Fractal measures for meromorphic functions of finite order

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Abstract. We prove several essential fractal properties, such as positivity, finiteness or local infinity, of Hausdorff and packing measures of radial Julia sets of large subclasses of entire and meromorphic functions considered in [MU]. Most of the results proven are shown to be optimal.

1. Introduction

In the paper [MU] we have developed the geometric thermodynamic formalism for a large class of transcendental entire and meromorphic functions. We have also brought up its frirst geometrical consequences such as Bowen's formula and real analyticity of the Hausdorff dimension function of radial Julia sets. Given a meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$, geometric thermodynamical formalism deals with potentials of the form $-t \log |f'|$, $t > 0$. In [MU] we have thoroughly explored the class of hyperbolic meromorphic functions $f : \mathbb{C} \to \hat{\mathbb{C}}$ satisfying the following:

Rapid derivative growth: There are $\alpha_2 > \max\{0, -\alpha_1\}$ and $\kappa > 0$ such that

$$|f'(z)| \geq \kappa^{-1}(1 + |z|^{\alpha_1})(1 + |f(z)|^{\alpha_2}) \tag{1.1}$$

for all finite $z \in J(f) \setminus f^{-1}(\infty)$,

and frequently the stronger:

Balanced growth condition: There are $\alpha_2 > \max\{0, -\alpha_1\}$ and $\kappa > 0$ such that

$$\kappa^{-1}(1 + |z|^{\alpha_1})(1 + |f(z)|^{\alpha_2}) \leq |f'(z)| \leq \kappa(1 + |z|^{\alpha_1})(1 + |f(z)|^{\alpha_2}) \tag{1.2}$$

for all finite $z \in J(f) \setminus f^{-1}(\infty)$.

Throughout the entire paper we use the notation

$$\alpha = \alpha_1 + \alpha_2.$$

Hyperbolicity means here that the map $f : \mathbb{C} \to \hat{\mathbb{C}}$ is topologically hyperbolic, i.e. the Julia set $J(f)$ stays within a positive Euclidean distance apart from the forward trajectory

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of the $sing(f^{-1})$ of singular points of f^{-1}, and that this map is expanding, i.e. there are $c > 0$ and $\lambda > 1$ such that

$$|(f^n)'(z)| \geq c\lambda^n \quad \textit{for all} \;\; z \in J(f) \setminus f^{-1}(\infty) \; .$$

Note (see Proposition 2.2 in [MU] that each topologically hyperbolic meromorphic function satisfying the rapid derivative growth condition with $\alpha_1 \geq 0$ is expanding, and consequently, hyperbolic.

The conditions (1.1) and (1.2) are comfortable to work with and easy to to verify (see [MU]) for a large natural class of functions which include the entire exponential family λe^z, certain other periodic functions ($sin(az+b)$, $\lambda \tan(z)$, elliptic functions...), the cosine-root family $\cos(\sqrt{az+b})$ and the composition of these functions with arbitrary polynomials.

In contrast to the research devoted to the thermodynamic formalism of some classes of entire and meromorphic functions (nearly all of them being periodic) done prior to [MU], we assummed in [MU] no periodicity and we needed no projection onto infinite cylinders or tori. Instead, we have worked with the conformal Riemannian metric σ defined by the formula

$$d\sigma(z) = (1+|z|^{\alpha_2})^{-1}|dz|.$$

The norm of the derivative $f'(z)$ is in this metric given by the formula

$$|f'(z)|_\sigma = |f'(z)|(1+|z|^{\alpha_2})(1+|f(z)|^{\alpha_2})^{-1}.$$

The associated Perron-Frobenius-Ruelle (or transfer) operator

$$\mathcal{L}_t\varphi(w) = \sum_{z\in f^{-1}(w)} |f'(z)|_\sigma^{-t}\varphi(z) \tag{1.3}$$

