-product endomorphisms. Then the function $\lambda\mapsto h_\lambda$ is continuous on $W$ and for all $x\in X$ there exists a Borel set $W_x\subset W$ such that $l_d(W\setminus W_x)=0$ and*

(a)
$$
\mathrm{HD}(Y_{\lambda,x})=\min\{h_\lambda,q\}\ \text{for all}\ \lambda\in W_x.
$$

(b)
$$
l_d\big(\{\lambda\in W: h_\lambda>q \text{ and } l_d(Y_{\lambda,x})>0\}\big)=l_d\big(\{\lambda\in W: h_\lambda>q\}\big).
$$

*Proof.* First of all we recall that $\tilde{f}:\tilde{X}\to\tilde{X}$ is an expanding homeomorphism.

Continuity of the function $\lambda\mapsto h_\lambda$ is an immediate consequence of the thermodynamic formalism for expanding maps ($\tilde{f}:\tilde{X}\to\tilde{X}$) and condition (bf). Inequality $\mathrm{HD}(Y_{\lambda,x})\le\min\{h_\lambda,q\}$ is known for all hyperbolic fiberwise conformal endomorphisms. Proving (a) suppose for the contrary that for some $x\in X$, $l_d(Z)>0$, where $Z=\{\lambda\in W:\mathrm{HD}(Y_{\lambda,x})<\min\{h_\lambda,q\}\}$. Then there is $\varepsilon>0$ such that $l_d(Z_\varepsilon)>0$, where $Z_\varepsilon=\{\lambda\in W:\mathrm{HD}(Y_{\lambda,x})<\min\{h_\lambda,q\}-2\varepsilon\}$. Let $\lambda_0$ be a Lebesgue density point of $Z_\varepsilon$. So, there exists $\delta_0>0$ such that for each $\delta\in(0,\delta_0]$,

$$
l_d(Z_\varepsilon\cap B(\lambda_0,\delta))>0. \tag{2.11}
$$

By the continuity of the function $\lambda \mapsto \min\{h_\lambda, q\}$ there exists $\delta_1 \in (0, \delta_0)$ such that $\min\{h_\lambda, q\} < \min\{h_{\lambda_0}, q\} + \varepsilon$ for all $\lambda \in B(\lambda_0, \delta_1)$. Combining this with (2.11), we conclude that

$$l_d\big(\{\lambda \in B(\lambda_0, \delta) : \mathrm{HD}(Y_{\lambda,x}) < \min\{h_{\lambda_0}, q\} - \varepsilon\}\big) > 0$$

for all $\delta \le \delta_1$. This directly contradicts item (a) of Lemma 2.7, and the proof of item (a) of our present theorem is complete. To finish the proof, that is to demonstrate item (b), note that it directly follows from item (b) of Lemma 2.7. We are done. □

An interesting question arises of when we can find a universal set $W'$ of full measure in $W$ such that item (a) holds for all $x \in X$ and all $\lambda \in W'$. We provide below two sufficient conditions.

**Corollary 2.9.** *Suppose that $\Phi = \{F_\lambda\}_{\lambda \in W}$ is a transversal family of hyperbolic fiberwise conformal skew-products and the function $x \mapsto \mathrm{HD}(Y_{\lambda,x})$, $x \in X$, is upper semi-continuous, for all $\lambda \in W$. Then the function $\lambda \mapsto h_\lambda$ is continuous on $W$ and there exists a measurable set $W' \subset W$ such that $l_d(W \setminus W') = 0$ and*

$$\mathrm{HD}(Y_{\lambda,x}) = \min\{h_\lambda, q\}$$

*for all $\lambda \in W'$ and all $x \in X$.*

*Proof.* Suppose on the contrary that there exists a measurable set $W_+$ such that $l_d(W_+) > 0$ and for every $\lambda \in W_+$ there exists $x_\lambda \in X$ such that $\mathrm{HD}(Y_{\lambda,x}) < \min\{h_\lambda, q\}$. Fix $\mathcal{B}$, a countable base of topology on $X$. Since the function $x \mapsto \mathrm{HD}(Y_{\lambda,x})$, $x \in X$, is upper semi-continuous, for every $\lambda \in W_+$ there exists a set $B_\lambda \in \mathcal{B}$ such that $\mathrm{HD}(Y_{\lambda,x}) < \min\{h_\lambda, q\}$ for all $x \in B_\lambda$. For every $B \in \mathcal{B}$, let $W_+(B) = \{\lambda \in W_+ : B = B_\lambda\}$. Since the family $\mathcal{B}$ is countable and $l_d(W_+) > 0$, either there exists $B \in \mathcal{B}$ such that $l_d(W_+(B)) > 0$ or $W_+(B)$ is not measurable. Thus, in any case, there exists $B \in \mathcal{B}$ and a measurable set $U \subset W_+(B)$ such that $l_d(U) > 0$. Fix $z \in B$. Then $\mathrm{HD}(Y_{\lambda,z}) < \min\{h_\lambda, q\}$ for all $\lambda \in U$ contrary to Theorem 2.8(a). We are done. □

Another way to guarantee the existence of a universal set $W'$ as in the corollary above, is to strenghten the transversality condition (cf) as follows.

(c'f) (Uniform Transversality Condition) There exists $C_2 > 0$ such that for all $x \in X$, $\forall \tilde{x}, \tilde{y} \in p_0^{-1}(x), x_1 \neq y_1$, and $\forall r > 0$, we have

$$l_d\big(\lambda \in W : ||\pi_\lambda(\tilde{x}) - \pi_\lambda(\tilde{y})|| \le r\big) \le C_2 r^q.$$

All that has to be done then, is to replace $R_x(\lambda)$ in formula (2.6) by $\sup_{x \in X} R_x(\lambda)$. We thus get the following.

