HIGHER-DIMENSIONAL MULTIFRACTAL VALUE SETS FOR CONFORMAL INFINITE GRAPH DIRECTED MARKOV SYSTEMS

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ABSTRACT. We give a description of the level sets in the higher dimensional multifractal formalism for infinite conformal graph directed Markov systems. If these systems possess a certain degree of regularity this description is complete in the sense that we identify all values with non-empty level sets and determine their Hausdorff dimension. This result is also new for the finite alphabet case.

1. INTRODUCTION AND STATEMENTS OF MAIN RESULTS

In this paper we study *Graph Directed Markov System* (GDMS) as defined in [1] consisting of a directed multigraph (V,E,i,t,A) with incidence matrix A together with a family of non-empty compact metric spaces $(X_v)_{v\in V}$, a number $s\in(0,1)$, and for every $e\in E$, an injective contraction $\phi_e : X_{t(e)} \to X_{i(e)}$ with Lipschitz constant not exceeding s. Briefly, the family

$$\Phi = \left(\phi_e : X_{t(e)} \to X_{i(e)}\right)_{e\in E}$$

is called a GDMS. Throughout this paper we will assume that the system is *conformal* (Def. 2.2), *finitely irreducible* (Def. 2.1), and *co-finitely regular* (Def. 2.7). The necessary details will be postponed to Section 2. Let E_A^∞ denote the set of admissible infinite sequences for A and $\sigma : E_A^\infty \to E_A^\infty$ the shift map given by $(\sigma(x))_i := (x_{i+1})_i$. With $\pi : E_A^\infty \to X := \bigoplus_{v\in V} X_v$ we denote the natural coding map from the subshift E_A^∞ to the disjoint union X of the compact sets X_v (see (2.1)). Its image $\Lambda := \Lambda_\Phi := \pi(E_A^\infty)$ denotes the limit set of Φ. An important tool for studying Φ is the following geometric potential function given by the conformal derivatives of the contractions $(\phi_i)_{i\in E}$

$$I = I_\Phi : E_A^\infty \to \mathbb{R}^+,\ I_\Phi(\omega) := -\log\left|\phi'_{\omega_1}(\pi(\sigma(\omega)))\right|.$$

We are going to set up a multifractal analysis for I with respect to another bounded Hölder continuous function

$$J : E_A^\infty \to \mathbb{R}^k.$$

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That is for $v \in V$ and $\alpha \in \mathbb{R}^k$ we investigate the level sets

$$\mathcal{F}_\alpha(v) := \left\{ \pi(\omega) : \omega \in E_A^\infty, i(\omega_1) = v, \text{ and } \lim_{n\to\infty} \frac{S_n J(\omega)}{S_n I(\omega)} = \alpha \right\},$$

and $\mathcal{F}_\alpha = \bigoplus_{v\in V} \mathcal{F}_\alpha(v) \subset \Lambda$. Let

$$Q : \mathcal{M}(E_A^\infty, \sigma) \to \mathbb{R}^k, \quad Q(\mu) := \frac{1}{\int I\, d\mu} \int J\, d\mu,$$

where $\mathcal{M}(E_A^\infty, \sigma)$ denotes the set of shift invariant Borel probability measures on E_A^∞. We are now interested in the following three subsets of $\mathbb{R}^k$.

$$\begin{aligned} K &:= \left\{ \alpha \in \mathbb{R}^k : \mathcal{F}_\alpha \neq \emptyset \right\}, \\ L &:= \left\{ Q(\mu) = \left(\int I\, d\mu \right)^{-1} \int J\, d\mu : \mu \in \mathcal{M}(E_A^\infty, \sigma) \right\}, \\ M &:= \nabla\beta\left(\mathbb{R}^k\right), \end{aligned} \tag{1.1}$$

where $\beta : \mathbb{R}^k \to \mathbb{R}$ is the convex differentiable function defined in terms of some pressure function within Proposition 3.1. Since $I > -\log s$, the sets $K, L, M \subset \mathbb{R}^k$ are all bounded. Since E_A^∞ is finitely irreducible (see Def. 2.1) we have for all $v \in V$

$$K = \{\alpha : \mathcal{F}_\alpha(v) \neq \emptyset\}.$$

Our first main theorem relates the three sets in (1.1).

Theorem 1.1. *The set K is compact and we have* $\operatorname{Int} L \subset M \subset L$ *and* $\overline{M} \subset K \subset \overline{L}$.

Remark. For the finite alphabet case (i.e. E is a finite set) the inclusion $K \subset L$ is well-known for the one dimensional situation (i.e. $k = 1$) and equality of K and L is also proved for $k \geq 1$ in [2]. The proof uses the fact that for $x \in \mathcal{F}_\alpha \neq \emptyset$ the set of measures $\left\{\mu_n := n^{-1} \sum_{i=0}^{n-1} \delta_{\sigma^i x} : n \in \mathbb{N}\right\}$ always possesses a weak convergent subsequence with limit measure μ such that $Q(\mu) = \alpha$. This gives $\alpha \in L$. Also, since in the finite alphabet case $\mathcal{M}(E_A^\infty, \sigma)$ is compact and Q is continuous with respect to the weak-* topology we have $\overline{\operatorname{Int} L} = L$. Hence the above theorem gives in this situtation $L = \overline{\operatorname{Int} L} \subset \overline{M} \subset K$. This shows that $L = K$ in the finite alphabet case.

If some additional regularity conditions are satisfied we get the following stronger results.

Theorem 1.2. *Suppose that J_i are linearly independent as cohomology classes. Then M is an open convex domain, $L \subset \overline{\operatorname{Int} L}$, and in particular*

$$\overline{L} = \overline{M} = K.$$

If additionally $0 \in M$ *then* $L = \overline{M} = K$.

Our third theorem gives the multifractal formalism in the higher-dimensional situation. If $0 \in M$ then our description is complete in the sense that the formula for the Hausdorff dimension holds not only for α form the interior of K but for all $\alpha \in \mathbb{R}^k$. This result is also new in the finite alphabet case. Recall that the (negative) Legendre transform $\widehat{\beta} : \mathbb{R}^k \to \overline{\mathbb{R}}$ of β is given by

$$\widehat{\beta}(a) := \inf_{t \in \mathbb{R}^k} \left(\beta(t) - \langle t, a \rangle\right).$$

For the following let $\mathrm{HD}(A)$ denotes the Hausdorff dimension of the set A.

Theorem 1.3. *Suppose that J_i are linearly independent as cohomology classes. Then we have for $\alpha \in M$ and $v \in V$*

$$\mathrm{HD}(\mathcal{F}_\alpha(v)) = \widehat{\beta}(\alpha)$$

and for all $\alpha \in \mathbb{R}^k$ we have

$$\mathrm{HD}(\mathcal{F}_\alpha(v)) \le \max\left\{\widehat{\beta}(\alpha), 0\right\}. \tag{1.2}$$

If additionally $0 \in M$ then equality holds in (1.2).

A detailed study of the multifractal level sets and variational formulae for the entropy in the finite alphabet setting can be found in [2]. Compairing the finite with the infinite alphabet case we mainly encounter the following obstacles. The shift space E_A^∞ is not even locally compact and hence also $\mathcal{M}(E_A^\infty, \sigma)$ is not compact with respect to the weak-* topology. The function Q is in general not continuous but only upper semi-continuous with respect to the weak-* topology on $\mathcal{M}(E_A^\infty, \sigma)$. This follows from the fact that

$$\mu \mapsto \int -I\, d\mu \tag{1.3}$$

is upper semi-continuous and $I \ge \text{const.} > 0$. In Remark 4.1 we take as an example measures $\mu, \mu_1, \mu_2, \ldots \in \mathcal{M}(E_A^\infty, \sigma)$ such that $\mu_n \xrightarrow{*} \mu$, $\int I\, d\mu < +\infty, \int I\, d\mu_n < +\infty$, $n \in \mathbb{N}$, but nevertheless $\liminf \int I\, d\mu_n > \int I\, d\mu$. Finally, the entropy map $\mu \mapsto h_\mu(\sigma)$ is not even upper semi-continuous (cf. [3]) and the pressure functions under consideration (cf. Proposition 3.1) is only defined in some open region.

