ERGODIC OPTIMIZATION FOR NON-COMPACT DYNAMICAL SYSTEMS

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Abstract. The purpose of this note is to initiate the study of ergodic optimization for general topological dynamical systems $T : X \to X$, where the topological space X need not be compact. Given $f : X \to \mathbb{R}$, four possible notions of largest ergodic average are defined; for compact metrisable X these notions coincide, while for general Polish spaces X they are related by inequalities, each of which may be strict.

We seek conditions on f which guarantee the existence of a normal form, in order to characterise its maximizing measures in terms of their support. For compact metrisable X it suffices to find a fixed point form. For general Polish X this is not the case, but an extra condition on f, essential compactness, is shown to imply the existence of a normal form. When $T : X \to X$ is the full shift on a countable alphabet, essential compactness yields an easily checkable criterion for the existence of a normal form.

Let X be a topological space, not necessarily compact. For a continuous transformation $T : X \to X$, let $\mathcal{M}$ denote the set of T-invariant Borel probability measures on X. In general $\mathcal{M}$ might be empty, though if X is a non-empty compact metrisable space then $\mathcal{M} \neq \emptyset$, by the Krylov-Bogolioubov Theorem [W, Cor. 6.9.1]. For a function $f : X \to \mathbb{R}$, define $S_n f := \sum_{i=0}^{n-1} f \circ T^i$. We will be interested in the *largest ergodic average* of the function f (see e.g. [J2] for an introduction to this optimization problem in the case of compact X). Four possible definitions for this are as follows.

Definition 1. By convention we define the supremum of the empty set to be $-\infty$. Let X be a topological space, and suppose that $f : X \to \mathbb{R}$ is continuous. For any Borel measure μ, the integral $\int f\, d\mu \in [-\infty, \infty]$ is defined provided $\int f^+\, d\mu$ and $\int f^-\, d\mu$ are not both infinite, where $f^{\pm} := \max(\pm f, 0)$. Let $\mathcal{M}_f := \{\mu \in \mathcal{M} : \int f\, d\mu \text{ is defined}\}$, and define

$$\alpha(f) = \alpha(f,T) = \sup_{m \in \mathcal{M}_f} \int f\, dm\,.$$

A point $x \in X$ is called (T,f)-*regular* if $\lim_{n\to\infty} \frac{1}{n} S_n f(x)$ exists (we allow divergence to either $-\infty$ or $+\infty$), and the set of (T,f)-regular points is denoted by $\mathrm{Reg}(X,T,f)$. Define

$$\beta(f) = \beta(f,T) = \sup_{x \in \mathrm{Reg}(X,T,f)} \lim_{n\to\infty} \frac{1}{n} S_n f(x)\,,$$

$$\gamma(f) = \gamma(f,T) = \sup_{x \in X} \limsup_{n\to\infty} \frac{1}{n} S_n f(x)\,,$$

$$\delta(f) = \delta(f,T) = \limsup_{n\to\infty} \frac{1}{n} \sup_{x \in X} S_n f(x)\,.$$

Remark 2.
(a) If f is bounded either above or below then $\mathcal{M}_f = \mathcal{M}$.
(b) If X is compact then $\mathcal{M}_f = \mathcal{M}$ for all continuous functions f.
(c) If $\mathcal{M}_f$ is empty then $\alpha(f) = -\infty$. In particular, if $\mathcal{M}$ is empty then $\alpha(f) = -\infty$ for all continuous functions f.
(d) If $\mathcal{M}$ is non-empty, and $f \in L^1(\mu)$ for some $\mu \in \mathcal{M}$, then $\alpha(f) > -\infty$.
(e) If f is bounded above then $\alpha(f) < \infty$.
(f) If f, g are continuous, and $f - g$ is bounded, then $\mathcal{M}_f = \mathcal{M}_g$.
(g) If $\delta(f) \in [-\infty, \infty)$ then $\sup_{x\in X} S_n f(x)$ is finite for all sufficiently large n, and is a subadditive sequence of reals, so in fact the limit $\lim_n \frac{1}{n} \sup_{x\in X} S_n f(x) \in [-\infty, \infty)$ exists and equals $\inf_n \frac{1}{n} \sup_{x\in X} S_n f(x)$.