**Theorem 2.10.** *Suppose that $\Phi = \{F_\lambda\}_{\lambda \in W}$ is a uniformly transversal family of hyperbolic fiberwise conformal skew-products. Then the function $\lambda \mapsto h_\lambda$ is continuous on $W$ and there exists a measurable set $W' \subset W$ such that $l_d(W \setminus W') = 0$ and*

$$\mathrm{HD}(Y_{\lambda,x}) = \min\{h_\lambda, q\}$$

*for all $\lambda \in W'$ and all $x \in X$.*

3. **Examples.** We shall now describe examples of transversal families of hyperbolic fiberwise conformal skew products.

First we will give a class of examples which generalize skew products obtained from iterated function systems, possibly with overlaps (for such iterated systems see for example [11]). For this class we will obtain a corollary which gives a good estimate for the Hausdorff dimension of the fiber. Here the dependence on parameters is not necessarily polynomial. These are parametrized families of maps of the form $F_\lambda(x, y) =$

$(f(x), \lambda_j + \Phi_j(x, y, \lambda))$ if $x \in X_j, j = 1, \ldots, d$, where $\lambda = (\lambda_1, \ldots, \lambda_d)$, $f : I_1 \cup \ldots \cup I_d \to I$ is expanding, with $f(I_j) = I$ and $X_j = I_j \cap \{x \in I_1 \cup \ldots \cup I_d, f^k(x) \in I_1 \cup \ldots \cup I_d, \forall k \geq 0\}$, for $j = 1, \ldots, d$.

Then, we focus on examples of complex skew products with polynomial-like behavior ([3]), for which the transversality condition can actually be verified. This will be done by estimating the coordinates of points from $X$ and then by obtaining a general formula for the projection $\pi_\lambda(\tilde{z}), \tilde{z} \in \tilde{X}$. For the base function $f$ we take the simple polynomial $z \to z^2 + c$ with $|c|$ small, which is expanding on its Julia set $J_c$ ($J_c$ is close to the unit circle $S^1$ for small $|c|$); then the family will be of the form $\{F_\lambda\}_\lambda, F_\lambda(z, w) := (f(z), g_\lambda(z, w))$. These skew products are important examples of endomorphisms of $\mathbb{C}^2$ (hence not necessarily invertible). One of these examples is linear in $w$ (in the second coordinate) and another example is a perturbation of a map which is quadratic in $w$ (in the second coordinate); we give also an example containing the term $zw^2$ in the second coordinate.

We begin with the following elementary auxiliary facts.

**Lemma 3.1.** *For all $\eta > 0$, $\theta > 0$ and $l > 0$ there exists a constant $C(\eta, \theta, l) \geq 1$ with the following property. If $g : \Delta \to \mathbb{R}$ is a $C^1$-differentiable function such that*

(a) *$\Delta$ is a closed segment of $\mathbb{R}$ with $|\Delta| \leq l$,*
(b) *$|g'(x)| \leq \theta$ for all $x \in \Delta$,*
(c) *if $x \in \Delta$ and $|g(x)| \leq \eta$, then $|g'(x)| \geq \eta$,*

*then for every $r > 0$,*

$$l_1\big(\{x \in \Delta : |g(x)| \leq r\}\big) \leq C(\eta, \theta, l) r.$$

*Proof.* We may assume without loss of generality that $r < \min\{\eta, l\}/2$. It follows from condition (c) that the set $g^{-1}(0)$ is finite. Let $a < b$ be a closest pair of points in this set. Assume without loss of generality that $g'(a) \geq \eta$. Since $g(a) = g(b) = 0$, using the continuity of the function $g'$, we deduce from (c) that there exists a point $w \in (a, b)$ such that $g(w) = \eta$. Fix a minimal $w$ with this property. it then follows from the Mean Value Theorem that $\eta = g(w) - g(a) \leq \theta|w - a| \leq \theta|b - a|$. Hence $|b - a| \geq \eta/\theta$, and therefore

$$\#g^{-1}(0) \leq \theta l/\eta. \tag{3.1}$$

Suppose now that $z \in \Delta$ and $|g(z)| \leq r$. Assume without loss of generality that $0 \leq g(z) \leq r$. Let $a \leq \xi \leq z$ be the largest number such that $g(\xi) = 0$ if such a number exists, or else, let $\xi = a$. In either case $0 \leq g(t) \leq r < \eta$ and $g'(t) \geq \eta$ for all $t \in [\xi, z]$. By the Mean Value Theorem there exists $u \in [\xi, z]$ such that $r \geq g(z) - g(\xi) = g'(u)(z - \xi) \geq \eta(z - \xi)$. Thus $z \in (\xi - \frac{r}{\eta}, \xi + \frac{r}{\eta})$ and therefore $g^{-1}([-r, r]) \subset B(\partial\Delta \cup g^{-1}(0), r/\eta)$. So we conclude that $l_1(g^{-1}([-r, r])) \leq 2\eta^{-1}(2 + \theta l \eta^{-1})r$. $\square$

As a straightforward consequence of this lemma, we get the following.

**Lemma 3.2.** *Let $U \subset \mathbb{R}^d$ be a compact convex set with* $\operatorname{diam}(U) \leq l$. *Suppose that $g : U \to \mathbb{R}$ is a $C^1$-differentiable function with the following properties.*

(a) *There exists $1 \leq i \leq d$ such that $\left|\frac{\partial g}{\partial x_i}(x)\right| \leq \theta$ for all $x \in U$.*
(b) *If $x \in U$ and $|g(x)| \leq \eta$, then $\left|\frac{\partial g}{\partial x_i}(x)\right| \geq \eta$.*

*Then for every $r > 0$,*

$$l_d\big(\{x \in U : |g(x)| \leq r\}\big) \leq (2l)^{d-1} C(\eta, \theta, l) r.$$

*Proof.* Assume without loss of generality that $i = d$. For every $x \in \mathbb{R}^{d-1}$ let $\Delta_x = \{t \in \mathbb{R} : (x, t) \in U\}$. Since $U$ is a convex compact set with $\operatorname{diam}(U) \leq l$ it follows that $\operatorname{diam}(\hat{U}) \leq l$, where $\hat{U} = \{x \in \mathbb{R}^{d-1} :$