We would finally like to remark that the study of higher dimensional multifractal value sets for infinite GDMS naturally arose in [4] where a multifractal formalism has been developed in order to study the distributions of limiting modular symbols introduced by Manin and Marcolli in [5].

2. Conformal Graph Directed Markov Systems

In this section we begin our study of graph directed Markov systems. Let us recall the definition of these systems taken from [1]. Graph directed Markov systems are based upon a directed multigraph and an associated incidence matrix, (V, E, i, t, A). The multigraph consists of a finite set V of vertices and a countable (either finite or infinite) set of directed edges $E \subset \mathbb{N}$ and two functions $i, t : E \to V$. For each edge e, $i(e)$ is the initial vertex of the edge e and $t(e)$ is the terminal vertex of e. The edge goes from $i(e)$ to $t(e)$. Also, a function $A : E \times E \to \{0,1\}$ is given, called an (edge) incidence matrix. It determines which edges may follow a given edge. So, the matrix has the property that if $A_{uv} = 1$, then $t(u) = i(v)$. We will consider finite and infinite walks through the vertex set consistent with the incidence matrix. Thus, we define the set of infinite admissible words E_A^∞ on an alphabet A,

$$E_A^\infty = \{\omega \in E^\infty : A_{\omega_i \omega_{i+1}} = 1 \text{ for all } i \geq 1\},$$

by E_A^n we denote the set of all subwords of E_A^∞ of length $n \geq 1$, and by E_A^* we denote the set of all finite subwords of E_A^∞. We will consider the left shift map $\sigma : E_A^\infty \to E_A^\infty$ defined by $\sigma(\omega_i) := (\omega_{i+1})_{i \geq 1}$. Sometimes we also consider this shift as defined on words of finite length. Given $\omega \in E^*$ by $|\omega|$ we denote the length of the word ω, i.e. the unique n such that $\omega \in E_A^n$. If $\omega \in E_A^\infty$ and $n \geq 1$, then

$$\omega|_n = \omega_1 \dots \omega_n.$$

For $\omega \in E_A^\infty$, or $\omega \in E_A^*$ with $|\omega| \geq n$ we will denote with

$$C_n(\omega) := \{x \in E_A^\infty : x|_n = \omega|_n\}$$

the *cylinder set of length n* containing ω.

Definition 2.1. E_A^∞ (or equivalently the GDMS Φ) is called *finitely irreducible* if there exists a finite set $W \subset E_A^*$ such that for each $\omega, \eta \in E$ we find $w \in W$ such that the concatenation $\omega w \eta \in E_A^*$.

We recall from the introduction that a *Graph Directed Markov System* (GDMS) now consists of a directed multigraph and incidence matrix together with a family of non-empty compact metric spaces $(X_v)_{v \in V}$, a number $s \in (0,1)$, and for every $e \in E$, a injective contraction $\phi_e : X_{t(e)} \to X_{i(e)}$ with a Lipschitz constant not exceeding s. We now describe its limit set. For each $\omega \in E_A^*$, say $\omega \in E_A^n$, we consider the map coded by ω,

$$\phi_\omega := \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n} : X_{t(\omega_n)} \to X_{i(\omega_1)}.$$

For $\omega \in E_A^\infty$, the sets $\left\{\phi_{\omega|_n}\left(X_{t(\omega_n)}\right)\right\}_{n \geq 1}$ form a descending sequence of non-empty compact sets and therefore $\bigcap_{n \geq 1} \phi_{\omega|_n}\left(X_{t(\omega_n)}\right) \neq \emptyset$. Since for every $n \in \mathbb{N}$, $\operatorname{diam}\left(\phi_{\omega|_n}\left(X_{t(\omega_n)}\right)\right) \leq s^n \operatorname{diam}\left(X_{t(\omega_n)}\right) \leq s^n \max\{\operatorname{diam}(X_v) : v \in V\}$, we conclude that the intersection

$$\bigcap \phi_{\omega|_n}\left(X_{t(\omega_n)}\right) \in X_{i(\omega_1)}$$

is a singleton and we denote its only element by $\pi(\omega)$. In this way we have defined the coding map

$$\pi = \pi_\Phi : E_A^\infty \to X := \bigoplus_{v \in V} X_v \tag{2.1}$$

from E^∞ to $\bigoplus_{v\in V} X_v$, the disjoint union of the compact sets X_v. The set

$$\Lambda = \Lambda_\Phi = \pi(E_A^\infty)$$

will be called the *limit set* of the GDMS Φ.

Definition 2.2. We call a GDMS *conformal* (CGDMS) if the following conditions are satisfied.

(a) For every vertex $v \in V$, X_v is a compact connected subset of a Euclidean space $\mathbb{R}^d$ (the dimension d common for all $v \in V$) and $X_v = \overline{\mathrm{Int}(X_v)}$.

(b) *(Open set condition (OSC))* For all $a, b \in E$, $a \neq b$,

$$\phi_a\left(\mathrm{Int}(X_{t(a)}\right) \cap \phi_b\left(\mathrm{Int}(X_{t(b)}\right) = \emptyset.$$

(c) For every vertex $v \in V$ there exists an open connected set $W_v \supset X_v$ such that for every $e \in I$ with $t(e) = v$, the map ϕ_e extends to a C^1 conformal diffeomorphism of W_v into $W_{i(e)}$.

(d) *(Cone property)* There exist $\gamma, l > 0$, $\gamma < \pi/2$, such that for every $x \in X \subset \mathbb{R}^d$ there exists an open cone $\mathrm{Con}(x, \gamma, l) \subset \mathrm{Int}(X)$ with vertex x, central angle of measure γ, and altitude l.

(e) There are two constants $L \geq 1$ and $\alpha > 0$ such that

$$\left||\phi_e'(y)| - |\phi_e'(x)|\right| \leq L \|(\phi_e')^{-1}\|^{-1} \|y - x\|^\alpha$$

for every $e \in I$ and every pair of points $x, y \in X_{t(e)}$, where $|\phi_\omega'(x)|$ means the norm of the derivative.

The following remarkable fact was proved in [1].

Proposition 2.3. *If $d \geq 2$ and a family $\Phi = (\phi_e)_{e\in I}$ satisfies conditions (a) and (c), then it also satisfies condition (e) with* $\alpha = 1$.

The following rather straightforward consequence of (4e) was proved in [1].

Lemma 2.4. *If $\Phi = (\phi_e)_{e\in I}$ is a CGDMS, then for all $\omega \in E^*$ and all $x, y \in W_{t(\omega)}$, we have*

$$\left|\log|\phi_\omega'(y)| - \log|\phi_\omega'(x)|\right| \leq \frac{L}{1-s}\|y - x\|^\alpha.$$

As a straightforward consequence of (e) we get the following.

(f) (Bounded distortion property). There exists $K \geq 1$ such that for all $\omega \in E^*$ and all $x, y \in X_{t(\omega)}$

$$|\phi_\omega'(y)| \leq K|\phi_\omega'(x)|.$$

Next we define the geometrical potential function associated with Φ by

$$I = I_\Phi : E_A^\infty \to \mathbb{R}^+, \, I_\Phi(\omega) := -\log \left| \phi'_{\omega_1}(\pi(\sigma(\omega))) \right|.$$

It was proved in [1] that for each $t \geq 0$ the following limit exists (possibly be equal to $+\infty$).

$$\mathfrak{p}(t) := \mathcal{P}(-tI) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in E_A^n} \exp(-tS_n I(\omega)) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in E_A^n} ||\phi'_\omega||^t,$$

where $S_n I(\omega) := \sup_{\{x : x|_n = \omega\}} \sum_{i=0}^{n-1} I(\sigma^i x)$. This number is called the *topological pressure* of the parameter t. The function $\mathfrak{p}$ is always non-increasing and convex. In [1] a useful parameter associate with a CGDMS has been introduced. Namely,

$$\theta(\Phi) := \inf\{t : \mathfrak{p}(t) < +\infty\} = \sup\{t : \mathfrak{p}(t) = +\infty\}.$$

Let $\mathrm{Fin}(E)$ denote the family of all finite subsets of E. The following characterization of $h_\Phi = \mathrm{HD}(\Lambda_\Phi)$ (also denoted by h_E), the Hausdorff dimension of the limit set Λ_Φ, being a variant of Bowen's formula, was proved in [1] as Theorem 4.2.13.