Theorem 3. *Let X be a Polish space. If $T : X \to X$ and $f : X \to \mathbb{R}$ are continuous then $\alpha(f) \le \beta(f) \le \gamma(f) \le \delta(f)$. If furthermore X is compact then $\alpha(f) = \beta(f) = \gamma(f) = \delta(f) \ne \pm\infty$.*

Proof. It is immediate from their definitions that $\beta(f) \le \gamma(f)$ and $\gamma(f) \le \delta(f)$, so it only remains to show that $\alpha(f) \le \beta(f)$. Suppose on the contrary that $\alpha(f) > \beta(f)$. Then there exists a measure $\mu \in \mathcal{M}_f$ for which $\int f\,d\mu > \beta(f)$. In particular $\int f\,d\mu > -\infty$. Since X is a Polish space then the triple $(X, \mathcal{B}, \mu)$ is a Lebesgue space, where $\mathcal{B}$ is the completion of the Borel σ-algebra by μ [R, p. 174]. So $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is a measure-preserving endomorphism of a Lebesgue space, and consequently admits an ergodic decomposition ([R] pp. 178, 194, [W] p. 34): there is a Borel probability measure P_μ on the set $\mathcal{E} \subset \mathcal{M}$ of T-ergodic measures, such that if $g \in L^1(\mu)$ then $g \in L^1(m)$ for P_μ almost every $m \in \mathcal{E}$, and

$$\int g\,d\mu = \int_{m\in\mathcal{E}} \int g\,dm\,dP_\mu(m)\,.$$

If $f \in L^1(\mu)$ this gives $\int f\,d\mu = \int_{m\in\mathcal{E}} \int f\,dm\,dP_\mu(m)$, where $f \in L^1(m)$ for P_μ-a.e. $m \in \mathcal{E}$. So there exists an ergodic measure μ' such that $\int f\,d\mu' \ge \int f\,d\mu > \beta(f)$ and $f \in L^1(\mu')$ (so in particular $\mu' \in \mathcal{M}_f$). If $f \notin L^1(\mu)$ then necessarily $\int f\,d\mu = +\infty$ (because $\mu \in \mathcal{M}_f$ and $\int f\,d\mu > -\infty$), so there exists $\mu' \in \mathcal{E}$ such that $\int f\,d\mu' = +\infty$.

In either case we may apply Birkhoff's ergodic theorem (see e.g. [K, p. 15] for the case where $\int f\,d\mu' = +\infty$) to see that μ'-almost every x satisfies $\lim_{n\to\infty} \frac{1}{n} S_n f(x) = \int f\,d\mu'$. In particular there is at least one $x \in \mathrm{Reg}(X, T, f)$ for which $\lim_{n\to\infty} \frac{1}{n} S_n f(x) = \int f\,d\mu' > \beta(f)$, contradicting the definition of $\beta(f)$. So in fact $\alpha(f) \le \beta(f)$.

To prove $\alpha(f) = \beta(f) = \gamma(f) = \delta(f)$ for compact X it suffices to show that $\alpha(f) \ge \delta(f)$. The compactness of X means that $\mathcal{M}$ is compact for the weak* topology [W, Thm. 6.10]. If δ_y denotes the Dirac measure concentrated at y, and $\mu_n := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i x_n}$, where x_n is such that

$$\sup_{x\in X} \frac{1}{n} S_n f(x) = \frac{1}{n} S_n f(x_n) = \int f\,d\mu_n\,,$$

then the sequence (μ_n) has a weak* accumulation point μ, with $\int f\,d\mu = \delta(f)$. It is easily seen that $\mu \in \mathcal{M}$, so $\delta(f) = \int f\,d\mu \le \alpha(f)$. The common largest ergodic average is finite because f is bounded. $\square$

The following example shows that the inequalities in Theorem 3 may all be strict.