$\Delta_x \neq \emptyset\}$. Applying Fubini's Theorem and Lemma 3.1, we then get that

$$\begin{aligned} l_d\big(\{x \in U : |g(x)| \le r\}\big) &= \int_U 1\!\!1_{g^{-1}([-r,r])}(z)dl_d(z) = \int_{\hat{U}} \int_{\Delta_x} 1\!\!1_{g^{-1}([-r,r])}(x,t)dtdl_{d-1}(x) \\ &= \int_{\hat{U}} l_1\big(\{t \in \Delta_x : |g(x,t)| \le r\}\big)dl_{d-1}(x) \le C(\eta,\theta,l)l_{d-1}(\hat{U})r \\ &\le (2\mathrm{diam}(\hat{U}))^{d-1}C(\eta,\theta,l)r \le (2l)^{d-1}C(\eta,\theta,l)r. \end{aligned}$$

We are done. $\square$

Passing to the actual examples, let $f : X \to X$ be a topologically exact open distance expanding map for which there exist closed mutually disjoint sets $X_1, X_2, \dots, X_d$ such that $X = \cup_{i=1}^d X_i$, $f(X_i) = X$ for all $i = 1, 2, \dots, d$ and $f|_{X_i}$ is injective for all $i = 1, 2, \dots, d$. The model that we have in mind here is that of an expanding map $f : I_1 \cup \dots \cup I_d \to [0,1]$ where $I_1, \dots, I_d$ are closed mutually disjoint subintervals of $[0,1]$, $f(I_j) = [0,1], \forall j$, and $f|_{I_j}$ is injective. Then we will take as the compact space $X$, the set $I_* = \{x \in I_1 \cup \dots \cup I_d, f^m(x) \in I_1 \cup \dots \cup I_d, \forall m \ge 0\}$. So, in this case, $X_i = I_* \cap I_i, i = 1, \dots, d$.

Returning to the general case of the dynamical system $f : X \to X$ as above, consider $\lambda = (\lambda_1, \dots, \lambda_d) \in B_d(0,\eta) \subset \mathbb{R}^d$, for some small enough $\eta > 0$, and fix Lipschitz continuous functions $\phi_1, \dots, \phi_d : X \times [0,1] \times B_d(0,\eta) \to (0,1)$. So $\phi_1, \dots, \phi_d$ are functions of $(x,y,\lambda) \in X^* := X \times [0,1] \times B_d(0,\eta)$. Let us assume also that $\phi_1(x,\cdot,\cdot), \dots, \phi_d(x,\cdot,\cdot)$ are $C^2$ differentiable functions of $(y,\lambda)$, with derivatives in $(y,\lambda)$ depending Lipschitz continuously on $(x,y,\lambda)$, and that there exist constants $\alpha, \alpha' > 0$ with $0 < \alpha' < |\frac{\partial}{\partial y}\phi_i| < \frac{1}{4}$ on $X^*$, for all $i = 1, \dots, d$ and $|\frac{\partial}{\partial \lambda_j}\phi_i| < \alpha$ on $X^*$, for all $i, j = 1, \dots, d$. If $\phi_i \le \beta$ on $X^*$, for $i = 1, \dots, d$, then we assume also that $\eta + \beta < 1$. We define now the parametrized maps $F_\lambda : X \times [0,1] \to X \times (0,1)$ by the formula

$$F_\lambda(x,y) = (f(x), \lambda_i + \phi_i(x,y,\lambda)),$$

if $x \in X_i, i = 1, \dots, d$. Due to the conditions that we imposed on the functions $\phi_1, \dots, \phi_d$, one can see that $F_\lambda$ is well defined and it is a hyperbolic fiberwise conformal skew product endomorphism. In this case, $\phi_x^\lambda(y) = \lambda_i + \phi_i(x,y,\lambda)$, for $x \in X_i, i = 1, \dots, d$. We see that $0 < \alpha' < |(\phi_x^\lambda)'| < \frac{1}{4}, x \in X, \lambda \in B_d(0,\eta)$, so condition (af) from the definition of a transversal family is satisfied automatically. For this family, the set of parameters is $W = B_d(0,\eta) \subset \mathbb{R}^d$.

**Theorem 3.3.** *The family $\{F_\lambda\}_{\lambda \in B_d(0,\eta)}$ is uniformly transversal, and therefore, the assertions of Theorem 2.10 hold.*

*Proof.* For every $w \in X$ let $i(w) \in \{1, \dots, d\}$ be uniquely determined by the property that $w \in X_{i(w)}$. Fix $1 \le k \le d$ and a prehistory $\tilde{w} \in \tilde{X}$. We have that

$$\pi_\lambda(\tilde{w}) = \lim_n \big(\phi_{w_1}^\lambda \circ \dots \circ \phi_{w_n}^\lambda\big)(\zeta) = \phi_{w_1}^\lambda \circ \dots \circ \phi_{w_n}^\lambda(\pi_\lambda(\tilde{w}_n)),$$

where $\tilde{w}_n = (w_n, w_{n+1}, \dots)$. Notice also that the limit above is uniform in $\zeta$. So,

$$\frac{\partial}{\partial \lambda_j}(\phi_{w_1}^\lambda \circ \phi_{w_2}^\lambda(\zeta)) = \frac{\partial}{\partial \lambda_j}(\lambda_{i(w_1)} + \phi_{i(w_1)}(w_1, \lambda_{i(w_2)} + \phi_{i(w_2)}(w_2, \zeta, \lambda), \lambda))$$