Theorem 2.5. *If the a CGDMS Φ is finitely irreducible, then*

$$\mathrm{HD}(\Lambda_\Phi) = \inf\{t \geq 0 : \mathfrak{p}(t) < 0\} = \sup\{h_F : F \in \mathrm{Fin}(I)\} \geq \theta(\Phi).$$

If $\mathfrak{p}(t) = 0$, then t is the only zero of the function $\mathfrak{p}(t)$, $t = \mathrm{HD}(\Lambda_\Phi)$ and the system Φ is called regular.

In fact it was assumed in [1] that the system Φ is finitely primitive but the proof can be easily improved to this slightly more general setting. It will be convenient for us to recall and make use of the following definitions.

Definition 2.6. A CGDMS is said to be *strongly regular* if there exists $t \geq 0$ such that $0 < \mathfrak{p}(t) < \infty$. A family $(\phi_i)_{i\in F}$ is said to be a *co-finite subsystem* of a system of $\Phi = (\phi_i)_{i\in E}$ if $F \subset E$ and the difference $E \setminus F$ is finite.

Definition 2.7. A CGDMS is said to be *co-finitely regular* if each of its co-finite subsystem is regular.

The following fact relating all these three notions can be found in [1].

Proposition 2.8. *Each co-finitely regular system is strongly regular and each strongly regular system is regular.*

Note that the system Φ is strongly regular if and only if $\mathrm{HD}(\Lambda_\Phi) > \theta(\Phi)$.

3. HIGHER-DIMENSIONAL THERMODYNAMIC FORMALISM

From now on let us assume that the CGDMS is infinite, i.e. $\operatorname{card}(E) = \infty$ and hence $\theta(\Phi) \geq 0$, and co-finitely regular. Recall that for $\omega, \tau \in E_A^\infty$, we define $\omega \wedge \tau \in E_A^\infty \cup E_A^*$ to be the longest initial block common to both ω and τ. We say that a function $g : E_A^\infty \to \mathbb{R}$ is *Hölder continuous* with an exponent $\alpha > 0$ if

$$v_\alpha(f) := \sup\{V_{\alpha,n}(f) : n \geq 1\} < \infty,$$

where

$$V_{\alpha,n}(f) = \sup\{|f(\omega) - f(\tau)| e^{\alpha(n-1)} : \omega, \tau \in E_A^\infty \text{ and } |\omega \wedge \tau| \geq n\}.$$

For every $\alpha > 0$ let $\mathcal{K}_\alpha$ be the set of all real-valued Hölder continuous (not necessarily bounded) functions on E_A^∞. Set

$$\mathcal{K}_\alpha^s := \left\{ f \in \mathcal{K}_\alpha : \sum_{e \in E} \exp\left(\sup\left(f|_{C_1(e)}\right)\right) < +\infty \right\}.$$

Each member of $\mathcal{K}_\alpha^s$ is called an α*-Hölder summable potential*.

For fixed $k \in \mathbb{N}$ let $J : E_A^\infty \to \mathbb{R}^k$ such that $J_i \in \mathcal{K}_\alpha$ is a *bounded* Hölder continuous function for $i = 1, \ldots, k$. The following proposition will be of central importance throughout this paper.

Proposition 3.1. *Each member of the family* $\left(\langle t, J\rangle - \beta I : t \in \mathbb{R}^k, \beta > \theta\right)$ *is an element of* $\mathcal{K}_\alpha^s$. *The pressure functional*

$$p : \mathbb{R}^k \times (\theta, \infty) \to \mathbb{R},\ p(t,\beta) := \mathcal{P}(\langle t, J\rangle - \beta I)$$

is a well-defined, real-analytic, convex function. For each $t \in \mathbb{R}^k$ *there exists a unique number* $\beta(t)$ *such that* $p(t, \beta(t)) = 0$*. Also* $t \mapsto \beta(t)$ *defines a real-valued, real-analytic convex function on* $\mathbb{R}^k$*. Its gradient is given by*

$$\nabla\beta(t) = \frac{1}{\int I\, d\mu_t} \int J\, d\mu_t, \tag{3.1}$$

where $\mu_t = \mu_{t,\beta(t)}$ *denotes the unique invariant Gibbs measure for the potential* $\langle t, J\rangle - \beta(t) I$*, i.e. there exists* $C > 0$ *such that for all* $\omega \in E_A^\infty$ *we have*

$$C^{-1} \leq \frac{\mu_t C_n(\omega)}{\exp S_n(\langle t, J\rangle - \beta(t) I)(\omega)} \leq C. \tag{3.2}$$

Proof. The properties of the family $\left(\langle t, J\rangle - \beta I : t \in \mathbb{R}^k, \beta > \theta\right)$ follows immediately from the boundedness of J and the Hölder continuity of I. From [1] we then know that p is a well-defined and real-analytic function. Since the system is infinite and co-finitely regular we have $\lim_{\beta \searrow \theta} p(t, \beta) = +\infty$. Furthermore for every $t \in \mathbb{R}^k$, we have

$$\partial_\beta p(t,\beta) = -\int I\, d\mu_{t,\beta} \leq \log s < 0.$$

Hence for every $t \in \mathbb{R}^k$, $\beta \mapsto p(t,\beta)$ is a strictly decreasing function and $\lim_{\beta\to+\infty} p(t,\beta) = -\infty$. From this we conclude that for each $t \in \mathbb{R}^k$ there exists a unique number $\beta(t) > \theta$ such that $p(t,\beta(t)) = 0$. By the implicit function theorem also $\beta : \mathbb{R}^k \to \mathbb{R}$ is real-analytic and convex. The formula for the gradient of β follows again from the implicit function theorem. □

Lemma 3.2. *Any set of measures $M \subset \mathcal{M}(E_A^\infty, \sigma)$ such that $\sup_{\mu\in M} \int I\, d\mu < +\infty$ is tight.*

Proof. For every $i, \ell \in \mathbb{N}$ put

$$E_{i,\ell} := \{\omega \in E_A^\infty : \omega_i \geq \ell\}.$$

Then we have for all $\mu \in M$

$$\text{const.} \geq \int I\, d\mu \geq \int_{E_{1,\ell}} I\, d\mu \geq \mu(E_{1,\ell}) \inf_{E_{1,\ell}} I.$$

Combining this with the fact that $E_{i,\ell} \subset \sigma^{-i+1}(E_{1,\ell})$ and that μ is σ–invariant we get

$$\mu(E_{i,\ell}) \leq \mu(E_{1,\ell}) \leq \frac{\text{const.}}{\inf_{E_{1,\ell}} I}.$$

Now, fix $\varepsilon > 0$. It follows from the above estimate that for every $i \geq 1$ there exists $\ell_i \geq 1$ such that $\mu(E_{i,\ell_i}) < 2^{-i}\varepsilon$. Then $\bigcup_{i=1}^\infty E_{i,\ell_i}$ is the complement of the compact set $\{\omega \in E_A^\infty : \forall i \in \mathbb{N} : \omega_i < \ell_i\}$ and

$$\mu\left(\bigcup_{i=1}^\infty E_{i,\ell_i}\right) \leq \sum_{i=1}^\infty \mu(E_{i,\ell_i}) \leq \sum_{i=1}^\infty 2^{-i}\varepsilon = \varepsilon \quad \text{for all} \quad \mu \in M.$$

From this the tightness of M follows. □

4. PROOFS

Clearly since $I > -\log s$ and J is bounded, the sets $K, L, M \subset \mathbb{R}^k$ are all bounded.

Proof of Theorem 1.1. The inclusion $M \subset L$ follows immediately from the definitions and (3.1).

