GEOMETRY AND ERGODIC THEORY OF NON-HYPERBOLIC EXPONENTIAL MAPS

MARIUSZ URBAŃSKI AND ANNA ZDUNIK

Abstract. We deal with all the maps from the exponential family $\{\lambda e^z\}$ such that the orbit of zero escapes to infinity sufficiently fast. In particular all the parameters $\lambda \in (1/e, +\infty)$ are included. We introduce as our main technical devices the projection F_λ of the map f_λ to the infinite cylinder $Q = \mathbb{C}/2\pi i\mathbb{Z}$ and an appropriate conformal measure m. We prove that $J_r(F_\lambda)$, essentially the set of points in Q returning infinitely often to a compact region of Q disjoint from the orbit of $0 \in Q$, has the Hausdorff dimension $h_\lambda \in (1,2)$, that the h_λ-dimensional Hausdorff measure of $J_r(F_\lambda)$ is positive and finite, and that the h_λ-dimensional packing measure is locally infinite at each point of $J_r(F_\lambda)$. We also prove the existence and uniqueness of a Borel probability F_λ-invariant ergodic measure equivalent to the conformal measure m. As a byproduct of the main course of our considerations, we reprove the result obtained independently by Lyubich and Rees that the ω-limit set (under f_λ) of Lebesgue almost every point in $\mathbb{C}$, coincides with the orbit of zero under the map f_λ. Finally we show that the the function $\lambda \mapsto h_\lambda$, $\lambda \in (1/e, +\infty)$, is continuous.

1. **Introduction**

Let $f_\lambda = \lambda \exp(z)$, $\lambda \in \mathbf{C}$, $\lambda \neq 0$ be a family of exponential maps. In this paper we deal with a set of parameters λ for which the trajectory of the singular value 0 tends to infinity exponentially fast. More precisely, let

$$\beta_n = f_\lambda^n(0),\ \ \alpha_n = \mathrm{Re}\beta_n,$$

We say that the parameter λ is super-growing if $\alpha_n \to +\infty$ and there exists a constant $c > 0$ such that for all n large enough

$$\alpha_{n+1} \geq ce^{\alpha_n} = \frac{c}{|\lambda|}|\beta_{n+1}| \tag{1.1}$$

Notice that this implies

$$|\beta_{n+1}| \geq |\lambda| \exp\left(\frac{c}{|\lambda|}|\beta_n|\right) \tag{1.2}$$

It is known that for these parameters $J(f_\lambda) = \mathbf{C}$, moreover it follows from [We] that the Hausdorff dimension of the set of super-growing parameters equals 2.

The research of the first author was supported in part by the NSF Grant DMS 0100078. The research of the second author was supported in part by the Polish KBN Grant 2 PO3A 034 25. The research of both authors was supported in part by the NSF/PAN grant INT-0306004.

In the papers [UZ1] and [UZ2] we have dealt with geometry and dynamics of the set J_r of points non-escaping to infinity under the iteration of a hyperbolic exponential map. In this paper we go beyond hyperbolicity, allowing in particular the singular value 0 to belong to the Julia set. We consider the projection F_λ of the map f_λ to the infinite cylinder $Q = \mathbb{C}/2\pi i\mathbb{Z}$, and we define $J_r(F_\lambda)$, essentially the set of points in Q returning infinitely often to a compact region of Q disjoint from the orbit of $0 \in Q$. This set turns out to carry on a reach geometric structure and intriguing dynamics. Its Hausdorff dimension h_λ lies strictly between 1 and 2, its h_λ-dimensional Hausdorff measure is positive and finite and its h_λ-dimensional packing measure is locally infinite at each point of $J_r(f_\lambda)$. The former fact, interesting itself, provides also a transparent geometric interpretation of the h_λ-conformal measure, the object defined by purely dynamical means. The latter fact is more interesting than it could seem at the first look. The reason is that it forms the main ingredients in the proofs of the following two results. That $\mathrm{HD}(J_r(f_\lambda)) < 2$ and that, consequently, the ω-limit set (under f_λ) of Lebesgue almost every point in $\mathbb{C}$, coincides with the orbit of zero under the map f_λ. So, as a byproduct of the main course of our considerations, we have reproved the celebrated result of M. Lyubich and M. Rees (see [Lyu], [Re]). In the last section we show that the function $\lambda \mapsto h_\lambda$, $\lambda \in (1/e, +\infty)$, is continuous. We also study the metric dynamics of the map F_λ, and starting of with M. Martens' approach (see [Ma]) and using a very useful old result of Hayman (see [Ha]) we prove the existence and uniqueness of a Borel probability F_λ-invariant ergodic measure equivalent to the conformal measure m, or equivalently to the Hausdorff measure $H^h|_{J_r(f_\lambda)}$. The just mentioned conformal measure m forms the basic tool to exhibit both geometrical and dynamical features of the set $J_r(f_\lambda)$. Already proving its existence (via tightness) requires new ideas and careful estimates. Other results described in this introduction also require very technical considerations and fresh methods.