is well defined and has all the required properties that made the thermodynamical formalism developed in [MU] work. Let (X, m) be a probability measure and $T : X \to X$ a measurable map. Recall that given a bounded above non-negative measurable function $g : X \to [0, +\infty)$, the measure m is called g-conformal provided that

$$m(T(A)) = \int_A g dm$$

for every measurable subset A of X such that $T|_X$ is injective. With these tools in hand we were then able to obtain dynamically and geometrically significant information about the Julia set $J(f)$ and about the radial (or conical or hyperbolic) Julia set

$$J_r(f) = \{z \in J(f) \; : \; \liminf_{n\to\infty} |f^n(z)| < \infty\} \; .$$

Its Hausdorff dimension is frequently called the hyperbolic dimension of the Julia set $J(f)$. The key result of [MU] was the following.

Theorem 1.1. *If $f : \mathbb{C} \to \hat{\mathbb{C}}$ is an arbitrary hyperbolic meromorphic function of finite order ρ that satisfies the rapid derivative growth condition (1.1), then for every $t > \frac{\rho}{\alpha}$ the following are true.*

(1) *The topological pressure $\mathrm{P}(t) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_t^n(1\!\!1)(w)$ exists and is independent of $w \in J(f) \cap \mathbb{C}$.*

(2) *There exists a unique* $\lambda|f'|_\sigma^t$*-conformal measure* m_t *and necessarily* $\lambda = e^{\mathrm{P}(t)}$*. Also, there exists a unique Gibbs state* μ_t*, i.e.* μ_t *is* f*-invariant and equivalent to* m_t*. Moreover, both measures are ergodic and supported on the radial (or conical) Julia set.*
(3) *The density* $\psi = d\mu_t/dm_t$ *is a continuous and bounded function on the Julia set* $J(f)$*.*

Throughout the paper we will also require that the following condition is satisfied.

Divergence type: The series $\Sigma(t,w) = \sum_{z\in f^{-1}(w)} |z|^{-t}$ diverges at the critical exponent (which is the order of the function $t=\rho$; w is any non Picard exceptional value).

The result linking the dynamics and geometry of f, proven in [MU] was this.

Theorem 1.2. *(Bowen's formula) If* $f:\mathbb{C}\to\hat{\mathbb{C}}$ *is a hyperbolic meromorphic function that is of finite order* $\rho>0$*, of divergence type and of balanced derivative growth with* $\alpha_1 \geq 0$*, then the pressure function* $\mathrm{P}(t)$ *has a unique zero* $h > \rho/\alpha$ *and*

$$\mathrm{HD}(J_r(f)) = h\ .$$

Starting of with this theorem, developing new ideas focused around the Riemannian metric σ, and borrowing from [UZ2], we have shown in [MU] the real analytic dependenceof the hyperbolic dimension in large classes of functions satisfying the assumptions of Theorem 1.2 (Bowen's formula).

In this paper our goal is to obtain further finer geometric properties of such maps as in Theorem 1.2. They will concern Hausdorff and packing measures and can be regarde as related and as extensions of results from [KU1], [KU2], [KU3], [UZ1], [UZ2], and [UZ3], comp. also the survey article [KU4].

2. Preliminaries

Let us mention that the Julia set of a hyperbolic function is never the whole sphere. We thus may and we do assume that the origin $0\in\mathcal{F}_f$ is in the Fatou set (otherwise it suffices to conjugate the map by a translation). This means that there exists $T>0$ such that

$$B(0,T)\cap J(f) = \emptyset. \tag{2.1}$$

The derivative growth condition (1.1) can then be reformulated in the following more convenient form:

There are $\alpha_2>0$, $\alpha_1>-\alpha_2$ *and* $\kappa>0$ *such that*

$$|f'(z)| \geq \kappa^{-1}|z|^{\alpha_1}|f(z)|^{\alpha_2} \quad \textit{for all finite } z\in J(f)\setminus f^{-1}(\infty)\ . \tag{2.2}$$

Similarly, the balanced condition (1.2) becomes

$$\kappa^{-1}|z|^{\alpha_1}|f(z)|^{\alpha_2} \leq |f'(z)| \leq \kappa|z|^{\alpha_1}|f(z)|^{\alpha_2} \quad \textit{for all finite } z\in J(f)\setminus f^{-1}(\infty) \tag{2.3}$$

and the metric

$$d\sigma(z) = |z|^{-\alpha_2}|dz|.$$

As a complement to Theorem 1.2 (Bowen's formula), we have proved in [MU], Corollary 7.2, the following result, crucial for our further theoretical considerations.