Let us first show that $\alpha = 0$ is always an element of K and L. To see this we construct a Bernoulli measure μ_p (which is invariant and ergodic) with probability vector $p := (p_i)$ chosen in such a way that $\sum p_i \inf I|_{C_1(i)} = +\infty$, which is always possible. Then $|Q(\mu_p)| = \left|\frac{\int J\, d\mu_p}{\int I\, d\mu_p}\right| \leq \frac{\text{const.}}{+\infty} = 0$ and also for μ_p–almost all points $\omega \in E_A^\infty$ we have by the ergodic theorem that $\lim_{n\to\infty} \frac{S_n J(\omega)}{S_n I(\omega)} = Q(\mu_p) = 0$.

Proof of '$\operatorname{Int} L \subset M$**'.**

Here we follow some ideas from [2]. Let $\alpha \in \operatorname{Int} L$. Then there exists $r > 0$ such that

$B_r(\alpha) \subset L$. Let

$$D_\alpha := \sup_{\mu \in Q^{-1}(\alpha)} \left\{ \frac{h(\mu)}{\int I \, d\mu} \right\}, \alpha \neq 0.$$

By the variational principle (cf. [1, Theorem 2.1.7]) it follows that

$$0 = \mathcal{P}(\mathrm{HD}(\Lambda_\Phi) I) \geq h(\mu) - \mathrm{HD}(\Lambda_\Phi) \int I \, d\mu$$

and hence D_α is always dominated by the Hausdorff dimension $\mathrm{HD}(\Lambda_\Phi)$. Let us now consider the family of potentials

$$(\langle q, J \rangle - (\langle q, \alpha \rangle + D_\alpha) I : q \in G_\alpha),$$

where $G_\alpha := \left\{ x \in \mathbb{R}^k : \langle x, \alpha \rangle + D_\alpha > \theta \right\}$. Firstly, we show for all $q \in G_\alpha$

$$p_\alpha(q) := P(\langle q, J \rangle - (\langle q, \alpha \rangle + D_\alpha) I) \geq 0.$$

Indeed, by the variational principle we have

$$\begin{aligned} p_\alpha(q) &\geq \sup_{\mu \in Q^{-1}(\alpha)} \left\{ h_\mu + \left\langle q, \int J \, d\mu \right\rangle - (\langle q, \alpha \rangle + D_\alpha) \int I \, d\mu \right\} \\ &\geq \sup_{\mu \in Q^{-1}(\alpha)} \left\{ \int I \, d\mu \left(\frac{h_\mu}{\int I \, d\mu} - D_\alpha \right) \right\} \\ &\geq \text{const.} \sup_{\mu \in Q^{-1}(\alpha)} \left\{ \frac{h_\mu}{\int I \, d\mu} - D_\alpha \right\} = 0. \end{aligned}$$

Here we used the fact that $\sup_{\mu \in Q^{-1}(\alpha)} \int I \, d\mu$ is bounded above by some constant for $\alpha \neq 0$.

Next we show that the infimum $\inf_{q \in G_\alpha} p_\alpha(q)$ is attained at some point $q \in G_\alpha$. This follows from the fact that $p_\alpha(q_n)$ diverges to infinity whenever either $\langle q_n, \alpha \rangle + D_\alpha \to \theta$, $n \to \infty$, which is clear, or $|q_n| \to \infty$ which can be seen as follows. For $q = (q_1, \ldots, q_k)$ let $\beta_i := \alpha_i + r/2 \operatorname{sign} q_i$, $i = 1, \ldots, k$ and $\mu \in Q^{-1}(\beta)$. Then we have

$$\begin{aligned} p_\alpha(q) &\geq h_\mu + \left\langle q, \int (J - \alpha I) \, d\mu \right\rangle - D_\alpha \int I \, d\mu \\ &= h_\mu + \langle q, (\beta - \alpha) \rangle \int I \, d\mu - D_\alpha \int I \, d\mu \\ &= h_\mu + (\langle q, (\beta - \alpha) \rangle - D_\alpha) \int I \, d\mu \\ &\geq \left(\frac{r}{2} \sum_{i=1}^{k} |q_i| - \mathrm{HD}(\Lambda_\Phi) \right) (-\log s). \end{aligned} \tag{4.1}$$

Now the right hand side diverges to infinity for $|q| \to \infty$ showing that the infimum must be attained in some uniformly bounded region, say in $q_\alpha \in G_\alpha$. Since p_α is real-analytic on G_α we have

$$0 = \nabla p_\alpha(q_\alpha) = \int J - \alpha I \, d\mu_{q_\alpha} \implies \alpha = \frac{\int J \, d\mu_{q_\alpha}}{\int I \, d\mu_{q_\alpha}}, \tag{4.2}$$

where μ_{q_α} is the Gibbs measure for the potential $(\langle q_\alpha, J\rangle - (\langle q_\alpha, \alpha\rangle + D_\alpha) I)$. And hence, by the variational principle, we have

$$p(q_\alpha) = h_{\mu_{q_\alpha}} - D_\alpha \int I \, d\mu_{q_\alpha} \geq 0 \implies D_\alpha \leq \frac{h_{\mu_{q_\alpha}}}{\int I \, d\mu_{q_\alpha}}.$$

By definition of D_α we actually have equality and hence $p(q_\alpha) = 0$. This, (4.2) and Proposition 3.1 show that $\alpha = \nabla\beta(q_\alpha)$ for $\alpha \neq 0$.

Now we consider the case $\alpha = 0$. If $0 \in \partial L$ then nothing has to be shown. If $0 \in \operatorname{Int}(L)$ then also $B_r(0) \subset \operatorname{Int}(L)$ for some small $r > 0$ and by the above we have $B_r(0) \setminus \{0\} \subset M$. The convexity of β then implies that β has a minimum in $\mathbb{R}^d$ and since β is real-analytic this minimum is unique, say in t_0, with $\nabla\beta(t_0) = 0$. This shows that also $0 \in M$.

Proof of '$K \subset \overline{L}$'.

Let $\alpha \in K$ and without loss of generality not equal to zero. Then there exist $\omega \in E_A^\infty$ such that $\lim_{n\to\infty} \frac{S_n J(\omega)}{S_n I(\omega)} = \alpha$. Now, for every $n \in \mathbb{N}$ there exists by the finite irreducibility condition a word $w_n \in W$ such that the periodic element $x_n := (\omega|_n w_n)^\infty$ belongs to E_A^∞. The point x_n gives then rise to the invariant probability measure

$$\mu_n := \frac{1}{k_n} \sum_{j=0}^{k_n - 1} \delta_{\sigma^j x_n} \in \mathcal{M}(E_A^\infty, \sigma),$$

where $k_n := n + |w_n|$. By the Hölder continuity (bounded distortion) and the finiteness of W we estimate

$$\frac{\int J \, d\mu_n}{\int I \, d\mu_n} = \frac{S_{k_n} J(x_n)}{S_{k_n} I(x_n)} = \frac{S_n J(x_n) + \mathrm{O}(1)}{S_n I(x_n) + \mathrm{O}(1)} = \frac{S_n J(\omega) + \mathrm{O}(1)}{S_n I(\omega) + \mathrm{O}(1)}, \tag{4.3}$$

where O denotes the Landau symbol. Since $S_n I$ growth at least like $-n \log s$ the above quotient converges to α as $n \to \infty$. This shows $\alpha \in \overline{L}$.

Proof of compactness of K.

Since we know that K is bounded, we are left to show that the set is closed. As mentioned above $0 \in K$, hence we may consider without loss of generality a sequence $(\alpha_k) \in K^{\mathbb{N}}$ converging to $\alpha \in \mathbb{R}^k \setminus \{0\}$. We are going to construct inductively an element $\omega \in E_A^\infty$ such that $\lim \frac{S_n J(\omega)}{S_n I(\omega)} = \alpha$. Fix a sequence $\varepsilon_k \searrow 0$ such that $|\alpha_k - \alpha| < \varepsilon_k/2$. Using the observation in (4.3) we find for each $k \in \mathbb{N}$ a periodic element $x_k = p_k^\infty \in E_A^\infty$, $m_k := |p_k|$, such that $\left| \frac{S_{m_k} J(x_k)}{S_{m_k} I(x_k)} - \alpha_k \right| < \frac{\varepsilon_k}{2}$ which gives

$$\left| \frac{S_{m_k} J(x_k)}{S_{m_k} I(x_k)} - \alpha \right| < \varepsilon_k$$

We begin the induction by defining $\omega_1 := p_1^{l_1} w_1$ with $l_1 = 1$.

