PSEUDO-MARKOV SYSTEMS AND INFINITELY GENERATED SCHOTTKY GROUPS

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Abstract. In this paper we extend the theory of conformal graph directed Markov systems to what we call conformal pseudo-Markov systems. These systems may have a countable infinite set of edges and, unlike graph directed Markov systems, they may also have a countable infinite set of vertices. Our first goal is to develop suitable symbolic dynamics, which we then use to analyse conformal pseudo-Markov systems by giving extensions of various aspects of the thermodynamic formalism and of fractal geometry. Most important, by establishing the existence of a unique conformal measure along with its invariant version, we obtain a generalization of Bowen's formula concerning the Hausdorff dimension of the limit set of a conformal pseudo-Markov system. Here, we also obtain an interesting formula for the closure of the limit set. Finally, we give some applications of our analysis to the theory of Kleinian groups. We show that there exists a rather exotic class of infinitely generated Schottky groups of the second kind (acting on $(d+1)$-dimensional hyperbolic space), containing groups with limit sets of Hausdorff dimension equal to any given number $t \leq d$, whereas their Poincaré exponent can be less than any given positive number $s < t$. Moreover, we show that the dissipative part of the limit sets of these groups has further interesting properties.

1. Introduction

In this paper we extend the theory of conformal graph directed Markov systems to what we call conformal pseudo-Markov systems. These systems can have a countable infinite set of edges, and unlike graph directed Markov systems (see [13] for a fairly complete exposition of these), they may also have a countable infinite set of vertices. Also, in comparison with graph directed Markov systems, these systems come with significantly weaker distortion conditions. After having developed suitable symbolic dynamics for pseudo-Markov systems, we analyse them by giving extensions of various aspects of the thermodynamic formalism and of fractal geometry. Most important, by establishing the existence and uniqueness of conformal measures along with their (up to a multiplicative constant) unique invariant versions, we obtain a generalization of Bowen's formula [3] concerning the Hausdorff dimension of the limit set of a conformal pseudo-Markov system. Here, we also obtain an interesting formula for the closure of the limit set. Finally, we apply our formalism to the theory of Kleinian groups and obtain the existence of a rather exotic class of infinitely generated Schottky groups of the second kind. Let us emphasize that there is a significant qualitative difference between pseudo-Markov systems with a countable infinite set of vertices and graph directed Markov systems. In particular, the step from a finite to an infinite set of vertices is certainly much

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more demanding than the step from iterated function systems (one vertex) to graph directed Markov systems (finitely many vertices). Note that for the definitions and some discussions of Hausdorff dimension, Hausdorff measure and related concepts we refer to [6] and [11]. Also, note that ergodic theoretical studies of systems with infinitely many branches (states) have been performed before by various authors, and since it seems impossible to list them all, let us at least mention the papers [5], [18], [19] and [12]. However, to the best of our knowledge, the development of a thermodynamic formalism for shift spaces of the type in this paper is completely new and seems not to have been considered anywhere in the literature so far.

For a slightly more detailed description of our applications to Kleinian groups, recall that Patterson [17], making use of a careful computational study of Poincaré series of free products of finitely generated Schottky groups acting on $(d+1)$-dimensional hyperbolic space $\mathbb{H}^{d+1}$, showed that there are infinitely generated Schottky groups of the first kind with critical exponent of their Poincaré series arbitrarily close to zero (recall that a Kleinian group is of the first kind if its limit set is equal to the whole boundary of $\mathbb{H}^{d+1}$, and therefore in this case the Hausdorff dimension of the limit set is equal to d). By applying the theory of conformal pseudo-Markov systems developed in this paper, we extend this class of groups to infinitely generated Schottky groups G of the second kind with limit sets $L(G)$ of Hausdorff dimension $\mathrm{HD}(L(G))$ equal to any prescribed number $t \in (0, d]$, whereas the critical exponent $\delta(G)$ of their Poincaré series can be less than any given positive number $s < t$. Moreover, these groups have the remarkable additional property that $\mathrm{HD}(L(G))$ is equal to the Hausdorff dimension of the set $\Delta(G)$ of accumulation points of the generating balls, whereas the complement within $L(G)$ of the G-orbit of $\Delta(G)$ already has Hausdorff dimension equal to $\delta(G)$. In particular, we will see that for these groups the dimension gap between the radial limit set of G and $L(G)$ cannot be filled smoothly by subsets of $L(G) \setminus \Delta(G)$.

However, the main goal of this paper is the development of the theory of conformal pseudo-Markov systems. This development was originally motivated by the combinatorics and geometry of the type of Schottky groups just mentioned, and one should mark that for these groups the theory of graph directed Markov systems as developed for instance in [12] and [13] is not applicable. Our analysis of conformal pseudo-Markov systems is structured as follows. In Section 2 we extend various aspects of the thermodynamic formalism to shift spaces generated by an infinite alphabet. For instance, by studying the Perron-Frobenius operator and some relevant pressure functions for these systems, we obtain the existence of pseudo-conformal measures and their invariant versions. In Section 3 we introduce the concept of pseudo-Markov systems and give a first analysis of the limit sets of these systems. We then investigate certain families of functions F and introduce the notion 'pseudo-conformality' for a pseudo-Markov systems. Using the results of Section 2, we eventually obtain under very weak conditions on the family F, that for pseudo-conformal pseudo-Markov systems there always exists an F-pseudo-conformal measure. Finally, in Section 4 we introduce conformal pseudo-Markov systems. After having discussed some of their main basic properties, especially their pseudo-conformality, we introduce the concepts of thinness and weak thinness for a conformal pseudo-Markov system. The idea here is that thinness guarantees a rather rapid decay of the sizes of the generators. We then derive the generalization of Bowen's Formula as well as the existence and uniqueness of conformal measures. In particular, we also obtain that for a thin conformal pseudo-Markov system the Hausdorff dimension of its limit set is always equal to its hyperbolic dimension, that is the supremum of the set of Hausdorff dimensions of all those limit sets which arise from finite subsystems.

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2. Thermodynamic Formalism for Systems with Countable Alphabets

2.1. Symbolic Dynamics.

Throughout let E be a countable alphabet which is either finite or infinite, and let

$$A = \big(A_{ij}\big) : E \times E \to \{0,1\}$$

be some arbitrary *incidence matrix*. The set of infinite admissible words is given by

$$E_A^\infty := \{\omega = \omega_1\omega_2 \ldots \in E^\infty : A_{\omega_i\omega_{i+1}} = 1 \text{ for all } i \in I\!N\}.$$

Also, let E_A^* refer to the set of all finite admissable subwords of E_A^∞. The left shift map $\sigma : E_A^\infty \to E_A^\infty$ is given for $\omega_1\omega_2 \ldots \in E_A^\infty$ by

$$\sigma(\omega_1\omega_2\omega_3 \ldots) := \omega_2\omega_3 \ldots .$$

Occasionally, we will consider this shift to be defined on words of finite length. For $\omega = \omega_1\omega_2 \ldots \omega_n \in E^*$, let $|\omega| := n$ denote the word length of ω, and let E_A^n refer to the set of all words in E_A^* of length n. If $\omega = \omega_1\omega_2 \ldots \in E_A^\infty$ and $n \in I\!N$, then

$$\omega|_n := \omega_1 \ldots \omega_n.$$

refers to the initial segment of ω of length n. Also, we define a metric d on E^∞ which is given for $\omega, \tau \in E^\infty$ by

$$d(\omega, \tau) := (1/2)^{l(\omega,\tau)},$$

where $l(\omega, \tau)$ refers to the length (possibly 0 or $+\infty$) of the longest common initial word of ω and τ, and where we make the usual convention $(1/2)^{+\infty} := 0$. Clearly, the space (E^∞, d) is a metric space, and consequently so is E_A^∞. The topology induced by d is the product (Tychonov) topology on E^∞. Finally, for $\tau \in E_A^n$ we define

$$[\tau] := \{\omega \in E_A^\infty : \omega|_n = \tau\}.$$

Throughout the entire paper we will always assume that the matrix A is *finitely irreducible*, meaning that there exists a finite set $\Xi \subset E_A^*$ such that for all $a, b \in E$ there exists $\alpha \in \Xi$ such that $a\alpha b \in E_A^*$.

2.2. **Topological Pressure, Perron-Frobenius Operators and Pseudo-Conformal measures on Symbolic Spaces.**
With the notation from the previous section, consider some arbitrary set $D \subseteq E$ and some arbitrary function $f : D_A^\infty \to I\!R$. For $n \in I\!N$, let $S_n f := \sum_{j=0}^{n-1} f \circ \sigma^j$ and define the *n-th partition function* $Z_n(D, f)$ by

$$Z_n(D, f) := \sum_{\omega \in D_A^n} \exp\Big(\sup_{\tau \in [\omega] \cap D_A^\infty} \Big(S_n f(\tau) \Big) \Big).$$

In case $D = E$, then we simply write $Z_n(f)$ instead of $Z_n(E, f)$. One immediately verifies that the sequence $(\log Z_n(D, f))_{n \in I\!N}$ is subadditive. This allows to define the *topological pressure* of f with respect to the shift map $\sigma : D_A^\infty \to D_A^\infty$ by

$$\mathrm{P}_D(f) := \lim_{n \to \infty} \frac{1}{n} \log Z_n(D, f) = \inf \left\{ \frac{1}{n} \log Z_n(D, f) : n \in I\!N \right\}. \tag{2.1}$$

In case $D = E$, then we simply write $\mathrm{P}(f)$ instead of $\mathrm{P}_E(f)$. Also, we put

$$\mathrm{P}_*(f) := \sup\{ \mathrm{P}_D(f) : D \subset E \text{ finite} \}.$$

Note that we immediately have that

$$\mathrm{P}_*(f) \le \mathrm{P}(f). \tag{2.2}$$

For the thermodynamic formalism in this paper, the following notions will be crucial.

Definition 2.1. *A function $f : E_A^\infty \to I\!R$ is called* summable *if*

$$\sum_{e \in E} \exp(\sup(f|_{[e]})) < \infty. \tag{2.3}$$

Definition 2.2. *A function $f : E_A^\infty \to I\!R$ is called* boundedly distorted *if and only if there exists a constant $Q = Q_{f,E} > 0$ such that*

$$\text{if } \tau|_{n+1} = \omega|_{n+1} \text{ for some } \omega, \tau \in E_A^\infty \text{ and } n \in I\!N, \text{ then } |S_n f(\omega) - S_n f(\tau)| \le Q. \tag{2.4}$$

Frequently, in case there is no confusion concerning which potential is in use, we will skip the subscript E and/or f, writing Q or Q_E instead of $Q_{f,E}$.

Throughout, we will always assume that $f : E_A^\infty \to I\!R$ is a summable, boundedly distorted and continuous function, and we will abbreviate this by writing SBDC.

Remark 2.3. *Note that the notion 'boundedly distorted' is significantly weaker than the type of bounded distortion used in connection with graph directed Markov systems. More precisely, the analysis of graph directed Markov systems in* [13] *required that if the first $n+1$ entries of τ and ω coincide then $|S_{n+1} f(\omega) - S_{n+1} f(\tau)| \le Q$. In contrast to this, despite the fact that* *we even allow an infinite set of vertices, in the present paper we only require that coincidence of the first $n+1$ entries of τ and ω implies that $|S_nf(\omega)-S_nf(\tau)| \leq Q$. Also, we remark that to our knowledge the currently available literature on subshifts on infinite alphabets always makes assumption on the function f such as for instance local Hölder continuity or at least uniformly bounded oscillation on cylinders of length one. Let us emphasize that in this paper we do not have to assume anything of this kind.*

Remark 2.4. *In case that the alphabet E is finite and $f : E_A^\infty \to I\!\!R$ is a boundedly distorted continuous function, then f is boundedly distorted in the above mentioned stronger sense, that is if the first $n+1$ entries of τ and ω coincide then $|S_{n+1}f(\omega) - S_{n+1}f(\tau)| \leq Q_E$.*

With $C_b(E_A^\infty)$ referring to the space of bounded continuous functions equipped with the supremums norm $||\cdot||_\infty$, the *Perron-Frobenius operator* $\mathcal{L}_f : C_b(E_A^\infty) \to C_b(E_A^\infty)$ is defined for $g \in C_b(E_A^\infty)$ and $\omega = \omega_1\omega_2\ldots \in E_A^\infty$ by

$$\mathcal{L}_f(g)(\omega) := \sum_{\substack{e\in E \\ A_{e\omega_1}=1}} \exp f(e\omega)g(e\omega).$$

Note that summability of f gives that $||\mathcal{L}_f||_\infty \leq \sum_{e\in E}\exp(\sup(f|_{[e]})) < \infty$. Also, for each $n \in I\!\!N$ we have

$$\mathcal{L}_f^n(g)(\omega) = \sum_{\substack{\tau\in E_A^n \\ A_{\tau_n\omega_1}=1}} \exp\Big(S_nf(\tau\omega)\Big)g(\tau\omega).$$

The dual operator $\mathcal{L}_f^* : C_b^*(E_A^\infty) \to C_b^*(E_A^\infty)$ is given by

$$\mathcal{L}_f^*(\mu)(g) := \mu(\mathcal{L}_f(g)) = \int \mathcal{L}_f(g)d\mu.$$

Throughout this section we will always assume that $\tilde{m}$ is an f*-pseudo-conformal measure*. This means by definition that $\tilde{m}$ is an eigenmeasure of $\mathcal{L}_f^*$ which is in particular a Borel probability measure on E_A^∞. Since $\mathcal{L}_f$ is a positive operator, we have for the eigenvalue $\lambda = \lambda(f,\tilde{m})$ associated with $\tilde{m}$ that $\lambda \geq 0$. Also, since $\mathcal{L}_f^{*n}(\tilde{m}) = \lambda^n\tilde{m}$, we have for each $g \in C_b(E_A^\infty)$,

$$\int \sum_{\substack{\tau=\tau_1\ldots\tau_n\in E_A^n \\ A_{\tau_n\omega_1}=1}} \exp\Big(S_nf(\tau\omega)\Big)g(\tau\omega)d\tilde{m}(\omega) = \lambda^n \int g d\tilde{m}. \tag{2.5}$$

Note that (2.5) immediately extends to the space of all bounded Borel functions on E_A^∞. For $\omega \in E_A^n$, let $B \subset E_A^\infty$ be some Borel set such that $A_{\omega_n\tau_1} = 1$ for each $\tau = \tau_1\tau_2\ldots \in B$. By choosing $g = 1\!\!1_{\omega B}$ in (2.5), we obtain

$$\begin{aligned}\lambda^n \tilde{m}(\omega B) &= \int \sum_{\substack{\tau\in E_A^n \\ A_{\tau_n\rho_1}=1}} \exp\Big(S_n f(\tau\rho)\Big) 1\!\!1_{\omega B}(\tau\rho) d\tilde{m}(\rho) \\ &= \int_{\{\rho\in B: A_{\omega_n\rho_1}=1\}} \exp\Big(S_n f(\omega\rho)\Big) d\tilde{m}(\rho) \\ &= \int_B \exp\Big(S_n f(\omega\rho)\Big) d\tilde{m}(\rho).\end{aligned} \tag{2.6}$$

The following straight forward lemma will turn out to be useful in the proof of the preceeding lemma.

Lemma 2.5. *For each $e \in E$ we have that $\tilde{m}([e]) > 0$.*

Proof. Since $E_A^\infty = \bigcup_{e\in E}[e]$, there exists $a \in E$ such that $\tilde{m}([a]) > 0$. Also, since A is finitely irreducible, we have for each $e \in E$ that there exists $\alpha \in \Xi$ such that $e\alpha a \in E_A^*$. It then follows from (2.6) that

$$\tilde{m}([e]) \geq \tilde{m}([e\alpha a]) = \lambda(f,\tilde{m})^{-n} \int_{[a]} \exp\Big(S_n f(e\alpha\rho)\Big) d\tilde{m}(\rho)0.$$

■

Lemma 2.6. *If $f : E_A^\infty \to I\!\!R$ is an SBDC function and $\tilde{m}$ is an f-pseudo-conformal measure, then* $\mathrm{P}_*(f) \leq \log \lambda(f,\tilde{m}) \leq \mathrm{P}(f)$.