Corollary 2.1. *If $f : \mathbb{C} \to \hat{\mathbb{C}}$ is a hyperbolic divergence type meromorphic function of finite order $\rho > 0$ and of balanced derivative growth (condition (2.3)), then f has a $|f'|_\sigma^h$-conformal measure m_h. This measure will be in the sequel simply refered to as h-conformal.*

The definitions of Hausdorff and packing measures as well as Hausdorff dimension can be found for example in [Mat] or [PU]. The symbols H_σ^t and P_σ^t refer respectively to the t-dimensional Hausdorff and packing measure evaluated with respect to the Riemannian metric $d\sigma$. Fix $t > \rho/\alpha$. By Theorem 1.1(2) there exists m_t, an $e^{\mathrm{P}(t)}|f'|_\sigma^t$-conformal measure, and let m_t^e be its Euclidean version defined by the requirement that

$$dm_t^e(z) = |z|^{\alpha_2 t} dm_t(z).$$

Note that this measure is $e^{\mathrm{P}(t)}|f'|^t$-conformal and a straightforward calculation shows that

$$\frac{dm_t^e \circ f}{dm_t^e}(z) = e^{\mathrm{P}(t)}|f'(z)|^t, \ \ z \in J(f). \tag{2.4}$$

Let $\delta(f)$ be the quarter of the Euclidean distance of the Julia set of f the the forward trajectory of the set of singular points of f^{-1}. Fix any radius

$$R \in (0, \delta(f)).$$

So, if $z \in J(f)$, $n \geq 0$, and $z \in f^{-n}(w)$, then there exists a unique holomorphic inverse branch $f_z^{-n} : B(w, 4R) \to \mathbb{C}$ of f^n sending w to z.

We will also need the following distortion theorem which is a straightforward consequence of hyperbolicity of the function f, Koebe's Distortion Theorem, and the fact that $0 \notin J(f)$.

Lemma 2.2. *For every hyperbolic meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ satisfying the rapid derivative growth condition there exists a constant $K_\sigma \geq 1$, called σ-adjusted Koebe constant, such that if $R > 0$ is sufficiently small, then for every integer $n \geq 0$, every $w \in J(f)$, every $z \in f^{-n}(w)$ and all $x, y \in B_\sigma(w, R|w|^{-\alpha_2}) \cup B(w, R)$, we have that*

$$K_\sigma^{-1} \leq \frac{|(f_z^{-n})'(y)|_\sigma}{|(f_z^{-n})'(x)|_\sigma} \leq K_\sigma. \tag{2.5}$$

It follows from this lemma that

$$B_\sigma\big(z, K_\sigma^{-1} R|w|^{-\alpha_2}|(f^n)'(z)|_\sigma^{-1}\big) \subset f_z^{-n}\big(B_\sigma(w, R|w|^{-\alpha_2})\big) \subset B_\sigma\big(z, K_\sigma R|w|^{-\alpha_2}|(f^n)'(z)|_\sigma^{-1}\big) \tag{2.6}$$

and that

$$m_t\big(f_z^{-n}\big(B_\sigma(w, R|w|^{-\alpha_2})\big)\big) \asymp e^{-\mathrm{P}(t)n}|(f^n)'(z)|_\sigma^{-t} m_t\big(B_\sigma(w, R|w|^{-\alpha_2})\big). \tag{2.7}$$