In what follows, we shall frequently use the Koebe distortion theorem: given $r < 1$ there exists a constant K_r such that for every univalent function f defined in $B(0,1)$ and for every $x, y \in B(0,r)$ we have $\left|\frac{f'(x)}{f'(y)}\right| \le K_r$. We shall denote by K the Koebe constant $K_{\frac{1}{2}}$.
We shall use the notation $a \preceq b$ to compare the variables a and b; $a \preceq b$ if there is a constant C such that $a \le Cb$.

2. Preliminaries

From now on throughout the entire paper we fix a super-growing parameter $\lambda \in \mathbb{C} \setminus \{0\}$, and we denote the map $f_\lambda : \mathbb{C} \to \mathbb{C}$ by f. We define the equivalence relation $\sim$ on $\mathbb{C} \times \mathbb{C}$ by saying that $z \sim w$ if and only if there exists $k \in \mathbb{Z}$ such that $z - w = 2\pi i k$. We denote the quotient space $\mathbb{C}/\sim$ by Q, which is an infinite cylinder, and by $\pi : \mathbb{C} \to Q$ we denote the corresponding quotient map, i.e. $\pi(z)$ is the equivalence class of z with respect to the equivalence relation $\sim$. Since the maps $f : \mathbb{C} \to \mathbb{C}$ and $\pi \circ f : \mathbb{C} \to Q$ are constant on equivalence classes of the relation $\sim$, they canonically induce respective conformal maps

$$f : Q \to \mathbb{C} \quad \text{and} \quad F : Q \to Q.$$

Definition 2.1. *For every $n \geq 0$ we put*

$$\beta_n = f^n(0),\ \alpha_n = \mathrm{Re}\beta_n,\ \beta_n^\infty = \{\beta_k : k \geq n\},\ \hat{\beta}_n = \pi(\beta_n),\ \text{and } \hat{\beta}_n^\infty = \{\hat{\beta}_k : k \geq n\}.$$

Fix $M > 0$ and consider two sets

$$Q_M = \{z \in Q : |\mathrm{Re}z| \leq M\}$$

and

$$J_M = \{z \in Q : F^n(z) \in Q_M \text{ for all } n \geq 0\} = \bigcap_{n\geq 0} F^{-n}(Q_M).$$

Obviously the set J_M is compact and forward invariant under F. Since λ is super-growing, $\beta_n \to \infty$. Thus $0 \notin J_M$ and

$$\delta_M = \frac{1}{2}\mathrm{dist}(J_M, \hat{\beta}_0^\infty) > 0.$$

Since $\delta_M > 0$, for every $z \in J_M$ and every $n \geq 1$, all the holomorphic inverse branches of F^{-n} are well-defined on the ball $B(z, 2\delta_M)$. We shall prove the following.

