INSTABILITY OF EXPONENTIAL COLLET - ECKMANN MAPS

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ABSTRACT. Given $\lambda \in \mathbb{C} \setminus \{0\}$ let the entire function $f_\lambda : \mathbb{C} \to \mathbb{C}$ be defined by the formula

$$f_\lambda(z) = \lambda e^z.$$

The question of structural stability within this family is one of the most important problems in the theory of iterates of entire functions. The natural conjecture is that f_λ is stable iff f_λ is hyperbolic, i.e. if the only singular value 0 is attracted by a an attracting periodic orbit. We present some results positively contributing towards this conjecture. More precisely, we give some sufficient conditions of summability type which guarantee that the map f_λ is unstable.

1. Introduction

Structural stability is one of the most important issues in the theory of dynamical systems. It is well-known that systems with strongly hyperbolic features of dynamics are structurally stable. It is widely believed that in a sense these are only structurally stable systems. More precisely, the hyperbolic systems are frequently expected to form a dense subset in an appropriate class of systems in question. In this paper we deal with the class of exponential functions on the complex plane, i.e. with maps $f_\lambda(z) = \lambda e^z$, where $\lambda \in \mathbb{C} \setminus \{0\}$ is a fixed complex parameter, whereas $z \in \mathbb{C}$ is a variable. We want to contribute positively to the conjecture that the parameters λ for which f_λ is hyperbolic (there is an attracting periodic cycle) coincide with those λ's for which f_λ are structurally stable (within this class). It is known that exponential maps either with a rationally indifferent periodic point, a Siegel disk, and those with finite orbit of zero are unstable. We aim to show that Collet-Eckmann exponential maps, systems which exhibit some weak hyperbolicity features are still unstable. Our general approach is motivated by the works [Le], [M1], [M2] and [DMS]. We make an extensive use of the Beltrami, Ruelle and Perron-Frobenius operators and we prove the following.

Theorem. *If the series* $\sum_{k=1}^\infty \frac{1}{|(f_\lambda^k)'(0)|}$ *converges and either* $\overline{O_\lambda(0)}$*, the closure of the orbit of* 0*, is a nowhere dense set with* $\mathrm{Leb}(\overline{O_\lambda(0)}) = 0$ *or if the orbit of* 0 *is non-recurrent, then the parameter* λ *is unstable.*

We would like to add that Makienko at al have always dealt with transcendental functions having critical singularities and they made use of hyperbolic behaviour of trajectories of

The research of the first author was supported in part by the NSF Grant DMS 0400481. The research of the second author was supported in part by the Polish KBN Grant 2 PO3A 034 25. The research of both authors was supported in part by the NSF/PAN grant INT-0306004.

critical singularities only. We find it interesting that the operator method (construction of a fixed point of Ruelle's operator) works also for trajectories of essential singularities. Our concluding arguments are entirely different than those used in [Le], [M1], [M2] and [DMS]. Making use of the existence of invariant line field they lead to a contradiction by showing that an exponential function would be globally holomorphically conjugate to an affine map.

2. Numerical condition for (in)stability

Definition 2.1. *A parameter λ_0 is called stable if there exists a neighbourhood U of λ_0 in $\mathbb{C}$ such that for every $\lambda \in U$, the map f_λ is topologically conjugate to f_{λ_0}.*

For every $\lambda \in \mathbb{C}$ and every $z \in \mathbb{C}$ put

$$O_\lambda(z) = \{f_\lambda^n(z) : n \geq 0\}.$$

Set also

$$g^n(\lambda) = f_\lambda^n(0).$$

We shall prove the following

Proposition 2.2. *If $\lambda_0 \in \mathbb{C} \setminus \{0\}$, $\lim_{n\to\infty}(f_{\lambda_0}^n)'(0) = \infty$ and the series $\sum_{n=0}^\infty \frac{1}{(f_{\lambda_0}^n)'(0)}$ does not converge to 0, then the parameter λ_0 is unstable.*

Proof. First, notice that we can assume that the point 0 is not eventually periodic under iterates of f_{λ_0}. Indeed, if 0 is eventually periodic and the parameter λ_0 is stable then the equation $f_\lambda^n(0) - f_\lambda^k(0) = 0$ is satisfied on some open neighborhood of λ_0 with some fixed positive integers n and k. But, since the left hand side of the above equation defines a holomorphic function of $\lambda \in \mathbb{C}$, we conclude that the equation above is satisfied in the whole $\mathbb{C}$, which is impossible.

Abusing notation slightly, put $f(\lambda, z) = f_\lambda(z)$ and, more generally, $f^n(\lambda, z) = f_\lambda^n(z)$. Then $f^{n+1}(\lambda, z) = f(\lambda, f^n(\lambda, z))$ and, differentiating with respect to λ, we get

$$\begin{aligned}\frac{\partial}{\partial\lambda}f^{n+1}(\lambda, z) &= \frac{\partial f}{\partial\lambda}(\lambda, f^n(\lambda, z)) + \frac{\partial}{\partial w}_{|f^n(\lambda,z)} f(\lambda, w) \cdot \frac{\partial}{\partial\lambda}f^n(\lambda, z) \\ &= \frac{1}{\lambda}f^{n+1}(\lambda, z) + f^{n+1}(\lambda, z)\frac{\partial}{\partial\lambda}f^n(\lambda, z) = f^{n+1}(\lambda, z)\left(\frac{1}{\lambda} + \frac{\partial}{\partial\lambda}f^n(\lambda, z)\right)\end{aligned}$$

Setting $z = 0$, this gives

$$g_{n+1}'(\lambda) = \left(\frac{1}{\lambda} + g_n'(\lambda)\right) f_\lambda^{n+1}(0)$$

or, equivalently,

$$\lambda g_{n+1}'(\lambda) = (1 + \lambda g_n'(\lambda)) f_\lambda^{n+1}(0) \tag{2.1}$$

We claim that for every $n \geq 1$,

$$\lambda(g^n)'(\lambda) = (f^n)'(0) \sum_{k=0}^{n-1} ((f_\lambda^k)'(0))^{-1}, \tag{2.2}$$

where f^0 is the identity map. Indeed, for $n = 1$ this equality follows by a trivial computation. So, suppose it is true for some $n \geq 1$. Then, using (2.1), we get

$$\begin{aligned}
\lambda g'_{n+1}(\lambda) &= \left(1 + (f_\lambda^n)'(0) \sum_{k=0}^{n-1} ((f_\lambda^k)'(0))^{-1}\right) f_\lambda^{n+1}(0) \\
&= (f_\lambda^n)'(0) f_\lambda^{n+1}(0) \left(\frac{1}{(f_\lambda^n)'(0)} + \sum_{k=0}^{n-1} (f_\lambda^k)'(0))^{-1})\right) \\
&= (f_\lambda^{n+1})'(0) \sum_{k=0}^{n} ((f_\lambda^k)'(0))^{-1}.
\end{aligned}$$

Hence, (2.2) is proved by induction. Since the series $\sum_{k=0}^\infty \frac{1}{(f_{\lambda_0}^k)'(0)}$ does not converge to 0 and $\lim \frac{1}{(f_{\lambda_0}^k)'(0)} = 0$, there exist $\theta > 0$ and an increasing to ∞ sequence $\{n_j\}_{j=1}^\infty$ of positive integers such that

$$\left|\sum_{k=0}^{n_j} \left((f_{\lambda_0}^k)'(0)\right)^{-1}\right| \geq \theta$$

for all $j \geq 1$. Since $\lim_{n\to\infty} (f_{\lambda_0}^n)'(0) = \infty$, using (2.2) we conclude that

$$\lim_{j\to\infty} |g'_{n_j}(\lambda_0)| = +\infty \quad \text{and} \quad \lim_{j\to\infty} |g'_{n_j-1}(\lambda_0)| = +\infty \tag{2.3}$$

Let log be a holomorphic branch of logarithm defined in $B(\lambda_0, |\lambda_0|)$. It follows from (2.3) that

$$\lim_{j\to\infty} (g_{n_j} + \log)'(\lambda_0) = \infty. \tag{2.4}$$

Now, we consider two cases. If for every $r \in B(0, \lambda_0)$ the family of maps $\{g_{n_j} + \log : B(\lambda_0, r) \to \mathbb{C}\}$ is not normal, then, by Montel's theorem, for every $r > 0$ there are $j = j(r) \geq 1$ and $\lambda_r \in B(\lambda_0, r)$ such that

$$g_{n_j}(\lambda_r) + \log(\lambda_r) = \log(2\pi) + 2\pi i l + i\frac{\pi}{2} \tag{2.5}$$

for some $l \in \mathbb{Z}$. If, on the other hand, there exists $R < \frac{|\lambda_0|}{2}$ such that the sequence $\{g_{n_j} + \log : B(\lambda_0, 2R) \to \mathbb{C}\}$ is normal, then it follows from (2.4) that $\lim_{j\to\infty} g_{n_j} = \infty$. This in turn implies (using $f_\lambda(z) = \lambda \exp(z)$) that $\mathrm{Re}(g_{n_j-1}(\lambda) + \log\lambda)$ converges uniformly to $+\infty$ on $B(\lambda_0, R)$. By Bloch's theorem, for every $r \in (0, |\lambda_0|)$ and every $j \geq 1$ sufficiently large (depending on r), the image $(g_{n_j-1} + \log)(B(\lambda_0, r))$ contains a disc $D \subset \{z : \mathrm{Re} z > 0\}$ of radius 2π. Therefore, there exist $\lambda \in B(\lambda_0, r)$ and $j = j(r)$ such that

$$g_{n_j-1}(\lambda) + \log\lambda = \log(2k\pi) + 2\pi i l + i\frac{\pi}{2}$$

where $k \geq 1$ and l are integers. Notice that (2.5) has the same form with $k = 1$ and $n_j - 1$ replaced by n_j. So, in the first case we get

$$g_{n_j+1}(\lambda) = \lambda \exp(g_{n_j}(\lambda)) = \lambda \exp(-\log \lambda + \log(2k\pi) + i\frac{\pi}{2} + 2\pi i l) = 2k\pi i.$$