Finally, using Nevanlinna's theory we have proved in [MU] the following

Proposition 2.3. *Let f be a meromorphic function of finite order ρ and suppose that $0 \notin J(f)$. Then, for every $t > \rho$, there is $M_t > 0$ such that*

$$\sum_{f(z)=a} \frac{1}{|z|^t} \le M_t \quad \textit{for all} \;\; a \in J(f)\ .$$

3. Hausdorff and Packing Measures.

We now analyze the behavior of the Hausdorff and Packing measures on the radial Julia set of a function f which is again supposed to be hyperbolic, of divergence type and of balanced growth (condition (2.3)). First note that it was proved in [MU] that

$$\mathrm{H}^h_\sigma(J_r(f)) < +\infty. \tag{3.1}$$

The corresponding result for packing measures is this.

Proposition 3.1. *The packing measure $\mathrm{P}^h_\sigma(J_r(f)) > 0$.*

Proof. By Theorem 1.1 there exists a Borel probability f-invariant ergodic measure μ_h equivalent to m_h. Fix $M > T$ so large that $\mu_h(B(0,M)) > 0$. It then follows from Poincare's Return Theorem that there exists a Borel set $X \subset J(f) \cap B(0,M)$ such that $\mu_h(X) = \mu_h(B(0,M)) > 0$ and for every $z \in X$ there exists an infinite incerasing sequence $\{n_j\}_{j=1}^\infty$ such that $f^{n_j}(z) \in B(0,M)$ for all $j \ge 1$. In particular $X \subset J_r(f)$. Using (2.6) and (2.7) we therefore obtain for every $j \ge 1$ that

$$m_h\big(B_\sigma\big(z, K_\sigma^{-1}R|(f^{n_j}(z)|^{-\alpha_2}|(f^{n_j})'(z)|_\sigma^{-1}\big)\big) \le K_\sigma|(f^{n_j})'(z)|_\sigma^{-h}$$

Consequently,

$$\begin{aligned}\liminf_{r\to 0}\frac{m_h(B_\sigma(z,r))}{r^h} &\le \liminf_{j\to\infty}\frac{m\big(B_\sigma\big(z, K_\sigma^{-1}R|(f^{n_j}(z)|^{-\alpha_2}|(f^{n_j})'(z)|_\sigma^{-1}\big)\big)}{\big(K_\sigma^{-1}R|(f^{n_j}(z)|^{-\alpha_2}|(f^{n_j})'(z)|_\sigma^{-1}\big)^h}\\ &\preceq |(f^{n_j})'(z)|^{\alpha_2 h} \le M^{\alpha_2 h}.\end{aligned}$$

Hence $\mathrm{P}^h_\sigma(J_r(f)) \ge \mathrm{P}^h_\sigma(X) > 0$, and we are done. $\square$

A meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ is said to be of finite injectivity radius if there exists $R_* > 0$ such that for every $z \in J(f)$ the function $f|_{B(z,R_*)}$ is injective. A meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ is said to be of bounded local multiplicity if there exists $R_* > 0$ and $P > 0$ such that for every $z \in J(f)$ the function $f|_{B(z,R_*)}$ is at most P-to-1. Call P a local multiplicity of f. Of course each meromorphic function of finite injectivity radius is of bounded local multiplicity. We shall prove the following.

Theorem 3.2. *If f is of bounded local multiplicity and if $\alpha_1 > 0$, then the packing measure $\mathrm{P}^h|_{J_r(f)}$ is locally infinite at every point of $J_r(f)$.*