Lemma 2.2. *For every $M > 0$ there exists $n_M \geq 1$ such that $|(F^k)'(x)| \geq 2$ for all $x \in J_M$ and all $k \geq n_M$.*

Proof. Suppose on the contrary that there exist a sequence $\{x_i\}_{i=1}^\infty \subset J_M$ and $\{n_i\}_{i=1}^\infty$, an unbounded increasing sequence of positive integers such that

$$|(F^{n_i})'(x_i)| \leq 2. \tag{2.1}$$

Consider inverse branches $F_i^{-n_i} : B(F^{n_i}(x_i), 2\delta_M) \to Q$ of F^{n_i} sending $F^{n_i}(x_i)$ to x_i. Let $y \in Q$ be an accumulation point of the sequence $\{F^{n_i}(x_i)\}_{i=1}^\infty$. Passing to a subsequence, we may assume that $|y - F^{n_i}(x_i)| < \delta_M/2$ for all $i \geq 1$. Then all the inverse branches $F_i^{-n_i}$ are well-defined on $B(y, 3\delta_M/2)$, and applying Koebe's distortion theorem, it follows from (2.1) that $|(F_i^{-n_i})'(y)| \geq \kappa$ for some $\kappa > 0$ and all $i \geq 1$. Applying now $\frac{1}{4}$-Koebe distortion theorem we see that there exists a non-empty open set $B \subset Q$ such that $F_*^{-n_i}(B(y, 3\delta_M/2)) \supset B$ for all $i \geq 1$. Hence $F^{n_i}(B) \subset B(y, 3\delta_M/2)$, which means that $f^{n_i}(B) \subset \bigcup_{n=-\infty}^{+\infty}(B(y, 3\delta_M/2) + 2k\pi i)$ (recall that $F^n = \pi \circ f^n \circ \pi^{-1}$). In particular the family $\{f^{n_i} : B \to \mathbb{C}\}$ is normal. This is a contradiction with the fact that $J(f) = \mathbb{C}$ and we are done. ■

3. The existence of a conformal measure

Given $t \geq 0$ a Borel probability measure m on Q is said to be t-conformal (or: conformal with exponent t) if and only if

$$m(F(A)) = \int_A |F'|^t dm \tag{3.1}$$

for every Borel set $A \subset Q$ such that $F|_A$ is 1-to-1. Our main goal in this section is to prove the existence of a conformal measure. Obviously, one conformal (but infinite) measure already exists; this is simply the Lebesgue measure. We shall construct another measure, which will be finite and conformal with an exponent smaller than 2. First, following [DU1], for every $M > 0$ we shall build a probability Borel measure m_M, with support contained in J_M, which will be "almost conformal" for some $t_M \geq 0$, i.e.

$$m_M(F(A)) \geq \int_A |F'|^{t_M} dm_M \tag{3.2}$$

for every Borel set $A \subset Q$ such that $F|_A$ is 1-to-1 and (3.1) holds if we assume in addition that $A \cap \{z \in Q : |\mathrm{Re} z| \geq M\} = \emptyset$. In the sequel, we shall need to refer to some details of the construction, so we make it more specific now. So, let δ_M be defined as above. For every $M > 0$ we choose a collection of points $E^M = \{x_1, \ldots, x_{q_M}\} \subset J_M$ such that the balls $B(x_i, \delta_M)$ cover the set J_M. Consider the function

$$c_M(t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x \in E^M} \sum_{w \in F^{-n}_{|J_M} x} |(F^n)'|^{-t}(w)$$

(Notice that the summation is take over only those preimages of x which are in J_M.) The function $t \mapsto c_M(t)$ has three important properties. First, notice that it follows from Hölder inequality that it is convex in $I\!R$, so it is continuous. Next, it follows easily from Lemma 2.2 that it is strictly decreasing. Finally, it follows from [UZ1], Theorem 2.1 that one construct an expanding Cantor repeller whose limit set X is invariant under F^2, contained in J_M for M large, and $c_M(0) \geq \frac{1}{2} h_{top}(F^2_{|X}) > 0$. Thus,we conclude that there exists a unique value $t = t_M$ such that $c_M(t_M) = 0$. Following the general construction in [DU1] (see also [PU], Chapter 10), and using the sets $E_n = F^{-n}_{|J_M}(E^M)$ we construct the measure m_M, for which $m_M(J_M) = 1$ and which is "almost conformal" with exponent t_M.