Hence, $f^{n_j+2}(0) = f_\lambda(g_{n_j+1}(\lambda)) = \lambda e^{2k\pi i} = \lambda$. In the second case we end up with the same conclusion, with n_j replaced by $n_j - 1$. Since $f_\lambda(0) = \lambda$, we see that 0 is eventually periodic for f_λ. Since we have assumed that 0 is not eventually periodic for f_{λ_0}, we conclude that λ_0 is an unstable parameter. ■

3. The operator T and its fixed point φ

From now on, to simplify the notation, we put $f = f_\lambda$.
Given any function $g : \mathbb{C} \to \mathbb{C}$, we put

$$Tg(z) = \frac{1}{z} \sum_{w \in f^{-1}(z)} \frac{g(w)}{w} \tag{3.1}$$

for all those $z \in \mathbb{C} \setminus \{0\}$ for which the series $\sum_{w \in f^{-1}(z)} |\frac{g(w)}{w}|$ converges. For every $a \in \mathbb{C} \setminus \{0\}$ define the function $\varphi_a : \mathbb{C} \to \mathbb{C}$ by

$$\varphi_a(z) = \frac{1}{z - a}.$$

Then, formally, without taking care of the convergence of the series defining $T\varphi_a$, we can write

$$T\varphi_a(z) = \frac{1}{z} \sum_{w \in f^{-1}(z)} \frac{1}{w(w-a)}.$$

Notice that, since $f^{-1}(\{z\}) = \{w_0 + 2k\pi i\}_{k \in \mathbb{Z}}$, the function $T\varphi_a$ is well-defined in $\mathbb{C} \setminus \{0, f(0), f(a)\}$, because the corresponding series converges absolutely in $\mathbb{C} \setminus \{0, f(0), f(a)\}$. We shall prove

Lemma 3.1. *The function $T\varphi_a$ extends to a meromorphic function in $\mathbb{C}$ given by the formula*

$$z \mapsto \frac{1}{a}\left(\frac{1}{z - f(a)} - \frac{1}{z - f(0)}\right).$$

Proof. Since $\lim_{z \to 0} \mathrm{Re}(f^{-1}(z)) = -\infty$, we see that $\lim_{z \to 0} \sum_{w \in f^{-1}(z)} \frac{1}{w(w-a)} = 0$. So, the function $z \mapsto \sum_{w \in f^{-1}(z)} \frac{1}{w(w-a)}$ extends holomorphically to some neighbourhood of zero and it takes the value 0 at 0. This implies that our function $z \mapsto \frac{1}{z} \sum_{w \in f^{-1}(z)} \frac{1}{w(w-a)}$ also extends holomorphically to some neighbourhood of 0. Let f_0^{-1} be the holomorphic branch of f^{-1} sending $f(0)$ to 0. Then

$$\lim_{z \to f(0)} (z - f(0))\varphi_a(z) = \lim_{z \to f(0)} \frac{z - f(0)}{z} \left(\frac{1}{f_0^{-1}(z)(f_0^{-1}(z) - a)} + \sum_{w \in f^{-1}(z) \setminus f_0^{-1}(z)} \frac{1}{w(w-a)}\right).$$

Now,

$$\lim_{z\to f(0)}\frac{z-f(0)}{zf_0^{-1}(z)(f_0^{-1}(z)-a)}=\lim_{z\to f(0)}\frac{1}{z(f_0^{-1}(z)-a)}\frac{f(f_0^{-1}(z))-f(0)}{f_0^{-1}(z)-0}=\frac{f'(0)}{f(0)(-a)}=-\frac{1}{a}.$$

If $a\notin f^{-1}(f(0))$, then

$$\lim_{z\to f(0)}\sum_{w\in f^{-1}(z)\setminus f_0^{-1}(z)}\frac{1}{w(w-a)}=\sum_{w\in f^{-1}(f(0))\setminus\{0\}}\frac{1}{w(w-a)}\in\mathbb{C}.$$

and consequently,

$$\lim_{z\to f(0)}(z-f(0))T\phi_z(z)=-\frac{1}{a}\in\mathbb{C}.$$

If, on the other hand, $a\in f^{-1}(f(0))$, then let $f_a^{-1}:B(f(0),|f(0)|)$ be the holomorphic inverse branch of f^{-1} mapping $f(a)$ to a. Then

$$\lim_{z\to f(0)}\frac{z-f(0)}{zf_a^{-1}(z)(f_a^{-1}(z)-a)}=\lim_{z\to f(0)}\frac{1}{zf_a^{-1}(z)}\frac{f(f_a^{-1}(z))-f(a)}{f_a^{-1}(z)-a}=\frac{f'(a)}{f(0)a}=\frac{f'(0)}{f(0)a}=\frac{1}{a}.$$

Since

$$\lim_{z\to f(0)}\sum_{w\in f^{-1}(z)\setminus\{f_0^{-1}(z),f_a^{-1}(z)\}}\frac{1}{w(w-a)}=\sum_{w\in f^{-1}(f(0))\setminus\{0,a\}}\frac{1}{w(w-a)}\in\mathbb{C},$$

we conclude that

$$\lim_{z\to f(0)}(z-f(0))T\phi_a(z)=-\frac{1}{a}+\frac{1}{a}=0.$$

So, in either case, $T\phi_a$ has a simple pole at $f(0)$ and

$$Res_{f(0)}T\phi_a=\begin{cases}\frac{1}{a}\text{ if } f(a)\neq f(0)\\ 0\text{ if } f(a)=f(0).\end{cases}\tag{3.2}$$

Dealing with the behavior of the function $T\phi_a$ around the point a, let $f_a^{-1}:B(f(a),|f(a)|)\to\mathbb{C}$ be the holomorphic inverse branch of f sending $f(a)$ to a. We then have

$$\lim_{z\to f(a)}(z-f(a))\frac{1}{zf_a^{-1}(z)(f_a^{-1}(z)-a)}=\lim_{z\to f(a)}\frac{1}{zf_a^{-1}(z)}\cdot\frac{f(f_a^{-1}(z))-f(a)}{f_a^{-1}(z)-a}=\frac{f'(a)}{f(a)a}=\frac{1}{a}.$$

Suppose now that $f(0)\neq f(a)$. Then

$$\lim_{z\to f(a)}\sum_{w\in f^{-1}(z)\setminus\{f_a^{-1}(z)\}}\frac{1}{w(w-a)}=\sum_{w\in f^{-1}(f(a))\setminus\{a\}}\frac{1}{w(w-a)}\in\mathbb{C}.$$

Consequently,

$$\lim_{z\to f(a)}(z-f(a))T\phi_a(z)=\frac{1}{a}.$$

If, on the other hand, $f(0) = f(a)$, denote by f_0^{-1} the holomorphic branch sending $f(a) = f(0)$ to 0. Then

$$\lim_{z \to f(a)} \frac{z - f(a)}{z f_0^{-1}(z)(f_0^{-1}(z) - a)} = \lim_{z \to f(a)} \frac{1}{z(f_0^{-1}(z) - a)} \cdot \frac{f(f_0^{-1}(z)) - f(0)}{f_0^{-1}(z) - 0} = \frac{f'(0)}{f(a)(-a)}$$
$$= \frac{f'(0)}{-af(0)} = -\frac{1}{a}.$$

Since

$$\lim_{z \to f(a)} \sum_{w \in f^{-1}(z) \setminus \{f_a^{-1}(z), f_0^{-1}(z)\}} \frac{1}{w(w-a)} = \sum_{w \in f^{-1}(f(a)) \setminus \{0,a\}} \frac{1}{w(w-a)} \in \mathbb{C},$$

we conclude that in this case

$$\lim_{z \to f(a)} (z - f(a)) T\phi_a(z) = \frac{1}{a} - \frac{1}{a} = 0.$$

So, in either case, $T\phi_a$ has a simple pole at $f(a)$ and

$$Res_f(a) T\phi_a = \begin{cases} \frac{1}{a} \text{ if } f(a) \neq f(0) \\ 0 \text{ if } f(a) = f(0) \end{cases} \tag{3.3}$$