Suppose we have already defined $\omega_k := p_1^{l_1} w_1 \cdots p_k^{l_k} w_k$ for some $k \in \mathbb{N}$ and let

$$N_k := \sum_{i=1}^{k} l_i m_i + |w_i| .$$

Then choose $w_{k+1} \in W$ such that $\omega_k w_{k+1} p_{k+1} \in E_A^*$ and $l_{k+1} \in \mathbb{N}$ large enough such that

$$\frac{1}{-\log s} \cdot \frac{1}{l_{k+1} k_{k+1}} \max \left\{ S_{m_{k+2}} I\left(x_{k+2}\right), S_{m_{k+2}} \left|J\left(x_{k+2}\right)\right| \right\} \leq \varepsilon_{k+1}.$$

In this way we define inductively the infinite word $\omega := \left(p_i^{l_i} w_i\right)_{i=1}^{\infty} \in E_A^{\infty}$.

We will need the following observation. Suppose we have two sequences $(a_n) \in \left(\mathbb{R}^d\right)^{\mathbb{N}}$ and $(b_n) \in (\mathbb{R}^+)^{\mathbb{N}}$ such that $b_n^{-1} a_n \to \alpha$ and $\liminf_n b_n > 0$. Define $A_N := \sum_{k=1}^{N} a_k$ and $B_N := \sum_{k=1}^{N} b_k$. Let o denote the Landau symbol. Then for any two sequences (c_n) and (d_n) given by $c_n := A_{k_n} + \mathrm{o}(B_{k_n})$ and $d_n := B_{k_n} + \mathrm{o}(B_{k_n})$ for some sequence $(k_n) \in \mathbb{N}^{\mathbb{N}}$ tending to infinity, we have $d_n^{-1} c_n \to \alpha$.

For $n \in \mathbb{N}$ we define a sequence $(k_n) \in \mathbb{N}^{\mathbb{N}}$ such that $N_{k_n} \leq n < N_{k_n+1}$, and r_n, ℓ_n such that $n = N_{k_n} + q_n \cdot m_{k_n+1} + r_n$ with $0 \leq r_n \leq m_{k_n+1}$ and $0 \leq q_n \leq \ell_{k_n+1}$. Then applying the above observation to

$$S_n J(\omega) = \sum_{i=1}^{k_n} l_i S_{m_i} J\left(x_i\right) + q_n S_{m_{k_n+1}} J\left(x_{k_n+1}\right) + \mathrm{O}\left(S_{r_n} J\left(x_{k_n+1}\right)\right) + \mathrm{O}\left(k_n\right)$$

and

$$S_n I(\omega) = \sum_{i=1}^{k_n} l_i S_{m_i} I\left(x_i\right) + q_n S_{m_{k_n+1}} I_n\left(x_{k_n+1}\right) + \mathrm{O}\left(S_{r_n} I\left(x_{k_n+1}\right)\right) + \mathrm{O}\left(k_n\right)$$

and observing the definition of (ℓ_k) the claim follows.

Proof of '$\overline{M} \subset K$'.
Since we have already seen that K is closed it suffices to prove $M \subset K$. Let $\alpha = \nabla\beta(t) \in M$. Then for the Gibbs measure μ_t we have $\alpha = \left(\int I\, d\mu_t\right)^{-1} \int J\, d\mu_t$. By the ergodicity of μ_t we have for μ_t-a.e. $x \in E_A^{\infty}$ that $\lim_{n\to\infty} \frac{S_n J(x)}{S_n I(x)} = \alpha$ and hence $\alpha \in K$. □

Proof of Theorem 1.2. For J_i linearly independent as cohomology classes it is well known that β is strictly convex, or equivalently the Hessian $\mathrm{Hess}(\beta)$ is strictly positive definite. From this it follows that $\nabla\beta : \mathbb{R}^k \to \nabla\beta\left(\mathbb{R}^k\right)$ is a diffeomorphism and hence $M := \nabla\beta\left(\mathbb{R}^k\right)$ is an open and connected subset of $\mathbb{R}^k$.

Proof of '$L \subset \overline{\mathrm{Int} L}$'.
Since $\emptyset \neq M \subset \mathrm{Int} L$ we may use similar arguments as in [2] to prove that in this situation we have $L \subset \overline{\mathrm{Int} L}$. Let $\alpha = Q(m_0) \in L$. Since $\mathrm{Int} L$ is not empty we find measures $m_1, \ldots, m_k$ such that $\left(\frac{\int J\, dm_1}{\int I\, dm_1} - \alpha, \ldots, \frac{\int J\, dm_k}{\int I\, dm_k} - \alpha\right)$ form a basis of $\mathbb{R}^k$. For $p \in \Delta :=$

$\left\{u \in \mathbb{R}^k : u_i \geq 0 \wedge \sum_l u_l \leq 1\right\}$ we define $\mu_p := \sum_{l=1}^k p_l m_l + (1 - \sum p_l)\, m_0$. Then the derivative of $b : \Delta \to \mathbb{R}^k$, $b(p) = Q(\mu_p)$ is given by

$$\begin{aligned}\frac{\partial b_i}{\partial p_j}(0) &= \frac{\left(\int J_i\, dm_j - \int J_i\, dm_0\right)}{\int I\, dm_0} - \frac{\int J_i\, dm_0 \left(\int I\, dm_j - \int I\, dm_0\right)}{\left(\int I\, dm_0\right)^2} \\ &= \frac{\int J_i\, dm_j - \alpha_i \int I\, dm_j}{\int I\, dm_0}.\end{aligned}$$

By our assumption also $\left(\frac{\int I\, dm_1}{\int I\, dm_0}\left(\frac{\int J\, dm_1}{\int I\, dm_1} - \alpha\right), \ldots, \frac{\int I\, dm_k}{\int I\, dm_0}\left(\frac{\int J\, dm_k}{\int I\, dm_k} - \alpha\right)\right)$ are linearly independent, and hence $\frac{db}{dp}$ is invertible. This shows that there exists an open set $U \subset \Delta$ such that $0 \in \overline{U}$ and $b : U \to b(U)$ is an diffeomorphism. We finish the argument by observing that $\alpha \in \overline{b(U)} \subset \overline{\operatorname{Int} L}$.

Proof of '$L = \overline{M}$ for $0 \in M$'.
Since $\operatorname{Int} L \subset M \subset L \subset \overline{\operatorname{Int} L}$ we have $\overline{M} = \overline{L}$. It now suffices to show that $\overline{M} \subset L$ since the inclusion $\overline{L} = \overline{M} \subset L \subset \overline{L}$ would then imply that $L = \overline{M}$. To see that indeed $\overline{M} \subset L$ we proceed as follows. Recall that by our assumptions $0 \in M$. Let $\alpha \in \partial M$, which is then necessary different from 0. Let $(\alpha_n) \in M^{\mathbb{N}}$ be a sequence converging to α. For this sequence we find a sequence of Gibbs measures $\left(\mu_{s_k}\right)$ (for the potential $\langle s_k, J\rangle - \beta(s_k)\, I$) such that $\left(\int I\, d\mu_{s_k}\right)^{-1} \int J\, d\mu_{s_k}$ converges to α and $|s_k| \to \infty$. In particular, we have that $\left(\int I\, d\mu_{s_k}\right)$ is bounded . By Lemma 3.2 this sequence of measures is tight and hence there is a weak convergent subsequence $\mu_{t_k} \to \mu \in \mathcal{M}(E_A^\infty, \sigma)$. Now we have to show that $Q(\mu) = \alpha$. We clearly have $\int J\, d\mu_{t_k} \to \int J\, d\mu = v$ since J is bounded and by (1.3) we also have

$$\liminf_{k \to \infty} \int I\, d\mu_{t_k} \geq \int I\, d\mu. \tag{4.4}$$

To show that also $\limsup_{k \to \infty} \int I\, d\mu_{t_k} \leq \int I\, d\mu$ we make use of the variational principle. Indeed we have

$$0 = h\left(\mu_{t_k}\right) + \left\langle t_k, \int J\, d\mu_{t_k}\right\rangle - \beta(t_k) \int I\, d\mu_{t_k} \geq h_\mu + \left\langle t_k, \int J\, d\mu\right\rangle - \beta(t_k) \int I\, d\mu$$