Example 4. Equip the integers $\mathbb{Z}$ with the discrete topology. Define $T:\mathbb{Z}\to\mathbb{Z}$ by

$$T(i)=\begin{cases} i+1 & \text{if } i<0 \\ i & \text{if } i=0 \\ i+2 & \text{if } i>0\,. \end{cases}$$

Define $f:\mathbb{Z}\to\mathbb{R}$ by

$$f(i)=\begin{cases} 2 & \text{if } i<0 \\ -1 & \text{if } i=0 \\ 0 & \text{if } i>0 \text{ is odd} \\ \frac{i}{2} & \text{if } i>0 \text{ and } i\equiv 2 \pmod 4 \\ 1-\frac{i}{2} & \text{if } i>0 \text{ and } i\equiv 0 \pmod 4\,. \end{cases}$$

The only T-invariant probability measure is the Dirac measure at 0, so $\alpha(f)=f(0)=-1$.

If $i>0$ is odd then $\lim_{n\to\infty}\frac{1}{n}S_nf(i)=0$. The only other (T,f)-regular points are the non-positive integers i; each such i is eventually iterated onto the fixed point 0, and therefore satisfies $\lim_{n\to\infty}\frac{1}{n}S_nf(i)=f(0)=-1<0$. Therefore

$$\beta(f)=\sup_{i\in\mathrm{Reg}(\mathbb{Z},T,f)}\lim_{n\to\infty}\frac{1}{n}S_nf(i)=0\,.$$

If $i>0$ is even then

$$\liminf_{n\to\infty}\frac{1}{n}S_nf(i)=0<1=\limsup_{n\to\infty}\frac{1}{n}S_nf(i)\,,$$

and therefore $\gamma(f)=\sup_{i\in\mathbb{Z}}\limsup_{n\to\infty}\frac{1}{n}S_nf(i)=1$.

If $i=-n<0$ then $\frac{1}{n}S_nf(i)=2$. So $\sup_{i\in\mathbb{Z}}\frac{1}{n}S_nf(i)=2$ for all $n\in\mathbb{N}$, and therefore $\delta(f)=\lim_{n\to\infty}\frac{1}{n}\sup_{i\in\mathbb{Z}}S_nf(i)=2$.

Definition 5. A measure $\mu\in\mathcal{M}_f$ is called *f-maximizing* if

$$\int f\,d\mu=\alpha(f)\,.$$

Let $\mathcal{M}_{\max}(f)$ denote the set of f-maximizing measures.

If X is compact then $\mathcal{M}_{\max}(f)$ is non-empty, because the map $\mu\mapsto\int f\,d\mu$ is continuous for the weak* topology. If X is non-compact then $\mathcal{M}_{\max}(f)$ may be empty, even if $\mathcal{M}_f$ is non-empty (e.g. this occurs if T is the identity map on $\mathbb{R}$, and $f:\mathbb{R}\to\mathbb{R}$ is strictly increasing). Even when $\mathcal{M}_{\max}(f)$ is non-empty, the possible non-coincidence of $\alpha(f)$, $\beta(f)$, $\gamma(f)$, $\delta(f)$ means that maximizing measures may be less canonical objects of investigation.

In the following we shall consider the set of f-maximizing measures in situations where $\alpha(f)=\delta(f)\neq\pm\infty$. One approach to understanding $\mathcal{M}_{\max}(f)$ is to attempt to modify f by a coboundary in such a way that the set of suprema of the resulting function contains the support of some invariant measure (cf. [B1, B2, CG, CLT, J1, J2]).

Definition 6. For a topological space X, let $CB(X)$ denote the set of real-valued bounded continuous functions on X. For a continuous map $T:X\to X$, a function of the form $\varphi-\varphi\circ T$, where $\varphi\in CB(X)$, is called a *coboundary.* Two functions f,g which differ by a coboundary are called *cohomologous*, and we write $f \sim g$. A continuous function $\tilde{f} \sim f$ is called a *normal form*[1] for f if $\tilde{f}^{-1}(\sup \tilde{f})$ contains the support (i.e. the smallest closed subset of X with full measure) of some T-invariant probability measure.