Since $a \neq 0$, it follows from (3.2) and (3.3) that in either case

$$T\phi_a(z) - \frac{1}{a}\left(\frac{1}{z - f(a)} - \frac{1}{z - f(0)}\right)$$

is an analytic function in $\mathbb{C}$, and, since

$$\lim_{z \to \infty} \left(T\phi_a(z) - \frac{1}{a}\left(\frac{1}{z - f(a)} - \frac{1}{z - f(0)}\right)\right) = 0$$

(the limit of each term is zero) we therefore conclude from Liouville's theorem, that $T\phi_a(z) - \frac{1}{a}(\frac{1}{z-f(a)} - \frac{1}{z-f(0)})$ is identically equal to zero. ■

Since T is a linear operator, it follows from Lemma 3.1 that for all $k \geq 1$,

$$T\left(\frac{1}{(f^{k-1})'(0)} \frac{1}{z - f^k(0)}\right) = \frac{1}{(f^{k-1})'(0)} \frac{1}{f^k(0)} \left(\frac{1}{z - f^{k+1}(0)} - \frac{1}{z - f(0)}\right)$$
$$= \frac{1}{(f^k)'(0)} \left(\frac{1}{z - f^{k+1}(0)} - \frac{1}{z - f(0)}\right)$$

Hence, using linearity again, we get for every $n \geq 1$ that

$$T\left(\sum_{k=1}^{n} \frac{1}{(f^{k-1})'(0)} \frac{1}{z - f^k(0)}\right) = \sum_{k=1}^{n} \frac{1}{(f^k)'(0)} \frac{1}{z - f^{k+1}(0)} - \frac{1}{z - f(0)} \sum_{k=1}^{n} \frac{1}{(f^k)'(0)}$$
$$= \sum_{k=1}^{n+1} \frac{1}{(f^{k-1})'(0)} \frac{1}{z - f^k(0)} - \frac{1}{z - f(0)} \sum_{k=0}^{n} \frac{1}{(f^k)'(0)}. \tag{3.4}$$

We want to let $n \to \infty$ and to obtain a similar equation for the infinite sum. To do this, we prove first lemmas 3.2 and 3.3 below.

Lemma 3.2. *If* $\xi \in \mathbb{C} \setminus \overline{O_\lambda(0)}$*, then* $\mathrm{dist}(f^{-1}(B(\xi, r)), \overline{O_\lambda(0)}) > 0$ *for every* $r < \mathrm{dist}(\xi, \overline{O_\lambda(0)})$

Proof. Because of the choice of the radius r we have

$$\overline{B(\xi, r)} \cap \overline{O_\lambda(0)} = \emptyset. \tag{3.5}$$

Suppose now that $\mathrm{dist}(f^{-1}(B(\xi, r)), \overline{O_\lambda(0)}) = 0$. Then there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset f^{-1}(B(\xi, r))$ such that $\lim_{n\to\infty} \mathrm{dist}(x_n, \overline{O_\lambda(0)}) = 0$. Consequently, there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset \overline{O_\lambda(0)}$ such that

$$\lim_{n\to\infty} |z_n - x_n| = 0 \tag{3.6}$$

Since $f(\{x_n\}_{n=1}^{\infty}) \subset B(\xi, r)$, passing to subsequence we may assume that

$$\lim_{n\to\infty} f(x_n) = y \tag{3.7}$$

for some $y \in \overline{B(\xi, r)}$. But then, for every $n \geq 1$ there exists $y_n \in f^{-1}(y)$ such that $\lim_{n\to\infty} |x_n - y_n| = 0$. Combining this and (3.6), we see that $\lim_{n\to\infty} |z_n - y_n| = 0$. But $\mathrm{Re} y_n = \log|y| - \log|\lambda|$ for all $n \geq 1$. Then for all $n \geq 1$ so large that $z_n \in B(y_n, 1)$, we get

$$|y - f(z_n)| = |f(y_n) - f(z_n)| \leq \exp(\log|y| - \log|\lambda| + 1)|y_n - z_n|.$$

Therefore, $y = \lim_{n\to\infty} f(z_n)$, and consequently $y \in \overline{O_\lambda(0)}$. This however contradicts (3.5) and (3.7). We are done. ■

Let $\sim$ be an equivalence relation on $\mathbb{C} \times \mathbb{C}$ determined by the requirement that $w \sim z$ iff $z - w \in 2\pi i\mathbb{Z}$. Denote by $[z]$ the equivalence class of z. For every $R > 0$ let

$$w(R) = \{(a, z) \in \mathbb{C} \times \mathbb{C} : \mathrm{dist}(0, [z]) \geq R \text{ and } \mathrm{dist}(a, [z]) \geq R\}.$$

Define the function $\alpha : w(R) \to [0, \infty)$ by the formula

$$\alpha(a, z) = \sum_{w \in [z]} \frac{1}{|w||w - a|}.$$

We shall need the following technical lemma. The proof is rather straightforward, but technically involved. It is therefore postponed to Section 6.

Lemma 3.3. *For every $R > 0$ the supremum*

$$M(R) = \sup\{\alpha(a,z) : (a,z) \in w(R)\}$$

is finite.

Since $f^{-1}(\mathbb{C} \setminus \overline{O_\lambda(0)}) \subset \mathbb{C} \setminus (\overline{O_\lambda(0)})$, we can consider the "operator" T defined by formula (3.1) acting on functions $g : \mathbb{C} \setminus \overline{O_\lambda(0)} \to \mathbb{C}$. We shall prove the following.

Lemma 3.4. *If $\sum_{n=0}^{\infty} |(f^n)'(0)|^{-1} < \infty$, $\sum_{n=0}^{\infty}((f^n)'(0))^{-1} = 0$ and $\overline{O}_\lambda(0)$ is a nowhere dense subset of $\mathbb{C}$, then the function*

$$\phi(z) = \sum_{k=1}^{\infty} \frac{1}{(f^{k-1})'(0)} \frac{1}{z - f^k(0)}$$

is well-defined on $\mathbb{C} \setminus \overline{O}_\lambda(0)$, $T(\phi)$ is also well-defined on $\mathbb{C} \setminus \overline{O_\lambda(0)}$ and $T(\phi) = \phi$.

Proof. The fact that ϕ is well-defined on $\mathbb{C} \setminus \overline{O_\lambda(0)}$ follows from absolute convergence of the series $\sum_{k=1}^{\infty}((f^{k-1})'(0))^{-1}$ and from the fact that if $z \in \mathbb{C} \setminus \overline{O_\lambda(0)}$, then $\mathrm{dist}(z, \overline{O_\lambda(0)}) > 0$. Suppose now that $z \in \mathbb{C} \setminus \overline{O_\lambda(0)}$, and let $r = \mathrm{dist}(z, \overline{O_\lambda(0)})$. Then $r > 0$ and, in view of Lemma 3.2, $R = \mathrm{dist}(f^{-1}(z), \overline{O_\lambda(0)}) > 0$. It therefore follows from Lemma 3.3 and our first assumption that the series

$$\sum_{k=1}^{\infty} \sum_{w \in f^{-1}(z)} \frac{1}{(f^{k-1})'(0)} \frac{1}{w(w - f^k(0))}$$

converges absolutely. Hence, for every $n \geq 1$ we can apply the operator T to the function $\sum_{n+1}^{\infty} \frac{1}{(f^{k-1})'(0)} \cdot \frac{1}{z - f^k(0)}$, and we get

$$\begin{aligned}
T\left(\sum_{n+1}^{\infty} \frac{1}{(f^{k-1})'(0)} \cdot \frac{1}{z - f^k(0)}\right) &= \frac{1}{z} \sum_{w \in f^{-1}(z)} \frac{1}{w} \sum_{k=n+1}^{\infty} \frac{1}{(f^{k-1})'(0)} \frac{1}{w - f^k(0)} \\
&= \frac{1}{z} \sum_{w \in f^{-1}(z)} \sum_{k=n+1}^{\infty} \frac{1}{(f^{k-1})'(0)} \frac{1}{w(w - f^k(0))} \\
&= \frac{1}{z} \sum_{k=n+1}^{\infty} \sum_{w \in f^{-1}(z)} \frac{1}{(f^{k-1})'(0)} \frac{1}{w(w - f^k(0))}.
\end{aligned}$$

Since the sum $\sum_{w \in f^{-1}(z)} \frac{1}{w(w - f^k(0))}$ is bounded by a constant (depending on z), we conclude that

$$\lim_{n \to \infty} T\Big(\sum_{k=n+1}^{\infty} \frac{1}{(f^{k-1})'(0)} \frac{1}{z - f^k(0)}\Big) = 0. \tag{3.8}$$