Proof. Put $\lambda := \lambda(f,\tilde{m})$. In order to show that $\log\lambda \leq \mathrm{P}(f)$, we apply (2.5) once more with $g = 1\!\!1$, and obtain for each $n \in I\!\!N$,

$$\begin{aligned}\lambda^n &= \sum_{\tau\in E_A^n} \sum_{\substack{e\in E \\ A_{\tau_n e}=1}} \int_{[e]} \exp\Big(S_n f(\tau\omega)\Big) d\tilde{m}(\omega) \\ &\leq \sum_{\tau\in E_A^n} \sum_{e\in E} \exp\Big(\sup\Big(S_n f|_{[\tau]}\Big)\Big) \tilde{m}([e]) \\ &\leq \sum_{\tau\in E_A^n} \exp\Big(\sup\Big(S_n f|_{[\tau]}\Big)\Big) = Z_n(f).\end{aligned}$$

Inserting this estimate into the definition of $\mathrm{P}(f)$, the assertion follows.

In order to show that $\log\lambda \geq \mathrm{P}_*(f)$, let D be a finite subset of E which contains Ξ. Observe that by Lemma 2.5 we have

$$M_D := \min\{\tilde{m}([j]) : j \in D\} > 0.$$

Also, without loss of generality we can assume that for each $a \in D$ there exists $b \in D$ such that $A_{ab} = 1$. Therefore, an application of (2.5) with $g = 1\!\!1$ then gives for each $n \in I\!\!N$, with

Q_D defined as in Remark 2.4,

$$\begin{aligned}\lambda^n &= \int \sum_{\substack{\tau\in D_A^n\\ A_{\tau_n\omega_1}=1}} \exp\Big(S_n f(\tau\omega)\Big) d\tilde{m}(\omega)\\ &\geq e^{-Q_D}\int \sum_{\substack{\tau\in D_A^n\\ A_{\tau_n\omega_1}=1}} \exp\Big(\sup\Big(S_n f|_{[\tau]}\Big)\Big) d\tilde{m}(\omega)\\ &\geq e^{-Q_D}\int \sum_{\tau\in D_A^n} \exp\Big(\sup\Big(S_n f|_{[\tau]}\Big)\Big) A_{\tau_n\omega_1} d\tilde{m}(\omega)\\ &\geq e^{-Q_D} \sum_{\tau\in D_A^n} \exp\Big(\sup\Big(S_n f|_{[\tau]}\Big)\Big) M_D\\ &= M_D e^{-Q_D} Z_n(f,D).\end{aligned}$$

This implies that $\log\lambda \geq \mathrm{P}_D(f)$, and hence by taking the supremum over all possible finite D such that $\Xi \subset D \subset E$, the assertion follows. ■

For the proof of the main result of this section we require that a pseudo-conformal measure already exists under reasonably mild conditions. This will be guaranteed by the following result, which is essentially due to Bowen ([3]).

Lemma 2.7. *For E finite, A finitely irreducible, and $f : E_A^\infty \to \mathbb{R}$ a SBDC function, we have that there exists an f-pseudo-conformal measure.*

Proof. Consider the map given by

$$\mu \mapsto \frac{\mathcal{L}_f^*(\mu)}{\mathcal{L}_f^*(\mu)(\mathbb{1})}.$$

Clearly, this is a continuous map of the compact convex space of all Borel probability measures on E_A^∞ into itself. The Schauder-Tychonov Theorem then guarantees the existence of a fixed point, and one immediately verifies that this fixed point is an f-pseudo-conformal measure. ■

As an immediate corollary of the previous two lemmata, we obtain the following well known result. Note that in the theorem to come we will establish the same result without assuming that the set E is finite.

Corollary 2.8. *For E finite, A finitely irreducible, and $f : E_A^\infty \to \mathbb{R}$ a SBDC function, we have that the eigenvalue $\lambda(f,\tilde{m})$ is independent of the eigenmeasure $\tilde{m}$, and hence can be denoted by $\lambda(f)$. Also, we have that* $\mathrm{P}_*(f) = \log\lambda(f) = \mathrm{P}(f)$.

We are now in the position to come to the main result of this section, in which we show that even if the alphabet is infinite we still have under relatively mild conditions on the potential function f that the three quantities $\mathrm{P}_*(f), \mathrm{P}(f)$ and $\log\lambda(f)$ do coincide. Note that our proof here is completely different from the proof of the corresponding result for graph directed Markov systems given in [13].

Theorem 2.9. *If $f : E_A^\infty \to I\!R$ is an SBDC functions and A is finitely irreducible, then*

$$\mathrm{P}_*(f) = \mathrm{P}(f).$$

Proof. Since we trivially have that $\mathrm{P}_*(f) \le \mathrm{P}(f)$, it is sufficient to show that $\mathrm{P}(f) \le \mathrm{P}_*(f)$. For this we can assume without loss of generality that $E = I\!N$. Since A is finitely irreducible, there exists $p \in I\!N$ such that for every $n \ge p$ we have $D_n := \{1, 2, \ldots, n\} \supset \Xi$. Note, for this choice of n we in particular have that the restriction of A to $D_n \times D_n$ is irreducible. For ease of notation, let us put $(D_n)_A^l := D_n^l$, for $1 \le l \le \infty$ and $n \in I\!N$. Using Lemma 2.7, it then follows that there exists an eigenmeasure $\tilde{m}_n$ of the conjugate $\mathcal{L}_n^*$ of the Perron-Frobenius operator

$$\mathcal{L}_n : C(D_n^\infty) \to C(D_n^\infty)$$

associated to the function $f|_{D_n^\infty}$. Also, let $\mathrm{P}_n := \mathrm{P}\Big(\sigma|_{D_n^\infty}, f|_{D_n^\infty}\Big)$, and note that we obviously have that $\mathrm{P}_*(f) \ge \mathrm{P}_n \ge \mathrm{P}_1$, for all $n \in I\!N$. Since the function $f : E_A^\infty \to I\!R$ is summable, there exists $q \ge p$ sufficiently large such that

$$\sum_{i>q} \exp\Big(\sup\Big(f|_{[i]}\Big) - \mathrm{P}_1\Big) < 1/2.$$

Using the pseudo-conformality of the measure $\tilde{m}_k$, for some arbitrary $k \ge q$, we obtain

$$\begin{aligned} 1 = \sum_{i=1}^{k} \tilde{m}_k([i]) &= \sum_{i=1}^{q} \tilde{m}_k([i]) + \sum_{i=q+1}^{k} \tilde{m}_k([i]) \le \sum_{i=1}^{q} \tilde{m}_k([i]) + \sum_{i=q+1}^{k} \exp\Big(\sup\Big(f|_[i]\Big) - \mathrm{P}_k\Big) \\ &\le \sum_{i=1}^{q} \tilde{m}_k([i]) + \sum_{i=q+1}^{k} \exp\Big(\sup\Big(f|_[i]\Big) - \mathrm{P}_1\Big) \le \frac{1}{2} + \sum_{i=1}^{q} \tilde{m}_k([i]). \end{aligned}$$

This implies that $\sum_{i=1}^{q} \tilde{m}_k([i]) \ge 1/2$, and hence there exists $a \in \{1, 2, \ldots, q\}$ such that

$$\tilde{m}_k([a]) \ge \frac{1}{2q}. \tag{2.7}$$

Since $k \ge q \ge p$, we have that for each $i \in \{1, 2, \ldots, k\}$ there exists $\rho \in \Xi$ such that $i\rho a \in (D_k)_A^*$. Therefore, invoking Remark 2.3 and (2.7), and putting $Q := Q_{f,E}$, we obtain

$$\begin{aligned} \tilde{m}_k([i]) \ge \tilde{m}_k([i\rho a]) &\ge e^{-Q} \exp\Big(\sup\Big(S_{|\rho|+1} f\|_{[i\rho]}\Big)\Big) - (|\rho|+1)\mathrm{P}_k\Big)\tilde{m}_k([a]) \\ &\ge (2qe^{Q})^{-1} \exp\Big(\sup\Big(S_{|\rho|+1} f\|_{[i\rho]}\Big)\Big) - (|\rho|+1)\mathrm{P}_k\Big) \\ &\ge (2qe^{Q})^{-1} \exp\Big(\sup\Big(S_{|\rho|+1} f\|_{[i\rho]}\Big)\Big) - (|\rho|+1)\mathrm{P}_*(f)\Big). \end{aligned}$$

Since for each $i \in E$ we always have that there are at most $\#\Xi$ possible paths of the form $i\rho$, for $\rho \in \Xi$, the latter estimate immediately implies that $\inf\{\tilde{m}_k([i]) : k \geq q\} > 0$. Consequently, it follows

$$T_q := \min_{1 \leq i \leq q} \inf\{\tilde{m}_k([i]) : k \geq q\} > 0. \tag{2.8}$$

Now, let $k \geq q$ be fixed. Combining the observations above, we now have for every $n \geq 0$ that, where for ease of notation we let $Z_n(l, f) := Z_n(D_l, f)$,

$$\begin{aligned}
\exp(-\mathrm{P}_*(f)(n+1)) Z_{n+1}(k,f) &= \sum_{\omega \in D_k^n} \sum_{\substack{i \in E \\ A_{\omega_n i}=1}} \exp\Big(\sup\Big(S_{n+1} f|_{[\omega i]}\Big) - \mathrm{P}_*(f)(n+1)\Big) \\
&\leq \sum_{\omega \in D_k^n} \sum_{\substack{i \in E \\ A_{\omega_n i}=1}} \exp\Big(\sup\Big(S_n f|_{[\omega i]}\Big) - \mathrm{P}_*(f) n\Big) \exp\Big(\sup\Big(f|_{[i]}\Big) - \mathrm{P}_*(f)\Big) \\
&\leq \sum_{\omega \in D_k^n} \sum_{\substack{i \leq q \\ A_{\omega_n i}=1}} \exp\Big(\sup\Big(S_n f|_{[\omega i]}\Big) - \mathrm{P}_k n\Big) \exp\Big(\sup\Big(f|_{[i]}\Big) - \mathrm{P}_*(f)\Big) \\
&\quad + \sum_{\omega \in D_k^n} \exp\Big(\sup\Big(S_n f|_{[\omega i]}\Big) - \mathrm{P}_*(f) n\Big) \cdot \sum_{i > q} \exp\Big(\sup\Big(f|_{[i]}\Big) - \mathrm{P}_1(f)\Big) \\
&\leq \sum_{\omega \in D_k^n} \sum_{\substack{i \leq q \\ A_{\omega_n i}=1}} e^Q \frac{\tilde{m}_k([\omega i])}{\tilde{m}_k([i])} \exp\Big(\sup\Big(f|_{[i]}\Big) - \mathrm{P}_*(f)\Big) + \frac{1}{2} \sum_{\omega \in D_k^n} \exp\Big(\sup\Big(S_n f|_{[\omega i]}\Big) - \mathrm{P}_*(f) n\Big) \\
&\leq T_q^{-1} e^Q \sum_{\omega \in D_k^n} \sum_{\substack{i \leq q \\ A_{\omega_n i}=1}} \tilde{m}_k([\omega i]) \exp\Big(\sup\Big(f|_{[i]}\Big) - \mathrm{P}_*(f)\Big) + \frac{1}{2} Z_n(k,f) \exp(-\mathrm{P}_*(f) n) \\
&\leq T_q^{-1} e^Q \sum_{i \leq q} \exp\Big(\sup\Big(f|_{[i]}\Big) - \mathrm{P}_*(f)\Big) \sum_{\substack{\omega \in D_k^n \\ A_{\omega_n i}=1}} \tilde{m}_k([\omega i]) + \frac{1}{2} Z_n(k,f) \exp(-\mathrm{P}_*(f) n) \\
&\leq T_q^{-1} e^Q \sum_{i=1}^{\infty} \exp\Big(\sup\Big(f|_{[i]}\Big) - \mathrm{P}_*(f)\Big) + \frac{1}{2} Z_n(k,f) \exp(-\mathrm{P}_*(f) n) \\
&= T_q^{-1} e^Q \exp(-\mathrm{P}_*(f)) Z_1(E,f) + \frac{1}{2} \exp(-\mathrm{P}_*(f) n) Z_n(k,f).
\end{aligned}$$

Hence, a straight forward inductive argument now gives

$$\begin{aligned}
\exp(-\mathrm{P}_*(f) n) Z_n(k,f) &\leq 2 T_q^{-1} e^Q \exp(-\mathrm{P}_*(f)) Z_1(E,f) + 2^{-(n-1)} \exp(-\mathrm{P}_*(f)) Z_1(k,f) \\
&\leq 2 T_q^{-1} e^Q \exp(-\mathrm{P}_*(f)) Z_1(E,f) + \exp(-\mathrm{P}_*(f)) Z_1(E,f) \\
&= (2 T_q^{-1} e^Q + 1) \exp(-\mathrm{P}_*(f)) Z_1(E,f) < \infty.
\end{aligned}$$

Thus, by letting k tend to infinity, we obtain for each $n \in \mathbb{N}$,

$$\exp(-\mathrm{P}_*(f) n) Z_n(E,f) \leq (2 T_q^{-1} e^Q + 1) \exp(-\mathrm{P}_*(f)) Z_1(E,f).$$

Here, the right hand side does not depend on n, and therefore,

$$\mathrm{P}(f) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(E,f) \le \mathrm{P}_*(f).$$

This finishes the proof. ■

2.3. **Pseudo-Conformal and Invariant Measures.**
We start with the main result of this section which generalizes Lemma 2.7 to systems with an infinite alphabet. Note that the proof is inspired by the proof of the corresponding result for graph directed Markov systems given in [13] (Theorem 2.7.3).

Theorem 2.10. *For A finitely irreducible and $f : E_A^\infty \to I\!R$ a SBDC function, we have that there exists an f-pseudo-conformal measure.*

Proof. Again, we can assume without loss of generality that $E = I\!N$. We use the notation as introduced in the proof of Theorem 2.9. Recall that by Lemma 2.7 we have that there exists an eigenmeasure $\tilde{m}_n$ of the conjugate $\mathcal{L}_n^*$ of the Perron-Frobenius operator $\mathcal{L}_n$. Occasionally, we will view $\mathcal{L}_n$ as acting on $C(E_A^\infty)$, and $\mathcal{L}_n^*$ as acting on $C^*(E_A^\infty)$. Now, our first aim is to show tightness for the sequence $\{\tilde{m}_n\}_{n\ge p}$. For this, note that Corollary 2.8 gives that e^{P_n} is the eigenvalue of $\mathcal{L}_n^*$ corresponding to the eigenmeasure $\tilde{m}_n$. Therefore, an application of (2.6) gives for each $n, k \in I\!N$ and every $e \in E$, where $\pi_k : E_A^\infty \to E$ refers to the projection onto the k-th coordinate given by $\pi_k(e_1e_2\ldots) := e_k$,

$$\begin{aligned}
\tilde{m}_n(\pi_k^{-1}(e)) &= \sum_{\substack{\omega\in D_n^k\\ \omega_k=e}} \tilde{m}_n([\omega]) \le \sum_{\substack{\omega\in D_n^k\\ \omega_k=e}} \exp\Big(\sup(S_kf|_{[\omega]}) - \mathrm{P}_n k\Big)\\
&\le \mathrm{e}^{-\mathrm{P}_n k} \sum_{\substack{\omega\in D_n^k\\ \omega_k=e}} \exp\Big(\sup(S_{k-1}f|_{[\omega]}) + \sup(f|_{[e]})\Big)\\
&\le \mathrm{e}^{-\mathrm{P}_1 k}\left(\sum_{i\in E} \mathrm{e}^{\sup(f|_{[i]})}\right)^{k-1} \mathrm{e}^{\sup(f|_{[e]})}.
\end{aligned}$$

Therefore,

$$\tilde{m}_n(\pi_k^{-1}([e+1,\infty))) \le \mathrm{e}^{-\mathrm{P}_1 k}\left(\sum_{i\in E} \mathrm{e}^{\sup(f|_{[i]})}\right)^{k-1} \sum_{j>e} \mathrm{e}^{\sup(f|_{[j]})}.$$

Now, let $\epsilon > 0$ be fixed. Also, for each $k \in I\!N$ let $n_k \in I\!N$ be chosen such that

$$\mathrm{e}^{-\mathrm{P}_1 k}\left(\sum_{i\in E} \mathrm{e}^{\sup(f|_{[i]})}\right)^{k-1} \sum_{j>n_k} \mathrm{e}^{\sup(f|_{[j]})} \le \frac{\epsilon}{2^k}.$$

We then have $\tilde{m}_n(\pi_k^{-1}([n_k+1,\infty))) \le \epsilon/2^k$ for all $n, k \in I\!N$, and hence,

$$\tilde{m}_n\left(E_A^\infty \cap \prod_{k\in I\!N}[1,n_k]\right) \ge 1 - \sum_{k\in I\!N} \tilde{m}_n(\pi_k^{-1}([n_k+1,\infty))) \ge 1 - \sum_{k\in I\!N} \frac{\epsilon}{2^k} = 1-\epsilon.$$