The following result implies that if f has a normal form, its maximizing measures are completely characterised by their support.

Lemma 7. *Suppose $T : X \to X$ is a continuous map on the topological space X, and the continuous function $f : X \to \mathbb{R}$ has a normal form $\tilde{f}$. Then $\mathcal{M}_f = \mathcal{M}$, and*

$$\mathcal{M}_{max}(f) \;=\; \left\{ m \in \mathcal{M} : supp(m) \subset \tilde{f}^{-1}(\sup \tilde{f}) \right\} \;\neq\; \emptyset\,.$$

Proof. The normal form $\tilde{f}$ is bounded above, so $\mathcal{M}_{\tilde{f}} = \mathcal{M}$ (cf. Remark 2 (a)). Moreover $f \sim \tilde{f}$, so $\mathcal{M}_f = \mathcal{M}_{\tilde{f}} = \mathcal{M}$, by Remark 2 (f).

Now $\int f\,dm = \int \tilde{f}\,dm \le \sup \tilde{f}$ for all $m \in \mathcal{M}$. If $\mu \in \mathcal{M}$ satisfies $\mathrm{supp}(\mu) \subset \tilde{f}^{-1}(\sup \tilde{f})$ then $\int f\,d\mu = \int \tilde{f}\,d\mu = \sup \tilde{f}$, so $\alpha(f) = \sup \tilde{f}$, and μ is f-maximizing. But $\tilde{f}$ is a normal form, so there exists at least one such μ, hence there exists at least one f-maximizing measure.

If $m \in \mathcal{M}$ is such that $\mathrm{supp}(m) \not\subset \tilde{f}^{-1}(\sup \tilde{f})$, then in fact $\int f\,dm = \int \tilde{f}\,dm < \sup \tilde{f}$, because $\tilde{f} \le \sup \tilde{f}$ and $m(\{x : \tilde{f}(x) < \sup \tilde{f}\}) > 0$, so m is not f-maximizing. ☐

Not every continuous function has a normal form, even when X is compact; indeed the absence of normal forms is in a sense typical (cf. [B2, BJ, J3]). However if X is compact, T enjoys some hyperbolicity, and f is sufficiently regular, then f is known to have a normal form (see e.g [B1, B2, CG, CLT, J1, J2]). One approach to proving this is to search for fixed points of a certain nonlinear map M_f, defined below. This map was introduced by Bousch [B1] to study maximizing measures in the case where X is compact.

Definition 8. Let $T : X \to X$ be a surjection on a non-empty set X, and $f : X \to \mathbb{R}$ any function. If $\varphi : X \to \mathbb{R}$ then for each $x \in X$, define $M_f\varphi(x) \in (-\infty, \infty]$ by

$$M_f\varphi(x) := \sup_{y \in T^{-1}x} (f + \varphi)(y)\,. \tag{1}$$

If f is bounded (respectively bounded above) then M_f preserves the set of bounded (respectively bounded above) functions. Iterates of M_f can be expressed as

$$M_f^n\varphi(x) = \sup_{y \in T^{-n}(x)} \left(S_n f + \varphi\right)(y)\,.$$

Lemma 9. *Let $T : X \to X$ be a surjection on a non-empty set X, and $f : X \to \mathbb{R}$ any function. If there exists $c \in \mathbb{R}$, and a bounded function $\varphi : X \to \mathbb{R}$, such that*

$$M_f\varphi = \varphi + c\,,$$

then

$$c = \delta(f) = \lim_{n\to\infty} \frac{1}{n} \sup_{x \in X} S_n f(x)\,.$$

[1] This terminology arises because $\tilde{f}$ is a privileged element in the equivalence class of functions which are cohomologous to f.