Combining this along with (3.4), linearity of T, and our second assumption ($\sum_{n=0}^{\infty}((f^n)'(0))^{-1} = 0$), we see that

$$\begin{aligned} T\phi(z) &= T\left(\sum_{k=1}^{\infty} \frac{1}{(f^{k-1})'(0)} \frac{1}{z-f^k(0)}\right) \\ &= T\left(\sum_{k=1}^{n} \frac{1}{(f^{k-1})'(0)} \frac{1}{z-f^k(0)}\right) + T\left(\sum_{k=n+1}^{\infty} \frac{1}{(f^{k-1})'(0)} \frac{1}{z-f^k(0)}\right) \\ &= \sum_{k=1}^{n+1} \frac{1}{(f^{k-1})'(0)} \frac{1}{z-f^k(0)} - \frac{1}{z-f(0)} \sum_{k=0}^{n} \frac{1}{(f^k)'(0)} + T\left(\sum_{k=n+1}^{\infty} \frac{1}{(f^{k-1})'(0)} \frac{1}{z-f^k(0)}\right). \end{aligned}$$

Passing to the limit with $n \to \infty$ and using (3.8), we get

$$T\left(\sum_{k=1}^{\infty} \frac{1}{(f^{k-1})'(0)} \frac{1}{z-f^k(0)}\right) = \sum_{k=1}^{\infty} \frac{1}{(f^{k-1})'(0)} \frac{1}{z-f^k(0)} - \frac{1}{z-f(0)} = \phi(z)$$

We are done. ■

4. The Ruelle operator R and its fixed point ψ

Given any function $g : \mathbb{C} \setminus \overline{O_\lambda(0)}$, put

$$Rg(z) = \frac{1}{z^2} \sum_{w \in f^{-1}(z)} g(w)$$

for all those $z \in \mathbb{C} \setminus \overline{O_\lambda(0)}$ for which the series $\sum_{w\in f^{-1}(0)} g(w))$ converges. The function

$$\psi(z) = \frac{1}{z}\phi(z) = \frac{1}{z} \sum_{k=1}^{\infty} \frac{1}{(f^{k-1})'(0)} \frac{1}{z-f^k(0)} \tag{4.1}$$

is well-defined throughout $\mathbb{C} \setminus \overline{O_\lambda(0)}$. Lemma 3.4 easily implies the following.

Corollary 4.1. *The function $R\psi$ is well-defined on $\mathbb{C} \setminus \overline{O_\lambda(0)}$ and $R(\psi) = \psi$*

Proof. Take $z \in \mathbb{C} \setminus \overline{O_\lambda(0)}$. Applying Lemma 3.4, we get

$$R\psi(z) = \frac{1}{z^2} \sum_{w \in f^{-1}(z)} \frac{\phi(w)}{w} = \frac{1}{z}\phi(z) = \psi(z).$$

The proofs of the following important propositions are postponed to Section 6.

Proposition 4.2. *If the series $\sum_{k=1}^{\infty} \frac{1}{(f^{k-1})'(0)}$ converges absolutely and its sum is equal to zero, then the function $\psi : \mathbb{C} \setminus \overline{O_\lambda(0)} \to \mathbb{C}$ given by formula 4.1 is integrable with respect to the Lebesgue measure on $\mathbb{C} \setminus \overline{O_\lambda(0)}$.*

Proposition 4.3. *Assume that the series $\sum_{n=1}^{\infty} \frac{1}{(f^{n-1})'(0)}$ converges absolutely and its sum is equal to zero. If $\overline{O_\lambda(0)}$ is a nowhere dense set with* $\mathrm{Leb}(\overline{O_\lambda(0)} = 0$ *or the trajectory of* 0 *is non-recurrent then the function $\psi : \mathbb{C} \setminus \overline{O_\lambda(0)} \to \mathbb{C}$ is not equal to zero identically.*

5. Conclusion: Instability

We show the instability in two cases: if the trajectory of 0 is non-recurrent or if $\overline{O_\lambda(0)}$ is a nowhere dense set with $\mathrm{Leb}(\overline{O_\lambda(0)}) = 0$. In both cases we show that the function ψ cannot exist. This implies that the sum $\sum_{k=1}^{\infty} \frac{1}{(f^{k-1})'(0)}$ is not equal to zero, thus, by Proposition 2.2, the parameter λ is unstable. Let us modify the function ψ slightly. Put

$$\hat{\psi} = \begin{cases} \psi(z) \text{ if } z \in \mathbb{C} \setminus \overline{O_\lambda(0)} \\ 0 \text{ if } z \in \overline{O_\lambda(0)} \end{cases}$$

We see that if $z \in \mathbb{C} \setminus \overline{O_\lambda(0)}$, then $R(\hat{\psi})(z) = \psi(z) = \hat{\psi}(z)$, while for $z \in \overline{O_\lambda(0)}$ we have $|R(\hat{\psi}(z))| \geq |\hat{\psi}(z)|$. Let $|R|$ be the usual Ruelle operator given by the formula

$$|R|(g)(z) = \frac{1}{|z^2|} \sum_{w \in f^{-1}(z)} g(w).$$

Since $|R(\hat{\psi})| \geq |\hat{\psi}|$, we conclude that $|R|(|\psi|) \geq |\psi|$. But, on the other hand, the Ruelle operator $|R|$ preserves the integral, thus $|R|(|\psi|) = |\psi|$ a.e. Since $|\psi|$ and $|R|(|\psi|)$ are continuous in $\mathbb{C} \setminus \overline{O_\lambda(0)}$, we have $|R|(|\psi|) = |\psi|$ everywhere in $\mathbb{C} \setminus \overline{O_\lambda(0)}$. Let $z \in \mathbb{C} \setminus \overline{O_\lambda(0)}$. Since $\psi(z) = \frac{1}{z^2} \sum_{w \in f^{-1}(z)} |\psi(w)|$ for every $z \notin \overline{O_\lambda(0)}$ and

$$|\psi(z)| = \frac{1}{|z^2|} \sum_{w \in f^{-1}(z)} |\psi(w)|$$

almost everywhere, thus (by continuity) everywhere in $\mathbb{C} \setminus \overline{O_\lambda(0)}$, we conclude that

$$\psi(w) = z^2 \psi(z) k(w) \tag{5.1}$$

with some $0 \leq k(w) \leq 1$, for every $z \notin \overline{O_\lambda(0)}$ and for every $w \in f^{-1}(\{z\})$.

Let us assume that $\psi(z) = 0$ for some $z \notin \overline{O_\lambda(0)}$. Then using (5.1) we conclude that $\psi(w) = 0$ for every w such that$f(w) = z$ and, by induction, $\psi \equiv 0$ on the set $\Lambda = \bigcup_n f^{-n}(\{z\})$. But, since $z \neq 0$, the set Λ is dense in $\mathbb{C} = J(f_\lambda)$, which implies that $\psi \equiv 0$ everywhere in $\mathbb{C} \setminus \overline{O_\lambda(0)}$. By Proposition 4.2 this is impossible. Now, we are ready to prove the following

Proposition 5.1. *If the trajectory of* 0 *is non recurrent and the series $\sum_{k=1}^{\infty} \frac{1}{(f^{k-1})'(0)}$ converges absolutely then the parameter λ is unstable.*

Proof. We shall show that the function ψ cannot exist. Indeed, by the above reasoning we would have

$$k(w) z^2 \psi(z) = \psi(w),$$

where $z = f(w)$ and the function $k(w)$ takes only real values. On the other hand the equation (5.1) shows that the function k is holomorphic on every component of $\mathbb{C} \setminus f^{-1}(\overline{O_\lambda(0)})$. Therefore k is constant on every component of $\mathbb{C} \setminus f^{-1}(\overline{O_\lambda(0)})$. Since

$$z^2\psi(z) = \sum_{w \in f^{-1}(z)} \psi(w) = \sum_w k(w) z^2 \psi(z),$$

we see that

$$\sum_{w \in f^{-1}(z)} k(w) = 1. \tag{5.2}$$

Now, since the trajectory of 0 is non-recurrent there exists $\varepsilon > 0$ such that for $z \in B(0, \varepsilon)$ the set $\{w : w \in f^{-1}(\{z\})\}$ is contained in the same component of $\mathbb{C} \setminus f^{-1}(\overline{O_\lambda(0)})$. Thus, the number $k(w)$ is the same for all $w \in f^{-1}(z)$. Obviously, this implies that (5.2) cannot be satisfied, since the set $f^{-1}(z)$ is infinite. ■

Proposition 5.2. *If the series $\sum_{k=1}^{\infty} \frac{1}{|(f^k)'(0)|}$ converges and $\overline{O_\lambda(0)}$ is a nowhere dense set with* $\mathrm{Leb}(\overline{O_\lambda(0)}) = 0$*, then the parameter λ is unstable.*

Proof. Again, we check that the function ψ cannot exist. Since $\psi(z) \neq 0$ for every $z \in \mathbb{C} \setminus \overline{O_\lambda(0)}$, the harmonic function $\eta(z) = \arg\psi(z)$ is defined (locally) in a neighbourhood of every point $z_0 \in \mathbb{C} \setminus \overline{O_\lambda(0)}$.