For the derivative of $\phi_{w_1}^\lambda \circ \dots \circ \phi_{w_n}^\lambda$ with respect to $\lambda_j$ we obtain a similar formula, and then using that $|\frac{\partial}{\partial y}\phi_i| < \frac{1}{4}, i = 1, \dots, d$, one proves that the map $\lambda \to \pi_\lambda(\tilde{w})$ is differentiable for every $\tilde{w} \in \tilde{X}$, and the derivative is continuous with respect to $\tilde{w}$. Let us assume first that $i(w_n) \neq k, \forall n \ge 1$. We have then $\pi_\lambda(\tilde{w}) = \lambda_{i(w_1)} + \phi_{i(w_1)}(w_1, \pi_\lambda(\tilde{w}_1), \lambda)$. Therefore

$$\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w}) = \frac{\partial}{\partial y}\phi_{i(w_1)}(w_1, \pi_\lambda(\tilde{w}_1), \lambda) \cdot \frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w}_1) + \frac{\partial}{\partial \lambda_k}\phi_{i(w_1)}(w_1, \pi_\lambda(\tilde{w}_1), \lambda).$$

Hence $|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w})| \le \frac{1}{4}|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w}_1)| + |\frac{\partial}{\partial \lambda_k}\phi_{i(w_1)}(w_1, \pi_\lambda(\tilde{w}_1), \lambda)|$. Thus by induction we get

$$\begin{aligned}|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w})| &\le \frac{1}{4}|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w}_1)| + |\frac{\partial}{\partial \lambda_k}\phi_{i(w_1)}(w_1, \pi_\lambda(\tilde{w}_1), \lambda)| \\ &\le \frac{1}{4}(\frac{1}{4}|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w}_2)| + |\frac{\partial}{\partial \lambda_k}\phi_{i(w_2)}(w_2, \pi_\lambda(\tilde{w}_2, \lambda)|) + \alpha \\ &\le \alpha + \frac{1}{4}\alpha + \frac{1}{4^2}\alpha + \dots \\ &= \alpha \cdot \frac{4}{3}\end{aligned}$$

Let us consider now the case when there exists $n \ge 1$ with $i(w_n) = k$, and assume that $n$ is chosen as the smallest integer with this property (for $k \ge 1$ fixed). If $i(w_1) = k$, then

$$|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w})| \le 1 + \frac{1}{4}|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w}_1)| + \alpha \le 1 + \frac{1}{4}(1 + \frac{1}{4}|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w}_2)| + \alpha) + \alpha \le \dots \le (1+\alpha)\cdot\frac{4}{3},$$

as one can see by induction, and using the fact that the derivative of the function $\lambda \to \pi_\lambda(\tilde{w})$ is bounded in $\tilde{w} \in \tilde{X}$. In the case when $i(w_1) \ne k$, but there exists $n \ge 2$ with $i(w_n) = k$, we obtain similarly that

$$|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w})| \le \frac{1}{4}|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w}_1)| + \alpha \le \alpha + \frac{1}{4}(1+\alpha)\cdot\frac{4}{3} = \alpha + \frac{1+\alpha}{3}.$$

In conclusion, in all cases we get

$$|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w})| \le \frac{4}{3}(1+\alpha)$$

for all $k = 1, \dots, d$ and all $\tilde{w} \in \tilde{X}$. Consider now $\tilde{x}, \tilde{z} \in p_0^{-1}(x)$, with $x_1 \ne z_1$, and define, for $\lambda \in B_d(0, \eta)$,

$$g(\lambda) := \pi_\lambda(\tilde{z}) - \pi_\lambda(\tilde{x}) = \lambda_{i(z_1)} + \phi_{i(z_1)}(z_1, \pi_\lambda(\tilde{z}_1), \lambda) - \lambda_{i(x_1)} - \phi_{i(x_1)}(x_1, \pi_\lambda(\tilde{x}_1), \lambda)$$

Let us put $k := i(z_1)$ and $j := i(x_1)$. Then using the estimate obtained above, we infer that $|\frac{\partial}{\partial \lambda_k}g(\lambda)| \le (1+\alpha)\cdot\frac{8}{3}$. On the other hand, from the formula

$$g(\lambda) = \lambda_k + \phi_k(z_1, \pi_\lambda(\tilde{z}_1), \lambda) - \lambda_j - \phi_j(x_1, \pi_\lambda(\tilde{x}_1), \lambda),$$

we obtain

$$\begin{aligned}\frac{\partial}{\partial \lambda_k}g(\lambda) = 1 &+ \frac{\partial}{\partial y}\phi_k(z_1, \pi_\lambda(\tilde{z}_1), \lambda)\cdot\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{z}_1) + \frac{\partial}{\partial \lambda_k}\phi_k(z_1, \pi_\lambda(\tilde{z}_1), \lambda) - \\ &- \frac{\partial}{\partial y}\phi_j(x_1, \pi_\lambda(\tilde{x}_1), \lambda)\cdot\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{x}_1) - \frac{\partial}{\partial \lambda_k}\phi_j(x_1, \pi_\lambda(\tilde{x}_1), \lambda).\end{aligned}$$

Hence using the above estimate on the supremum of $|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w})|$, we have

$$|\frac{\partial}{\partial \lambda_k}g(\lambda)| \ge 1 - \frac{1}{4}\cdot\frac{4}{3}\cdot(1+\alpha) - \alpha - \frac{1}{4}\cdot\frac{4}{3}\cdot(1+\alpha) - \alpha = 1 - \frac{2}{3}(1+4\alpha)$$

We want $1 > \frac{2}{3}(1+4\alpha)$, so it is enough to take $\alpha < \frac{1}{8}$. Thus we have verified the hypothesis of Lemma 3.2, and the parametrized family $\{F_\lambda\}_{\lambda \in B_d(0,\eta)}$ is uniformly transversal. □

Therefore, we can apply the conclusion of Theorem 2.10 in order to obtain an estimate for the Hausdorff dimension of the fibers $Y_{\lambda,x}$ of $F_\lambda$; recall that for this family, $W = B_d(0, \eta)$.