This gives

$$\int I\, d\mu \geq \frac{h_\mu - h_{\mu_{t_k}}}{\beta(t_k)} + \left\langle \beta(t_k)^{-1} \cdot t_k, \int J\, d\mu - \int J\, d\mu_{t_k}\right\rangle + \int I\, d\mu_{t_k}. \tag{4.5}$$

Since

$$0 \leq h_{\mu_t} \leq \mathrm{HD}(\Lambda_\Phi) \int I\, d\mu_t$$

and $\int I\, d\mu_{t_k}$ is bounded we conclude that h_{μ_t} is bounded. The assumption $0 \in M$ implies that $|t_k| / \beta(t_k)$ is bounded (cf. [6, Theorem 3.26]). As J is a bounded function we also have $\int J\, d\mu - \int J\, d\mu_{t_k} \to 0$ and hence taking limits in (4.5) gives $\limsup_{k \to \infty} \int I\, d\mu_{t_k} \leq \int I\, d\mu$. Combining this with (4.4) finally proves $Q(\mu) = \alpha$.

The remaining parts of the theorem are an immediate application of Theorem 1.1. □

Remark 4.1. We would like to emphasis that, unlike above, for $\mu,\mu_1,\mu_2,\ldots\in\mathcal{M}\left(\Sigma_E^\infty,\sigma\right)$ with $\mu_n\xrightarrow{*}\mu$ and $\int I\,d\mu<+\infty$, $\int I\,d\mu_n<+\infty$, $n\in\mathbb{N}$, the convergence $\int I\,d\mu_n\to\int I\,d\mu$ does not hold in general. For a counter example we consider the iterated function system generated by continued fractions, i.e.

$$\Phi:=\left(\phi_k:[1,0]\to[1,0],x\mapsto 1/(x+k):k\in\mathbb{N}\right).$$

Fix $M>0$ to be large; $c_n:=1-M/\log(n)$, $n>\exp(M)$; $S:=\sum_{k=1}^\infty k^{-2}$ and $\left(p_k^{(n)}\right)_{k\in\mathbb{N}}$ a probability vector such that

$$p_k^{(n)}:=\begin{cases}0 & \text{if}\quad k>n,\\ S^{-1}c_nk^{-2} & \text{if}\quad k<n,\\ \left(1-S^{-1}c_n\sum_{j=1}^{n-1}j^{-2}\right) & \text{if}\quad k=n.\end{cases}$$

Let μ_n be the Bernoulli measure associated with $\left(p_k^{(n)}\right)_{k\in\mathbb{N}}$. Then $(\mu_n)_{n>\exp M}$ converges weakly to the Bernoulli measure associated with the probability vector $\left(S^{-1}k^{-2}\right)_{k\in\mathbb{N}}$. We then have

$$\begin{aligned}\int I\,d\mu &\le 2S^{-1}\sum_{k\in\mathbb{N}}\frac{\log(k+1)}{k^2}<+\infty\\ \int I\,d\mu_n &\le 2\left(1-S^{-1}c_n\sum_{j=1}^{n-1}j^{-2}\right)\log(n+1)+\sum_{k=1}^{n-1}2S^{-1}c_n\frac{\log(k+1)}{k^2}\\ &\le 2\left(1-S^{-1}c_n\left(S-\frac{c}{n}\right)\right)\log(n+1)+2S^{-1}\sum_{k=1}^{n-1}c_n\frac{\log(k+1)}{k^2}\\ &\le 2\left(\frac{M}{\log n}+\frac{cS^{-1}}{n}\right)\log(n+1)+2S^{-1}\sum_{k=1}^{n-1}c_n\frac{\log(k+1)}{k^2}\\ &\le 4M+2S^{-1}\sum_{k=1}^{n-1}c_n\frac{\log(k+1)}{k^2}<+\infty,\end{aligned}$$

for some constant $c>0$ and M sufficiently large. On the other hand

$$\begin{aligned}\int I\,d\mu_n &\ge \int_{C_1([1,\ldots])}I\,d\mu\ge 2\left(1-S^{-1}c_n\sum_{j=1}^{\infty}j^{-2}\right)\log n\\ &\ge 2\left(1-c_n\right)\log n=2M\ge 2\int I\,d\mu,\end{aligned}$$

for M large enough. Hence in this example, we have

$$\liminf_{k\to\infty}\int I\,d\mu_n>\int I\,d\mu.$$

For the proof of Theorem 1.3 we need the following lemma from convex analysis adapted to our situation. For an extended real-valued function $f:\mathbb{R}^k\to\overline{\mathbb{R}}$ we define the effective domain

$$\operatorname{dom}f:=\left\{x\in\mathbb{R}^k:f(x)>-\infty\right\}.$$

Lemma 4.2. *Under the conditions of Theorem 1.3 we have $M = \operatorname{Int}\left(\operatorname{dom}\widehat{\beta}\right)$ is a non-empty convex set. Furthermore for each $a \in M$ and $\alpha \in \partial M$ we have*

$$\lambda\alpha + (1-\lambda)\,a \in M \text{ for all } \lambda \in [0,1), \text{ and } \lim_{\lambda\to 1}\widehat{\beta}\left(\lambda\alpha + (1-\lambda)\,a\right) = \widehat{\beta}(\alpha). \tag{4.6}$$

In particular, we have

$$\overline{M} = \operatorname{dom}\widehat{\beta}. \tag{4.7}$$

Proof. Since $\operatorname{Int}\left(\operatorname{dom}\widehat{\beta}\right) \subset M \subset \operatorname{dom}\widehat{\beta}$ (cf. [7, Theorem 23.4]) and M is open we have $M = \operatorname{Int}\left(\operatorname{dom}\widehat{\beta}\right)$ and hence the convexity of $\operatorname{dom}\widehat{\beta}$ implies the convexity of M. Consequently, (4.6) immediately follows from [7, Theorem 6.1] and [7, Corollary 7.5.1].

Clearly by the definition of $\operatorname{dom}\widehat{\beta}$, we have $\widehat{\beta}(\alpha) = -\infty$ for $\alpha \notin \overline{M} = \overline{\operatorname{dom}\widehat{\beta}}$. The finiteness of $\widehat{\beta}$ on $\overline{M}$ follows from (4.6) and the fact that $0 \le \widehat{\beta}(a) \le \operatorname{HD}(\Lambda_\Phi)$ for all $a \in M$. This shows (4.7). □

Now we are in the position to give a proof of the first part of Theorem 1.3.

Proof of the first part of Theorem 1.3. We split the proof of this theorem in two parts - upper bound and lower bound.