Lemma 5.3. *For every $z_0 \in \overline{O_\lambda}$, $z_0 \neq 0$ there exists a point w such that $f^n(w) = z_0$ for some n and $w \notin \overline{O_\lambda(0)}$.*

Proof. Indeed, the set $\bigcup f^{-n}(z_0)$ is dense in $\mathbb{C} \setminus \overline{O_\lambda(0)}$ while $\overline{O_\lambda(0)}$ is nowhere dense. ■

Next, we show that the function η can be extended in a nice way.

Proposition 5.4. *For every $z_0 \in \overline{O_\lambda(0)}$ there exists a neighbourhood $V = V(z_0)$ and a harmonic function θ defined in V such that $\theta(z) - \eta(z) = 2l(z)\pi i$ where $l(z)$ is an integer and the function $l(z)$ is constant on every component of $V \cap (\mathbb{C} \setminus \overline{O_\lambda(0)})$.*

Proof. Let $z_0 \in \overline{O_\lambda(0)}$ and assume that there exists a point w_0 such that $f(w_0) = z_0$ and $w_0 \notin \overline{O_\lambda(0)}$. Let f_0^{-1} be the branch of f^{-1} mapping the point z_0 to w_0. Then the equation (5.1) shows that the formula

$$\eta(z) = \eta(f_0^{-1}(z)) - 2\mathrm{Arg} z \tag{5.3}$$

defines the harmonic function in a neighbourhood of z_0 such that

$$\mathrm{Arg}\psi(z) = [\eta(z)]_{\mod 2\pi} \tag{5.4}$$

In general, let k be the smallest positive integer for which there exists a point w_0 such that $f^k(w_0) = z_0$ and $w_0 \notin \overline{O_\lambda(0)}$. Using consecutive branches of f^{-i}, $i \le k$ we define the function η in a neighbourhood of $f^i(w_0)$, $i \le k$ such that (5.3) and (5.4) are satisfied. Thus, the conclusion is the following: For every $z_0 \in \mathbb{C}$, $z_0 \neq 0$ there exists a neighbourhood (a ball with center at z_0) V_{z_0} and a function η defined in V_{z_0} such that for every $z \in (\mathbb{C} \setminus \overline{O_\lambda(0)}) \cap V_{z_0}$, $\eta(z)$ is an argument of $\psi(z)$. Looking at the equation (5.1) again, we see that the formula (5.3) defines also the function η in the neighbourhood of 0; for w close to 0 we put $\eta(w) = \eta(f(w)) + 2\mathrm{Arg} f(w)$ ($f(w)$ is close to $f(0)$ so the argument is well-defined). Let γ be the harmonic conjugate to η; more precisely: for every z_0 and the corresponding neighbourhood V_{z_0} we consider the holomorphic function $\tau_{z_0} = \gamma + i\eta$ defined in V_{z_0}. Now, if $V_{z_0} \cap V_{z_1} \neq \emptyset$ then we have two functions on $V_{z_0} \cap V_{z_1}$: $\tau_{z_0} = \gamma_0 + i\eta_0$ and $\tau_{z_1} = \gamma_1 + i\eta_1$. Consider the difference $\tau_{z_0} - \tau_{z_1}$. Since each function η is an argument of ψ we conclude that

$$\mathrm{Im}(\tau_{z_1} - \tau_{z_1}) \in \{2k\pi,\ k \in \mathbb{Z}\}.$$

But this implies that $\tau_{z_0} - \tau_{z_1}$ is constant in $V_{z_0} \cap V_{z_1}$. Using the Monodromy Theorem we see that there exists a globally defined function $\tau : \mathbb{C} \to \mathbb{C}$ such that for $z \in \mathbb{C} \setminus \overline{O_\lambda(0)}$, $\mathrm{Im}\tau(z)$ is an argument of $\psi(z)$. Consider the function $G = \exp(\frac{1}{2}\tau)$. Notice that there is a close relation between G and ψ. Namely,

$$\frac{G^2}{\psi} = \frac{\exp(\tau)}{\psi} = \frac{\exp(\gamma + i\eta)}{|\psi| \exp(i\mathrm{Arg}\psi)} = \frac{\exp(\gamma)}{|\psi|} \cdot \exp(i\eta - i\mathrm{Arg}\psi) = \frac{\exp(\gamma)}{|\psi|}$$

Thus, the function $\frac{G^2}{\psi}$ takes only real values. Consequently, it is constant on every connected component of $\mathbb{C} \setminus \overline{O_\lambda(0)}$. This also implies, using the formula (5.1), that the function

$$w \mapsto \left(\frac{G(f(w))}{G(w)} \cdot f'(w)\right)^2 = \left(\frac{G(f(w))}{G(w)} \cdot f(w)\right)^2 \tag{5.5}$$

takes only real values in $\mathbb{C} \setminus \overline{O_\lambda(0)}$. Since this function is globally holomorphic and the set $\mathbb{C} \setminus \overline{O_\lambda(0)}$ is dense, we conclude that the function $\frac{G(f(w))}{G(w)} \cdot f'(w)$ is, actually, constant. Let $\hat{G}$ be the the primitive function of G. Then $\hat{G}'(z) \neq 0$ for every $z \in \mathbb{C}$. We shall consider two cases:

Case I. $\hat{G}(\mathbb{C}) = \mathbb{C}$. Then $\hat{G}$ is a conformal covering, thus a conformal homeomorphism and it must be of the form $\hat{G}(z) = Cz + D$ for some $C, D \in \mathbb{C}$. However,

$$\left((\hat{G} \circ f \circ \hat{G}^{-1})'(z)\right)^2 = \left(\frac{G(f \circ \hat{G}^{-1}(z))}{G(\hat{G}^{-1}(z))} \cdot f'(\hat{G}^{-1}(z))\right)^2$$

and we see that $(\hat{G} \circ f \circ \hat{G}^{-1})'(z)$ would be constant and, consequently, $\hat{G} \circ f \circ \hat{G}^{-1}(z) = az + b$ for some $a, b \in \mathbb{C}$. Clearly, this is impossible.

Case II. $\hat{G}(\mathbb{C}) \neq \mathbb{C}$. The only possibility is that $\hat{G}(\mathbb{C}) = \mathbb{C} \setminus \{p\}$ for some p. Again, $\hat{G}$ is a covering. The map $\pi : \mathbb{C} \to \mathbb{C} \setminus \{p\}$, $\pi(z) = \exp(z) + p$ is another covering. Thus, there exists a lift $\tilde{G} : \mathbb{C} \to \mathbb{C}$ such that $\pi \circ \tilde{G} = \hat{G}$. Again, $\tilde{G}$ is a conformal homeomorphism, thus

$\tilde{G}(z) = Cz + D$ and $\hat{G}(z) = \pi \circ \tilde{G}(z) = \exp(Cz + D) + p$. Thus, $\hat{G}'(z) = C\exp(Cz + D) = \exp(Cz + D + \log C) = \exp 2(Cz + D')$ for some constants $C, D' \in \mathbb{C}$. On the other hand, by construction, $\hat{G}' = \exp(\frac{1}{2}\tau)$ and we conclude that $\tau(z) = Cz + D$ for some constants C, D. But we already know that the function $\frac{G(f(w))(f'(w))^2}{G(w)}$ is constant. This cannot be true in this case since

$$\frac{G(f(w))(f'(w))^2}{G(w)} = \frac{\exp(\frac{1}{2}\tau(f(w)))(f'(w))^2}{\exp(\frac{1}{2}\tau(w))} = \frac{\exp(\frac{1}{2}(C\lambda\exp(w) + D))(\lambda\exp(w))^2}{\exp(\frac{1}{2}(Cw + D))}$$

is, obviously, not constant. This contradiction ends the proof. ■

6. Postponed proofs

Proof of Lemma 3.3. The proof is rather straightforward (although technically involved). First notice that if $(a, z) \in w(R)$ then $[a] \times [z] \subset w(R)$. Then observe that the function α is constant on each set of the form $\{a\} \times [z]$, $(a, z) \in w(R)$. Therefore

$$M(R) = \sup\{\alpha(a, z) : (a, z) \in \tilde{w}(R)\},$$

where

$$\tilde{w}(R) = \{(a, z) \in \mathbb{C} \times Q : \text{dist}(0, [z]) \geq R \text{ and } \text{dist}(a, [z]) \geq R$$

and $Q = \mathbb{R} \times [-\pi, \pi]$. Now, fix $R \in \mathbb{R}$ and $z \in Q$ with $\text{dist}(0, [z]) \geq R$. Define

$$A_+(v, z) = \{t \in [\text{Re}z, +\infty) : \text{dist}(t + iv, [z]) \geq R\}$$

and

$$A_-(v, z) = \{t \in [-\infty, \text{Re}z) : \text{dist}(t + iv, [z]) \geq R\}.$$