We start with the following.

Lemma 3.1. *It holds* $\mathrm{HD}(J_M) \geq t_M$.

Proof. Fix a point $x \in J_M$ and an integer $n \geq 1$. Let $F_x^{-n} : B(F^n(x), 2\delta_M) \to Q$ be the holomorphic inverse branch of F^n sending $F^n(x)$ to x. Applying now $\frac{1}{4}$-Koebe distortion theorem and the standard Koebe distortion theorem, it follows from (3.2) that

$$\begin{aligned} m_M\left(B\Big(x, \frac{1}{4}|(F^n)'(x)|^{-1}\delta_M\Big)\right) &\leq m_M\Big(F_x^{-n}(B(F^n(x), \delta_M))\Big) \\ &\leq K^{t_M} |(F^n)'(x)|^{-t_M} m_M(B(F^n(x), \delta_M)) \\ &\leq (4K\delta_M^{-1})^{t_M} \left(\frac{1}{4}\delta_M |(F^n)'(x)|^{-1}\right)^{t_M}. \end{aligned} \tag{3.3}$$

Since, by Lemma 2.2, $\lim_{n\to\infty} |(F^n)'(x)| = \infty$ uniformly in J_M, we conclude that for every $r > 0$ small enough there exists $n \geq 1$ such that

$$\frac{1}{4}\delta_M |(F^{n+1})'(x)|^{-1} \leq r \leq \frac{1}{4}\delta_M |(F^n)'(x)|^{-1}.$$

Using (3.3), we therefore get

$$
\begin{aligned}
m_M(B(x,r)) &\le m_M\Big(B(x,\frac{1}{4}\delta_M|(F^n)'(x)|^{-1})\Big)\\
&\le (4K\delta_M^{-1})^{t_M}(4^{-1}\delta_M|(F^{n+1})'(x)|^{-1})^{t_M}|F'(F^n(x))|^{t_M} \le \Big[(4KT\delta_M^{-1})^{t_M}\Big] r^{t_M},
\end{aligned}
$$

where $T=\sup\{|F'(y)|: y\in J_M\}$ is finite since J_M is bounded. This inequality implies in a standard way that $\mathrm{HD}(J_M)\ge t_M$ (see e.g [PU]). ■

Lemma 3.2. *For every M large enough there exists $p_0>0$ such that for all $p>p_0$ $\mathrm{HD}(J_M)\le t_{M+p}$.*

Proof. It easily follows from Lemma 2.2 and the absence of critical points of F in Q that

$$L=\inf\{|(F^n)'(w)|: w\in J_M, n\ge 1\}>0 \quad\text{and}\quad \lim_{n\to\infty}|(F^n)'(z)|=\infty$$

for all $z\in J_M$. Let us fix $p>0$ so that $KL^{-1}<p$ and let us consider the set J_{M+p}. Following the construction described above, we choose a finite collection of points $E^{M+p}\subset J_{M+p}$ such that the balls $B(x,\delta_{M+p}), x\in E^{M+p}$ cover the set J_{M+p}. Let $y\in J_M\subset J_{M+p}$. Given $n\ge 0$ there exists $x\in E^{M+p}$ such that $F^n(y)\in B(x,\delta_{M+p})$. By our definition of δ_{M+p}, all holomorphic branches F^{-i} are defined in $B(x,2\delta_{M+p})$. Let F_y^{-i} be the branch sending the point $F^n(y)$ to $F^{n-i}(y)$. Then, by Koebe distortion theorem, for all $z\in B(x,\delta_{M+p})$ we get