Proof. The equation $\varphi + c = M_f\varphi$ is equivalent to $M_{f-c}\varphi = \varphi$, which implies that

$$\varphi(x) = M_{f-c}^n\varphi(x) = -nc + \sup_{y\in T^{-n}(x)} (S_nf(y) + \varphi(y))$$

for all $n \in \mathbb{N}$, $x \in X$. Now φ is bounded, and writing $a = \inf\varphi$, $b = \sup\varphi$ we have

$$\frac{a-b}{n} + c \ \le\ \frac{1}{n}\sup_{y\in T^{-n}(x)} S_nf(y) \ \le\ c + \frac{b-a}{n}$$

for all $n > 0$, $x \in X$. Therefore for all $n > 0$,

$$\frac{a-b}{n} + c \ \le\ \frac{1}{n}\sup_{x\in X}\sup_{y\in T^{-n}(x)} S_nf(y) \ \le\ c + \frac{b-a}{n}\,,$$

which is equivalent to

$$\frac{a-b}{n} + c \ \le\ \frac{1}{n}\sup_{y\in X} S_nf(y) \ \le\ c + \frac{b-a}{n} \quad \text{for all } n > 0\,.$$

Letting $n \to \infty$ gives the result. □

Definition 10. Let X be a topological space. Suppose that $T : X \to X$ is a continuous surjection, and $f : X \to \mathbb{R}$ is continuous. If $\varphi \in CB(X)$ is a fixed point of $M_{f-\delta(f)}$, then the function $\tilde{f} := f + \varphi - \varphi\circ T$ is called a *fixed point form* for f.

Lemma 11. *Let X be a topological space. Suppose that $T : X \to X$ is a continuous surjection, and $f : X \to \mathbb{R}$ is continuous.*
(i) *If $\tilde{f}$ is a fixed point form for f, then $\tilde{f} \le \delta(f)$.*
(ii) *If moreover X is compact and metrisable, then any fixed point form for f is also a normal form.*

Proof. (i) We know that $\tilde{f} = f + \varphi - \varphi\circ T$ for some $\varphi \in CB(X)$ which satisfies

$$\varphi(x) + \delta(f) = \sup_{y\in T^{-1}(x)} (f+\varphi)(y) \tag{2}$$

for all $x \in X$. Replacing x by $T(x)$ in (2) gives

$$\begin{aligned}(f+\varphi-\varphi\circ T)(x) &= \delta(f) - \left(\sup_{y\in T^{-1}(Tx)} (f+\varphi)(y) - (f+\varphi)(x)\right)\\ &\le \delta(f)\,.\end{aligned}$$

(ii) Combining (i) above with Theorem 3 we see that $\sup\tilde{f} \le \alpha(f)$. But clearly $\alpha(f) = \alpha(\tilde{f}) \le \sup\tilde{f}$, so in fact $\alpha(f) = \sup\tilde{f}$. Let μ be an f-maximizing measure. We claim that $\tilde{f}^{-1}(\sup\tilde{f})$ contains $\mathrm{supp}(\mu)$. Were this not the case, the fact that $\tilde{f} \le \sup\tilde{f}$ and $\mu(\{x : \tilde{f}(x) < \sup\tilde{f}\}) > 0$ would imply that $\int f\,d\mu = \int\tilde{f}\,d\mu < \sup\tilde{f}$, contradicting the fact that μ is f-maximizing. □

So Lemma 11 (ii) implies that, when X is compact and metrisable, to find a normal form for f it is sufficient to find a fixed point form. Unfortunately, if X is non-compact then this is not the case, even when $\alpha(f) = \delta(f) \ne \pm\infty$:

Example 12. Let X denote the countable full (unilateral) shift, i.e. the set of all sequences $x = (x_n)_{n=1}^{\infty}$ of strictly positive integers. The shift map $T : X \to X$ defined by $(Tx)_n = x_{n+1}$ is continuous with respect to the metric $\delta(x, y) = 2^{-\min\{n: x_n \neq y_n\}}$. If $w = w_1 \dots w_k$, then the corresponding length-k cylinder is the set $[w] := \{x \in X : x_i = w_i \text{ for } 1 \leq i \leq k\}$.