Proof. Fix
$$R = \min\{R_*, T/2, \mathrm{dist}(\mathrm{P}_f, J(f))/2\}.$$
Since the measure μ_h is ergodic and its topological support is equal to $J(f)$, the set
$$J_{r\infty}(f) = \{z \in J_r(f) : \limsup_{n\to\infty} |f^n(z)| = +\infty\}$$
is of measure μ_h equal to 1 and is dense in $J_r(f)$. It therefore suffices to prove the proposition for all points from $J_{r\infty}(f)$. Let $P < \infty$ be a local multiplicity of f. Making use of the Perron-Frobenius operator $\mathcal{L}_h$ we get for every $\xi \in J(f)$ with $|\xi| > 2R$ that

$$\begin{aligned} m_h\big(B_\sigma(\xi, R|\xi|^{-\alpha_2})\big) &\le m_h(B(\xi, 2R)) \le \kappa^h \int_{J(f)} \sum_{z\in f^{-1}(w)\cap B(\xi,2R)} |z|^{-\alpha h} dm_h(w) \\ &\le \kappa^h (|\xi| - 2R)^{-\alpha h} \int_{J(f)} P dm_h(w) = P\kappa^h(|\xi| - 2R)^{-\alpha h}. \end{aligned} \tag{3.2}$$

Now fix $z \in J_{r\infty}(f)$ and let $\{n_j\}_{j=1}^\infty$ be an increasing to infinity sequence of positive integers such that $|f^{n_j}(z)| \ge 4R$ for all $j \ge 1$ and $\lim_{j\to\infty} |f^{n_j}(z)| = +\infty$. Using (2.7) and (3.2) we get for every $j \ge 1$ that

$$\begin{aligned} &m_h\big(B_\sigma\big(z, K_\sigma^{-1} R|f^{n_j}(z)|^{-\alpha_2} |(f^{n_j})'(z)|_\sigma^{-1}\big)\big) \le \\ &\qquad \le K_\sigma^h |(f^{n_j})'(z)|_\sigma^{-h} m\big(B_\sigma\big(f^{n_j}(z), R|f^{n_j}(z)|^{-\alpha_2}\big)\big) \\ &\qquad \le P(2^\alpha \kappa K_\sigma)^h |(f^{n_j})'(z)|_\sigma^{-h} |f^{n_j}(z)|^{-\alpha h} \\ &\qquad \asymp |(f^{n_j})'(z)|_\sigma^{-h} |f^{n_j}(z)|^{-\alpha h}. \end{aligned}$$

Hence,

$$\begin{aligned} \liminf_{r\to 0} \frac{m(B_\sigma(z,r))}{r^h} &\le \liminf_{j\to\infty} \frac{m\big(B_\sigma\big(z, K_\sigma^{-1} R|f^{n_j}(z)|^{-\alpha_2} |(f^{n_j})'(z)|_\sigma^{-1}\big)\big)}{\big(K_\sigma^{-1} R|f^{n_j}(z)|^{-\alpha_2} |(f^{n_j})'(z)|_\sigma^{-1}\big)^h} \\ &\preceq \liminf_{j\to\infty} |f^{n_j}(z)|^{-\alpha_1 h} = 0. \end{aligned}$$

We are done. □

Notice that the assumption $\alpha_1 > 0$ is sharp. For elliptic functions $\alpha_1 = 0$, $\alpha_2 > 1$ and, as was shown in [KU2], if they are hyperbolic (in fact it suffices for them to be non-recurrent without parabolic points) then $h = \mathrm{HD}(J_r(f)) = \mathrm{HD}(J(f))$ and the h-dimensional packing measure of the Julia set of f is finite. We can complete the picture by the following.

Theorem 3.3. *Suppose that $f : \mathbb{C} \to \hat{\mathbb{C}}$ is a hyperbolic meromorphic function of balanced rapid derivative growth with $\alpha_1 = 0$ and $\alpha_2 \le 1$. If f is of finite injectivity radius, then the packing measure $\mathrm{P}^h|_{J_r(f)}$ is locally infinite at every point of $J_r(f)$.*

Proof. Fix
$$0 < R \le \min\{R_*/4, T/2, (4\kappa K T^{2\alpha_1 - 1})^{-1}, \mathrm{dist}(\mathrm{P}_f, J(f))/2\}.$$