$$\frac{|(F_y^{-i})'(z)|}{|(F_y^{-i})'(F^n(y))|}<K.$$

So, $|(F_y^{-i}(z))'|\le K|(F^{-i})'(F^n(y))|\le KL^{-1}$ since $F^{(n-i)}(y)\in J_M$. Thus, by integrating, we conclude that $d(F_y^{-i}(x),F^{n-i}(y))<KL^{-1}<p$ and, finally, $F_y^{-i}(x)\in Q_{M+p}$ for all $0\le i\le n$ (since $F^{n-i}(y)\in Q_M$). This implies that the point $w=F_y^{-n}(x)$ belongs to the set J_{M+p}, i.e. $w\in F^{-n}_{|J_{M+p}}(\{x\})$. Let $\mathcal{F}_n(x)$ be the collection of branches F_ν^{-n} on $B(x,\delta_{M+p})$, satisfying $F_\nu^{-n}(x)\in J_{M+p}$. It follows from the above considerations that

$$J_M\subset\bigcup_{x\in E^{M+p}}\bigcup_{\nu\in\mathcal{F}_n(x)}F_\nu^{-n}(B(x,\delta_{M+p})).$$

Moreover, $\mathrm{diam}(F_\nu^{-n}(B(x,\delta_{M+p}))\to 0$ uniformly as $n\to\infty$ since $F_{|J_{M+p}}$ is expanding and

$$\sum_{x\in E_{M+p}}\sum_{\nu\in\mathcal{F}_n(x)}\Big(\mathrm{diam}(F_\nu^{-n}(B(x,\delta_{M+p}))\Big)^t \preceq \sum_{x\in E^{M+p}}\sum_{w\in F^{-n}_{|J_{M+p}}(x)}\frac{1}{|(F^n)'(w)|^t} \tag{3.4}$$

Fix now an arbitrary $t>t_{M+p}$. Then $c_{M+p}(t)<0$, so there exists $\varepsilon>0$ such that

$$\sum_{x\in E^{M+p}}\sum_{w\in F^{-n}_{|J_{M+P}}(x)}\frac{1}{|(F^n)'(w)|^t}<\exp(-n\varepsilon)$$

for all n large enough. Using (3.4) we conclude that $\mathrm{H}_t(J_M)=0$ for all $t>t_{M+p}$ and, consequently, $\mathrm{HD}(J_M)\le t_{M+p}$. ■

Corollary 3.3. *There exists $s > 1$ such that $t_M > s$ for all M large enough.*

Proof. It follows from Theorem 2.1 in [UZ1] that $HD(J_M) > 1$ for all M large enough. Fix one such M. Choose p as in the preceding lemma. Then $t_{M+q} \geq HD(J_M) = s > 1$ for all $q \geq p$ and we are done. ■

Given $M > 0$ we set

$$Y_M^+ = \{z \in Q : \mathrm{Re} z > M\},\ Y_M = \{z \in Q : |\mathrm{Re} z| > M\},\ Y_M^- = \{z \in Q : \mathrm{Re} z < -M\}.$$

The main technical result of this section is the following.

Lemma 3.4. *The sequence of measures $\{m_n\}_{n=1}^\infty$ is tight on Q.*

Proof. We are to check that for every $\varepsilon > 0$ there exists $M > 0$ such that for all n $m_n(Y_M) < \varepsilon$. We first estimate from above $m_n(Y_M^+)$ in essentially the same way as in [UZ1]. For the needs of the proof of Theorem 3.8 we shall establish a slightly more general result. Fix a Borel set $G \subset Q$. This set is mapped one to one by f onto some set in $\mathbb{C}$. But F is no longer one-to-one in G since two points in the image are identified if they differ by $2k\pi i$ for some $k \in \mathbb{Z}$. We have

$$m_n(G \cap Y_M^+) = m_n\Big(\{z \in G \cap Y_M^+ : F(z) \in Y_{\exp(M/2)}\}\Big) + m_n\Big(\{z \in G \cap Y_M^+ : F(z) \in Q_{\exp(M/2)}\}\Big).$$