Define $f : X \to \mathbb{R}$ to be constant on length-2 cylinders, with $f[m, n] = \frac{-1}{n(n+1)}$ if $m = n+1$ and $f[m, n] = -1$ otherwise. Let $\varphi \in CB(X)$ be constant on length-1 cylinders, defined by $\varphi[n] = -1/n$ for all $n \in \mathbb{N}$. A short calculation reveals that $f + \varphi - \varphi \circ T = 0$ on cylinder sets of the form $[n+1, n]$, whereas $(f + \varphi - \varphi \circ T)([m, n]) = -1 - \frac{1}{m} + \frac{1}{n} < 0$ if $m \neq n+1$, so φ is a fixed point for $M_{f-\delta(f)}$, and $\delta(f) = 0$. Now $\alpha(f) \leq \delta(f) = 0$, by Theorem 3. If ν_n denotes the unique invariant measure supported on the periodic orbit generated by $x^{(n)} := \overline{(n, n-1, \dots, 1)}$, then $\sup_{n \geq 0} \int f \, d\nu_n = 0$. So in fact $\alpha(f) = 0$. Clearly f has no maximizing measures, since $f < 0$ implies that $\int f \, dm < 0$ for any (invariant) probability measure m. In particular, by Lemma 7, f does not have a normal form.

Definition 13. Let $T : X \to X$ be a continuous surjection on the topological space X. A continuous function $f : X \to \mathbb{R}$ is *essentially compact* (with respect to T) if there is a fixed point $\varphi \in CB(X)$ of $M_{f-\delta(f)}$, and a subset $Y \subset X$ such that

(a) $\widehat{Y} := \cap_{n=0}^{\infty} T^{-n} Y$ is non-empty and compact,
(b) $T(Y) = X$,
(c) for each $x \in X$,

$$\varphi(x) + \delta(f) = \sup_{y \in T^{-1}(x) \cap Y} (f + \varphi)(y) \, . \tag{3}$$

Essential compactness guarantees that a fixed point form is in fact a normal form:

Theorem 14. *Let $T : X \to X$ be a continuous surjection on a Polish space X. If the continuous function $f : X \to \mathbb{R}$ is essentially compact, and $\varphi \in CB(X)$ is as in Definition 13, then $\tilde{f} = f + \varphi - \varphi \circ T$ is a normal form for f, and hence*

$$\mathcal{M}_{max}(f) \; = \; \left\{ m \in \mathcal{M} : supp(m) \subset \tilde{f}^{-1}(\sup \tilde{f}) \right\} \; \neq \; \emptyset \, .$$

Proof. It suffices to show that $\tilde{f}$ is a normal form; the characterisation of $\mathcal{M}_{\max}(f)$ then follows by Lemma 7. Condition (3) implies that if $x \in \widehat{Y}$ then

$$\varphi(x) + \delta(f) = \sup_{y \in T^{-1}(x) \cap Y} (f + \varphi)(y) = \sup_{y \in T^{-1}(x) \cap \widehat{Y}} (f + \varphi)(y) \, , \tag{4}$$

the second equality following because $Y \cap T^{-1} \widehat{Y} = \widehat{Y}$. Now (4) says that $\tilde{f}|_{\widehat{Y}}$ is a fixed point form for $f|_{\widehat{Y}}$ (with respect to $T|_{\widehat{Y}}$), and

$$\delta(f|_{\widehat{Y}}, T|_{\widehat{Y}}) = \delta(f) \tag{5}$$

by Lemma 9. The compactness of $\widehat{Y}$ means that $\tilde{f}|_{\widehat{Y}}$ is actually a normal form for $f|_{\widehat{Y}}$, by Lemma 11 (ii), so

$$\sup \tilde{f}|_{\widehat{Y}} = \delta(f|_{\widehat{Y}}, T|_{\widehat{Y}}) \, . \tag{6}$$