Since $E_A^\infty \cap \prod_{k\in I\!N}[1,n_k]$ is a compact subset of E_A^∞, it follows that the sequence $\{\tilde m_n\}_{n\in I\!N}$ is tight. This finishes the first part of the proof. In order to obtain the existence of an f-pseudo-conformal measure, we apply Prokhorov's Theorem to the sequence $\{\tilde m_n\}_{n\in I\!N}$. This gives that this sequence of measures must have a weak limit $\tilde m$. Hence, we are left with to show that $\mathcal{L}_f^*\tilde m = \mathrm{e}^{\mathrm{P}(f)}\tilde m$. For this, let $g \in C_b(E_A^\infty)$ and $\epsilon > 0$ be fixed, and define the normalized Perron-Frobenius operator $\mathcal{L}_0 := \mathrm{e}^{-\mathrm{P}(f)}\mathcal{L}_f$. For the remainder of the proof we will assume that $n \in I\!N$ is chosen sufficiently large such that the following four inequalities are satisfied.

$$\sum_{i>n} \|g\|_0 \exp\Big(\sup(f|_{[i]}) - \mathrm{P}(f)\Big) \le \frac{\epsilon}{6}, \tag{2.9}$$

$$\sum_{i\in E} \|g\|_0 \exp\Big(\sup(f|_{[i]})\Big)\mathrm{e}^{-\mathrm{P}_1}|e^{\mathrm{P}(f)} - e^{\mathrm{P}_n}| \le \frac{\epsilon}{6}, \tag{2.10}$$

$$|\tilde m_n(g) - \tilde m(g)| \le \frac{\epsilon}{3}, \tag{2.11}$$

and

$$\left|\int \mathcal{L}_0(g)d\tilde m - \int \mathcal{L}_0(g)d\tilde m_n\right| \le \frac{\epsilon}{3}. \tag{2.12}$$

Note that (2.10) holds, since by Theorem 2.9 we have that $\lim_{n\to\infty}\mathrm{P}_n = \mathrm{P}(f)$. For $n \ge p$ in this range, we define $g_n := g|_{(D_n)_A^\infty}$ and consider the normalized Perron-Frobenius operator $\mathcal{L}_{0,n} := \mathrm{e}^{-\mathrm{P}_n}\mathcal{L}_n$ associated with D_n. Let us make the following two observations. First,

$$\begin{aligned}
\mathcal{L}_{0,n}^*\tilde m_n(g) &= \int_{E^\infty} \sum_{\substack{i\le n\\ A_{i\omega_n}=1}} g(i\omega)\exp(f(i\omega) - \mathrm{P}_n)d\tilde m_n(\omega) \\
&= \int_{D_n^\infty} \sum_{\substack{i\le n\\ A_{i\omega_n}=1}} g(i\omega)\exp(f(i\omega) - \mathrm{P}_n)d\tilde m_n(\omega) \\
&= \int_{D_n^\infty} \sum_{\substack{i\le n\\ A_{i\omega_n}=1}} g_n(i\omega)\exp(f(i\omega) - \mathrm{P}_n)d\tilde m_n(\omega) \\
&= \mathcal{L}_{0,n}^*\tilde m_n(g_n) = \tilde m_n(g_n),
\end{aligned} \tag{2.13}$$

and secondly,

$$\tilde m_n(g_n) - \tilde m_n(g) = \int_{D_n^\infty}(g_n - g)d\tilde m_n = \int_{D_n^\infty} 0\ d\tilde m_n = 0. \tag{2.14}$$

Hence, using the triangle inequality, it follows

$$\begin{aligned}
|\mathcal{L}_0^*\tilde m(g) - \tilde m(g)| \le{}& |\mathcal{L}_0^*\tilde m(g) - \mathcal{L}_0^*\tilde m_n(g)| + |\mathcal{L}_0^*\tilde m_n(g) - \mathcal{L}_{0,n}^*\tilde m_n(g)| + \\
&+ |\mathcal{L}_{0,n}^*\tilde m_n(g) - \tilde m_n(g_n)| + |\tilde m_n(g_n) - \tilde m_n(g)| + |\tilde m_n(g) - \tilde m(g)|.
\end{aligned} \tag{2.15}$$

In here we obtain for the second summand, by applying (2.9) and (2.10),

$$
\begin{aligned}
|\mathcal{L}_0^*\tilde{m}_n(g) - \mathcal{L}_{0,n}^*\tilde{m}_n(g)| = \\
&= \left| \int_{E_A^\infty} \sum_{\substack{i\leq n \\ A_{i\omega_n}=1}} g(i\omega)\Big(\exp(f(i\omega) - \mathrm{P}(f)) - \exp(f(i\omega) - \mathrm{P}_n)\Big) d\tilde{m}_n(\omega) \right. \\
&+ \left. \int_{E_A^\infty} \sum_{\substack{i>n \\ A_{i\omega_n}=1}} g(i\omega)\exp(f(i\omega) - \mathrm{P}(f)) d\tilde{m}_n(\omega) \right| \\
&\leq \sum_{i\leq n} \|g\|_0 \mathrm{e}^{f(i\omega)}\mathrm{e}^{-\mathrm{P}_n}|e^{\mathrm{P}(f)} - e^{\mathrm{P}_n}| + \sum_{i>n} \|g\|_0 \exp\Big(\sup(f|_{[i]}) - \mathrm{P}(f)\Big) \qquad (2.16) \\
&\leq \sum_{i\in E} \|g\|_0 \exp\Big(\sup(f|_{[i]})\Big)\mathrm{e}^{-\mathrm{P}_1}|e^{\mathrm{P}(f)} - e^{\mathrm{P}_n}| + \frac{\epsilon}{6} \\
&\leq \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{3}.
\end{aligned}
$$

Combining (2.15) with (2.11), (2.12), (2.13), (2.14) and (2.16), it now follows that

$$|\mathcal{L}_0^*\tilde{m}(g) - \tilde{m}(g)| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

By letting ϵ tend to 0, we conclude that $\mathcal{L}_0^*\tilde{m}(g) = \tilde{m}(g)$, or alternatively $\mathcal{L}_f^*\tilde{m}(g) = \mathrm{e}^{\mathrm{P}(f)}\tilde{m}(g)$. Since $g \in C_b(E_A^\infty)$ was assumed to be arbitrary, we have now shown that $\mathcal{L}_f^*\tilde{m} = \mathrm{e}^{\mathrm{P}(f)}\tilde{m}$.

■

We end this section showing that the f-pseudo-conformal measure $\tilde{m}$ is unique and that its measure class always contains a unique σ-finite shift-invariant representative $\tilde{\mu}$ which is ergodic and conservative. Note that a similar slightly stronger result was obtained in [13] for graph directed Markov systems, where in contrast to the situation in this paper the invariant measure is always finite.

Theorem 2.11. *Let A be finitely irreducible, and let $f : E_A^\infty \to \mathbb{R}$ be some SBDC function. Then the associated f-pseudo-conformal measure $\tilde{m}$ is unique, and there exists a shift-invariant σ-finite Borel measure μ on E_A^∞ absolutely continuous with respect to $\tilde{m}$. The measure μ is ergodic, conservative and unique (up to a multiplicative constant).*

Proof. Fix $e \in E$, $n \in \mathbb{N}$, and $\tau = \tau_1 \dots \tau_n \in E_A^n$ such that $A_{\tau_n e} = 1$. Let σ_τ^{-n} refer to the n-th inverse branch associated with τ, that is $\sigma_\tau^{-n}(\omega) := \tau\omega$ for $\omega \in \Sigma_A^\infty$ with $\tau\omega \in \Sigma_A^\infty$. By f-pseudo-conformality of $\tilde{m}$, we have for each Borel set $B \subseteq [e]$,

$$\tilde{m}(\sigma_\tau^{-n}(B)) = \int_B \exp\Big(S_nf(\tau\rho) - \mathrm{P}(f)n\Big) d\tilde{m}(\rho) \leq e^Q \exp\Big(\inf\Big(S_nf|_{[\tau e]} - \mathrm{P}(f)n\Big)\Big)\tilde{m}(B) \qquad (2.17)$$

and

$$\tilde{m}(\sigma_\tau^{-n}(B)) = \int_B \exp\Big(S_n f(\tau\rho) - \mathrm{P}(f)n\Big) d\tilde{m}(\rho) \geq \exp\Big(\inf\Big(S_n f|_{[\tau e]} - \mathrm{P}(f)n\Big)\Big)\tilde{m}(B). \tag{2.18}$$

Note that if in (2.17) we let $B = [e]$, then

$$\tilde{m}(\sigma_\tau^{-n}([e])) \leq e^Q \exp\Big(\inf\Big(S_n f|_{[\sigma_\tau^{-n}([e])} - \mathrm{P}(f)n\Big)\Big)\tilde{m}([e]).$$

Therefore, in this case we have

$$\exp\Big(\inf\Big(S_n f|_{[\tau e]} - \mathrm{P}(f)n\Big)\Big) \geq e^{-Q}\frac{\tilde{m}(\sigma_\tau^{-n}([e]))}{\tilde{m}([e])} = e^{-Q}\frac{\tilde{m}([\tau e])}{\tilde{m}([e])}. \tag{2.19}$$

Proceeding similar with (2.18), we obtain

$$\exp\Big(\inf\Big(S_n f|_{[\tau e]} - \mathrm{P}(f)n\Big)\Big) \leq \frac{\tilde{m}([\tau e])}{\tilde{m}([e])}. \tag{2.20}$$

Inserting (2.20) into (2.17) and (2.19) into (2.18), it follows

$$e^{-Q}\frac{\tilde{m}(B)}{\tilde{m}([e])}\tilde{m}([\tau e]) \leq \tilde{m}(\sigma_\tau^{-n}(B)) \leq e^{Q}\frac{\tilde{m}(B)}{\tilde{m}([e])}\tilde{m}([\tau e]). \tag{2.21}$$

By summing (2.21) over all $\tau \in E_A^n$ with $A_{\tau_n e} = 1$, we deduce that

$$e^{-Q}\frac{\tilde{m}([B])}{\tilde{m}([e])} \leq \frac{\tilde{m}(\sigma^{-n}(B))}{\tilde{m}(\sigma^{-n}([e]))} \leq e^{Q}\frac{\tilde{m}([B])}{\tilde{m}([e])}. \tag{2.22}$$

Again, without loss of generality we assume that $E = I\!N$. Also, define for each $k \in I\!N$,

$$I\!N_k := \{k+1, k+2, \dots\}.$$

Since the function $f : E_A^\infty \to I\!R$ is summable, there exists $k \in I\!N$ such that for every $e \in I\!N_k$,

$$\exp\Big(\sup\Big((f - \mathrm{P}(f))|_{[e]}\Big)\Big) \leq 1/2.$$

For each Borel set $B \subset E_A^\infty$, we then have

$$\begin{aligned}\tilde{m}\left(\sigma^{-1}(B) \cap I\!N_k\right) &= \tilde{m}\left(\bigcup_{j=k+1}^{\infty} \sigma_j^{-1}\left(\bigcup_{\substack{i\\A_{ji}=1}} B \cap [i]\right)\right)\\ &\leq \sum_{i=1}^{\infty}\sum_{\substack{j>k\\A_{ji}=1}} \exp\Big(\sup\Big((f - \mathrm{P}(f))|_{[j]}\Big)\Big)\tilde{m}(B \cap [i])\\ &\leq \frac{1}{2}\sum_{i=1}^{\infty}\sum_{\substack{j>k\\A_{ji}=1}} \tilde{m}(B \cap [i]) \leq \frac{1}{2}\tilde{m}(B).\end{aligned}$$

Consequently, it follows that for each $n \geq 0$,

$$\tilde{m}\left(\bigcap_{j=0}^{n+1} \sigma^{-j}(I\!N_k)\right) = \tilde{m}\left(\sigma^{-1}\left(\bigcap_{j=0}^{n} \sigma^{-j}(I\!N_k)\right) \cap I\!N_k\right) \leq \frac{1}{2}\tilde{m}\left(\bigcap_{j=0}^{n} \sigma^{-j}(I\!N_k)\right).$$

Therefore, we obtain by way of induction that $\tilde{m}\left(\bigcap_{j=0}^{n} \sigma^{-j}(I\!N_k)\right) \leq (1/2)^n \tilde{m}(I\!N_k)$. This shows that $\tilde{m}\left(\bigcap_{j=0}^{\infty} \sigma^{-j}(I\!N_k)\right) = 0$, and hence,

$$\tilde{m}\left(\bigcup_{n=0}^{\infty} \sigma^{-n}\left(\bigcap_{j=0}^{\infty} \sigma^{-j}(I\!N_k)\right)\right) = 0.$$

We hence have for $\tilde{m}$-almost every $\omega = \omega_1\omega_2 \ldots \in E_A^\infty$,

$$\liminf_{n\to\infty} \omega_n \leq k. \tag{2.23}$$

An immediate application of the Borel-Cantelli Lemma now gives

$$\sum_{n=0}^{\infty}\sum_{e=1}^{k} \tilde{m}(\sigma^{-n}([e])) = \sum_{n=0}^{\infty} \tilde{m}(\sigma^{-n}([1] \cup [2] \cup \ldots \cup [k])) = +\infty,$$

Hence, there exists $q \in D_k$ such that

$$\sum_{n=0}^{\infty} \tilde{m}(\sigma^{-n}([q])) = +\infty. \tag{2.24}$$

Now, let $e \in E$ be fixed, and choose $\alpha, \beta \in \Xi$ such that $q\alpha e$ and $e\beta q$ are both A-admissible. Using the first inequality in (2.22), we then have

$$\tilde{m}(\sigma^{-(n+1+|\alpha|)}([e])) \geq \tilde{m}(\sigma^{-n}([q\alpha e])) \geq e^{-Q}\frac{\tilde{m}([q\alpha e])}{\tilde{m}([q])}\tilde{m}(\sigma^{-n}([q]))$$

and

$$\tilde{m}(\sigma^{-(n+1+|\beta|)}([q])) \geq \tilde{m}(\sigma^{-n}([e\beta q])) \geq e^{-Q}\frac{\tilde{m}([e\beta q])}{\tilde{m}([e])}\tilde{m}(\sigma^{-n}([e])).$$

Combining the two latter formulae with (2.24), we obtain

$$e^{-Q}\frac{\tilde{m}([q\alpha e])}{\tilde{m}([q])} \leq \liminf_{n\to\infty} \frac{\sum_{j=0}^{n} \tilde{m}(\sigma^{-j}([e]))}{Z_n} \leq \limsup_{n\to\infty} \frac{\sum_{j=0}^{n} \tilde{m}(\sigma^{-j}([e]))}{Z_n} \leq e^{Q}\frac{\tilde{m}([e\beta q])}{\tilde{m}([e])}, \tag{2.25}$$

where $Z_n := \sum_{j=0}^{n} \tilde{m}(\sigma^{-j}([q])$. Now consider the Banach space ℓ_∞ of all bounded sequences in $I\!R$ endowed with the supremum norm $||\cdot||_\infty$, and let $L : \ell_\infty \to I\!R$ be some Banach limit. Using (2.22) and (2.25), it follows that for every $e \in E$ and every Borel set $B \subset [e]$,

$$\left(Z_n^{-1}\sum_{j=0}^{n} \tilde{m}(\sigma^{-j}(B))\right)_{n=0}^{\infty} \in \ell_\infty.$$

So, we can define

$$\mu_e(B) := L\left(\left(Z_n^{-1}\sum_{j=0}^{n}\tilde{m}(\sigma^{-j}(B))\right)_{n=0}^{\infty}\right). \tag{2.26}$$