To estimate the first summand, let us write $\{\mathrm{Re} z > \exp(\frac{M}{2})\} \cap f(G) = \bigcup S_k$ where $S_k = \{\mathrm{Re} z > \exp(\frac{M}{2})\} \cap f(G) \cap \{2k\pi \leq \mathrm{Im} z < 2(k+1)\pi\}$. Then the map F is one-to-one on each set $G \cap f^{-1}(S_k)$ and the derivative $|F'|$ on this set can be estimated from below by $\inf_{w \in S_k} |w| \geq |\exp(\frac{M}{2}) + 2k\pi i|$ if k is nonnegative and by $|\exp(\frac{M}{2}) + 2|(k+1)|\pi i|$ if k is negative. Thus,

$$\begin{aligned}
m_n\Big(F^{-1}(Y_{\exp(M/2)} \cap F(G))\Big) &\leq m_n\Big(Y_{\exp(M/2)} \cap F(G)\Big) \cdot 2\sum_{k=0}^{+\infty} |\exp(M/2) + 2k\pi i|^{-t_n} \\
&\leq m_n(F(G)) \cdot 2 \sum_{k=0}^{k=+\infty} |\exp(M/2) + 2k\pi i|^{-t_n} \\
&\preceq m_n(F(G)) \int_{\exp(M/2)}^{+\infty} x^{-t_n}\,dx \preceq (\exp(M/2))^{1-t_n} m_n(F(G)) \\
&\preceq (\exp(M/2))^{1-s} m_n(F(G)),
\end{aligned} \tag{3.5}$$

where $s > 1$ is the number produced in Corollary 3.3. In order to estimate the second summand put

$$A = \{z \in Y_M^+ \cap G : F(z) \in Q_{\exp(M/2)}\}.$$

Let

$$Z_A = \{k \in \mathbb{Z} : f(A) \cap \{z \in \mathbb{C} : 2k\pi \leq \mathrm{Im} z < 2(k+1)\pi\} \neq \emptyset\}.$$

Now, if $z \in A$ then $\mathrm{Re}z > M$ and therefore $|f(z)| = |\lambda|e^{\mathrm{Re}z} \geq |\lambda|e^M$. Hence, if $k \in Z_A$, then

$$|\lambda|^2e^{2M} \leq |f(z)|^2 \leq (\exp(M/2))^2 + 4\pi^2 \max\{|k+1|^2, |k-1|^2\}$$

and therefore

$$\Big(\max\{|k+1|, |k-1|\}\Big)^2 \geq \frac{1}{4\pi^2}(|\lambda|^2e^{2M} - e^M) = \frac{1}{4\pi^2}e^{2M}(|\lambda|^2 - e^{-M}) \succeq e^{2M}$$

assuming that M is large enough. Thus $\max\{|k+1|, |k-1|\} \succeq e^M$, and in consequence $|k| \succeq e^M$. Hence

$$m_n(A) \leq m_n(F(A)) \sum_{|k|\succeq e^M} |k|^{-t_n} \preceq e^{M(1-s)}m_n(F(A)) \preceq e^{M(1-s)}m_n(F(G)).$$

Combining this and (3.5), we get

$$m_n(Y_M^+ \cap G) \leq \exp\left(\frac{1}{2}M(1-s)\right) m_n(F(G)) \leq \exp\left(\frac{1}{2}M(1-s)\right). \tag{3.6}$$

In particular,

$$m_n(Y_M^+) \leq \exp\left(\frac{1}{2}M(1-s)\right). \tag{3.7}$$

We shall now estimate $m_n(Y_M^-)$. This will be more complicated, since the set Y_M^- is mapped by f onto the ball $B(0, |\lambda|\exp(-M))$ and $|F'(z)| = |F(z)|$. This means that even if we bound the measure m_n of $B\Big(0, |\lambda|\exp(-M)\Big)$ by the radius $\exp(-M)$ erased to power t_n, this will not by enough to conclude that $m_n(Y_M^-)$ is small. But, actually, due to our super-growing condition, the measure m_n of the ball $B(0, |\lambda|\exp(-M))$ is much smaller that $\exp(-t_nM)$ and we shall estimate it carefully. Keep the same set G. It follows from (1.2) that for every $k \geq 0$

$$|\beta_{k+1}| = |\lambda e^{\beta_k}| = |\lambda|e^{\alpha_k} \geq |\lambda| \exp\left(\frac{c}{|\lambda|}|\beta_k|\right). \tag{3.8}$$