For the upper bound we actually show a little more. Namely, for $\lambda \in (0,1)$ and $\alpha \in \overline{M}$ we consider $a_\lambda := \alpha_0 + \lambda(\alpha - \alpha_0)$, where $\alpha_0 := \nabla\beta(0)$ is the unique maximum of $\widehat{\beta}$. For

$$\mathcal{G}_{a_\lambda}(v) := \left\{\omega \in E_A^\infty : i(\omega_1) = v, \exists \ell \ge \lambda : \lim_{k\to\infty}\frac{S_k J(\omega)}{S_k I(\omega)} = \alpha_0 + \ell(\alpha - \alpha_0)\right\}$$

we prove

$$\operatorname{HD}\left(\mathcal{G}_{a_\lambda}(v)\right) \le \widehat{\beta}(a_\lambda). \tag{4.8}$$

Because then we have by the monotonicity of the Hausdorff dimension and (4.6)

$$\operatorname{HD}\left(\mathcal{F}_\alpha(v)\right) \le \operatorname{HD}\left(\mathcal{G}_{a_\lambda}(v)\right) \le \widehat{\beta}(a_\lambda) \to \widehat{\beta}(\alpha) \quad \text{for } \lambda \to 1.$$

To prove (4.8) let us first define $b : (0,1) \to \mathbb{R}$, $\lambda \mapsto \widehat{\beta}(a_\lambda)$. Then $b'(\lambda) = -\langle t(a_\lambda), \alpha - \alpha_0\rangle$ is non-positive since α_0 is the unique maximum of the strictly concave function $\widehat{\beta}$. Let $\mu_{t(a_\lambda)}$ denote the Gibbs measure for $\langle t(a_\lambda), J\rangle - \beta(t(a_\lambda))\,I$. Then by the above we have for every $\omega \in \mathcal{G}_{a_\lambda}(v)$, say $\lim_{k\to\infty}\frac{S_k J(\omega)}{S_k I(\omega)} = \alpha_0 + \ell(\alpha - \alpha_0)$ for some $\ell \ge \lambda$, and some $\varepsilon \ge 0$

$$\begin{aligned}
\mu_{t(a_\lambda)}(C_n(\omega)) &\gg \exp\left(\langle t(a_\lambda), S_n J(\omega)\rangle - \beta(t(a_\lambda))\,S_n I(\omega)\right) \\
&= \exp\left(-S_n I(\omega)\left(-\left\langle t(a_\lambda), \frac{S_n J(\omega)}{S_n I(\omega)}\right\rangle + \beta(t(a_\lambda))\right)\right) \\
&\gg \exp\left(-S_n I(\omega)\left(\widehat{\beta}(a_\lambda) + \varepsilon - (\ell - \lambda)\langle t(a_\lambda), \alpha - \alpha_0\rangle\right)\right) \\
&\gg \left|\pi(C_n(\omega))\right|^{\widehat{\beta}(a_\lambda)+\varepsilon}.
\end{aligned}$$

Now consider a sequence of balls $(B(\pi(\omega), r_n))_{n\in\mathbb{N}}$ with center in $\pi(\omega) \in X_v$ and radius $r_n := |\pi(C_n(\omega))|$, where $|A|$ denotes the diameter of the set $A \subset \mathbb{R}^k$. Then we have for all $\varepsilon > 0$

$$\mu_{t(a_\lambda)} \circ \pi^{-1}(B(\pi(\omega), r_n)) \quad \gg \quad \mu_{t(a_\lambda)}(C_n(\omega)) \gg r_n^{\widehat{\beta}(a_\lambda)+\varepsilon}.$$

Hence by standard arguments from geometric measure theory (cf. [8, 9]), the upper bound in (4.8) follows.

For the lower bound we first consider $\alpha = \nabla\beta(t) \in M$. Since $\int I \, d\mu_t < +\infty$ we have by [1, Theorem 4.4.2] and the variational principle that

$$\mathrm{HD}\left(\mu_t \circ \pi^{-1}\right) = \frac{h_{\mu_t}}{\int I \, d\mu_t} = \frac{\beta(t)\int I \, d\mu_t - \langle t, \int J \, d\mu_t\rangle}{\int I \, d\mu_t} = \widehat{\beta}(\alpha).$$

Since by Birkhoff's ergodic theorem we have $\mu_t\left(\pi^{-1}(\mathcal{F}_\alpha)\right) = 1$ the above equality gives

$$\widehat{\beta}(\alpha) = \mathrm{HD}\left(\mu_t \circ \pi^{-1}\right) \leq \mathrm{HD}(\mathcal{F}_\alpha).$$

The fact that by finite irreducibility $\mathrm{HD}(\mathcal{F}_\alpha) = \mathrm{HD}(\mathcal{F}_\alpha(v))$ for all $v \in V$ finishes the proof of the lower bound for $\alpha \in M$.

For $\alpha \in \complement\overline{M}$ we have on the one hand that $\widehat{\beta}(\alpha) = -\infty$ by (4.7) and on the other hand by Theorem 1.2 that $\mathrm{HD}(\mathcal{F}_\alpha(v)) = \mathrm{HD}(\emptyset) = 0$. This proves the theorem for $\alpha \in \complement\overline{M}$. $\square$

Before giving the proof of the remaining part of Theorem 1.3 we need the following proposition. Since from [3] we know that the entropy map is in our situation not upper semi-continuous in general this proposition might be of some interest for itself.

Proposition 4.3. *We assume that $0 \in M$ and let (t_k) be a sequence in $\mathbb{R}^k$ with $|t_k| \to +\infty$ such that the sequence of Gibbs measures $\mu_k := \mu_{t_k}$ converge weakly to some $\mu \in \mathcal{M}(E_A^\infty, \sigma)$. Then μ is supported on a subshift of finite type over a finite alphabet and we have*

$$\limsup_k h_{\mu_k} \leq h_\mu. \tag{4.9}$$

Proof. Since $0 \in M$ guarantees that $|t_k|/\beta(t_k)$ stays bounded for $k \to \infty$ (cf. "Proof of '$L = \overline{M}$ for $0 \in M$"') , J is a bounded, and I an unbounded function we find for $n \in \mathbb{N}$ and $D > 0$ an $N \in \mathbb{N}$ such that for

$$\omega \in \{x \in \Sigma_A^n : \exists m \in \{1, \ldots, n\}\, x_m \geq N\} =: \Sigma_A^n(N)$$

we have

$$\left(\left\langle \frac{1}{\beta(t_k)} t_k, S_n J(\omega)\right\rangle - S_n I(\omega)\right) < -D. \tag{4.10}$$

Using estimates from the proof of Theorem 2.3.3 in [1] (Gibbs property) one verifies that the constant C in (3.2) is always less or equal to $\exp(K(\beta(t) + |t|))$ for some positive constant K. Combining this fact, the Gibbs property (3.2) and (4.10) then gives that

$$\mu(C_n(\omega)) = 0 \text{ for all } \omega \in \Sigma_A^n(N). \tag{4.11}$$

Since μ is shift invariant it must be supported on a subshift contained in the full shift $\{1,\ldots,N-1\}^{\mathbb{N}}$.

To show (4.9) we set for $\nu \in \mathcal{M}(\Sigma_A^\infty,\sigma)$

$$H_n(\nu) := -\sum_{\omega\in\Sigma_A^n} \nu(C_n(\omega))\log\nu(C_n(\omega)).$$

It then suffices to verify for all $n \in \mathbb{N}$

$$H_n(\mu_k) \to H_n(\mu) \quad \text{for } k \to \infty. \tag{4.12}$$

To see this note that $h_\mu = \lim \frac{1}{n}H_n(\mu) = \inf \frac{1}{n}H_n(\mu)$. For $m,k \in \mathbb{N}$ we have

$$\begin{aligned} h_{\mu_k} &= \underbrace{\inf_n \frac{1}{n}H_n(\mu_k) - \frac{1}{m}H_m(\mu_k)}_{\leq 0} + \frac{1}{m}H_m(\mu_k) - \frac{1}{m}H_m(\mu) + \frac{1}{m}H_m(\mu) \\ &\leq \underbrace{\frac{1}{m}H_m(\mu_k) - \frac{1}{m}H_m(\mu)}_{\to 0,\text{ as } k\to\infty} + \frac{1}{m}H_m(\mu) \end{aligned}$$

implying $\limsup h_{\mu_k} \leq \frac{1}{m}H_m(\mu)$. Taking the infimum over $m \in \mathbb{N}$ gives (4.9). Indeed we have

$$\begin{aligned} H_n(\mu_k) &= -\sum_{\omega\in\Sigma_A^n} \mu_k(C_n(\omega))\log\mu_k(C_n(\omega)) \\ &= -\sum_{\omega\in\Sigma_A^n\setminus\Sigma_A^n(N)} \mu_k(C_n(\omega))\log\mu_k(C_n(\omega)) \\ &\qquad -\sum_{\omega\in\Sigma_A^n(N)} \mu_k(C_n(\omega))\log\mu_k(C_n(\omega)). \end{aligned}$$