Moreover

$$\tilde{f} \leq \delta(f) \, , \tag{7}$$

by Lemma 11 (i), because $\tilde{f}$ is a fixed point form for f. Combining (5), (6) and (7) gives

$$\tilde{f} \le \sup \tilde{f}|_{\widehat{Y}} .$$

Now $\tilde{f}|_{\widehat{Y}}$ is a normal form for $f|_{\widehat{Y}}$, so

$$(\tilde{f}|_{\widehat{Y}})^{-1}(\sup \tilde{f}|_{\widehat{Y}}) = (\tilde{f}|_{\widehat{Y}})^{-1}(\sup \tilde{f})$$

contains the support of a $T|_{\widehat{Y}}$-invariant probability measure. This measure is T-invariant, and contained in the larger set $\tilde{f}^{-1}(\sup \tilde{f})$, so $\tilde{f}$ is a normal form for f. □

Essential compactness is a rather abstract notion, so for specific systems it is useful to replace it by more readily verifiable conditions. For X the countable full shift, and $g : X \to \mathbb{R}$, we define $\mathrm{var}_0(g) = \sup g - \inf g$ and $\mathrm{var}_1(g) = \sup_{x_1=y_1} |g(x) - g(y)|$. For $x \in X$ and an integer $n \ge 1$, let nx denote the element $y \in X$ with $y_1 = n$ and $y_i = x_{i-1}$ for all $i \ge 2$.

Theorem 15. *Let X be the countable full shift. Suppose that $f : X \to \mathbb{R}$ is bounded above and constant on length-2 cylinders, and that there exists $n \ge 1$ such that*

$$var_1(f) < \inf f|_{[n]} - \sup f|_{[i]} \tag{8}$$

for all sufficiently large $i \ge 1$.

Then f is essentially compact, hence has a normal form $\tilde{f}$, and hence

$$\mathcal{M}_{max}(f) = \left\{ m \in \mathcal{M} : supp(m) \subset \tilde{f}^{-1}(\sup \tilde{f}) \right\} \ne \emptyset .$$

Proof. Since f is constant on length-2 cylinders, the map M_f preserves the space of functions which are constant on length-1 cylinders. Identifying this space with the sequence space ℓ^∞, equipped with its usual norm, we note that for $0 \le \lambda < 1$, the map $\varphi \mapsto M_f(\lambda\varphi)$ is λ-Lipschitz on ℓ^∞, so by the contraction mapping theorem has a unique fixed point $\varphi_\lambda \in \ell^\infty$. The fixed point equation implies that $\mathrm{var}_0(\varphi_\lambda) \le \mathrm{var}_1(f)$ for all $0 \le \lambda < 1$, so if $\varphi^*_\lambda := \varphi_\lambda - \inf \varphi_\lambda$ then $\|\varphi^*_\lambda\|_\infty = \mathrm{var}_0(\varphi^*_\lambda) \le \mathrm{var}_1(f)$ for all $0 \le \lambda < 1$. Therefore $(\varphi^*_\lambda)_{0 \le \lambda < 1}$ has an accumulation point $\varphi \in \ell^\infty$ which moreover satisfies

$$M_f \varphi = \varphi + \delta(f) \tag{9}$$

and

$$\mathrm{var}_0(\varphi) \le \mathrm{var}_1(f) . \tag{10}$$

If $J \ge n$ is such that (8) holds for all $i > J$, we claim that

$$M_f\varphi(x) = \max_{1 \le j \le J} (f + \varphi)(jx) \quad \text{for all } x \in X . \tag{11}$$

Note that (11) implies condition (c) of Definition 13 for the set $Y = \cup_{j=1}^{J} [j]$. Since Y clearly also satisfies conditions (a) and (b), it will follow from (11) that f is essentially compact, and therefore Theorem 14 will imply the desired result.