Since $\alpha_1 = 0$, $\alpha = \alpha_2$. Let $J_{r\infty}(f)$ have the same meaning as in the provious proposition. Similarly as there it suffices to prove our proposition for all points from $J_{r\infty}(f)$. Also, similarly as there, we get for every $\xi \in J(f)$ that

$$\begin{aligned} m_h\big(B_\sigma(\xi, R|\xi|^{-\alpha_2})\big) &\le \kappa^h \int_{J(f)} \sum_{z\in f^{-1}(w)\cap B(\xi,2R)} |z|^{-\alpha h} dm_h(w) \\ &= \kappa^h \int_{f(B(\xi,2R))} \sum_{z\in f^{-1}(w)\cap B(\xi,2R)} |z|^{-\alpha h} dm_h(w) \\ &\le \kappa^h(|\xi|-2R)^{-\alpha h} \int_{f(B(\xi,2R))} \#(f^{-1}(w)\cap B(\xi,2R)) dm_h(w) \\ &\le \kappa^h(|\xi|-2R)^{-\alpha h} m_h(f(B(\xi,2R))). \end{aligned}$$

Now, in view of the definition of R_*, Koebe's Distortion Theorem and the balanced rapid growth of the derivative, we get that

$$\begin{aligned} f(B(\xi,2R)) &\subset B(f(\xi), 2KR|f'(\xi)|) \subset B\big(f(\xi), 2\kappa KR|\xi|^{\alpha_1}|f(\xi)|^{\alpha_2}\big) \\ &\subset B\big(f(\xi), 2\kappa KT^{\alpha_1}T^{\alpha_1-1}R|f(\xi)|\big) \subset B(f(\xi), |f(\xi)|/2), \end{aligned}$$

where the last inclusion was written since $R \le (4\kappa KT^{2\alpha_1-1})^{-1}$. Hence

$$f(B(\xi,2R)) \subset \mathbb{C} \setminus B(0, |f(\xi)|/2),$$

and therefore

$$m_h\big(B_\sigma(\xi, R|\xi|^{-\alpha_2})\big) \le \kappa^h(|\xi|-2R)^{-\alpha h} m_h\big(\mathbb{C}\setminus B(0,|f(\xi)|/2)\big). \tag{3.3}$$

Now, as in the proof of the previous proposition, fix $z \in J_{r\infty}(f)$ and let $\{n_j\}_{j=1}^\infty$ be an increasing to infinity sequence of positive integers such that $|f^{n_j}(z)| \ge 4R$ for all $j \ge 1$ and $\lim_{j\to\infty}|f^{n_j}(z)| = +\infty$. Using (2.7) and (3.3), and remembering that $\alpha = \alpha_2$, we get for every $j \ge 1$ that

$$\begin{aligned} &m_h\big(B_\sigma\big(z, K_\sigma^{-1}R|f^{n_j}(z)|^{-\alpha}|(f^{n_j})'(z)|_\sigma^{-1}\big)\big) \le \\ &\qquad \le K_\sigma^h|(f^{n_j})'(z)|_\sigma^{-h} m_h\big(B_\sigma\big(f^{n_j}(z), R|f^{n_j}(z)|^{-\alpha}\big)\big) \\ &\qquad \le (2^\alpha\kappa K_\sigma)^h|(f^{n_j})'(z)|_\sigma^{-h}|f^{n_j}(z)|^{-\alpha h} m_h\big(\mathbb{C}\setminus B(0,|f^{n_j}(z)|/2\big) \\ &\qquad = (2^\alpha\kappa K_\sigma^2R^{-1})^h\big(K_\sigma^{-1}R|f^{n_j}(z)|^{-\alpha}|(f^{n_j})'(z)|_\sigma^{-1}\big)^h m_h\big(\mathbb{C}\setminus B(0,|f^{n_j}(z)|/2\big). \end{aligned}$$