**Corollary 3.4.** *If $f : I_1 \cup \ldots \cup I_d \to [0,1]$ and $X = I_*$ satisfy the assumptions of Theorem 3.3, and if there exist constants $a, b$ with $0 < a < b < \frac{1}{4}$ such that $a \le |\frac{\partial}{\partial y}\phi_i(x,y,\lambda)| \le b$ for all $(x,y,\lambda) \in X \times [0,1] \times B_d(0,\eta)$ and $i = 1, \dots, d$, then there exists a measurable set $W' \subset W$, with $l_d(W \setminus W') = 0$, such that for all $x \in X, \lambda \in W'$ we have:*

$$\min\left\{1, \frac{\log d}{|\log a|}\right\} \le \mathrm{HD}(Y_{\lambda,x}) \le \min\left\{1, \frac{\log d}{|\log b|}\right\}$$

*In particular, one obtains:*

(a) $\mathrm{HD}(Y_{\lambda,x}) > 0, x \in X, \lambda \in W'$.
(b) *if* $|a| \geq \frac{1}{d}$, *then* $\mathrm{HD}(Y_{\lambda,x}) = 1$, *for all* $x \in X, \lambda \in W'$.

*Proof.* We notice that, since $\zeta_\lambda(\tilde{x}) = \log |(\phi_x^\lambda)'(\pi_\lambda(\tilde{x}))|$, we get $\log a \leq \zeta_\lambda(\tilde{x}) \leq \log b$, hence

$$h_{top}(\tilde{f}|_{\tilde{X}}) + t \log a \leq P(\tilde{f}, t\zeta_\lambda) \leq h_{top}(\tilde{f}|_{\tilde{X}}) + t \log b$$

Now, let us recall that $h_{top}(\tilde{f}|_{\tilde{X}}) = h_{top}(f|_X)$. Also due to the fact that $f|_X$ is topologically conjugated to $\sigma_d : \Sigma_d^+ \to \Sigma_d^+$, the one-sided shift acting on the full symbol space $\Sigma_d^+$ generated by $d$ symbols, we have that $h_{top}(f|_X) = \log d$. Therefore, using Theorem 3.3, we obtain the announced estimates of $\mathrm{HD}(Y_{\lambda,x})$, for all $x \in X$, and $\lambda \in W'$. □

We will study in the sequel two other types of examples related to complex dynamics, which satisfy the uniform transversality condition, and hence Theorem 2.10 can be applied to them. The first such example is the family

$$F_\lambda(z, w) = (f(z), h(z) + \frac{1}{2}w + \lambda z)$$

Here we assume that $(z, w) \in U \times V \subset \mathbb{C} \times \mathbb{C}$, the set $U = \Delta(0, 2)$ is the disk of center 0 and radius 2 in $\mathbb{C}$, the set $V \subset \mathbb{C}$ is open, bounded and convex; assume also that the function $f(z)$ is close enough to a map of the form $z \to z^2 + c$, with $|c|$ small, and that $X = J(f)$, is the Julia set of $f$ (hence $f$ can be considered expanding on $X$). We will take also $h$ to be a complex valued Lipschitz continuous map defined in a neighbourhood of $X$; then since $|h|$ is bounded on $X$, we can take the bounded sets $V$ and $W \subset \mathbb{C}$ in such a way that the map $F_\lambda : U \times \overline{V} \to \mathbb{C} \times V$ is well defined for all $\lambda \in W$; for example one can take $W = \Delta(0, 1), V = \Delta(0, M)$, where $M > 2(\sup\limits_X |h| + 2)$.

**Theorem 3.5.** *The parametrized family* $\{F_\lambda\}_{\lambda \in W}$*, defined above, satisfies the uniform transversality condition.*

*Proof.* Recall that by our definition, $\pi_\lambda(\tilde{z}) = \lim_{n\to\infty} \phi_{z_1}^\lambda \circ \phi_{z_2}^\lambda \circ \ldots \circ \phi_{z_n}^\lambda(\zeta)$, where in general $\phi_z^\lambda(w) := h(z) + \frac{1}{2}w + \lambda z$. Hence

$$\phi_{z_1}^\lambda \circ \phi_{z_2}^\lambda(\zeta) = h(z_1) + \frac{1}{2}(h(z_2) + \frac{1}{2}\zeta + \lambda z_2) + \lambda z_1 = h(z_1) + \frac{1}{2}h(z_2) + \lambda z_1 + \frac{1}{2}\lambda z_2 + \frac{1}{4}\zeta.$$

It can be shown by induction that

$$\pi_\lambda(\tilde{z}) = [h(z_1) + \frac{1}{2}h(z_2) + \frac{1}{4}h(z_3) + \ldots] + \lambda(z_1 + \frac{1}{2}z_2 + \frac{1}{4}z_3 + \ldots).$$

Put

$$A(\tilde{z}) := h(z_1) + \frac{1}{2}h(z_2) + \frac{1}{4}h(z_3) + \ldots, \quad \text{and} \quad B(\tilde{z}) = z_1 + \frac{1}{2}z_2 + \frac{1}{4}z_3 + \ldots.$$

We shall consider now two prehistories $\tilde{z}, \tilde{z}' \in p_0^{-1}(z)$, with $z_1 \neq z_1'$. Let $g(\lambda) := \pi_\lambda(\tilde{z}) - \pi_\lambda(\tilde{z}') = A(\tilde{z}) + \lambda B(\tilde{z}) - A(\tilde{z}') - \lambda B(\tilde{z}')$. Let us notice now that since $f$ is close to the map $z \to z^2 + c$, we have $J(f)$ close to the circle $S^1$, if $c$ is small enough, and also it follows that $z_1'$ is close to $-z_1$; consequently $z_2' \approx iz_2$ or $z_2' \approx -iz_2$. This means that $|z_2' - z_2| \approx \sqrt{2}$. Hence $|z_2' - z_2 + \frac{1}{2}(z_3' - z_3) + \ldots| \leq \sqrt{2.2} + \frac{1}{2}(2.1 + \frac{1}{2}2.2 + \ldots) \leq \sqrt{2.2} + 2.2$, where we assumed that $f$ to be so close to $z^2 + c$, and $|c|$ to be so small that $|z_2' - z_2| < \sqrt{2.2}$ and $X \subset \Delta(0, 1.1)$. Thus