Notice that $A_+(v, z)$ and $A_-(v, z)$ are infinite intervals:

$$A_+(v, z) = [a_+(v, z), \infty),\ A_-(v, z) = (-\infty, a_-(v, z)] \tag{6.1}$$

The function $t \mapsto \alpha(t + iv, z)$, $t \geq a_+(v, z)$ is decreasing and the function $t \mapsto \alpha(t + iv, z)$, $t \leq a_-(v, z)$ is increasing. So, defining $M(v, z)$ to be the maximal value of $\alpha(a, z)$, where $\text{dist}(a, [z]) \geq R$ and the imaginary part of a is fixed (and equal to v), we see that

$$M(v, z) = \sup\{\alpha(t + iv, z) : t \in \mathbb{R}\} = \max\{\alpha(a_+(v, z), z), \alpha(a_-(v, z)z)\} \tag{6.2}$$

and both points $(a_+(v, z), z), (a_-(v, z)z)$ belong to $\tilde{w}(R)$. Now, given $z \in Q_+ = [0, +\infty) \times [-\pi, \pi]$, define

$$B_+(z) = \{t \in [0, +\infty) : \text{dist}(0, [t + i\text{Im}z]) \geq R\}$$

and

$$b_+(z) = \inf\Big(B_+(z)\Big) \in [0, R].$$

Consider now the function $t \mapsto \alpha(a + t - \mathrm{Re}z, t + i\mathrm{Im}z), t \in B_+(z)$. A straightforward calculation shows that it is decreasing. If $\mathrm{dist}(0, [z]) \geq R$ then $b_+(z) \leq \mathrm{Re}z$. Therefore, putting $t = \mathrm{Re}z$ and using the monotonicity mentioned above, we conclude that

$$\begin{aligned} \alpha(a, z) =& \alpha(a + \mathrm{Re}z - \mathrm{Re}z, \mathrm{Re}z + i\mathrm{Im}z) \leq M_+(a, z) \\ &= \sup\{\alpha(a + t - \mathrm{Re}z, t + i\mathrm{Im}z)) : t \in B_+(z)\} \\ &= \alpha(a + b_+(z), b_+(z) + i\mathrm{Im}z)) \end{aligned} \tag{6.3}$$

Similarly, given $z \in Q_- = (-\infty, 0] \times [0, 2\pi]$, we define

$$B_-(z) = \{t \in (-\infty, 0] : \mathrm{dist}(0, [t + i\mathrm{Im}z]) \geq R\}$$

and

$$b_-(z) = \sup B_-(z) \in [-R, 0].$$

In the same way, we obtain similar inequalities:

$$\begin{aligned} \alpha(a, z) \leq M_-(a, z) &:= \sup\{\alpha(a + t - \mathrm{Re}z, t + i\mathrm{Im}z) : t \in B_-(z)\} \\ &= \alpha(a + b_-(z) - \mathrm{Re}z, b_-(z) + i\mathrm{Im}(z)) \end{aligned} \tag{6.4}$$

Note that both pairs $(a+b_\pm(z)-\mathrm{Re}z, b_\pm(z)+i\mathrm{Im}(z))$ are in $\tilde{w}(R)$. Indeed, $b_\pm(z)$ was chosen so that $\mathrm{dist}([b_\pm(z)+i\mathrm{Im}z], 0) > R$ and $(a+b_\pm(z)-\mathrm{Re}z)-(b_\pm(z)+i\mathrm{Im}z) = \mathrm{Re}z+i\mathrm{Im}z-a = z-a$. The latter implies that $\mathrm{dist}([b_\pm(z) + i\mathrm{Im}z], a + b_\pm(z) - \mathrm{Re}z) = \mathrm{dist}([z], a) > R$.

Combining (6.3) and (6.2), for all $(a, z) \in W(R)$ with $\mathrm{Re}z \geq 0$, we get

$$\begin{aligned} &\alpha(a, z) \leq \alpha(a + b_+(z) - \mathrm{Re}z, b_+(z) + i\mathrm{Im}z) \leq \\ &\leq \max\Big\{\alpha\Big(a_+(\mathrm{Im}a, b_+(z) + i\mathrm{Im}z), b_+(z) + i\mathrm{Im}z\Big), \alpha\Big(a_-(\mathrm{Im}a, b_+(z) + i\mathrm{Im}z), b_+(z) + i\mathrm{Im}z\Big)\Big\}. \end{aligned}$$

Similarly, if $\mathrm{Re}z \leq 0$, then

$$\alpha\Big(a, z) \leq \max\Big\{\alpha\Big(a_+(\mathrm{Im}a, b_-(z) + i\mathrm{Im}z), b_-(z) + i\mathrm{Im}z\Big), \alpha\Big(a_-(\mathrm{Im}a, b_-(z) + i\mathrm{Im}z), b_-(z) + i\mathrm{Im}z\Big)\Big\}$$

Obviously, $|\mathrm{Re}(b_\pm(z) + i\mathrm{Im}z)| \leq R$ and, therefore, $|\mathrm{Re}(a_\pm(\mathrm{Im}a, b_\pm(z) + i\mathrm{Im}z))| \leq 2R$ (see (6.1)).

Hence, we have checked that the following holds:

Lemma 6.1.

$$M(R) = \sup\{\alpha(a, z) : (a, z) \in \tilde{w}(R) \text{ such that } -R \leq \mathrm{Re}z \leq R \text{ and } -2R \leq \mathrm{Re}a \leq 2R\} \tag{6.5}$$

Now, we shall prove the following.

Lemma 6.2. *If $w, a \in \mathbb{C}$ and $k \in \mathbb{Z}$ with $|k| \geq \frac{1}{\pi}|\mathrm{Im}(w - a)|$, then*

$$|w(w - a)| \leq |w||w - (a + 2k\pi i)|.$$

Proof. This is a straightforward computation. Indeed, we have $2|k|\pi \geq 2|\mathrm{Im}(w-a)|$. Therefore, $(2k\pi)^2 \geq 2(2k\pi\mathrm{Im}(w-a))$. Consequently,

$$(\mathrm{Im}(w-a)-2k\pi)^2 = \mathrm{Im}^2(w-a) - 2(2k\pi\mathrm{Im}(w-a)) + (2k\pi)^2 \geq \mathrm{Im}^2(w-a)$$

Hence

$$\begin{aligned}|w-a|^2 &= \mathrm{Re}^2(w-a) + \mathrm{Im}^2(w-a) \leq \mathrm{Re}^2(w-a) + (\mathrm{Im}(w-a)-2k\pi)^2 \\ &= \mathrm{Re}^2(w-(a+2k\pi i)) + \mathrm{Im}^2(w-(a+2k\pi i)) = |w-(a+2k\pi i)|^2\end{aligned}$$

■

Our second claim is the following.

Lemma 6.3. *If* $(a,w) \in \tilde{w}(R)$ *with* $a \in [0,2\pi] \times [0,2\pi]$ *and* $k \in \mathbb{Z}$ *with* $|k| \leq \frac{1}{\pi}|\mathrm{Im}(w-a)|$ *then*

$$|w-(a+2k\pi i)| \cdot |w| \geq C|(w-2k\pi i)-a| \cdot |w-2k\pi i|, \tag{6.6}$$

where $C = \left(2 \cdot (1+(1+2\pi\sqrt{2}R^{-1}))\right)^{-1}$.

Proof. Indeed, after cancelations, this inequality means that

$$|1-\frac{2k\pi i}{w}| \leq 2 \cdot \left(1+(1+2\pi\sqrt{2}R^{-1})\right). \tag{6.7}$$

Since $|w| \geq R$ and $|a| \leq 2\sqrt{2}\pi$, we get

$$\frac{|\mathrm{Im}(w-a)|}{|w|} \leq \frac{|w-a|}{|w|} = |1-\frac{a}{w}| \leq 1+\frac{|a|}{|w|} \leq 1+\frac{2\pi\sqrt{2}}{R}$$

So, using our hypothesis, we get that

$$|1-\frac{2k\pi i}{w}| \leq 1+\frac{2|k|\pi}{|w|} \leq 1+(1+2\pi\sqrt{2}R^{-1})\frac{2|k|\pi}{|\mathrm{Im}(w-a)|} \leq 2 \cdot \left(1+(1+2\pi\sqrt{2}R^{-1})\right)$$

Thus, (6.7) and, consequently also (6.6) are proved. ■

Take now an arbitrary point $(b,z) \in \tilde{w}(R)$ with $\mathrm{Re}b \in [-2R,2R]$, $z \in [-R,R] \times [0,2\pi]$. Write $b = a+2\pi ik$, $k \in \mathbb{Z}$, with $\mathrm{Im}(a) \in [0,2\pi]$. Note that $(a,z) \in \tilde{w}(R)$. Making use of (6.6), we obtain

$$\begin{aligned}\alpha(b,z) &= \sum_{w\in[z],|\mathrm{Im}(w-a)|<\pi k} \frac{1}{|w||w-(a+2\pi ik)|} + \sum_{w\in[z],|\mathrm{Im}(w-a)|\geq\pi k} \frac{1}{|w||w-(a+2\pi ik)|} \\ &\leq \sum_{w\in[z],|\mathrm{Im}(w-a)|<\pi k} \frac{1}{|w||w-a|} + C\sum_{w\in[z],|\mathrm{Im}(w-a)|\geq\pi k} \frac{1}{|w-2k\pi i| \cdot |(w-2k\pi i)-a|} \qquad (6.8) \\ &\leq \alpha(a,z) + C\alpha(a,z) = (1+C)\alpha(a,z)\end{aligned}$$

Therefore, we get that

$$M(R) \leq (1+C)\sup\{\alpha(a,z) : (a,z) \in \tilde{w}(R) \cap ([-2R,2R]\times[0,2\pi]) \times ([-R,R]\times[0,2\pi])\}.$$

Since the function $\alpha(a,z)$ is continuous and since the set over which the supremum is taken in the last formula is compact, we conclude that $M(R) < \infty$ and the proof of Lemma 3.3 is finished. ■

Next, we prove Propositions 4.2 and 4.3.