Let us show first that μ_e is a finite measure on $[e]$ which lies in the same measure class as $\tilde{m}|_{[e]}$. Obviously, we have that $\tilde{m}(\emptyset)=0$ and that μ_e is monotone, since $\tilde{m}$ and L are. Now consider some arbitrary sequence $(B_k)_{k=1}^{\infty}$ of pairwise disjoint Borel subsets of $[e]$. Since $L:\ell_\infty \to I\!R$ is a bounded linear operator, we obtain

$$\begin{aligned}
\mu_e\left(\bigcup_{k=1}^{\infty}B_k\right) &= L\left(\left(Z_n^{-1}\sum_{j=0}^{n}\tilde{m}\left(\sigma^{-j}\left(\bigcup_{k=1}^{\infty}B_k\right)\right)\right)_{n=0}^{\infty}\right) = L\left(\left(\sum_{j=0}^{n}Z_n^{-1}\sum_{k=1}^{\infty}\tilde{m}(\sigma^{-j}(B_k))\right)_{n=0}^{\infty}\right)\\
&= L\left(\sum_{k=1}^{\infty}\left(Z_n^{-1}\sum_{j=0}^{n}\tilde{m}(\sigma^{-j}(B_k))\right)_{n=0}^{\infty}\right) = \sum_{k=1}^{\infty}L\left(\left(Z_n^{-1}\sum_{j=0}^{n}\tilde{m}(\sigma^{-j}(B_k))\right)_{n=0}^{\infty}\right)\\
&= \sum_{k=1}^{\infty}\mu_e(B_k).
\end{aligned}$$

This shows that μ_e is σ-additive. Finiteness of μ_e is an immediate consequence of (2.25), whereas equivalence of μ_e and $\tilde{m}|_{[e]}$ can easily be deduced from (2.22), (2.25) and the monotonicity of L. We can now define a measure μ by, for arbitrary Borel sets $B\subset E_A^{\infty}$,

$$\mu(B) := \sum_{e\in E}\mu_e(B\cap[e]).$$

By construction, we immediately have that μ is a σ-finite Borel measure on E_A^{∞} which is equivalent to $\tilde{m}$. In order to show that μ is shift-invariant, consider $b\in E$ and let $B\subset[b]$ be some arbitrary Borel set. Using (2.25), we obtain for all $n\in I\!N$,

$$\begin{aligned}
\sum_{e\in E}\sum_{j=0}^{n}Z_n^{-1}\tilde{m}(\sigma^{-j}(\sigma^{-1}(B)\cap[e])) &= \sum_{j=0}^{n}\sum_{e\in E}Z_n^{-1}\tilde{m}(\sigma^{-j}(\sigma^{-1}(B)\cap[e]))\\
&= \sum_{j=0}^{n}Z_n^{-1}\tilde{m}(\sigma^{-j}(\sigma^{-1}(B)))\\
&\le \sum_{j=0}^{n}Z_n^{-1}\tilde{m}(\sigma^{-(j+1)}([b]))\\
&= \sum_{j=0}^{n}Z_n^{-1}\tilde{m}(\sigma^{-j}([b])) + Z_n^{-1}\left(\tilde{m}(\sigma^{-(n+1)}([b]-\tilde{m}([b]))\right)\\
&\le e^{Q}\frac{\tilde{m}([b\gamma q])}{\tilde{m}([b])} + Z_n^{-1},
\end{aligned}$$

where $\gamma\in\Xi$ is taken such that $b\gamma q$ is an A-admissible word. Since by (2.24) we have that $\lim_{n\to\infty}Z_n^{-1}=0$, we conclude that the series $\sum_{e\in E}\left(\sum_{j=0}^{n}Z_n^{-1}\tilde{m}(\sigma^{-j}(\sigma^{-1}(B)\cap[e]))\right)_{n=0}^{\infty}$

converges in ℓ_∞ with respect to the norm $||\cdot||_\infty$. Using the fact that L is a continuous linear operator and that $\lim_{n\to\infty} Z_n^{-1} = 0$, we obtain

$$\begin{aligned}
\mu(\sigma^{-1}(B)) &= \sum_{e\in E} \mu_e(\sigma^{-1}(B\cap[e])) = \sum_{e\in E} L\left(\Big(Z_n^{-1}\sum_{j=0}^{n} \tilde m\Big(\sigma^{-j}(\sigma^{-1}(B)\cap[e])\Big)\Big)_{n=0}^{\infty}\right)\\
&= L\left(\sum_{e\in E}\left(\sum_{j=0}^{n} Z_n^{-1}\tilde m(\sigma^{-j}(\sigma^{-1}(B)\cap[e]))\right)_{n=0}^{\infty}\right)\\
&= L\left(\left(\sum_{j=0}^{n}\sum_{e\in E} Z_n^{-1}\tilde m(\sigma^{-j}(\sigma^{-1}(B)\cap[e]))\right)_{n=0}^{\infty}\right)\\
&= L\left(\left(\sum_{j=0}^{n} Z_n^{-1}\tilde m(\sigma^{-j}(\sigma^{-1}(B)))\right)_{n=0}^{\infty}\right)\\
&= L\left(\left(\sum_{j=0}^{n} Z_n^{-1}\tilde m(\sigma^{-j}(B)) + Z_n^{-1}\Big(\tilde m(\sigma^{-(n+1)}(B)) - \tilde m(B)\Big)\right)_{n=0}^{\infty}\right)\\
&= L\left(\Big(Z_n^{-1}\sum_{j=0}^{n}\tilde m(\sigma^{-j}(B))\Big)_{n=0}^{\infty}\right) = \mu(B).
\end{aligned}$$

Summarizing the above, we have now shown that for each $b\in E$ and every Borel set $B\subset[b]$,

$$\mu(\sigma^{-1}(B)) = \sum_{e\in E}\mu(\sigma^{-1}((B\cap[e])) = \sum_{e\in E}\mu(B\cap[e]) = \mu(B).$$

This shows that μ is shift-invariant. Our next aim is to show that μ, or what is equivalent $\tilde m$, is conservative. For this consider some arbitrary backward invariant Borel set $G\subset E_A^\infty$, that is $\sigma^{-1}(G)\subset G$. Fix $\omega=\omega_1\ldots\omega_n\in E_A^n$ for some $n\in I\!N$, and let $e\in E$ such that $A_{\omega_n e}=1$. Using (2.22), we then obtain

$$\begin{aligned}
\frac{\tilde m(G)}{\tilde m([e])} &\le e^Q\frac{\tilde m\Big(\sigma_\omega^{-n}(G\cap[e])\Big)}{\tilde m(\sigma_\omega^{-n}([e]))} = e^Q\frac{\tilde m\Big(\sigma_\omega^{-n}(G\cap[e])\Big)}{\tilde m([\omega e])} \le e^Q\frac{\tilde m\Big(\sigma^{-n}(G)\cap\sigma_\omega^{-n}([e])\Big)}{\tilde m([\omega e])}\\
&= e^Q\frac{\tilde m\Big(\sigma^{-n}(G)\cap[\omega e]\Big)}{\tilde m([\omega e])} \le e^Q\frac{\tilde m(G\cap[\omega e])}{\tilde m([\omega e])}.
\end{aligned}\tag{2.27}$$

Note that by the Martingale Theorem we have for each Borel set $F\subset E_A^\infty$ and for $\tilde m$–almost every $\omega\in E_A^\infty$,

$$\lim_{n\to\infty}\frac{\tilde m(F\cap[\omega|_n])}{\tilde m([\omega|_n])} = \mathbb{E}_{\mathcal{A}(E_A^\infty)}(1\!\!1_F)(\omega) = 1\!\!1_F(\omega).$$

Here, $\mathbb{E}_{\mathcal{A}(E_A^\infty)}$ refers to the expectation over the Borel σ-algebra $\mathcal{A}(E_A^\infty)$ on E_A^∞ with respect to the measure $\tilde m$. Now suppose in addition that $\sigma(F)\subset F$ and $\tilde m(F)>0$. Taking (2.23) into account, we then have for $\tilde{m}|_F$–almost every $\omega = \omega_1\omega_2 \ldots \in F$,

$$\lim_{n\to\infty} \frac{\tilde{m}(F \cap [\omega|_{n+1}])}{\tilde{m}([\omega|_{n+1}])} = 1 \quad \text{and} \quad \liminf_{n\to\infty} \omega_{n+1} \leq k. \tag{2.28}$$

For ω of this type there then exists $e \in \{1, \ldots, k\}$ and an increasing sequence $(n_j)_{j=1}^{\infty}$ such that $\omega_{n_j+1} = e$, for all $j \in \mathbb{N}$. Using the first equality in (2.28), we obtain that

$$\lim_{j\to\infty} \frac{\tilde{m}\Big((E_A^\infty \setminus F) \cap [\omega|_{n_j+1}]\Big)}{\tilde{m}([\omega|_{n_j+1}])} = 0.$$

Combining this with the observation in (2.27) (where one should put $e = \omega_{n_j+1}$), and using the fact that $\sigma^{-1}(E_A^\infty \setminus F) \subset E_A^\infty \setminus F$, it follows that $\tilde{m}\Big((E_A^\infty \setminus F) \cap [e]\Big)/\tilde{m}([e]) = 0$. Consequently, we have that $\tilde{m}\Big((E_A^\infty \setminus F) \cap [e]\Big) = 0$, or what is equivalent,

$$\tilde{m}([e] \setminus F) = 0. \tag{2.29}$$

Next, note that σ is quasi-invariant (that is, $\tilde{m}(B) = 0$ implies $\tilde{m}(\sigma^{-1}(B)) = 0$) and non-singular (that is, $\tilde{m}(B) = 0$ implies $\tilde{m}(\sigma(B)) = 0$). Also, since A is finitely irreducible, we have that $E_A^\infty = \bigcup_{n=0}^{\infty} \sigma^n([e])$ (in fact, already finitely many terms in this union give rise to E_A^∞). Combining these observations with (2.29), it follows that $\tilde{m}(E_A^\infty \setminus F) = 0$, or equivalently $\tilde{m}(F) = 1$. Therefore, we have now shown that if $\sigma^{-1}(W) = W$ for some arbitrary Borel set $W \subset E_A^\infty$, then $\sigma(W) \subset W$ and consequently, either $\tilde{m}(W) = 0$ or $\tilde{m}(W) = 1$. This gives that $\tilde{m}$ and μ are both ergodic.

In order to prove conservativity of $\tilde{m}$, we need to show that for each Borel set $B \subset E_A^\infty$ with $\tilde{m}(B) > 0$ we have that $\tilde{m}(B') = 0$, where

$$B' := \{\omega \in E_A^\infty : \sum_{n\geq 0} \mathbb{1}_B(\sigma^n(\omega)) < +\infty\}.$$

Let us assume by way of contradiction that $\tilde{m}(B') > 0$, and let B_n be given for $n \geq 0$ by

$$B_n := \{\omega \in E_A^\infty : \sum_{k\geq n} \mathbb{1}_B(\sigma^n(\omega)) = 0\} = \{\omega \in E_A^\infty : \sigma^k(\omega) \notin B \;\text{ for all }\; k \geq n\}.$$

First, note that since $B' = \bigcup_{n\geq 0} B_n$, there exists $k \geq 0$ such that $m(B_k) > 0$. Secondly, we clearly have that $\sigma(B_n) \subset B_n$ for all $n \geq 0$, and therefore $m(B_k) = 1$. On the other hand we have $B_n \cap B = \emptyset$ and $\tilde{m}\Big(\sigma^{-n}(B)\Big) > 0$, for all $n \geq k$. Clearly, this gives a contradiction, and hence we obtain that $\tilde{m}$ is conservative. Note that by a standard result in ergodic theory (see e.g. [1]), we now also have in particular that up to a multiplicative constant, the measure μ is the unique σ-invariant measure absolutely continuous with respect to $\tilde{m}$.

Our next aim is to prove uniqueness of the f-pseudo-conformal measure $\tilde{m}$. For this, suppose by way of contradiction that ν is some other f-pseudo-conformal measure. Using (2.23), we have for $\tilde{m}$–almost every $\omega \in E_A^\infty$ that there exists $e \in E$ and an increasing sequence $(n_j)_{j=1}^{\infty}$

such that $\omega_{n_j+1} = e$, for all $j \in I\!N$. It then follows from (2.21) that

$$\tilde{m}([\omega|_{n_j+1}]) \leq e^{2Q} \frac{\tilde{m}([e])}{\nu([e])} \nu([\omega|_{n_j+1}]).$$

Thus $\tilde{m}$ is absolutely continuous with respect to ν, and by symmetry it therefore follows that $\nu \prec \tilde{m}$ (note that the proof only uses f-pseudo-conformality of $\tilde{m}$, and it does not use the actual construction of $\tilde{m}$). Now, a simple computation using the definition of a pseudo-conformal measure shows that the Radon-Nikodyn derivative $\rho = d\nu/d\tilde{m}$ is constant on grand orbits of σ. Therefore, the ergodicity of $\tilde{m}$ implies that ρ is constant $\tilde{m}$–almost everywhere. Since $\tilde{m}$ and ν are probability measures, it follows that $\rho = 1$ almost everywhere, and hence $\nu = \tilde{m}$. This finishes the proof. ■

3. Pseudo-Markov Systems

3.1. **Preliminaries for Pseudo-Markov Systems.**

In this section we first introduce the concept 'pseudo-Markov system'. This is then followed by a first analysis of the limit sets of these systems. Note that pseudo-Markov systems may have an infinite set of vertices, and therefore these systems represent a significant extension of the concept graph directed Markov system. In particular, we have that a pseudo-Markov system is a graph directed Markov system if and only if certain stronger conditions hold for the distortion of the maps involved, and if additionally the set $\{X_e : e \in E\}$ appearing in the following definition is a finite set of compact subsets of Y. Also, note that as in the previous section, throughout we will always assume that E is a countable alphabet which is either finite or infinite, as well as that $A : E \times E \to \{0, 1\}$ is some given finitely irreducible incidence matrix.

Definition 3.1. *Let* (Y, ρ) *be a bounded metric space and let* $\{X_e : e \in E\}$ *be a family of compact subsets of* Y*. A set* $\mathcal{S} = \{\phi_e : e \in E\}$ *of continuous injections* $\phi_e : X_e \to Y$ *is called a* pseudo-Markov system *if the following conditions are satisfied.*

(a) *The maps in* $\mathcal{S}$ *are uniform contractions. That is, there exists a constant* $0 < s < 1$ *such that*
$$\rho(\phi_e(y), \phi_e(x)) \leq s\rho(y, x) \text{ for all } e \in E,\ x, y \in X_e.$$
(b) *(Open Set Condition) For each* $a, b \in E$ *such that* $a \neq b$*, we have*
$$\phi_a(\mathrm{Int}(X_a)) \cap \phi_b(\mathrm{Int}(X_b)) = \emptyset.$$
(c) *(Markov Property) For each* $a, b \in E$*, we have that*
$$\text{either } \ \phi_a(\mathrm{Int}(X_a)) \cap \mathrm{Int}(X_b)) = \emptyset \ \text{ or } \ \phi_a(X_a) \subset X_b.$$
(d) *For each* $a, b \in E$ *such that* $A_{ab} = 1$*, we have*
$$\phi_b(X_b) \subset X_a.$$

Throughout we will use the notation, for $\omega = \omega_1\omega_2 \dots \omega_n \in E_A^*$,

$$\phi_\omega := \phi_{\omega_1} \circ \phi_{\omega_2} \dots \circ \phi_{\omega_n} : X_{\omega_n} \to \phi_{\omega_1}\Big(X_{\omega_1}\Big).$$

In order to define the limit set of a pseudo-Markov system $\mathcal{S}$, observe that by condition (d) in Definition 3.1, we have for each $\omega = \omega_1\omega_2 \ldots \in E_A^\infty$ that the sequence $\Big(\phi_{\omega|_n}\Big(X_{\omega_n}\Big)\Big)_{n\in\mathbb{N}}$ is a descending family of compact subsets of Y. Also, by condition (a) in Definition 3.1, we have $\lim_{n\to\infty} \operatorname{diam}\Big(\phi_{\omega|_n}\Big(X_{\omega_n}\Big)\Big) = 0$. Combining these two observations, it follows that the intersection $\bigcap_{n=1}^\infty \phi_{\omega|_n}\Big(X_{\omega_n}\Big)$ is a singleton, which will be referred to as $\pi(\omega)$. Obviously, this gives rise to the *coding map*

$$\pi : E_A^\infty \to Y.$$

Note that by assuming that E_A^∞ is equipped with a suitable metric, one immediately verifies, using (a), that π is Hölder continuous. The *limit set* $J_\mathcal{S}$ of a pseudo-Markov system $\mathcal{S}$ is then defined by

$$J_\mathcal{S} := \pi\Big(E_A^\infty\Big).$$

The remainder of this section is devoted to investigations of the geometry, dynamics and topology of this set. In the following a pseudo-Markov system $\mathcal{S}$ is said to be of *finite multiplicity* if and only if

$$T := \sup_{x\in Y} \{\operatorname{card}\{e \in E : x \in \phi_e(X_e)\}\} < \infty.$$

The following lemma shows that finite multiplicity is sufficient to ensure that the geometry of the iterations of a pseudo-Markov system is compatible with the geometry arising from the coding. For ease of exposition, in here we use the notation

$$X_\omega := X_{\omega_n}, \text{ for } \omega = \omega_1 \dots \omega_n \in E_A^n, n \in \mathbb{N}.$$

Lemma 3.2. *If $\mathcal{S}$ is a pseudo-Markov system of finite multiplicity, then*

$$J_\mathcal{S} = \bigcap_{n\in\mathbb{N}} \bigcup_{\omega\in E_A^n} \phi_\omega(X_\omega).$$

Proof. Let $x \in J_\mathcal{S}$ be fixed. Then there exists $\omega \in E_A^\infty$ such that $x = \pi(\omega)$. By definition of $J_\mathcal{S}$, it hence follows that $x \in \phi_{\omega|_n}(X_\omega) \subset \bigcup_{\tau\in E_A^n} \phi_\tau(X_\tau)$, for all $n \in \mathbb{N}$. This implies that

$$J_\mathcal{S} \subset \bigcap_{n\in\mathbb{N}} \bigcup_{\omega\in E_A^n} \phi_\omega(X_\omega).$$

(Note that so far we did not use the assumption 'finite multiplicity'). For the opposite inclusion, let $y \in \bigcap_{n=1}^\infty \bigcup_{\omega\in E_A^n} \phi_\omega(X_\omega)$, and for each $n \in \mathbb{N}$ consider the set

$$E_n(y) := \{\omega \in E_A^n : y \in \phi_\omega(X_\omega)\}.$$

We then have that $\omega|_n \in E_n(y)$, for every $\omega \in E_{n+1}(y)$. This shows that the set $\{E_n(y) : n \in \mathbb{N}\}$ can be viewed as a tree. Since $\operatorname{card}(E_n(y)) \le T^n < \infty$, Ramsey's Theorem gives that there exists $\omega \in E_A^\infty$ such that $\omega|_n \in E_n(y)$, for each $n \in I\!N$. This implies that $\omega \in E_A^\infty$ and that $y = \pi(\omega)$. ■

Finally, we now briefly comment on a topological aspect of the limit set of a pseudo-Markov system $\mathcal{S}$. For this we introduce the 'boundary set'

$$\Delta(\mathcal{S}) := \{\lim_{n\to\infty} \phi_{e_n}(x_n) : \lim_{n\to\infty} e_n = \infty \text{ for some arbitrary elements } x_n \in X_{e_n}\}.$$

The following result will be important in the final section where we will use pseudo-Markov systems to study limit sets of infinitely generated Schottky groups.