$$|B(\tilde{z}) - B(\tilde{z}')| \geq 1.9 - \frac{1}{2}(\sqrt{2.2} + 2.2) > 0.2,$$

if $\tilde{z}, \tilde{z}' \in \tilde{X}, z = z', z_1 \neq z_1'$. Therefore if $|g(\lambda)| = |A(\tilde{z}) - A(\tilde{z}') + \lambda(B(\tilde{z}) - B(\tilde{z}'))| < r$, then

$$\left|\lambda + \frac{A(\tilde{z}) - A(\tilde{z}')}{B(\tilde{z}) - B(\tilde{z}')}\right| < \frac{r}{|B(\tilde{z}) - B(\tilde{z}')|} < \frac{r}{0.2}$$

whenever $z = z'$ and $z_1 \neq z_1'$. This implies that $\lambda \in B(\frac{A(\tilde{z})-A(\tilde{z}')}{B(\tilde{z})-B(\tilde{z}')}, \frac{r}{0.2})$. Hence

$$l_2(\{\lambda : |g(\lambda)| < r\}) \leq 25\pi r^2$$

for all $r > 0$. Thus we proved that the Uniform Transversality Condition is satisfied for this family. $\square$

Another example, with a more complicated dynamics is presented below. Let us consider $f(z) = z^2 + c$, for $|c|$ small enough; thus $f$ has a Julia set denoted by $X$, close to the unit circle; then we have that $f$ is expanding on $X$. Assume also that $h$ is a complex valued Lipschitz continuous function defined on a neighbourhood of $X$, that $0.4 < |h(z)| < 0.6$, for $z \in X$, and that $|h(z)+h(z')| > \frac{3}{2}$ for $z^2 = -z'^2-2c, z \in X$, and $|c|$ small. We take then $\lambda$ to be a complex parameter with $|\lambda| < \frac{1}{6}$, and consider the parametrized family

$$F_\lambda(z, w) = (f(z), h(z) + \frac{1}{5}w^2 + \lambda z^2)$$

**Theorem 3.6.** *In the above setting, for any $\lambda$ from $W := \{\lambda \in \mathbb{C}, |\lambda| < \frac{1}{6}\}$ and $z \in X$, the map $F_\lambda(z, \cdot)$ defined above, invariates the domain $V := \{w \in \mathbb{C}, \frac{1}{30} < |w| < 1\}$, and $\{F_\lambda\}_{\lambda \in W}$ satisfies the Uniform Transversality condition.*

*Proof.* Without loss of generality we will assume that $c = 0$. Due to the way we defined $h$ and $X$, we have that $|h(z) + \frac{1}{5}w^2 + \lambda z^2| \le 0.6 + \frac{1}{5} + \frac{1}{6} < 1$ for $(z, w, \lambda) \in X \times V \times W$. Also, $|h(z) + \frac{1}{5}w^2 + \lambda z^2| \ge 0.4 - \frac{1}{5} - \frac{1}{6} = \frac{1}{30}$. Therefore $F_\lambda$ preserves the domain $V$. Let us check now the other conditions required for Uniform Transversality. Firstly, $|\frac{\partial}{\partial w}\phi_z^\lambda| = |\frac{2w}{5}| < \frac{2}{5}$, and $|\frac{\partial}{\partial w}\phi_z^\lambda| > \frac{1}{75} > 0$, for all $z \in X, w \in V$, where $\phi_z^\lambda(w) := h(z) + \frac{w^2}{5} + \lambda z^2$. We shall prove by induction that for all $n \ge 1$ there exist functions $A_n$, $B_n$ and $C_n$ such that for
all $\tilde{z} = (z, z_1, z_2, \ldots) \in \tilde{X}$, we have

$$\phi_{z_1}^\lambda \circ \ldots \circ \phi_{z_n}^\lambda(w) = A_n(z_n) + \lambda B_n(z_n, \lambda) + wC_n(z_n, w, \lambda).$$

For $n = 1$, we get $\phi_{z_1}^\lambda = h(z_1) + \lambda z_1^2 + \frac{w^2}{5}$, so $A_1(z) = h(z), B_1(z, \lambda) = z^2, C_1(z, w, \lambda) = \frac{w}{5}$. We want now to calculate the formula for $\phi_{z_1}^\lambda \circ \ldots \phi_{z_{n+1}}^\lambda$ and to get recurrence formulas for $A_n, B_n, C_n$. ¿From above,

$$\begin{aligned}
&\phi_{z_1}^\lambda \circ \ldots \circ \phi_{z_{n+1}}^\lambda(w) = \\
&\quad = \phi_{z_1}^\lambda(A_n(z_{n+1}) + \lambda B_n(z_{n+1}, \lambda) + wC_n(z_{n+1}, w, \lambda)) \\
&\quad = h(z_1) + \lambda z_1^2 + \frac{1}{5}[A_n(z_{n+1}) + \lambda B_n(z_{n+1}, \lambda) + wC_n(z_{n+1}, w, \lambda)]^2 \\
&\quad = h(z_1) + \lambda z_1^2 + \frac{1}{5}[A_n(z_{n+1})^2 + \lambda^2 B_n(z_{n+1}, \lambda)^2 + w^2 C_n(z_{n+1}, w, \lambda)^2 + \\
&\qquad + 2\lambda A_n(z_{n+1})B_n(z_{n+1}, \lambda) + 2\lambda w B_n(z_{n+1}, \lambda)C_n(z_{n+1}, w, \lambda) + 2A_n(z_{n+1})wC_n(z_{n+1}, w, \lambda)] \\
&\quad = h(z_1) + \frac{1}{5}A_n(z_{n+1})^2 + \lambda[z_1^2 + \frac{2}{5}A_n(z_{n+1})B_n(z_{n+1}, \lambda) + \frac{\lambda}{5}B_n(z_{n+1}, \lambda)^2] + \\
&\qquad + wC_n(z_{n+1}, w, \lambda) \cdot [\frac{2\lambda}{5}B_n(z_{n+1}, \lambda) + \frac{2}{5}A_n(z_{n+1}) + \frac{wC_n(z_{n+1}, w, \lambda)}{5}].
\end{aligned}$$