Consider now the balls $B_k = B(\hat{\beta}_k, |\beta_{k-1}|^{-1})$. The for all $k \geq 1$ large enough $2B_k$ is a topological disc and it follows from Koebe's distortion theorem followed by (1.1) that for all $k \geq 1$ large enough

$$\begin{aligned} f(B_k \cap G) &\subset B(\beta_{k+1}, K|\beta_{k+1}||\beta_{k-1}|^{-1}) \cap f(G) \\ &\subset \{z \in \mathbb{C} : \mathrm{Re}z \geq \alpha_{k+1} - K|\beta_{k+1}||\beta_{k-1}|^{-1}\} \cap f(G) \\ &\subset \{z \in \mathbb{C} : \mathrm{Re}z \geq \alpha_{k+1}(1 - K|\lambda|c^{-1}|\beta_{k-1}|^{-1}\} \cap f(G) \\ &\subset \{z \in \mathbb{C} : \mathrm{Re}z > \alpha_{k+1}/2\} \cap f(G). \end{aligned}$$

The map $F_{|B_k\cap G}$ is no longer one -to-one but, since the hight of the image $f(B_k)$ is bounded from above by $K|\beta_{k+1}||\beta_{k-1}|^{-1}$ every point in $F_{|B_k\cap G}$ has at most $\frac{1}{2\pi}K|\beta_{k+1}||\beta_{k-1}|^{-1}$ preimages in $B_k \cap G$. Using (3.6), we therefore get

$$\begin{aligned} m_n(B_k \cap G) &\le m_n\left(Y^+_{\alpha_{k+1}/2} \cap F(G)\right)\left(\inf\{|f'|_{|B_k}\}\right)^{-t_n} K|\beta_{k+1}||\beta_{k-1}|^{-1} \\ &\preceq \exp\left(\frac{\alpha_{k+1}}{4}(1-s)\right)\left(\inf\{|f'|_{|B_k}\}\right)^{-t_n}|\beta_{k+1}||\beta_{k-1}|^{-1} m_n(F^2(G)). \end{aligned}$$

Since $f(B_k) \subset B(\beta_{k+1}, K|\beta_{k+1}||\beta_{k-1}|^{-1})$, using (3.8) and Koebe's distortion theorem, we conclude that for all k large enough $\inf\{|f'|_{|B_k}\} \succeq |\beta_{k+1}|$. Therefore, using the "equality" part of (3.8), we obtain the following

$$\begin{aligned} m_n(B_k \cap G) &\preceq \exp\left(\frac{\alpha_{k+1}}{4}(1-s)\right)|\beta_{k+1}|^{-t_n}|\beta_{k+1}||\beta_{k-1}|^{-1} m_n(F^2(G)) \\ &\preceq |\beta_{k+1}|^{1-s}|\beta_{k-1}|^{-1}\left(\frac{|\beta_{k+2}|}{|\lambda|}\right)^{\frac{1-s}{4}} m_n(F^2(G)) \preceq |\beta_{k+2}|^{-\kappa} m_n(F^2(G)) \end{aligned} \tag{3.9}$$

for $\kappa = (s-1)/4$. We now consider the holomorphic inverse branch $F_0^{-k} : 2B_k \to Q$ sending $\hat\beta_k = F^k(\hat 0)$ to $\hat 0$. It follows from Koebe's distortion theorem that

$$K^{-1}(|\beta_1|\cdot|\beta_2|\cdot\ldots|\beta_k|)^{-1} \le |(F_0^{-k})'(z)| \le K(|\beta_1|\cdot|\beta_2|\cdot\ldots|\beta_k|)^{-1} \tag{3.10}$$

for all $z \in B_k$, and

$$\tilde B_k = F_0^{-k}(B_k) \subset B\Big(\hat 0, K(|\beta_1|\cdot|\beta_2|\cdot\ldots|\beta_k|)^{-1}|\beta_{k-1}|^{-1}\Big). \tag{3.11}$$