Since $\lim_{s\to\infty} m_h(\mathbb{C}\setminus B(0,s)) = 0$, we thus get

$$\begin{aligned} \liminf_{r\to 0}\frac{m_h(B_\sigma(z,r))}{r^h} &\le \liminf_{j\to\infty}\frac{m_h\big(B_\sigma\big(z, K_\sigma^{-1}R|f^{n_j}(z)|^{-\alpha}|(f^{n_j})'(z)|_\sigma^{-1}\big)\big)}{\big(K_\sigma^{-1}R|f^{n_j}(z)|^{-\alpha}|(f^{n_j})'(z)|_\sigma^{-1}\big)^h} \\ &\le \liminf_{j\to\infty}(2^\alpha\kappa K_\sigma^2R^{-1})^h m_h\big(\mathbb{C}\setminus B(0,|f^{n_j}(z)|/2\big) = 0. \end{aligned}$$

We are done. $\square$

A meromorphic function is called evenly distributed if

$$\forall\varepsilon > 0\,\exists C_\varepsilon \ge 1\,\forall \xi \in J(f)\,\forall w \in J(f)\,\forall R \ge 1 \quad \#(f^{-1}(w)\cap B(\xi,R)) \le C_\varepsilon R^{\rho+\varepsilon}. \tag{3.4}$$

Our last result is about positivity of the Hausdorff measure of $J_r(f)$. Notice that $\mathrm{H}^h_\sigma(J(f)) = 0$ for any hyperbolic elliptic function where again $h = \mathrm{HD}(J_r(f)) = \mathrm{HD}(J(f))$ (see [KU2]). In this case $h < \rho = 2$.

Theorem 3.4. *Suppose that a hyperbolic meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ satisfies the following conditions.*

(a) *f is of balanced rapid derivative growth with $\alpha_1 \geq 0$.*
(b) *$\rho < h$ (which is implied by (a) if $\alpha \leq 1$).*
(c) *The map f is of finite injectivity radius.*
(d) *The map f is evenly distributed.*

Then $0 < \mathrm{H}^h_\sigma(J_r(f)) < +\infty$.

Proof. Let

$$\theta = \min\{R_*, \mathrm{dist}(\mathrm{P}_f, J(f))\}.$$

First notice that (3.4) is satisfied for all $R \geq \theta/8$. Let ν be the Euclidean version of the h-conformal measure m_h. In view of (b) there exists $\varepsilon > 0$ so small that $\rho+\varepsilon \leq h$. Similarly as (3.2), we get for every $\xi \in J(f)$ and every $R \in [\theta/8, |\xi|/2)$ that

$$\begin{aligned}
\nu(B(\xi,R)) &\leq 2^{\alpha h}|\xi|^{\alpha h} m_h(B(\xi,R)) \leq 2^{\alpha h}\kappa^h|\xi|^{\alpha h}\int_{J(f)}\sum_{z\in f^{-1}(w)\cap B(\xi,R)}|z|^{-\alpha h}dm_h(w)\\
&\leq (2^\alpha\kappa)^h|\xi|^{\alpha h}2^{\alpha h}|\xi|^{-\alpha h}\int_{J(f)}\#(f^{-1}(w)\cap B(\xi,R))dm_h(w) \qquad (3.5)\\
&\leq (4^\alpha\kappa)^h C_\varepsilon R^{\rho+\varepsilon} \leq C^{(1)}R^h
\end{aligned}$$

with an absolute constant $C^{(1)} \geq 1$. For all $0 < r_1 < r_2$ denote by $A(r_1, r_2)$ the closed annulus centered at 0 with inner radius r_1 and outer radius r_2 and by

$$A_j = A(3R2^{-(j+1)}, 3R2^{-j}).$$

Keep ε and ξ as above. Fix $R \geq |\xi|/2$. Then $B(\xi, R) \subset B(0, 3R)$ and, using Proposition 2.3, we therefore get that