Thus we obtain the following recurrence formulas, with $\tilde{z} = (z, z_1, \ldots, z_n, \ldots) \in \tilde{X}$:

$$A_{n+1}(z_{n+1}) = h(z_1) + \frac{1}{5}A_n(z_{n+1})^2,$$

$$B_{n+1}(z_{n+1}, \lambda) = z_1^2 + \frac{2}{5}A_n(z_{n+1})B_n(z_{n+1}, \lambda) + \frac{\lambda}{5}B_n(z_{n+1}, \lambda)^2,$$

$$C_{n+1}(z_{n+1}, w, \lambda) = C_n(z_{n+1}, w, \lambda) \cdot (\frac{2\lambda}{5}B_n(z_{n+1}, \lambda) + \frac{2}{5}A_n(z_{n+1}) + \frac{wC_n(z_{n+1}, w, \lambda)}{5})$$

Now we want to prove that $\sup|A_n| < 0.7, \sup|B_n| < 1.5, \sup|C_{n+1}| < \frac{1}{2}\sup|C_n|$, for $z \in X, w \in V, \lambda \in W$. The first two inequlities are satisfied at the level $n = 1$, due to our assumptions on $F_\lambda$. If $|A_n| < 0.7$, then $|A_{n+1}| < 0.6 + \frac{1}{5}(0.7)^2 < 0.7$. So we proved the first inequality for all $n \ge 1$. Now, assume that $|B_n| < 1.5$; then $|B_{n+1}| < 1 + \frac{2}{5} \cdot 0.7 \cdot 1.5 + \frac{1}{30}(1.5)^2 = 1 + \frac{21}{50} + \frac{3}{40} < 1.5$. Thus we proved also the inequality $\sup|B_n| < 1.5, \forall n \ge 1$. Last, it is clear that $|C_1| < \frac{1}{5}$ on $V$. Assume that $|C_n| < \frac{1}{5}$; then $|C_{n+1}| < \frac{1}{5}(\frac{2}{30} \cdot \frac{3}{2} + \frac{2}{5} \cdot 0.7 + \frac{1}{25}) < \frac{1}{10}$, and from the recurrence formula for $C_{n+1}$, we obtain also

$\sup|C_{n+1}| < \frac{1}{2}\sup|C_n|, \forall n \geq 1$. This last inequality tells us that $\sup|C_n| \to 0$ when $n \to \infty$. Consequently, in general, for $\tilde{z} \in \tilde{X}$, we have

$$\pi_\lambda(\tilde{z}) = A(\tilde{z}) + \lambda B(\tilde{z}, \lambda)$$

Let us consider now $\tilde{z}, \tilde{z}' \in p_0^{-1}(z)$ and $g(\lambda) := \pi_\lambda(\tilde{z}) - \pi_\lambda(\tilde{z}') = A(\tilde{z}) - A(\tilde{z}') + \lambda(B(\tilde{z}, \lambda) - B(\tilde{z}', \lambda))$. We have from the recurrence formulas above that $B(\tilde{z}, \lambda) = z_1^2 + \frac{2}{5}A(\tilde{z}_1)B(\tilde{z}_1, \lambda) + \frac{\lambda}{5}B(\tilde{z}_1, \lambda)^2$; also we have $A(\tilde{z}_1) = h(z_2) + \frac{1}{5}A(\tilde{z}_2)^2$. Hence we can deduce

$$\begin{aligned}\pi_\lambda(\tilde{z}) &= A(\tilde{z}) + \lambda(z_1^2 + \frac{2}{5}A(\tilde{z}_1)B(\tilde{z}_1, \lambda) + \frac{\lambda}{5}B(\tilde{z}_1, \lambda)^2) \\ &= A(\tilde{z}) + \lambda[z_1^2 + \frac{2}{5}(h(z_2) + \frac{1}{5}A(\tilde{z}_2)^2) \cdot (z_2^2 + \frac{2}{5}A(\tilde{z}_2)B(\tilde{z}_2, \lambda) + \frac{\lambda}{5}B(\tilde{z}_2, \lambda)^2) + \frac{\lambda}{5}B(\tilde{z}_1, \lambda)^2]\end{aligned}$$

Similarly we show that

$$\pi_\lambda(\tilde{z}') = A(\tilde{z}') + \lambda[z_1'^2 + \frac{2}{5}(h(z_2') + \frac{1}{5}A(\tilde{z}_2')^2) \cdot (z_2'^2 + \frac{2}{5}A(\tilde{z}_2')B(\tilde{z}_2', \lambda) + \frac{\lambda}{5}B(\tilde{z}_2', \lambda)^2) + \frac{\lambda}{5}B(\tilde{z}_1', \lambda)^2]$$