The first sum has only finitely many summands and will therefore converge to the sum $-\sum_{\omega\in\Sigma_A^n\setminus\Sigma_A^n(N)} \mu(C_n(\omega))\log\mu(C_n(\omega))$ as k tends to infinity. Using the Gibbs property with constant $C = \exp(K(\beta(t)+|t|))$ one finds that the second sum is summable and dominated by

$$\begin{aligned} \exp(K(|t_k|+\beta(t_k))) \sum_{\omega\in\Sigma_A^n(N)} &(S_n(-\langle t_k,J\rangle + \beta(t_k)I)(\omega) + K(|t_k|+\beta(t_k))) \\ &\times \exp(S_n(\langle t_k,J\rangle - \beta(t_k)I)(\omega)) \end{aligned}$$

The inequality in (4.10) guarantees that for N sufficiently large this upper bound converges to 0 as $k \to \infty$. Hence, using (4.11) we conclude $H_n(\mu_k) \to H_n(\mu)$ for $k \to \infty$. □

Proof of the second part of Theorem 1.3. Now we consider the special situation in which $0 \in M$. Let $\alpha \in \partial M$. Then from the "Proof of '$L = \overline{M}$ for $0 \in M$' " we know that there exists be a sequence of Gibbs measures $\mu_k := \mu_{t_k}$ converging weakly to some $\mu \in \mathcal{M}\left(E_A^\infty, \sigma\right)$ such that $\left(\int I\, d\mu_k\right)^{-1} \int J\, d\mu_k =: \alpha_k$ lie on a line segment in M for all $k \in \mathbb{N}$ and converge to $\alpha = \left(\int I\, d\mu\right)^{-1} \int J\, d\mu$. Since μ is supported on a subshift of finite type over a finite alphabet we may apply results from [10, Appendix] (which generalize results from [11]). Let $\mathcal{G}(\mu) := \left\{\pi(x) : x \in \Sigma_A^\infty,\ \frac{1}{n}\sum_{k=0}^{n-1} \delta_{\sigma^k x} \xrightarrow{*} \mu\right\}$ be the set of the μ generic points. Then it has been shown in [10, Appendix] that there exists a Borel probability measure m on E_A^∞ and a Borel set $M \subset E_A^\infty$ such that $\pi(M) \subset \mathcal{G}(\mu)$, $m(M) = 1$ and for all $x \in M$ we have

$$\liminf_{n\to\infty} \frac{\log\left(m\left(C_n(x)\right)\right)}{\log\left|\pi\left(C_n(x)\right)\right|} = \frac{h_\mu}{\int I\, d\mu} \text{ and } \lim_n \frac{\log\left|\pi\left(C_n(x)\right)\right|}{-n} = \int I\, d\mu.$$

Now we argue similar as in the proof of Theorem 4.4.2 in [1] to conclude that $\mathrm{HD}(M) \geq \left(\int I\, d\mu\right)^{-1} h_\mu$. Fix $\varepsilon > 0$ small enough. Then by Egorov's Theorem, there exists a Borel set $M' \subset M$ and $n_0 \in \mathbb{N}$ such that $m(M') > 0$ and for all $n \geq n_0$ and $x \in M'$ we have

$$\frac{\log\left(m\left(C_n(x)\right)\right)}{\log\left|\pi\left(C_n(x)\right)\right|} \geq \frac{h_\mu}{\int I\, d\mu} - \varepsilon \text{ and } \frac{\log\left|\pi\left(C_n(x)\right)\right|}{-n} \leq \int I\, d\mu + \varepsilon.$$

This gives for all $n \geq n_0$, $x \in M'$

$$m\left(C_n(x)\right) \leq \left|\pi\left(C_n(x)\right)\right|^{\frac{h_\mu}{\int I d\mu} - \varepsilon} \text{ and } \mathrm{e}^{-n\left(\int I d\mu + \varepsilon\right)} \leq \left|\pi\left(C_n(x)\right)\right| \leq \mathrm{e}^{-n\left(\int I d\mu - \varepsilon\right)}.$$

We fix $0 < r < \exp\left(-n_0\left(\int I\, d\mu + \varepsilon\right)\right)$ and for $x \in M'$ let $n(x,r)$ be the least number n such that $\left|\pi\left(C_{n+1}(x)\right)\right| < r$. By the above estimates we have that $n(x,r) + 1 > n_0$ and hence $n(x,r) \geq n_0$ and $\left|\pi\left(C_{n(x,r)}(x)\right)\right| \geq r$. Lemma 4.2.6 in [1] guarantees that there exists a universal constant $L \geq 1$ such that for every $x \in M'$ and $0 < r < \exp\left(-n_0\left(\int I\, d\mu + \varepsilon\right)\right)$ there exist points $x_1, \ldots, x_k$ with $k \leq L$ such that $\pi\left(M' \cap B\left(\pi(x), r\right)\right) \subset \bigcup_{\ell=1}^k \pi\left(C_{n(x_\ell,r)}(x_\ell)\right)$. For $m' = m|_{M'}$ the restriction of m to the set M' we now have

$$\begin{aligned}
m' \circ \pi^{-1}\left(B\left(\pi(x), r\right)\right) &\leq \sum_{\ell=1}^k m\left(C_{n(x_\ell,r)}(x_\ell)\right) \leq \sum_{\ell=1}^k \left|\pi\left(C_{n(x_\ell,r)}(x_\ell)\right)\right|^{\frac{h_\mu}{\int I d\mu} - \varepsilon} \\
&\leq \sum_{\ell=1}^k \mathrm{e}^{\left(-n(x_\ell,r)\left(\int I d\mu - \varepsilon\right)\left(\frac{h_\mu}{\int I d\mu} - \varepsilon\right)\right)} \\
&= \sum_{\ell=1}^k \left(\mathrm{e}^{\left(-\left(n(x_\ell,r)+1\right)\left(\int I d\mu + \varepsilon\right)\right)}\right)^{\frac{n(x_\ell,r)\left(\int I d\mu - \varepsilon\right)}{\left(n(x_\ell,r)+1\right)\left(\int I d\mu + \varepsilon\right)}\left(\frac{h_\mu}{\int I d\mu} - \varepsilon\right)} \\
&\leq \sum_{\ell=1}^k \left|\pi\left(C_{n(x_\ell,r)+1}(x_\ell)\right)\right|^{\frac{n(x_\ell,r)\left(\int I d\mu - \varepsilon\right)}{\left(n(x_\ell,r)+1\right)\left(\int I d\mu + \varepsilon\right)}\left(\frac{h_\mu}{\int I d\mu} - \varepsilon\right)} \\
&\leq \sum_{\ell=1}^k r^{\frac{n(x_\ell,r)\left(\int I d\mu - \varepsilon\right)}{\left(n(x_\ell,r)+1\right)\left(\int I d\mu + \varepsilon\right)}\left(\frac{h_\mu}{\int I d\mu} - \varepsilon\right)} \\
&\leq L \cdot r^{\frac{\left(\int I d\mu - \varepsilon\right)}{\left(\int I d\mu + \varepsilon\right)}\left(\frac{h_\mu}{\int I d\mu} - 2\varepsilon\right)}
\end{aligned}$$

where the last inequality holds if we choose $n_0 > \varepsilon^{-1}\left(\frac{h_\mu}{\int I\,d\mu} - 2\varepsilon\right)$. By the mass distribution principle this shows that $\frac{h_\mu}{\int I\,d\mu} \leq \mathrm{HD}\left(\pi\left(M'\right)\right)$. Since $M' \subset M \subset \mathcal{G}(\mu) \subset \mathcal{F}_\alpha$ it follows that

$$\frac{h_\mu}{\int I\,d\mu} \leq \mathrm{HD}\left(\mathcal{F}_\alpha\right).$$

By (4.6) and Proposition 4.9 we therefore have

$$\widehat{\beta}(\alpha) = \lim_{k\to\infty} \widehat{\beta}(\alpha_k) = \lim_{k\to\infty} \frac{h_{\mu_k}}{\int I\,d\mu_k} \leq \frac{h_\mu}{\int I\,d\mu} \leq \mathrm{HD}\left(\mathcal{F}_\alpha\right).$$

This gives the lower bound for the Hausdorff dimension of $\mathcal{F}_\alpha$ also for $\alpha \in \partial M$. □

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