To prove (11), note that (8) and (10) imply that for $x \in X$ and $i > J$,

$$\varphi(ix) - \varphi(nx) \le \mathrm{var}_0(\varphi) \le \mathrm{var}_1(f) < \inf f|_{[n]} - \sup f|_{[i]} \le f(nx) - f(ix) .$$

In other words,

$$(f + \varphi)(nx) > (f + \varphi)(ix) \quad \text{for all } x \in X,\ i > J,$$

and this implies (11). □

Example 16. For X the countable full shift, let $f : X \to \mathbb{R}$ be constant on length-2 cylinders, defined by $f[i,j] \equiv \max g|_{C(i,j)}$, where $g : [0,1] \to \mathbb{R}$ is given by $g(x) = x(1-x)$ and

$$C(i,j) = \left[\frac{1}{i+1/j}, \frac{1}{i+1/(j+1)}\right] \subset [0,1]$$

for $i, j \geq 1$ (note that these intervals are the members of the level-2 refinement of the Markov partition for Gauss's continued fraction map $x \mapsto 1/x \pmod 1$).

Choosing $n = 2$, we claim that (8) holds. Now $\inf f|_{[2]} = g(1/3) = 2/9$, and $\sup f|_{[i]} \searrow 0$ as $i \to \infty$, so $\inf f|_{[2]} - \sup f|_{[i]} \nearrow 2/9$ as $i \to \infty$. In particular, if i is sufficiently large then $\inf f|_{[2]} - \sup f|_{[i]} > 1/36 = g(1/2) - g(2/3) = f[1,1] - f[1,2] = \mathrm{var}_1(f)$, so indeed (8) holds. Therefore, by Theorem 15, f is essentially compact, hence has a normal form, hence there exist f-maximizing measures, and they are characterised by whether or not their support lies in the set of maxima of the normal form.

Remark 17.
(a) The function f in Example 12 obviously does not satisfy (8): here $\mathrm{var}_1(f) = 1$, $\inf f|_{[n]} = -1$ for all n, and $\sup f|_{[i]} > -1$ for all i.
(b) Theorem 15 can be extended to more general countable alphabet subshifts of finite type and functions f of summable variation, at the expense of a considerably longer and more functional-analytic proof (see [JMU]).

References

[B1] T. Bousch, Le poisson n'a pas d'arêtes, *Ann. Inst. Henri Poincaré (Proba. et Stat.)* **36**, (2000), 489–508.

[B2] T. Bousch, La condition de Walters, *Ann. Sci. ENS*, **34**, (2001), 287–311.

[BJ] T. Bousch & O. Jenkinson, Cohomology classes of dynamically non-negative C^k functions, *Invent. Math.*, **148** (2002), 207–217.

[CG] J.-P. Conze & Y. Guivarc'h, Croissance des sommes ergodiques et principe variationnel, *manuscript*, circa 1993.

[CLT] G. Contreras, A. O. Lopes, & Ph. Thieullen, Lyapunov minimizing measures for expanding maps of the circle, *Ergod. Th. & Dyn. Sys.*, **21** (2001), 1379–1409.

[J1] O. Jenkinson, Rotation, entropy, and equilibrium states, *Trans. Amer. Math. Soc.*, **353** (2001), 3713–3739.

[J2] O. Jenkinson, Ergodic optimization, *Discr. & Cont. Dyn. Sys.*, **15** (2006), 197–224.

[J3] O. Jenkinson, Every ergodic measure is uniquely maximizing, *Discr. & Cont. Dyn. Sys.*, **16** (2006), 383–392.

[JMU] O. Jenkinson, R. D. Mauldin & M. Urbański, Ergodic optimization for countable alphabet subshifts of finite type, *Ergod. Th. & Dyn. Sys.*, **26** (2006), 1791–1803.

[K] U. Krengel, *Ergodic theorems*, de Gruyter, Berlin-New York, 1985.

[R] V. A. Rohlin, Selected topics in the metric theory of dynamical systems, *Uspekhi Mat. Nauk.*, **4** (1949), 57–128; also *Amer. Math. Soc. Transl. Ser. 2*, **49** (1966), 171–240.

[W] P. Walters, *An introduction to ergodic theory*, Springer, 1981.