Proof of Proposition 4.2. We need some preparation. Fix $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$, $b \neq a$. Let

$$g_a(b) = \int\int_{\mathbb{C}} \frac{|b-a|}{|z||z-b||z-a|} dA(z).$$

Since $\int_1^\infty \frac{dr}{r^2} < \infty$, it is easy to calculate, using polar coordinates, that $g_a(b)$ is finite for all $b \in \mathbb{C} \setminus \{0, a\}$. Notice that (using a new coordinate $v = z/a$)

$$g_a(b) = \left|1 - \frac{b}{a}\right| \int\int_{\mathbb{C}} \frac{1}{|v||v-1||v-\frac{b}{a}|} dA(v).$$

Thus, in order to estimate $b_a(b)$ it is enough to look at

$$g(b) := g_1(b) = |b-1| \int\int_{\mathbb{C}} \frac{1}{|z||z-b||z-1|} dA(z)$$

with $b \notin \{0,1\}$. Notice that (using a new coordinate $w = 1/z$) we get

$$g(b) = |1-b| \int\int \frac{dA(w)}{|\frac{1}{w}||\frac{1}{w}-1||\frac{1}{w}-b|} \cdot \frac{1}{|w|^4} = \left|\frac{1-b}{b}\right| \int\int \frac{dA(w)}{|w||1-w||\frac{1}{b}-w|} = g\left(\frac{1}{b}\right)$$

Thus, it is enough to consider b with $|b| \leq 1$. For every $\varepsilon > 0$ the function $g(b)$ is continuous in the compact set $L_\varepsilon = \overline{B}(0,1) \setminus (B(0,\varepsilon) \cup B(1,\varepsilon))$. So, for every $\varepsilon > 0$, $g_{|L_\varepsilon}$ is bounded by some constant C_ε and we are to estimate $g(b)$ for b close to 0 and b close to 1. Take $b \in B(0,\varepsilon)$. Write $g(b)$ as a sum of integrals over three regions: $\{|w| < 10|b|\}$, $\{10|b| \leq |w| < 2\}$, and

$\{|w| \geq 2\}$. We shall estimate these summands separately. First,

$$\begin{aligned}
|1-b| \int\int_{|w|<10|b|} \frac{1}{|w|} \frac{1}{|w-1|} \frac{1}{|w-b|} dA(w) &= \\
&= \int\int_{|w|<10|b|} \frac{1}{|w|} \left| \frac{1}{w-1} - \frac{1}{w-b} \right| dA(w) \\
&\leq \int\int_{|w|<10|b|} \frac{1}{|w||w-1|} dA(w) + \int\int_{|w|<10|b|} \frac{1}{|w||w-b|} dA(w) \\
&\leq \frac{1}{1-10|b|} \cdot 2\pi \cdot \int_{r=0}^{10|b|} \frac{1}{r} r dr + \frac{1}{|b|} \int\int_{|w|<10|b|} \left| \frac{1}{w} - \frac{1}{w-b} \right| dA(w) \qquad (6.9) \\
&\leq \frac{2\pi}{1-10\varepsilon} \cdot 10|b| + \frac{1}{|b|} \left(\int_{|w|<10|b|} \frac{1}{|w|} dA(w) + \int_{|w|<10b} \frac{1}{|w-b|} dA(w) \right) \\
&\leq \frac{2\pi}{1-10\varepsilon} \cdot 10|b| + \frac{1}{|b|} \cdot 2\pi \cdot (10|b| + 11|b|) \leq \text{const},
\end{aligned}$$

where the constant can be made independent of ε if, say, $\varepsilon < \frac{1}{20}$. Next, we estimate the second integral:

$$\begin{aligned}
|1-b| \int\int_{10|b|<|w|<2} \frac{dA(w)}{|w||w-1||w-b|} &= \int\int_{10|b|<|w|<2} \frac{dA(w)}{|w|} \left| \frac{1}{w-1} - \frac{1}{w-b} \right| \\
&\leq \int\int_{10|b|<|w|<2} \frac{dA(w)}{|w||w-1|} + \int\int_{10|b|<|w|<2} \frac{dA(w)}{|w||w-b|}.
\end{aligned}$$

The first integral in the above sum is bounded by

$$\begin{aligned}
\int\int_{10|b|<|w|<2} \frac{dA(w)}{|w||w-1|} &\leq \int\int_{|w|<2} \frac{dA(w)}{|w|} + \int\int_{|w|<2} \frac{dA(w)}{|w-1|} \\
&\leq \int\int_{|w|<2} \frac{dA(w)}{|w|} + \int\int_{|w|<3} \frac{dA(w)}{|w|} \\
&= 4\pi + 6\pi = 10\pi.
\end{aligned}$$

Write the second integral as

$$\int\int_{10|b|<|w|<2} \frac{dA(w)}{|w||w-b|} = \int\int_{10|b|<|w|<2} \frac{dA(w)}{|w|^2 \left|1 - \frac{b}{w}\right|}.$$

Now, since $|w| > 10|b|$, we see that $|1 - \frac{b}{w}| > \frac{9}{10}$, and finally we can estimate this integral by

$$\frac{10}{9} \int\int_{10|b|<|w|<2} \frac{dA(w)}{|w|^2} \leq \frac{10}{9} \cdot 2\pi \int_{r=10|b|}^{2} \frac{dr}{r} \leq C_1 + C_2 \log \frac{1}{|b|}.$$

where C_1, C_2 are some constants. It remains to estimate the integral over the region $\{|w| > 2\}$. This is simple: if, say $|b| < 1$, we can write

$$|1-b| \int\int_{|w|>2} \frac{1}{|w||w-1||w-b|} \leq 2 \int\int_{|w|>2} \frac{1}{|w|^3 |1-\frac{1}{w}||1-\frac{b}{w}|} \leq 2 \cdot 4 \cdot \int\int_{|w|>2} \frac{1}{|w|^3} = 8\pi.$$

Thus, we can write the following estimate, valid in the ball $b \in B(0,\varepsilon)$:

$$g(b) \le C_1 + C_2 \log \frac{1}{|b|},$$

where C_1 and C_2 are constants. It remains to look at the behaviour of the function $g(b)$ in the ball $B(1,\varepsilon)$. It is easy to see that $g(b)$ is bounded in the neighbourhood of 1 since, again, the integral

$$\int\int_{|w|>2} \frac{dA(w)}{|w||w-1||w-b|}$$

is bounded uniformly, while the remaining part

$$|1-b| \int\int_{|w|\le 2} \frac{dA(w)}{|w||w-b||w-1|}$$

can be written as

$$\int\int_{|w|\le 2} \frac{1}{|w|}\left|\frac{1}{w-b} - \frac{1}{w-1}\right| dA(w) \le \int\int_{|w|\le 2} \frac{1}{|w|}\left|\frac{1}{w-b}\right| dA(w) + \int\int_{|w|\le 2} \frac{1}{|w|}\left|\frac{1}{w-1}\right| dA(w).$$

Since both integrals above are bounded independently of $b \in B(1,\varepsilon)$, we are done.

We summarize the above considerations in the following lemma.