$$\begin{aligned}
\nu(B(\xi,R)) &\leq \nu(B(0,3R)) \leq \sum_{j=0}^{\log_2(R/T)}\nu\big(A_j\big) \leq \sum_{j=0}^{\log_2(R/T)}(3R2^{-j})^{\alpha h}m_h\big(A_j\big)\\
&\leq \sum_{j=0}^{\log_2(R/T)}(3R2^{-j})^{\alpha h}\int_{J(f)}\sum_{z\in f^{-1}(w)\cap A_j}|z|^{-\alpha h}dm_h(w) \qquad (3.6)\\
&= \sum_{j=0}^{\log_2(R/T)}(3R2^{-j})^{\alpha h}\int_{J(f)}\sum_{z\in f^{-1}(w)\cap A_j}|z|^{-(\rho+\varepsilon)}|z|^{\rho+\varepsilon-\alpha h}dm_h(w)
\end{aligned}$$

$$\leq (3R)^{\alpha h} \sum_{j=0}^{\log_2(R/T)} (3R)^{\rho+\varepsilon-\alpha h} 2^{-\alpha h j} 2^{-(j+1)(\rho+\varepsilon-\alpha h)} \int_{J(f)} \sum_{z\in f^{-1}(w)\cap A_j} |z|^{-(\rho+\varepsilon)} dm(w)$$

$$\leq M_{(\rho+\varepsilon)} 2^{\alpha h-\rho-\varepsilon} (3R)^{\rho+\varepsilon} \sum_{j=0}^{\log_2(R/T)} 2^{-(\rho+\varepsilon)j} \leq C^{(2)} R^{\rho+\varepsilon} \leq C^{(3)} R^h$$

with some absolute constants $C^{(2)}, C^{(3)} \geq 1$. Fix now $z \in J_r(f)$ and $r \in (0, \theta/32)$. Let $n \geq 0$ be the largest non-negative integer such that

$$r|(f^n)'(z)| \leq \theta/32. \tag{3.7}$$

Such an integer exists since $f : J(f) \to J(f)$ is an expanding map. Then

$$r|(f^{n+1})'(z)| > \theta/32. \tag{3.8}$$

It follows from the definition of θ that the holomorphic inverse branch $f_z^{-n} : B(f^n(z), \theta) \to \mathbb{C}$ of f^n sending $f^n(z)$ to z is well-defined. Since the restriction $f|_{B(f^n(z),\theta)}$ is injective and since, by Koebe's $\frac{1}{4}$-Distortion Theorem, $f(B(f^n(z), \theta)) \subset B\big(f^{n+1}(z), \frac{1}{4}\theta|f'(f^n(z))|\big)$, we conclude that the holomorphic inverse branch $f_z^{-(n+1)} : B\big(f^{n+1}(z), \frac{1}{4}\theta|f'(f^n(z))|\big) \to \mathbb{C}$ of f^{n+1} sending $f^{n+1}(z)$ to z is well-defined. Since, by (3.7),

$$4r|(f^{n+1})'(z)| = 4r|(f^n)'(z)| \cdot |f'(f^n(z))| \leq \frac{\theta}{8}|f'(f^n(z))|,$$

applying Koebe's $\frac{1}{4}$-Distortion Theorem again, we get that

$$f_z^{-(n+1)}\big(B\big(f^{n+1}(z), 4r|(f^{n+1})'(z)|\big)\big) \subset B(z, r).$$

Noting (3.8) and applying (3.5), (3.6) along with Koebe's Distortion Theorem, we thus get that

$$\begin{aligned}\nu(B(z,r)) &\leq K^h|(f^{n+1})'(z)|^{-h}\nu\big(B\big(f^{n+1}(z), 4r|(f^{n+1})'(z)|\big)\big) \\ &\leq K^h|(f^{n+1})'(z)|^{-h}\max\{C^{(1)}, C^{(3)}(4r|(f^{n+1})'(z)|)^h = K^h\max\{C^{(1)}, C^{(3)}\}r^h.\end{aligned}$$

Thus, $\mathrm{H}_e^h(J_r(f)) > 0$ and consequently, $\mathrm{H}_\sigma^h(J_r(f)) > 0$. Since the inequality $\mathrm{H}_\sigma^h(J_r(f)) < \infty$ was established in Proposition 3.1, we are done. $\square$

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