Therefore, recalling that $z_1^2 = z_1'^2$, we obtain that:

$$\begin{aligned}g(\lambda) &= A(\tilde{z}) - A(\tilde{z}') + \lambda\{\frac{2}{5}[(h(z_2)z_2^2 - h(z_2')z_2'^2) + h(z_2)(\frac{2}{5}A(\tilde{z}_2)B(\tilde{z}_2, \lambda) + \\ &\quad + \frac{\lambda}{5}B(\tilde{z}_2, \lambda)^2) - h(z_2')(\frac{2}{5}A(\tilde{z}_2')B(\tilde{z}_2', \lambda) + \frac{\lambda}{5}B(\tilde{z}_2', \lambda)^2) + \frac{1}{5}A(\tilde{z}_2)^2(z_2^2 + \frac{2}{5}A(\tilde{z}_2)B(\tilde{z}_2, \lambda) + \\ &\quad + \frac{\lambda}{5}B(\tilde{z}_2, \lambda)^2) - \frac{1}{5}A(\tilde{z}_2')^2(z_2'^2 + \frac{2}{5}A(\tilde{z}_2')B(\tilde{z}_2', \lambda) + \frac{\lambda}{5}B(\tilde{z}_2', \lambda)^2)] + \frac{\lambda}{5}(B(\tilde{z}_1, \lambda)^2 - B(\tilde{z}_1', \lambda)^2)\} \\ &= A(\tilde{z}) - A(\tilde{z}') + \lambda\{\frac{2}{5}[(h(z_2)z_2^2 - h(z_2')z_2'^2) + D(\tilde{z}, \tilde{z}', \lambda) + E(\tilde{z}, \tilde{z}', \lambda)] + \frac{\lambda}{5}G(\tilde{z}, \tilde{z}')\},\end{aligned}$$

where

$$D(\tilde{z}, \tilde{z}', \lambda) := h(z_2)(\frac{2}{5}A(\tilde{z}_2)B(\tilde{z}_2, \lambda) + \frac{\lambda}{5}B(\tilde{z}_2, \lambda)^2) - h(z_2')(\frac{2}{5}A(\tilde{z}_2')B(\tilde{z}_2', \lambda) + \frac{\lambda}{5}B(\tilde{z}_2', \lambda)^2),$$

$$E(\tilde{z}, \tilde{z}', \lambda) := \frac{1}{5}A(\tilde{z}_2)^2(z_2^2 + \frac{2}{5}A(\tilde{z}_2)B(\tilde{z}_2, \lambda) + \frac{\lambda}{5}B(\tilde{z}_2, \lambda)^2) - \frac{1}{5}A(\tilde{z}_2')^2(z_2'^2 + \frac{2}{5}A(\tilde{z}_2')B(\tilde{z}_2', \lambda) + \frac{\lambda}{5}B(\tilde{z}_2', \lambda)^2),$$

and

$$G(\tilde{z}, \tilde{z}', \lambda) = B(\tilde{z}_1, \lambda)^2 - B(\tilde{z}_1', \lambda)^2$$

But we can estimate $|D(\tilde{z}, \tilde{z}', \lambda)|$ as follows:

$$|D(\tilde{z}, \tilde{z}', \lambda)| \leq 2 \cdot 0.6 \cdot (\frac{2}{5} \cdot 0.7 \cdot 1.5 + \frac{1}{30} \cdot \frac{9}{4}) < 0.6.$$

Also we obtain:

$$\begin{aligned}|E(\tilde{z}, \tilde{z}', \lambda)| &= 2\sup|\frac{1}{5}A(\tilde{z}_2)^2(z_2^2 + \frac{2}{5}A(\tilde{z}_2)B(\tilde{z}_2, \lambda) + \frac{\lambda}{5}B(\tilde{z}_2, \lambda)^2)| \\ &\leq 2 \cdot \frac{1}{5} \cdot (0.7)^2(1 + \frac{2}{5} \cdot 0.7 \cdot 1.5 + \frac{1}{5} \cdot \frac{1}{6} \cdot (1.5)^2) < \frac{2}{5}.\end{aligned}$$

Notice that

$$|G(\tilde{z}, \tilde{z}', \lambda)| \leq 3,$$

for all $\tilde{z}, \tilde{z}' \in \tilde{X}, \lambda \in W$, from the estimate for $|B_n|$. Combining all the above we obtain $|\frac{2}{5}[(h(z_2)z_2^2 - h(z_2')z_2'^2) + D(\tilde{z}, \tilde{z}', \lambda) + E(\tilde{z}, \tilde{z}', \lambda)] + \frac{\lambda}{5}G(\tilde{z}, \tilde{z}', \lambda)| \geq \frac{2}{5}(|h(z_2)z_2^2 - h(z_2')z_2'^2| - 0.6 - \frac{2}{5}) - \frac{1}{5} \cdot \frac{1}{6} \cdot 3 = \frac{2}{5}(|h(z_2)z_2^2 - h(z_2')z_2'^2| - 1) - 0.1$. Thus, since $z_2^2 = z_1 - c$, we will obtain

$$|B(\tilde{z}, \lambda) - B(\tilde{z}', \lambda)| \geq \frac{2}{5}(|h(z_2)z_2^2 - h(z_2')z_2'^2| - 1) - 0.1 > \gamma > 0,$$

where $\gamma > 0$ is small enough, when $|c|$ is small enough and $|h(z_2) + h(z_2')| > \frac{6}{4}$ for $z_2^2 = -z_2'^2 - 2c, z_2 \in X$. This means that now we can prove Uniform Transversality for $F_\lambda$, as in the previous Theorem. $\square$

Another example of a complex parametrized family with Uniform Transversality is

$$F_\lambda(z, w) = (z^2, z^2 + \lambda_1 z + \lambda_2 z w^2),$$

with $W = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2, |\lambda_1| < \frac{1}{50}, \frac{1}{10} < |\lambda_2| < \frac{1}{8}\}, V := \{w \in \mathbb{C}, \frac{1}{2} < |w| < 1.5\}$. Then it can be shown that $F_\lambda(z, \cdot) : \overline{V} \to V$ is well defined for $z \in S^1, \lambda \in W$, and $\exists\ \kappa, \underline{\kappa} \in (0, 1)$ such that $\underline{\kappa} \le |(\phi_z^\lambda)'| \le \kappa$ on $V$. For this example it can be proved similarly that $\{F_\lambda\}_{\lambda \in W}$ is a parametrized family with Uniform Transversality.

Therefore, for all the examples we have given in this section, the conclusions of Theorem 2.10 apply, and we can write, for almost all parameters $\lambda$, the Hausdorff dimension of **all** fibers (thus the stable dimension in our case), by means of the thermodynamic formalism on $\tilde{X}$.

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