Lemma 6.4. *There are constants C_1, C_2 such that*

$$g(b) \le C_1 + C_2 \left|\log|b|\right|.$$

Similarly,

$$g_a(b) \le C_1 + C_2 \left|\log\left|\frac{b}{a}\right|\right|.$$

Now, integrability of the function $\psi : \mathbb{C} \setminus \overline{O_\lambda(0)}$ is easy. Indeed, since $\sum_{k=0}^{\infty}((f^{k-1})'(0))^{-1} = 0$, it follows from (4.1) that

$$\psi(z) = \frac{1}{z}\sum_{k=1}^{\infty} \frac{1}{(f^{k-1})'(0)}\left(\frac{1}{z-f^k(0)} - \frac{1}{z-f(0)}\right) = \sum_{k=2}^{\infty} \frac{1}{(f^{k-1})'(0)} \frac{f^k(0)-f(0)}{z(z-f^k(0))(z-f(0))}.$$

So, $\int_{\mathbb{C}} |\psi| dA$ can be estimated by

$$\sum_{k=1}^{\infty} \frac{1}{|(f^{k-1})'(0)|} \cdot g_a(f^k(0)),$$

where $a = f(0)$. Using Lemma 6.4, we can write $g_a(f^k(0)) \le \left(C_1 + C_2 \log|f^k(0)|\right)$. But $|f^k(0)| = |\lambda||\exp(f^{k-1}(0))| = |\lambda| \exp \mathrm{Re}(f^{k-1}(0))|$. So,

$$\log|f^k(0)| = \log|\lambda| + \mathrm{Re}(f^{k-1}(0))$$

and

$$\left|\log|f^k(0)\right| \le |\log|\lambda|| + \left|\mathrm{Re} f^{k-1}(0)\right| \le |\log|\lambda|| + |f^{k-1}(0)|.$$

This gives us the following estimate

$$g_a\left(f^k(0)\right) \le C_1 + C_2\left(|\log|\lambda|| + |f^{k-1}(0)|\right) \le C_3 + C_2|f^{k-1}(0)|,$$

where C_3 is another constant. Finally,

$$\int\int|\psi|dA \le \sum_{k=1}^{\infty}|\frac{1}{(f^{k-1})'(0)}|\left(C_3 + C_2|f^{k-1}(0)|\right) = \sum_{k=1}^{\infty}\left(\frac{C_3}{|(f^{k-1})'(0)|} + \frac{C_2}{|(f^{k-2})'(0)|}\right).$$

Since the sum $\sum_{k=1}^{\infty}|\frac{1}{(f^{k-1})'(0)}|$ is finite, the proof of integrability is finished. ■

Proof of Proposition 4.3. The proof of Proposition 4.3 is particularly simple in the case of non-recurrent trajectory of 0, so we give it separately:

Proposition 6.5. *If the series $\sum_{n=1}^{\infty}\frac{1}{(f^{n-1})'(0)}$ converges absolutely, its sum is equal to zero and the point 0 is non-recurrent i.e. $0 \notin \overline{O_\lambda(0)}$, then the function ψ is not identically equal to zero.*

Proof. Recall that $\phi(z) = \sum_{k=1}^{\infty}\frac{1}{(f^{k-1})'(0)}\frac{1}{z-f^k(0)}$. So,

$$\phi(0) = -\sum_{k=1}^{\infty}\frac{1}{(f^{k-1})'(0)}\frac{1}{f^k(0)} = -\sum_{k=1}^{\infty}\frac{1}{(f^k)'(0)} = 1$$

since $\sum_{k=0}^{\infty}\frac{1}{(f^k)'(0)} = 0$. Thus, ψ has a simple pole with residuum equal to 1 at zero and, obviously, it is not identically equal to zero. ■

Now, we present the proof of Proposition 4.3 in the case when $\overline{O_\lambda(0)}$ is a nowhere dense set with $\mathrm{Leb}(\overline{O_\lambda(0)}) = 0$. Define $\hat{\phi} : \mathbb{C}\setminus\overline{O_\lambda(0)} \to \mathbb{R}$ by

$$\hat{\phi}(z) = \sum_{k=1}^{\infty}\frac{1}{|(f^{k-1})'(0)|}\frac{1}{|z-f^k(0)|}$$

and put $\hat{\psi}(z) = \frac{1}{|z|}\hat{\phi}(z)$. Extend the functions ϕ, $\hat{\phi}$,ψ, $\hat{\psi}$ to the whole complex plane by declaring them to be identically equal to zero on $\overline{O_\lambda(0)}$. Since the series $\sum_{n=1}^{\infty}\frac{1}{(f^{n-1})'(0)}$ converges absolutely, we see that

$$\int\int_{B(0,|f(0)|)}\hat{\phi}(z)|dz|^2 < \infty. \tag{6.10}$$

and there exists $k \ge 1$ such that

$$\sum_{j=k+1}^{\infty}\frac{1}{|(f^{j-1})'(0)|} \le 1/9. \tag{6.11}$$

Since $f(0)$ is not a periodic point of f, there exists $R \in (0, |f(0)|)$ so small that

$$B(f(0), R) \cap O_\lambda(0) \subset \{f(0)\} \cup \{f^j(0) : j \ge k+1\}. \tag{6.12}$$

For every $n \geq 1$ put

$$\phi_n(z) = \sum_{k=1}^{n} \frac{1}{(f^{k-1})'(0)} \frac{1}{(z - f^k(z))}$$

for all $z \in \mathbb{C} \setminus \overline{O_\lambda(0)}$ and $\hat{\phi}_n(z) = 0$ for all $z \in \overline{O_\lambda(0)}$. It follows form (6.10) and Fubini's theorem that there exists a measurable set $L \subset [0, R]$ such that the linear measure of L equals R and

$$\int_{\partial B(f(0),r)} \hat{\phi}(z)|dz| < \infty \tag{6.13}$$

for every $r \in L$. Since $|\phi_n(z)| \leq \hat{\phi}(z)$ and ϕ_n converges pointwise to ϕ in $\mathbb{C}$, in particular in $B(F(0), R)$, applying (6.13) and Lebesgue dominated convergence theorem, we conclude that

$$\lim_{n\to\infty} \int_{\partial B(f(0),r)} |\phi_n(z) - \phi(z)| dl(z) = 0. \tag{6.14}$$

It also follows from Fubini's theorem that there exists a measurable set $T \subset (0, R)$ of full measure so that

$$\mathrm{Leb}_1(\overline{O_\lambda(0)}) \cap \partial B(f(0), r)) = 0 \tag{6.15}$$

for all $r \in T$, where Leb_1 is the the 1-dimensional Lebesgue measure on the circle $\partial B(f(0), r)$. Fix a radius $r \in L \cap T$ such that

$$O_\lambda(0) \cap \partial B(f(0), r)) = \emptyset. \tag{6.16}$$

By (6.14) there exists $n \geq 1$ such that

$$\int_{\partial B(f(0),r)} |\phi_n(z) - \phi(z)| dl(z) \leq \frac{\pi}{2}. \tag{6.17}$$

Now, in view of (6.12) there exists a set $I_n \subset [k+1, k+2, \ldots, n]$ such that $f^j(0) \in B(f(0), r)$ for all $j \in I_n$ and

$$\begin{aligned} \int_{\partial B(f(0),r)} \phi_n(z) dz &= \int_{\partial B(f(0,r)} \frac{dz}{z - f(0)} + \sum_{j\in I_n} \frac{1}{(f^{j-1})'(0)} \int_{\partial B(f(0),r)} \frac{dz}{z - f^j(0)} \\ &= 2\pi i \left(1 + \sum_{j\in I_n} \frac{1}{(f^{j-1})'(0)} \right). \end{aligned} \tag{6.18}$$

Since I_n is a finite set and since the intersection $\overline{O_\lambda(0)} \cap \partial B(f(0), r)$ is compact, it follows from (6.15) that there exists a set $\overline{O_\lambda(0)} \cap \partial B(f(0), r) \subset \Delta \subset \partial B(f(0), r)$, being a finite union of closed arcs and such that

$$\int_\Delta \frac{|dz|}{|z - f^j(0)|} \leq \frac{\pi}{4}$$

for all $j \in I_n \cup \{1\}$. Hence, using (6.18) and (6.11), we get that

$$\begin{aligned}\left|\int_{\partial B(f(0),r)\setminus\Delta} \phi_n(z)dz - 2\pi i\right| &= \left|\int_{\partial B(f(0),r)\setminus\Delta} \phi_n(z) - \int_{\partial B(f(0),r)} \frac{dz}{z-f(0)}\right| \\ &= \left|-\int_\Delta \frac{dz}{z-f(0)} + \sum_{j\in I_n} \frac{1}{(f^{j-1})'(0)}\left(2\pi i - \int_\Delta \frac{dz}{z-f^j(0)}\right)\right| \\ &\le \int_\Delta \frac{|dz|}{|z-f(0)|} + \sum_{j\in I_n} \frac{9\pi}{4|(f^{j-1})'(0)|} \\ &\le \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.\end{aligned}$$

Thus

$$\int_{\partial B(f(0),r)\setminus\Delta} |\phi_n(z)||dz| \ge |\int_{\partial B(f(0),r)\setminus\Delta} \phi_n(z)dz| \ge \frac{3}{2}\pi.$$

Therefore, using (6.17), we get

$$\begin{aligned}\int_{\partial B(f(0),r)\setminus\Delta} |\phi(z)|dl(z) &\ge \int_{\partial B(f(0),r)\setminus\Delta} |\phi_n(z)|dl(z) - \int_{\partial B(f(0),r)\setminus\Delta} |\phi_n(z) - \phi(z)|dl(z) \\ &= \int_{\partial B(f(0),r)\setminus\Delta} |\phi_n(z)||dz| - \int_{\partial B(f(0),r)\setminus\Delta} |\phi_n(z) - \phi(z)|dl(z) \\ &\ge \frac{3}{2}\pi - \frac{\pi}{2} = \pi.\end{aligned}$$

Since $\partial B(f(0),r) \setminus \Delta \subset \mathbb{C} \setminus \overline{O_\lambda(0)}$, we are therefore done. ■

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