Proposition 3.3. *If $\mathcal{S}$ is a pseudo-Markov system such that $\lim_{e\in E} \operatorname{diam}\Big(\phi_e(X_e)\Big) = 0$, then*

$$\overline{J_\mathcal{S}} = J_\mathcal{S} \cup \bigcup_{\omega\in E_A^*} \phi_\omega\Big(\Delta(\mathcal{S}) \cap X_\omega\Big).$$

Proof. Since the matrix A is finitely irreducible, for each $e \in E$ there exists $\omega \in E_A^\infty$ such that $e\omega \in E_A^\infty$. Therefore, using the assumption $\lim_{e\in E} \operatorname{diam}\left(\phi_e(X_{t(e)})\right) = 0$, we deduce that $\Delta(\mathcal{S}) \subset \overline{J_\mathcal{S}}$. Combining this observation and the continuity of the maps ϕ_ω (which implies that $\phi_\omega\Big(\Delta(\mathcal{S}) \cap X_\omega\Big) \subset \phi_\omega\Big(\overline{J_\mathcal{S}} \cap X_\omega\Big) \subset \overline{\phi_\omega\Big(J_\mathcal{S} \cap X_\omega\Big)} \subset \overline{J_\mathcal{S}}$), it follows that

$$\overline{J_\mathcal{S}} \supset J_\mathcal{S} \cup \bigcup_{\omega\in E_A^*} \phi_\omega\Big(\Delta(\mathcal{S}) \cap X_\omega\Big).$$

In order to prove the reverse inclusion, let $x \in \overline{J_\mathcal{S}}$ be fixed. Then there exists a sequence $\Big(\omega^{(n)}\Big)_{n\in I\!N}$ of elements $\omega^{(n)} \in E_A^\infty$ such that $x = \lim_{n\to\infty} \pi(\omega^{(n)})$. For each $k \in I\!N$, define

$$E'_k(x) := \{\omega^{(n)}|_k : n \geq k\} \text{ and } E'_0(x) := \emptyset,$$

and note that if $\tau \in E'_{k+1}(x)$ then there exists $\gamma \in E'_k(x)$ such that $\tau|_k = \gamma$. This shows that the set $\{E'_k(x) : k \in I\!N\}$ can be viewed as a tree rooted at the vertex $E'_0(x)$. We now distinguish the following two cases. First, suppose that there exists $k \in I\!N$ such that $E'_k(x)$ has infinitely many elements. Then define

$$q := \min\{k \geq 0 : E'_k(x) \text{ is infinite}\}.$$

Note that $q \geq 1$, and that the set $E'_{q-1}(x)$ is finite and non-empty (although it might be equal to the singleton $\{\emptyset\}$). Then there exists $\tau \in E'_{q-1}(x) \subset E_A^*$ and an infinite sequence $\Big(\omega_q^{(n_j)}\Big)_{j\in I\!N}$ of distinct elements of E such that $\tau\omega_q^{(n_j)} = \omega^{(n_j)}|_q$, for all $j \in I\!N$. By passing to a subsequence if necessary, we may assume without loss of generality that the sequence $\Big(\pi(\sigma^{q-1}(\omega^{(n_j)}))\Big)_{j\in I\!N}$ converges to a point $y \in X_\tau$. It follows that $y \in \Delta(\mathcal{S})$, and furthermore,

$$\begin{aligned} x &= \lim_{j\to\infty} \pi\left(\omega^{(n_j)}\right) = \lim_{j\to\infty} \pi\left(\tau(\sigma^{q-1}(\omega^{(n_j)}))\right) = \lim_{j\to\infty} \phi_\tau\left(\pi(\sigma^{q-1}(\omega^{(n_j)}))\right) \\ &= \phi_\tau\left(\lim_{j\to\infty} \pi(\sigma^{q-1}(\omega^{(n_j)}))\right) = \phi_\tau(y) \in \phi_\tau\left(\Delta(\mathcal{S}) \cap X_\tau\right). \end{aligned}$$

This gives the assertion in this first case. Now suppose that the set $E'_k(x)$ is finite for each $k \in I\!N$. Since, as mentioned before, these sets form a tree rooted at $E'_0(x)$, Ramsey's Theorem implies that there exists an infinite path $\omega \in E_A^\infty$ such that $\omega|_k \in E'_k(x)$, for each $k \in I\!N$. Hence, there exists an increasing sequence $(n_k)_{k\in I\!N}$ such that $\omega|_k = \omega^{(n_k)}|_k$, for all $k \in I\!N$. It follows that $\pi(\omega), \pi(\omega^{(n_k)}) \in \phi_{\omega|_k}(X_{\omega_k})$. Combining this with the fact that $x = \lim_{k\to\infty} \pi\left(\omega^{(n_k)}\right)$ and $\lim_{k\to\infty} \operatorname{diam}\left(\phi_{\omega|_k}(X_{\omega_k})\right) = 0$, we conclude that $x = \pi(\omega) \in J_{\mathcal{S}}$. This completes the proof of the proposition. ■

3.2. **Summable Families of Functions and F-Pseudo-Conformal Measures.** Throughout this section, let $\mathcal{S} = \{\phi_e : e \in E\}$ be a pseudo-Markov system based on a bounded metric space (Y, ρ).

Definition 3.4. *A family $F = \{f^{(e)} : X_e \to I\!R \,|\, e \in E\}$ is said to be* continuous *if the function $f^{(e)}$ is continuous, for each $e \in E$. Moreover, let $f : E_A^\infty \to I\!R$ be given by*

$$f(\omega) := f^{(\omega_1)}(\pi(\sigma(\omega))), \text{ for all } \omega = \omega_1\omega_2 \ldots \in E_A^\infty.$$

The function f will be referred to as the amalgamated function induced by *F.*

Our convention will be to use lower case letters for the amalgamated function induced by a given summable family of functions. Since the coding map $\pi : E_A^\infty \to Y$ is (Hölder) continuous, we immediately obtain the following result.

Proposition 3.5. *If a family of funtions F is continuous, then the amalgamated function f induced by F is also continuous.*

In analogy to the previous paragraph we introduce the following concepts for $F = \{f^{(e)} : X_e \to I\!R \,|\, e \in E\}$. The family F is called *summable* if and only if

$$\sum_{e\in E} \exp\Big(\sup(f^{(e)})\Big) < \infty. \tag{3.1}$$

Also, we define the function $S_\omega(F) : X_\omega \to I\!R$ by, for $\omega = \omega_1\omega_2 \ldots \in E_A^\infty$,

$$S_\omega(F) := \sum_{j=1}^{n} f^{(\omega_j)} \circ \phi_{\sigma^j\omega}.$$

Note that for each $\omega \in E_A^n$ with $n \in I\!N$ arbitrary, and for every $\tau = \tau_1\tau_2 \ldots \in E_A^\infty$ such that $A_{\omega_n\tau_1} = 1$, we have

$$S_\omega(F)(\pi(\tau)) = \sum_{j=1}^{n} f^{(\omega_j)} \circ \phi_{\sigma^j\omega}(\pi(\tau)) = \sum_{j=1}^{n} f(\sigma^{j-1}\omega\tau) = \sum_{j=0}^{n-1} f(\sigma^j\omega\tau) = S_n f(\omega\tau). \tag{3.2}$$

The topological pressure $\mathrm{P}(F)$ of the family F is given by

$$\mathrm{P}(F) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in E_A^n} \exp\Big(\sup\Big(S_\omega(F)\Big)\Big),$$

A straight forward calculation shows that if F is summable, then so is the amalgamated function f, and

$$\mathrm{P}(F) \le \log \sum_{e\in E} \exp(\sup(f^{(e)})) < \infty.$$

Likewise, in analogy to the previous paragraph, F is called *boundedly distorted* if and only if there exists a constant $Q = Q_F > 0$ such that for all $e \in E$ and $\omega \in E_A^*$ with $A_{\omega e} = 1$,

$$\sup\Big(S_\omega(F)|_{\phi_e(X_e)}\Big) - \inf\Big(S_\omega(F)|_{\phi_e(X_e)}\Big) \le Q. \tag{3.3}$$

Finally, similar as before we will say that F is an SBDC family if F is summable, boundedly distorted and continuous.

Lemma 3.6. *If F is an SBDC family of functions, then the amalgamated function f induced by F is a summable, boundedly distorted and countinuous.*

Proof. Summability and continuity of f have already been established. The fact that f is boundedly distorted is an immediate consequence of the above definition together with the observation in (3.2). ■

A link between pseudo-Markov systems and the symbolic dynamics of the previous paragraph is given by the following.

Proposition 3.7. *If F is an SBDC family and f is the amalgamated function induced by F, then* $\mathrm{P}(F) = \mathrm{P}(f)$.

Proof. Since the proof is very similar to the proof of Theorem 2.9, we only sketch it. In fact, here the proof is even more simple, since here we only have one universal f-pseudo-conformal measure $\tilde{m}$. As before, let us assume without loss of generality that $E = I\!N$. First note that by the definitions of the pressure functions we immediately have that $\mathrm{P}(f) \le \mathrm{P}(F)$. For the proof of the reverse inequality, note that since the amalgamated function $f : E_A^\infty \to I\!R$ is summable, there exists $q \ge 1$ such that

$$\sum_{i>q} \exp\Big(\sup\Big(f|_{[i]}\Big) - \mathrm{P}(f)\Big) < 1/2.$$

Also, by Lemma 2.5 we immediately have

$$T_q := \min\{\tilde{m}([i]) : 1 \le i \le q\} > 0. \tag{3.4}$$

Putting $Z_n(F) := \sum_{\omega\in E_A^n} \exp\Big(\sup\Big(S_\omega(F)\Big)$, an estimate almost identical to the one given in the proof of Theorem 2.9 shows that we have for each $n \ge 0$,

$$\exp(-\mathrm{P}(f)(n+1))Z_{n+1}(F) \le T_q^{-1} e^Q \exp(-\mathrm{P}(f))Z_1(F) + \frac{1}{2}\exp(-\mathrm{P}(f)n)Z_n(F).$$

Hence, as in the proof of Theorem 2.9, a straight forward finite inductive argument then gives

$$\exp(-\mathrm{P}(f)n)Z_n(F) \leq (2T_q^{-1}e^{2Q+1}\exp(-\mathrm{P}(f))Z_1(F) < \infty.$$

Consequently, it follows that

$$\mathrm{P}(F) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(F) \leq \mathrm{P}(f),$$

which finishes the proof of the proposition. ■

Similar as in the previous paragraph, we require the following notion of an F-pseudo-conformal measure for a family F based on a pseudo-Markov system $\mathcal{S}$. Here, two elements of E_A^* are called *incomparable* if none of them is an extension of the other.

Definition 3.8. *A Borel probability measure m on Y is called F-*pseudo-conformal *if m is supported on the limit set $J_{\mathcal{S}}$ and the following two conditions are satisfied.*

(i) *For every $\omega \in E_A^*$ and each Borel set $B \subset X_\omega$, we have*

$$m(\phi_\omega(B)) = \int_B \exp\Big(S_\omega(F) - \mathrm{P}(F)\,|\omega|\Big)\,dm. \tag{3.5}$$

(ii) *For all incomparable words $\omega, \tau \in E_A^*$, we have*

$$m\Big(\phi_\omega(X_\omega) \cap \phi_\tau(X_\tau)\Big) = 0. \tag{3.6}$$

A simple inductive argument shows that for m to be an F-pseudo-conformal measure, instead of (3.5) and (3.6) it is sufficient to check that for every $e \in E$ and for each Borel set $B \subset X_e$, we have

$$m(\phi_e(B)) = \int_B \exp\Big(f^{(e)} - \mathrm{P}(F)\Big)\,dm, \tag{3.7}$$

as well as that for all distinct $a, b \in E$,

$$m(\phi_a(X_a) \cap \phi_b(X_b)\Big) = 0. \tag{3.8}$$

In the following auxiliary result we use the notation

$$\hat{X}_a := \bigcup_{\substack{b\in E \\ A_{ab}=1}} \phi_b(X_b) \subset X_a, \text{ for } a \in E.$$

Lemma 3.9. *With the notation above, let m be a Borel probability measure on the limit set $J_{\mathcal{S}}$ of the pseudo-conformal Markov system $\mathcal{S} = \{\phi_e : e \in E\}$. If m satisfies (3.6) and if (3.5) holds for all Borel sets $A \subset \hat{X}_e$ with $e \in E$, then m is an F-pseudo-conformal measure.*

Proof. Since $(X_e \setminus \hat{X}_e) \cap J_{\mathcal{S}} = \emptyset$ for each $e \in E$, we have

$$m(X_e \setminus \hat{X}_e) = 0. \tag{3.9}$$

Let $x \in \phi_e\Big(X_e \setminus \hat{X}_e\Big) \cap J_{\mathcal{S}}$ be fixed, for some $e \in E$. Then $x = \pi(\omega) = \phi_{\omega_1}(\pi(\sigma(\tau)))$, for some $\omega \in E_A^\infty$ with $\omega_1 \neq e$. Hence, it follows that $J_{\mathcal{S}} \cap \phi_e\Big(X_e \setminus \hat{X}_e\Big) \subset \bigcup_{b \neq e} \phi_b(X_b) \cap \phi_e\Big(X_e\Big)$, and consequently (3.8) implies

$$m\Big(\phi_e\Big(X_e \setminus \hat{X}_e\Big)\Big) = m\Big(J_{\mathcal{S}} \cap \phi_e\Big(X_e \setminus \hat{X}_e\Big)\Big) \leq \sum_{b \neq e} m\Big(\phi_b(X_b) \cap \phi_e(X_e)\Big) = 0.$$

Therefore, using (3.9), we obtain by induction, for each $\omega \in E_A^*$ and every $B \subset X_\omega$,

$$\begin{aligned} m(\phi_\omega(B)) = m\Big(\phi_\omega\Big(B \cap \hat{X}_\omega\Big)\Big) &= \int_{B \cap X_\omega} \exp\Big(S_\omega(F) - \mathrm{P}(F)|\omega|\Big)\, dm \\ &= \int_B \exp\Big(S_\omega(F) - \mathrm{P}(F)\,|\omega|\Big)\, dm. \end{aligned} \tag{3.10}$$

■

A finite word $\omega\tau \in E^*$ is called a *pseudo-code* of an element $y \in Y$ if and only if $\omega, \tau \in E_A^*$ such that $\phi_\tau(X_\tau) \subset X_\omega$ and $y \in \phi_\omega(\phi_\tau(X_\tau))$. Note that the word $\omega\tau$ is actually not required to belong to E_A^*. If in here we do not want to specify the element y, we simply say that $\omega\tau$ is a pseudo-code. Similar as for elements of E_A^*, two pseudo-codes are called *comparable* if one of them is an extension of the other. Also, two pseudo-codes $\omega\tau$ and $\omega\rho$ of an element $y \in Y$ are said to form an *essential pair of pseudo-codes* of y if $\tau \neq \rho$ and $|\tau| = |\rho|$. Finally, the *essential length* of the essential pair of pseudo-codes $\omega\tau$ and $\omega\rho$ of y is defined to be equal to $|\omega|$. With these preparations we can now define the notion of a conformal-like pseudo-Markov system. Namely, a pseudo-Markov system S is said to be *conformal-like* if and only if no element of Y admits essential pairs of pseudo-codes of arbitrarily long essential lengths. One easily verifies that for instance a system which satisfies the strong separation condition (that is $\phi_a(X_a) \cap \phi_b(X_b) = \emptyset$, for all distinct $a, b \in E$) is always conformal-like. Also, in the next section we will see that every conformal pseudo-Markov systems is conformal-like. Now, the following theorem gives the main result about conformal-like systems and summable families of functions.

Theorem 3.10. *Let $\mathcal{S} = \{\phi_e : e \in E\}$ be a conformal-like pseudo-Markov system. If F is an SBDC family of functions, then there exists a unique F-pseudo-conformal measure m_F on $J_{\mathcal{S}}$. Moreover, we have that $m_F = \tilde{m}_F \circ \pi^{-1}$, where $\tilde{m}_F$ is the f-pseudo-conformal measure whose existence follows from Theorem 2.10 and Lemma 3.6.*

Proof. Let $m_F := \tilde{m}_F \circ \pi^{-1}$. In order to show that m_F is an F-pseudo-conformal measure, assume by way of contradiction that there exist two incomparable words $\rho, \tau \in E_A^*$ such that $m_F(\phi_\rho(X_\rho) \cap \phi_\tau(X_\tau)) > 0$. Without loss of generality we can assume that ρ and τ are of the same length, say $q \in I\!N$. For $n \in I\!N$, we then define

$$V := \phi_\rho(X_\rho) \cap \phi_\tau(X_\tau)) \text{ and } V_n := \bigcup_{\omega \in E_A^n} \phi_\omega(X_\omega \cap V).$$

Since each element of V_n admits at least one essential pair of pseudo-codes of essential length n, and since the system $\mathcal{S}$ is conformal-like, it follows that

$$\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} V_n = \emptyset.$$

On the other hand, we have $V_n \supset \pi(\sigma^{-n}(\pi^{-1}(V)))$, which implies that $\pi^{-1}(V_n) \supset \sigma^{-n}(\pi^{-1}(V))$. By Theorem 2.11 we know that there exists a σ-invariant Borel probability measure $\tilde{\mu}_F$ equivalent to $\tilde{m}_F$. Hence, $\tilde{\mu}_F(\pi^{-1}(V_n)) \geq \tilde{\mu}_F(\sigma^{-n}(\pi^{-1}(V))) = \tilde{\mu}_F \circ \pi^{-1}(V) > 0$, and therefore,

$$\tilde{\mu}_F \circ \pi^{-1}\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} V_n\right) \geq \tilde{\mu}_F \circ \pi^{-1}(V) > 0.$$

This gives in particular that $\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} V_n \neq \emptyset$, and hence we derive a contradiction. We conclude that property (3.6) is satisfied. In order to show that (3.5) holds, fix $\omega = \omega_1 \dots \omega_n \in E_A^n$ as well as a Borel set $B \subset X_\omega$, and define

$$\pi_\omega^{-1}(B) := \{\tau \in \pi^{-1}(B) : A_{\omega_n \tau_1} = 1\}.$$

Let $\tau = \tau_1\tau_2 \dots \in \pi^{-1}(B) \setminus \pi_\omega^{-1}(B)$ be fixed. We then have $A_{\omega_n\tau_1} = 0$ and $\pi(\tau) \in B \subset X_\omega$. This implies that $\pi(\tau) = \phi_e(X_e)$, for some $e \in E$ with $A_{\omega_n e} = 1$. Hence, $e \neq \tau_1$ and $\pi(\tau) \in \phi_e(X_e) \cap \phi_{\tau_1}(X_{\tau_1})$, and therefore,

$$\pi\Big(\pi^{-1}(B) \setminus \pi_\omega^{-1}(B)\Big) \subset \bigcup_{e \neq b} \phi_e(X_e) \cap \phi_b(X_b).$$

By combining the latter inclusion with (3.6), we obtain

$$\tilde{m}_F\Big(\pi^{-1}(B) \setminus \pi_\omega^{-1}(B)\Big) \leq \tilde{m}_F \circ \pi^{-1}\left(\bigcup_{e \neq b} \phi_e(X_e) \cap \phi_b(X_b)\right) = m_F\left(\bigcup_{e \neq b} \phi_e(X_e) \cap \phi_b(X_b)\right) = 0. \tag{3.11}$$

Note that one immediately verifies that

$$\pi^{-1}(\phi_\omega(B)) \supset \sigma_\omega^{-n} \circ \pi_\omega^{-1}(B).$$

Now, let $\rho \in \pi^{-1}(\phi_\omega(B)) \setminus (\sigma_\omega^{-n} \circ \pi_\omega^{-1}(B))$ be fixed. We then have $\pi(\rho) = \phi_{\omega e}(x)$, for $A_{\omega_n e} = 1$, $\phi_e(x) \in B$ and $\rho \notin \sigma_\omega^{-n} \circ \pi_\omega^{-1}(B)$. If $\rho|_{n+1} = \omega e$, then $x = \pi(\sigma^{n+1}\rho)$, and hence $\pi(e\sigma^{n+1}\rho) = \phi_e(\pi(\sigma^{n+1}\rho)) = \phi_e(x)$. Consequently, $\rho = \omega e \sigma^{n+1}(\rho) \in \sigma_\omega^{-n} \circ \pi_\omega^{-1}(B)$, and hence $\rho|_{n+1} \neq \omega i$. This shows that

$$\pi^{-1}(\phi_\omega(B)) \setminus \Big(\sigma_\omega^{-n} \circ \pi_\omega^{-1}(B)\Big) \subset \pi^{-1}\left(\bigcup_{\substack{\tau,\eta \in E_A^{n+1} \\ \tau \neq \eta}} \phi_\tau(X_\tau) \cap \phi_\eta(X_\eta)\right).$$

Using (3.6) and the definition of the measure m_F, it follows that

$$\tilde{m}_F\left(\pi^{-1}(\phi_\omega(B)) \setminus \sigma_\omega^{-n} \circ \pi_\omega^{-1}(B)\right) = 0.$$

Using the latter equality, (3.11) and (2.6), we now obtain

$$\begin{aligned} m_F(\phi_\omega(B)) &= \tilde{m}_F \circ \pi^{-1}(\phi_\omega(B)) = \tilde{m}_F\Big(\sigma_\omega^{-n} \circ \pi_\omega^{-1}(B)\Big) \\ &= \int_{\pi_\omega^{-1}(B)} \exp\Big(S_n f(\omega\rho) - \mathrm{P}(f)n\Big) d\tilde{m}_F(\rho) \\ &= \int_{\pi_\omega^{-1}(B)} \exp\Big(S_\omega F(\pi(\rho)) - \mathrm{P}(F)n\Big) d\tilde{m}_F(\rho) \\ &= \int_{\pi^{-1}(B)} \exp\Big(S_\omega F(\pi(\rho)) - \mathrm{P}(F)n\Big) d\tilde{m}_F(\rho) \\ &= \int_B \exp\Big(S_\omega F(x) - \mathrm{P}(F)n\Big) d\tilde{m} \circ \pi^{-1}(x) = \int_B \exp\Big(S_\omega F - \mathrm{P}(F)n\Big) dm_F. \end{aligned}$$

This shows that (3.5) is satisfied, and hence an application of Lemma 3.9 then completes the proof of the existence part of the theorem. In order to obtain uniqueness, consider some arbitrary F-pseudo-conformal measure ν, and let $\tilde{\nu}$ be defined by the formula $\tilde{\nu}([\omega]) := \nu(\phi_\omega(J_{\mathcal{S}}))$ for all $\omega \in E_A^\infty$. Clearly, $\tilde{\nu}$ is an f-pseudo-conformal measure such that $\nu = \tilde{\nu} \circ \pi^{-1}$. Using the uniqueness part of Theorem 2.11, it then immediately follows that $\tilde{\nu} = \tilde{m}$. Hence, we have $\nu = m_F$, which then finishes the proof. ■

4. Conformal Pseudo-Markov Systems

4.1. **Preliminaries for Conformal Pseudo-Markov Systems.**

In this section we consider a particular class of pseudo-Markov systems which we call conformal pseudo-Markov systems. These systems are represented within some Euclidean space $\mathbb{R}^d$, and this then allows to study their limit sets from a fractal geometric point of view. Before giving the definition of a conformal pseudo-Markov system, recall that a set $X \subset \mathbb{R}^d$ is called K-*quasi-convex*, for some $K \geq 1$, if for every pair of points $x, y \in X$ there exists a piecewise smooth path in X joining x and y of length less than or equal to $K|x-y|$.

Definition 4.1. *Let $\mathcal{S} = \{\phi_e : X_e \to Y \,|\, e \in E\}$ be a pseudo-Markov system. Then $\mathcal{S}$ is called* conformal pseudo-Markov system *if and only if Y is a compact subset of the Euclidean space $\mathbb{R}^d$, for some $d \in \mathbb{N}$, and if the following conditions are satisfied.*

1. *(Quasi-Convexity) There exists a constant $K \geq 1$ such that X_e and $\phi_e(X_e)$ are K-quasi-convex, for all $e \in E$.*
2. *(Cone Condition) There exists $\theta \in (0, \pi/2)$ such that for all $e \in E$ and $x \in X_e$, there exists an open cone $Cone(x, \theta)$ with vertex x and central angle θ such that $Cone(x, \theta) \subset \mathrm{Int}\big(\phi_e(X_e)\big)$; here, the interior is taken with respect to the Euclidean topology in $\mathbb{R}^d$.*
3. *(Conformal Extension) For each $e \in E$, there exists an open connected set $V_e \subset Y$ such that $X_e \subset V_e$, and such that the map $\phi_e : X_e \to Y$ extends to a $C^{1+\epsilon}$-conformal diffeomorphism from V_e to Y.*
4. *(Derivative Decay) $\lim_{e \in E} ||\phi_e'|| = 0$.*

(5) *(Pre-Distortion) There exist $L \geq 1$ and $\alpha > 0$ such that for all $ab \in E_A^2$ and all $x, y \in \phi_b(X_b)$,*

$$\big|\,|\phi_a'(y)| - |\phi_a'(x)|\,\big| \leq L\, \frac{|y-x|^\alpha}{\left\| \left(\phi_a'|_{\phi_b(X_b)}\right)^{-1} \right\|}.$$

Let us first collect a few geometric observations which follow from the Pre-Distortion condition.

Lemma 4.2. *If $\mathcal{S} = \{\phi_e : X_e \to Y \mid e \in E\}$ is a conformal pseudo-Markov system, then we have for all $\omega \in E_A^*$ and $b \in E$ with $\omega b \in E_A^*$,*

$$\Big|\log|\phi_\omega'(y)| - \log|\phi_\omega'(x)|\Big| \leq L(1-s^\alpha)^{-1}|y-x|^\alpha, \text{ for all } x, y \in \phi_b(X_b).$$

Proof. Let $n := |\omega|$, and define $z_0 := z$ as well as $z_k := \phi_{\omega_{n-k+1}} \circ \phi_{\omega_{n-k+2}} \circ \cdots \circ \phi_{\omega_n}(z)$, for some arbitrary $z \in \phi_b(X_b)$ and for all $1 \leq k \leq n$. Using the Pre-Distortion condition (5) in Definition 4.1, it follows for $x, y \in \phi_b(X_b)$,

$$\begin{aligned}
\Big|\log(|\phi_\omega'(y)|) - \log(|\phi_\omega'(x)|)\Big| &= \left|\sum_{j=1}^n \log\left(1 + \frac{|\phi_{\omega_j}'(y_{n-j})| - |\phi_{\omega_j}'(x_{n-j})|}{|\phi_{\omega_j}'(x_{n-j})|}\right)\right| \\
&\leq \sum_{j=1}^n \left\|(\phi_{\omega_j}')^{-1}\right\| \Big|\,|\phi_{\omega_j}'(y_{n-j})| - |\phi_{\omega_j}'(x_{n-j})|\,\Big| \\
&\leq \sum_{j=1}^n L|y_{n-j} - x_{n-j}|^\alpha \qquad (4.1)\\
&\leq L\sum_{j=1}^n s^{\alpha(n-j)}|y-x|^\alpha \leq L(1-s^\alpha)^{-1}|y-x|^\alpha.
\end{aligned}$$

■

Let us define

$$Q := L(1-s^\alpha)^{-1}(\mathrm{diam}(Y))^\alpha.$$

An immediate consequence of the previous lemma is that each conformal pseudo-Markov system has the following property.

Bounded Distortion Property. For all $\omega \in E_A^*$ and $b \in E$ such that $\omega b \in E_A^*$, we have

$$e^{-Q} \leq \frac{|\phi_\omega'(y)|}{|\phi_\omega'(x)|} \leq e^Q, \text{ for all } x, y \in \phi_b(X_b). \qquad (4.2)$$

The following lemma gives a sufficient condition for the Pre-Distortion property to hold.

Lemma 4.3. *Suppose that $\mathcal{S} = \{\phi_e : X_e \to Y \mid e \in E\}$ is a pseudo-Markov system in $I\!R^d$, for $d \geq 2$, which satisfies the conditions* (1)-(4) *of Definition* 4.1. *Moreover, suppose that there exists $\kappa > 0$ such that*

$$\operatorname{dist}\Big(\phi_b(X_b), I\!R^d \setminus X_a\Big) \geq \kappa \operatorname{diam}\Big(\phi_b(X_b)\Big), \text{ for all } ab \in E_A^2.$$

Then $\mathcal{S}$ satisfies the Pre-Distortion property (5), *and consequently $\mathcal{S}$ is a conformal pseudo-Markov system.*

Proof. The assertion follows immediately from Theorem 4.1.2 and Theorem 4.1.3 in [13]. ■

Proposition 4.4. *If $\mathcal{S}$ is a conformal pseudo-Markov system, then*

$$\overline{J_{\mathcal{S}}} = J_{\mathcal{S}} \cup \bigcup_{\omega \in E_A^*} \phi_\omega\Big(\Delta(\mathcal{S}) \cap X_\omega\Big).$$

Proof. Using the Mean Value Inequality, conditions (1) and (4) in Definition 4.1, and the assumption that Y is bounded, we immediately obtain that

$$\lim_{e \in E} \operatorname{diam}\Big(\phi_e(X_e)\Big) = 0. \tag{4.3}$$

Hence, we can apply Proposition 3.3, which then gives the assertion. ■

Lemma 4.5. *There exists $\beta > 0$ such that for each $y \in Y$ and every pseudo-code $\omega\tau$ of y, we have that $\phi_\omega\Big(\phi_\tau\Big(X_\tau\Big)\Big)$ contains an open cone with vertex $\phi_\omega(\phi_\tau(y))$ and central angle β.*

Proof. By using the Cone Condition (2) in Definition 4.1, the proof is exactly the same as the proof of formula (4.14) in [13], and will therefore be omitted. ■

One easily verifies that $\phi_\omega \circ \phi_\rho\Big(\operatorname{Int}\Big(X_\rho\Big)\Big) \cap \phi_\tau \circ \phi_\gamma\Big(\operatorname{Int}\Big(X_\gamma\Big)\Big) = \emptyset$, for any two incomparable pseudo-codes $\omega\rho$ and $\tau\gamma$. Hence, as an immediate consequence of the previous lemma we obtain the following corollary. In here λ_{d-1} refers to the $(d-1)$-dimensional Lebesgue measure on the unit sphere $\mathbb{S}^{d-1} \subset I\!R^d$.

Corollary 4.6. *With β as in the previous lemma, we have that for each $y \in Y$ there are at most $\lambda_{d-1}(\mathbb{S}^{d-1})/\beta$ mutually incomparable pseudo-codes of y.*

Lemma 4.7. *Every conformal pseudo-Markov system $\mathcal{S}$ is a conformal-like system.*

Proof. Suppose on the contrary that there exists a point $y \in Y$ with the following properties. There exists an infinite increasing sequence $(n_k)_{k \in I\!N}$ of positive integers, and there exist words

$\omega^{(k)}, \tau^{(k)}, \rho^{(k)} \in E_A^*$, for each $k \in I\!N$, such that $\phi_{\tau^{(k)}}\left(X_{\tau^{(k)}}\right) \cup \phi_{\rho^{(k)}}\left(X_{\tau^{(k)}}\right) \subset X_{\omega^{(k)}}$, such that the words $\tau^{(k)}$ and $\rho^{(k)}$ are incomparable, and

$$\lim_{k\to\infty} |\omega^{(k)}| = \infty, \tag{4.4}$$

as well as

$$y \in \phi_{\omega^{(k)}} \circ \phi_{\tau^{(k)}}\left(X_{\tau^{(k)}}\right) \cap \phi_{\omega^{(k)}} \circ \phi_{\rho^{(k)}}\left(X_{\rho^{(k)}}\right) \text{ for all } k \in I\!N.$$

We now construct by induction for each $n \in I\!N$ a set C_n which contains at least $n+1$ mutually incomparable pseudo-codes of y. The existence of such a set for large n will then clearly contradict the statement in Corollary 4.6, and hence will finish the proof. Define

$$C_1 := \{\omega^{(1)}\tau^{(1)}, \omega^{(1)}\rho^{(1)}\},$$

and suppose that the set C_n has been obtained, for some $n \in I\!N$. In view of (4.4), there exists $k_n \in I\!N$ such that

$$|\omega^{(k_n)}| > \max\{|\xi| : \xi \in C_n\}. \tag{4.5}$$

If $\omega^{(k_n)}\rho^{(k_n)}$ does not extend any word from C_n, then we have by (4.5) that $\omega^{(k_n)}\rho^{(k_n)}$ is not comparable with any element of C_n. We then obtain C_{n+1} from C_n by adding the word $\omega^{(k_n)}\rho^{(k_n)}$ to C_n. Similarly, if $\omega^{(k_n)}\tau^{(k_n)}$ does not extend any word from C_n, then form C_{n+1} by adding $\omega^{(k_n)}\tau^{(k_n)}$ to C_n. On the other hand, if $\omega^{(k_n)}\rho^{(k_n)}$ extends an element $\alpha \in C_n$ and $\omega^{(k_n)}\tau^{(k_n)}$ extends an element $\beta \in C_n$, then we obtain from (4.5) that $\alpha = \omega^{(k_n)}|_{|\alpha|}$ and $\beta = \omega^{(k_n)}|_{|\beta|}$. Since C_n consists of mutually incomparable words, this implies that $\alpha = \beta$. Now, form C_{n+1} by removing $\alpha = \beta$ from C_n and then adding both $\omega^{(k_n)}\rho^{(k_n)}$ and $\omega^{(k_n)}\tau^{(k_n)}$. Note that no element $\gamma \in C_n \setminus \{\alpha\}$ is comparable with $\omega^{(k_n)}\rho^{(k_n)}$ or $\omega^{(k_n)}\tau^{(k_n)}$, since otherwise $\gamma = \omega^{(k_n)}|_{|\gamma|}$, and consequently, γ would be comparable with α. Since also $\omega^{(k_n)}\rho^{(k_n)}$ and $\omega^{(k_n)}\tau^{(k_n)}$ are not comparable, it follows that C_{n+1} consists of mutually incomparable pseudo-codes of y. This completes our inductive construction, and hence finishes the proof. ■

4.2. **Pressure Functions, Conformal Measures and Bowen's Formula.**

Throughout this section let $\mathcal{S} = \{\phi_e : X_e \to Y : e \in E\}$ be a given conformal pseudo-Markov system. For $t \geq 0$, consider the following family of geometric potential functions

$$\boldsymbol{Log}_t := \{t \log |\phi_e'| : e \in E\}.$$

Lemma 4.8. *For each $t \geq 0$, we have that $\boldsymbol{Log}_t$ is a boundedly distorted, continuous family.*

Proof. The assertion is an immediate consequence of Lemma 4.2. ■

Let $\boldsymbol{log}_t : E_A^\infty \to I\!R$ be the amalgamated function induced by the family $\boldsymbol{Log}_t$, that is for every $t \geq 0$ and $\omega = \omega_1\omega_2 \ldots \in E_A^\infty$ we have

$$\boldsymbol{log}_t(\omega) := \log |\phi_{\omega_1}'(\pi(\sigma\omega))|.$$

By combining Proposition 3.7 and Lemma 4.8, it immediately follows that

$$\mathrm{P}(\boldsymbol{log}_t) = \mathrm{P}(\boldsymbol{Log}_t),$$

and in the sequel we will simply write $\mathcal{P}(t)$ to denote this common value of the two pressure functions. For the following theorem note that by condition (2) (Derivative Decay) in Definition 4.1, we have that there exists a unique number $\theta(\mathcal{S}) \in [0,\infty)$ such that the family $\boldsymbol{Log}_t$ is summable for all $t > \theta(\mathcal{S})$, and not summable for all $t < \theta(\mathcal{S})$. Clearly, $\mathcal{P}(\theta(\mathcal{S})) < \infty$ if and only if $\boldsymbol{Log}_{\theta(\mathcal{S})}$ is summable. Finally, recall that a Borel probability measure m_t on $J_{\mathcal{S}}$ is a $\boldsymbol{Log}_t$-pseudo-conformal measure if and only if

$$m_t\Big(\phi_\omega(X_\omega) \cap \phi_\tau(X_\tau)\Big) = 0, \text{ for all incomparable words } \omega, \tau \in E_A^*, \tag{4.6}$$

and

$$m_t(\phi_\omega(B)) = \int_B e^{-\mathcal{P}(t)|\omega|} |\phi_\omega'|^t dm_t, \text{ for all } \omega \in E_A^*, \text{ and } B \subset X_\omega \text{ Borel.} \tag{4.7}$$

In the special situation in which $\mathcal{P}(t) = 0$, the measure m_t is called *t-conformal*.

Theorem 4.9. *If $\mathbf{Log}_t$ is a summable family, and hence in particular if $t > \theta(S)$, then there exists a unique $\mathbf{Log}_t$-pseudo-conformal measure m_t on $J_{\mathcal{S}}$.*

Proof. The assertion follows immediately from combining Theorem 3.10, Lemma 4.7 and Lemma 4.8. ■

The following proposition collects some of the basic properties of the pressure function $\mathcal{P}$. These properties are immediate consequences of the Hölder inequality and the fact that $||\phi_e'|| \leq s < 1$ for all $e \in E$, and therefore we omit the proof.

Proposition 4.10. *For the pressure function $\mathcal{P} : [0,\infty) \to I\!R$ of a conformal pseudo-Markov system $\mathcal{S}$, the following holds.*

(a) *$\mathcal{P}$ is decreasing.*
(b) *$\mathcal{P}|_{[\theta(\mathcal{S}),\infty)}$ is strictly decreasing.*
(c) $\lim_{t\to\infty} \mathcal{P}(t) = -\infty$.
(d) *$\mathcal{P}|_{[\theta(\mathcal{S}),\infty)}$ is convex, and hence continuous.*

In the sequel we will need the following three parameters. Here, let $\mathcal{S}_D := \{\phi_e : e \in D\}$ for $D \subset E$, and recall that HD refers to the Hausdorff dimension.

$h(\mathcal{S}) := \mathrm{HD}(J_{\mathcal{S}})$,
$h_0(\mathcal{S}) := \inf\{s \geq 0 : \mathcal{P}(s) \leq 0\}$,
$h^*(\mathcal{S}) := \sup\{\mathrm{HD}(J_{\mathcal{S}_D}) : D \subset E \text{ finite}\}$.

We now first introduce the concepts thinness and weak thinness, which both will turn out to be crucial in the following.

Definition 4.11. *A conformal pseudo-Markov system $\mathcal{S}$ is called* weakly thin *if and only if the family $\mathbf{Log}_{h^*(\mathcal{S})}$ is summable. Moreover, $\mathcal{S}$ is called* thin *if and only if $\theta(\mathcal{S}) < h^*(\mathcal{S})$.*

We remark that we clearly have that if the system $\mathcal{S}$ is weakly thin then $\theta(\mathcal{S}) \le h^*(\mathcal{S})$. This shows that a thin system is always weakly thin.
The following auxiliary result will turn out to be useful in the proof of the theorem to come. For this recall that a system is called *irreducible* if each element of the alphabet can be connected by an admissable path to any other element of the alphabet.

Lemma 4.12. *For each finite and irreducible conformal pseudo-Markov system $\mathcal{S}$, we have that $\mathcal{P}(h(\mathcal{S})) = 0$.*

Proof. Since $0 \le \mathcal{P}(0) < \infty$, Proposition 4.10 implies that there exists a unique $u \ge 0$ such that $\mathcal{P}(u) = 0$. Let m_u be the associated u-conformal measure, whose existence is guaranteed by Theorem 4.9. Note that the system $\mathcal{S}' = \{\phi_a : \phi_b(X_b) \to \phi_a(X_a) \,|\, a, b \in E, A_{ab} = 1\}$ satisfies all the requirements of a graph directed Markov system (for the details, we refer to [13], page 71-72). Moreover, note that $J_{\mathcal{S}'} = J_{\mathcal{S}}$, and that the measure m_u is an u-conformal measure also for $\mathcal{S}'$. The proof now follows from Theorem 4.2.11 of [13], where primitivity can easily be replaced by irreducibility.

■

The following theorem represents the main result of this section. In there we obtain an extension of Bowen's formula [3] to the setting of weakly thin conformal pseudo-Markov systems. As we will see, the proof turns out to be an almost immediate consequence of the thermodynamic formalism developed in this paper.

Theorem 4.13. *If $\mathcal{S}$ is a weakly thin conformal pseudo-Markov system, then*

$$h(\mathcal{S}) = h_0(\mathcal{S}) = h^*(\mathcal{S}).$$

Moreover, if $\mathcal{S}$ is thin, then $h(\mathcal{S})$ is the unique zero of the pressure function $\mathcal{P}$.

Proof. Let us first show that $h(\mathcal{S}) \le h_0(\mathcal{S})$. For this, let $t > h_0(\mathcal{S})$ be fixed, and observe that by Proposition 4.10 we have that $\mathcal{P}(t) < 0$. Since we have by definition that X_e is K-quasi-convex for all $e \in E$, the definition of $\mathcal{P}$ immediately implies that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \ge N$,

$$\begin{aligned} \sum_{\omega \in E_A^n} \Big(\mathrm{diam}\Big(\phi_\omega(X_\omega)\Big)\Big)^t &\le K^t \sum_{\omega \in E_A^n} ||\phi_\omega'||^t \Big(\mathrm{diam}(X_\omega)\Big)^t \\ &\le K^t \Big(\mathrm{diam}(Y)\Big)^t \sum_{\omega \in E_A^n} ||\phi_\omega'||^t \le K^t \Big(\mathrm{diam}(Y)\Big)^t \exp\left(n(\mathcal{P}(t) + \epsilon)\right). \end{aligned}$$

Hence, by letting n tend to infinity and using the fact that by Lemma 3.2 we have that $\{\phi_\omega(X_\omega) : \omega \in E_A^n\}$ is a covering of $J_{\mathcal{S}}$, we conclude that the t-dimensional Hausdorff measure of $J_{\mathcal{S}}$ vanishes. This shows that $h(\mathcal{S}) \le t$, and hence by letting t tend to $h_0(\mathcal{S})$ from above, it follows that

$$h(\mathcal{S}) \leq h_0(\mathcal{S}). \tag{4.8}$$

Our next aim is to show that

$$h_0(\mathcal{S}) \leq h^*(\mathcal{S}).$$

For this, let $t > h^*(\mathcal{S})$ be fixed. In view of Lemma 4.12 and Proposition 4.10 we have that $\mathrm{P}_D(\boldsymbol{Log}_t) \leq 0$ for every finite subset D of E. Here, P_D refers to the pressure function associated with the subsystem of $\mathcal{S}$ arising from the finite alphabet D. Since $\mathcal{S}$ is weakly thin, we have that $\boldsymbol{Log}_t$ is an SBDC family. Therefore, Theorem 2.9 implies that $\mathcal{P}(t) = \mathrm{P}_*(\boldsymbol{Log}_t) \leq 0$. This gives that $t \geq h_0(\mathcal{S})$, and consequently $h^*(\mathcal{S}) \geq h_0(\mathcal{S})$. Summarizing the above, we have now shown that $h^*(\mathcal{S}) = h_0(\mathcal{S}) = h(\mathcal{S})$. Finally, note that if $\mathcal{S}$ is thin, then the equality $\mathcal{P}(h) = 0$ is an immediate consequence of combining $h^*(\mathcal{S}) = h_0(\mathcal{S}) = h(\mathcal{S})$ and Proposition 4.10.

■

5. Applications to infinitely generated Schottky group

5.1. **Preliminaries for Kleinian Limit Sets.** Hyperbolic geometry in the upper-half-space model $\left(\mathbb{H}^{d+1}, d_{_{hyp}}\right)$ and conformal geometry on the boundary $\partial\mathbb{H}^{d+1} = \mathbb{R}^d \cup \{\infty\}$ have the same automorphism group, namely the group $\mathrm{Isom}(d+1)$ of hyperbolic isometries is isomorphic to the group $\mathrm{Con}(d)$ of conformal maps in $\partial\mathbb{H}^{d+1}$. Therefore, any transformation in $\mathrm{Isom}(d+1)$ gives rise to a conformal map in $\partial\mathbb{H}^{d+1}$, and vice versa. It is well known that this isomorphism arises naturally from the principle of Poincaré extension, based on the elementary observation that a $(d+1)$-dimensional hyperbolic half-space H in $\mathbb{H}^{d+1}$ corresponds in a unique way to a d-dimensional ball $D := \overline{H} \cap \partial\mathbb{H}^{d+1}$.

Recall that a Kleinian group G is a discrete subgroup of $\mathrm{Con}(d)$, and that the limit set $L(G)$ of G is the derived set of some arbitrary G-orbit, that is $L(G) := \overline{G(z)} \setminus G(z)$, for some arbitrary $z \in \mathbb{H}^{d+1} \cup \partial\mathbb{H}^{d+1}$. It is well known that $L(G)$ can always be decomposed into the set $L_r(G)$ of radial limit points and the set $L_t(G)$ of transient limit points, where

$$L_r(G) := \left\{\xi \in L(G) : \liminf_{T\to\infty} \Delta(\xi_T) < \infty\right\} \text{ and } L_t(G) := \left\{\xi \in L(G) : \lim_{T\to\infty} \Delta(\xi_T) = \infty\right\}.$$

In here, ξ_T refers to the point on the hyperbolic geodesic ray from $i_d := (0, \dots, 0, 1) \in \partial\mathbb{H}^{d+1}$ to ξ for which $d_{hyp}(i_d, \xi_T) = T$, and $\Delta(\xi_T)$ refers to the hyperbolic distance of ξ_T to the orbit $G(i_d)$, that is $\Delta(\xi_T) := \inf\{d_{hyp}(\xi_T, g(i_d)) : g \in G\}$. Important subsets of $L(G)$ are the set $L_{ur}(G)$ of uniformly radial limit points and the set $L_J(G)$ of Jørgensen limit points. These are given as follows (cf. [7] [14]).

$$L_{ur}(G) := \left\{\xi \in L(G) : \limsup_{T\to\infty} \Delta(\xi_T) < \infty\right\},$$

$$L_J(G) := \{\xi \in L(G) : \text{ there exists a geodesic ray towards } \xi \text{ which is fully contained in some Dirichlet fundamental domain of } G\}.$$

One easily verifies that $L_{ur}(G) \subseteq L_r(G)$ and that $L_J(G) \subseteq L_t(G)$. We remark that in [2] it was shown for any arbitrary non-elementary Kleinian group G that the Hausdorff dimension of $L_{ur}(G)$ is equal to the exponent of convergence $\delta(G)$ of the Poincaré series $\sum_{g\in G} \exp(-sd_{hyp}(i_d, g(i_d)))$ associated with G (see also [20]). Finally, we briefly comment on how $L_J(G)$ relates to some of the concepts used in the study of rigidity of Kleinian groups acting on $\mathbb{H}^3$ (for the proofs and further details we refer to [22] (see also [9])). With $F_z(G)$ referring to the closure in $\mathbb{R}^3$ of the Dirichlet polyhedron centred at $z \in \mathbb{H}^3$, let $F_z^\infty(G) := F_z(G) \cap \partial\mathbb{H}^3$. We then have that $F_z^\infty(G)$ is a wandering set for G, that is we have $\lambda_2 \left(F_z^\infty(G) \cap g\left(F_z^\infty(G)\right)\right) = 0$ for all $g \in G \setminus \{id.\}$ (where λ_2 refers to the 2-dimensional Lebesgue measure). Note that if $\xi \in F_z^\infty(G)$ then the orbit $G(z)$ does not intersect the interior of the horoball at ξ through z. Therefore, the Dirichlet set $D_z(G) := \bigcup_{\gamma\in G} g\left(F_z^\infty(G)\right)$ does not contain any horospherical limit points (recall that a point ξ is called horospherical if every horoball at ξ contains infinitely many elements of $G(z)$). One then immediately verifies that $L_J(G) \subseteq D_z(G)$.

5.2. **Infinitely Generated Schottky Groups of Rapid Decay and Discrepancy Type.** We now restrict the discussion to a very special class of Kleinian groups which we will call infinitely generated classical Schottky groups of rapid decay. In order to define these groups, let $\mathcal{D} := \{D_n^i : n \in I\!N, i \in \{0,1\}\}$ be an infinite set of pairwise disjoint open d-dimensional balls in $\partial\mathbb{H}^{d+1}$. For ease of exposition we assume that $\operatorname{diam}(D_n^i) = \operatorname{diam}(D_n^{i\oplus 1})$ for all $n \in I\!N$, where "$\oplus$" refers to addition modulo 2. Furthermore, let $\{g_n^i : n \in I\!N, i \in \{0,1\}\}$ be a set of hyperbolic transformations in $\mathrm{Con}(d)$ such that for all $n \in \mathbb{N}$ and $i \in \{0,1\}$,

$$g_n^i\left(\mathrm{Ext}(D_n^i)\right) = \mathrm{Int}(D_n^{i\oplus 1}) \text{ for all } n \in I\!N, i \in \{0,1\}. \tag{5.1}$$

In here Int and Ext denote respectively the interior and exterior with respect to the Euclidean metric in $\mathbb{R}^d$. We will always assume that $\mathcal{D}$ satisfies the following rapid decay condition, where for $\alpha > 0$ and for a ball $D \in \mathcal{D}$ of radius $r > 0$ we let αD denote the ball of radius αr with the same centre as D.

Rapid Decay Condition. There exists an increasing sequence $(\alpha_n)_{n\in I\!N}$ of real numbers $\alpha_n > 2$ such that $\lim_{n\to\infty} \alpha_n = \infty$ and

$$\alpha_n D_n^i \cap \alpha_m D_m^j = \emptyset \textit{ for all distinct pairs } (n,i),(m,j) \in I\!N \times \{0,1\}. \tag{5.2}$$

Finally, we always assume that there exists $R > 0$ such that $\alpha_n D_n^i \subset B(0,R)$ for all $(n,i) \in I\!N \times \{0,1\}$, where $B(0,R)$ refers to the closed d-dimensional ball with centre $0 \in I\!R^d$ and radius R. Then put $Y(G) := B(0,3R)$. Now, let G be the group generated by $\{g_n^i : n \in I\!N, i \in \{0,1\}\}$. One easily verifies that $\bigcap_{D\in\mathcal{D}} \mathrm{Ext}(D)$ is a fundamental domain for the action of G on $\partial\mathbb{H}^{d+1}$ (cf. [8], Proposition VIII **A.4**). This then gives that G is a Kleinian group, and we will refer to G as an *infinitely generated Schottky group of rapid decay.* We remark that for $d = 2$, by viewing the g_n^i as elements of $\mathrm{Isom}(3)$, the set $\mathbb{H}^3/G$ of equivalence classes is topologically a 3-manifold whose interior is homeomorphic to a 3-dimensional ball to which infinitely many solid handles are attached, and whose boundary is the surface of that handle-body. We will now see that every infinitely generated Schottky group of rapid decay gives rise to some conformal pseudo-Markov system.

Theorem 5.1. *Let G be an infinitely generated Schottky group G of rapid decay as defined above. Define $X_n^i := Y(G) \setminus \alpha_n D_n^i$ and $\phi_n^i := g_n^i|_{X_n^i}$, for each $(n,i) \in I\!N \times \{0,1\}$. Put $E = E(G) := I\!N \times \{0,1\}\}$ and define the incidence matrix $A : E \times E \to \{0,1\}$ by $A_{(n,i)(m,j)} = 1$ if and only if $(n,i) \neq (m, j \oplus 1)$. Then $\mathcal{S}(G) := \{\phi_n^i : (n,i) \in E\}$ is a conformal pseudo-Markov system.*

Proof. Consider the system $\mathcal{S}(G)$ given as stated in the theorem, and note that by (5.1) we have

$$\phi_n^i(X_n^i) \subset D_n^{i\oplus 1}. \tag{5.3}$$

One immediately verifies that A is finitely irreducible (take for instance $\Xi := \{(1,0),(1,1)\}$). In order to show that $\mathcal{S}(G)$ is a a conformal pseudo-Markov system, note that (b), (c) and (d) in the definition of a pseudo-Markov system are immediate consequences of (5.2) and (5.3). Also, (3) and (4) in the definition of a conformal pseudo-Markov system are trivially satisfied, and the pseudo-convexity in (1) of Definition 4.1 is satisfied with $K = \pi/2$. In order to show that the Pre-Distortion Property (5) in Definition 4.1 is satisfied, let $(n,i),(m,j) \in E$ be fixed such that $A_{(n,i)(m,j)} = 1$. By (5.3) we have $\phi_m^j(X_m^j) \subset D_m^{j\oplus 1} \subset B(0,2R)$. Since $I\!R^d \setminus X_n^i \subseteq \left(I\!R^d \setminus Y(G)\right) \cup \alpha_n D_n^i$, since $(n,i) \neq (m, j\oplus 1)$ and since $\alpha_n > 2$, (5.2) implies that

$$\mathrm{dist}\left(\phi_m^j(X_m^j), I\!R^d \setminus X_n^i\right) \geq \alpha_m \frac{\mathrm{diam}\left(\phi_m^j(X_m^j)\right)}{2} > \mathrm{diam}\left(\phi_m^j(X_m^j)\right).$$

Hence, we can apply Lemma 4.3 with $\kappa = 1$, which gives that the system $\mathcal{S}(G)$ has the Pre-Distortion Property. In order to verify the remaining properties, let $(n,i) \in I\!N \times \{0,1\}$ be fixed. We now first give an upper estimate for the conformal derivative of ϕ_n^i. Note that we can assume without loss of generality that $i_d = (0,\ldots,0,1) \in \mathbb{H}^{d+1}$ is not contained in the Poincaré extensions of the balls in $\{D_k^j : (k,j) \in I\!N \times \{0,1\}\}$. By recalling that the conformal derivative is given by the Poisson kernel and then using elementary Euclidean and hyperbolic geometry, we obtain for each $z \in \mathrm{Ext}(\alpha_n D_n^i)$,

$$\begin{aligned} |(g_n^i)'(z)| &= \frac{p_{d+1}\left((g_n^i)^{-1}(i_d)\right)}{\left|z - (g_n^i)^{-1}(i_d)\right|^2} \leq \frac{p_{d+1}\left((g_n^i)^{-1}(i_d)\right)}{\left((\alpha_n \mathrm{diam}(D_n^i) - \mathrm{diam}(D_n^i))/2\right)^2} \\ &\leq \frac{1}{(\alpha_n - 1)^2} \frac{p_{d+1}\left((g_n^i)^{-1}(i_d)\right)}{\left(\mathrm{diam}(D_n^i)/2\right)^2} \leq \frac{1}{(\alpha_n - 1)^2}. \end{aligned}$$

Here, $p_{d+1} : \mathbb{H}^{d+1} \to I\!R^+$ refers to the projection onto the $(d+1)$-th coordinate. From this we deduce that

$$||(\phi_n^i)'|| \leq (\alpha_n - 1)^{-2} < 4\alpha_n^{-2}. \tag{5.4}$$

This formula immediately implies that conditions (2) in Definition 4.1 and (a) in Definition 3.1 are satisfied. This finishes the proof of the theorem. ■

One immediately verifies that for an infinitely generated Schottky group G of rapid decay there is the following dictionary translating the most important subsets of the limit set of G into the corresponding subsets of the limit set of the associated conformal pseudo-Markov system $\mathcal{S}(G)$. In here, $\mathcal{S}_D(G)$ refers to the subsystem of $\mathcal{S}(G)$ giving rise to all admissable words built from the finite alphabet $D \subset E(G)$.

$L(G) = \overline{J_{\mathcal{S}(G)}}$;
$L(G) \setminus L_J(G) = J_{\mathcal{S}(G)}$;
$L_{ur}(G) = \bigcup_{D \subset E(G) \text{ finite}} J_{\mathcal{S}_D(G)}$;
$L_J(G) = \bigcup_{\omega \in E_A^*} \phi_\omega\Big(\Delta(\mathcal{S}(G)) \cap X_\omega\Big)$.

We therefore have in particular $h^*(\mathcal{S}(G)) = \mathrm{HD}(L_{ur}(G))$ and $h(\mathcal{S}(G)) = \mathrm{HD}(L(G) \setminus L_J(G))$. Hence, by combining Theorem 5.1 and Theorem 4.13, we obtain the following corollary.

Corollary 5.2. *Let G be an infinitely generated Schottky group G of rapid decay such that the associated conformal pseudo-Markov system $\mathcal{S}(G)$ is thin. We then have*

$$\delta(G) = \mathrm{HD}(L_{ur}(G)) = h^*(\mathcal{S}(G)) = h(\mathcal{S}(G)) = \mathrm{HD}(L(G) \setminus L_J(G)).$$

Apriori, the existence of infinitely generated Schottky groups as considered in the latter corollary is not clear. In fact, this will follow from our final theorem in which we obtain the existence of a very special type of infinitely generated Schottky groups of rapid decay fulfilling the thinness condition. Before stating this theorem, let us recall the notion of a discrepancy group, introduced in [7]. Namely, a Kleinian group G is called *discrepancy group* if and only if $\mathrm{HD}(L_r(G)) < \mathrm{HD}(L(G))$. Kleinian groups of this type were first shown to exist by Patterson in [16], Theorem 4.4 (see also [15] [17]). (For further discussions and examples of Kleinian discrepancy groups we refer to [4] [7] [10]). We remark that the discrepancy groups considered in [16] and [17] are Schottky groups of the first kind, that is their limit set is the whole boundary of the underlying hyperbolic space. Therefore, a natural question to ask is if there are Schottky discrepancy groups which are of the second kind. A particular outcome of the following theorem will be to give an affirmative answer to this question. Also, note that a Kleinian group G considered in the following theorem has the remarkable property that $\mathrm{HD}(L_{ur}(G)) = \mathrm{HD}(L(G) \setminus L_J(G))$. Therefore, a further particular outcome of the following theorem is that for discrepancy groups we can not expect in general that the dimension gap between $L_{ur}(G)$ and $L(G)$ can be filled smoothly by subsets of $L(G) \setminus L_J(G)$.

Theorem 5.3. *For all $0 < s < t \leq d$ there exists an infinitely generated Schottky group $G < \mathrm{Con}(d)$ of rapid decay such that G is a discrepancy group with $\delta(G) \leq s < t = \mathrm{HD}(L(G))$.*

Furthermore, the conformal pseudo-Markov system associated with G fulfills the thinness property, and therefore,

$$\delta(G) = \mathrm{HD}(L_{ur}(G)) = \mathrm{HD}(L(G) \setminus L_J(G)) < \mathrm{HD}(L_J(G)) = \mathrm{HD}(L(G)) = t.$$

Proof. The proof consists of giving the explicit construction of an infinitely generated Schottky group $G < \mathrm{Con}(d)$ of rapid decay which has the properties as stated in the theorem. Suppose that $W \subset \mathbb{R}^d$ is some compact set such that $\mathrm{HD}(W) = t$ and such that there exists a countable set $V = \{v_n^i : n \in \mathbb{N}, i \in \{0,1\}\} \subset \mathbb{R}^d \setminus W$ whose derived set is equal to W, that is $W = \overline{V} \setminus V$. Also, let $(\alpha_n)_{n\in\mathbb{N}}$ be some increasing sequence of real numbers $\alpha_n > 2$ such that

$$\sum_{n\in\mathbb{N}} \alpha_n^{-2s} < \frac{1}{4} \text{ and } \sum_{n\in\mathbb{N}} \alpha_n^{-\theta} < \infty \text{ for all } \theta > 0. \tag{5.5}$$

Furthermore, fix some decreasing sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of positive numbers such that

$$\lim_{n\to\infty} \alpha_n \epsilon_n = 0 \text{ and } \sup\{\alpha_n \epsilon_n : n \in \mathbb{N}\} \leq 1. \tag{5.6}$$

We will now construct a sequence $\{D_n^i : n \in \mathbb{N}, i \in \{0,1\}\}$ of d-dimensional open balls in $\mathbb{R}^d$ by induction as follows. For $n = 1$ and $i \in \{0,1\}$, choose D_1^i to be centred at v_1^i such that

$$V \cap \alpha_1 D_1^i = \{v_1^i\}, \alpha_1 D_1^0 \cap \alpha_1 D_1^1 = \emptyset \text{ and } \mathrm{diam}(D_1^0) = \mathrm{diam}(D_1^1) \leq \epsilon_1.$$

For the inductive step suppose that the $2n$ balls $\{D_k^i : 1 \leq k \leq n, i \in \{0,1\}\}$ have already been constructed. For each $i \in \{0,1\}$, we then choose the ball D_{n+1}^i to be centred at v_{n+1}^i such that $V \cap \alpha_{n+1} D_{n+1}^i = \{v_{n+1}^i\}$ and

$$\alpha_{n+1} D_{n+1}^0 \cap \alpha_{n+1} D_{n+1}^1 = \emptyset, \alpha_{n+1} D_{n+1}^i \cap \alpha_k D_k^j = \emptyset$$

for all $1 \leq k \leq n, j \in \{0,1\}$, as well as

$$\mathrm{diam}(D_{n+1}^0) = \mathrm{diam}(D_{n+1}^1) \leq \epsilon_{n+1}.$$

We then choose the set $\{g_n^i : n \in \mathbb{N}, i \in \{0,1\}\}$ of hyperbolic elements of $\mathrm{Con}(d)$ such that

$$g_n^i\left(\mathrm{Ext}(D_n^i)\right) = \mathrm{Int}(D_n^{i\oplus 1}) \text{ for all } n \in \mathbb{N}, i \in \{0,1\}.$$

One immediately verifies that the group G generated by $\{g_n^i : n \in \mathbb{N}, i \in \{0,1\}\}$ is an infinitely generated Schottky group of rapid decay. (Note that the conditions in (5.6) are necessary to ensure that $Y(G)$ is bounded). By Theorem 5.1 we then have that G gives rise to a conformal pseudo-Markov system $\mathcal{S}(G)$ for which $\overline{J_{\mathcal{S}(G)}} = L(G)$. Furthermore, by combining (5.4) with the second condition in (5.5), we obtain that $\sum_{n,i} ||\phi_n^i||^\theta$ converges for every $\theta > 0$. So, the system $\mathcal{S}(G)$ is thin. Combining in turn the first part of (5.5), (5.4) (which both imply that $\mathcal{P}(s) < 0$), Proposition 4.10 (a) and the first part of Theorem 4.13, we obtain that $\delta(G) = \mathrm{HD}(J_{\mathcal{S}(G)}) \leq s$. Since $L_J(G) = \bigcup_{\omega\in E_A^*} \phi_\omega\Big(\Delta(\mathcal{S}(G)) \cap X_\omega\Big)$ and since by construction $\Delta(\mathcal{S}(G)) = W$, we obtain that $\mathrm{HD}(L_J(G)) = \mathrm{HD}(W) = t$. Combining these observations with Proposition 4.4, it follows that $\mathrm{HD}(L(G)) = \mathrm{HD}(L_J(G)) = t > s \geq \mathrm{HD}(J_{\mathcal{S}(G)}) = \delta(G)$. This finishes the proof of the theorem.

■

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