Applying in turn Koebe's $\frac14$-distortion theorem, we get

$$\tilde B_k \supset B\Big(\hat 0, 4^{-1}(|\beta_1|\cdot|\beta_2|\cdot\ldots|\beta_k|)^{-1}|\beta_{k-1}|^{-1}\Big). \tag{3.12}$$

Using (3.10) and (3.9), we obtain

$$\begin{aligned} m_n\Big(F_0^{-k}(B_k \cap G)\Big) &\le K^{t_n}(|\beta_1|\cdot|\beta_2|\cdot\ldots|\beta_k|)^{-t_n} m_n(B_k \cap G) \\ &\preceq (|\beta_1|\cdot|\beta_2|\cdot\ldots|\beta_k|)^{-s} m_n(B_k \cap G) \le |\beta_{k+2}|^{-\kappa} m_n(F^2(G)). \end{aligned} \tag{3.13}$$

Notice that, in particular, in this way we get the estimate of the measure of $\tilde B_k$ by $|\beta_{k+2}|^{-\kappa}$. Looking now at (3.11) and (3.12) with k replaced by $k+1$ we conclude for all k large enough

$$\mathrm{cl}\tilde B_{k+1} \subset \tilde B_k$$

Let W_k be the unbounded connected component of $F^{-1}(\tilde B_k)$ and let

$$V_k = W_k \setminus \overline{W_{k+1}}.$$

In view of (3.12) with k replaced by $k+1$, in view of (3.13), and in view of Lemma 3.1, we can estimate as follows.

$$
\begin{aligned}
m_n(V_k) &\le \left(\inf_{V_k}\{|f'|\}\right)^{-t_n} m_n\left(\tilde{B}_k \setminus \tilde{B}_{k+1}\right) \\
&\le \left(4|\lambda|(|\beta_1|\cdot|\beta_2|\cdot\ldots|\beta_k||\beta_{k+1}|)|\beta_k|\right)^{t_n} m_n\left(\tilde{B}_k\right) \\
&\preceq \left(4|\lambda|(|\beta_1|\cdot|\beta_2|\cdot\ldots|\beta_k||\beta_{k+1}|)|\beta_k|\right)^{2} |\beta_{k+2}|^{-\kappa} \\
&\preceq |\beta_{k+2}|^{-\gamma}
\end{aligned}
\tag{3.14}
$$

for an arbitrary $\gamma \in (0,\kappa)$ and all $k \ge 1$ sufficiently large (depending on γ). The latter follows from the following simple

Lemma 3.5. *If the sequence $(\alpha_i)_{i=1}^\infty$ satisfies $\alpha_i \to \infty$ and $\alpha_{n+1} > c\exp\alpha_n$ for some positive c then for every $\varepsilon > 0$ there exists n_0 such that for every $n > n_0$, $\alpha_1 + \cdots + \alpha_n < \varepsilon\alpha_{n+1}$*

For every $M > 0$ let $l(M) \ge 1$ be the largest integer such that

$$Y_M^- \subset \bigcup_{k=l(M)}^{\infty} V_k.$$

Since $\lim_{M\to+\infty} l(M) = +\infty$, it follows from (3.14) that for all $n \ge 1$ and all $M > 0$ sufficiently large

$$m_n(Y_M^-) \le \sum_{k=l(M)}^{\infty} |\beta_{k+2}|^{-\gamma} \longrightarrow 0 \ \text{ as } M \to +\infty.$$

The proof of tightness is finished. ■

Since, in view of Lemma 3.1, $t_n \in [0,2]$, we can choose a subsequence $\{n_k\}_{k=1}^\infty$ such that $\{t_{n_k}\}_{k=1}^\infty$ converges. Denote its limit by h. It follows from Lemma 3.4 and Prokhorov's theorem that passing yet to another subsequence, we may assume that the sequence $\{m_{n_k}\}_{k=1}^\infty$ converges weakly, say to a measure m on Q. Since there is a problem with conformality of measures m_{n_k} only on sets $\{z \in Q : |\mathrm{Re} z| = n_k\}$ (cf. the paragraph preceding Lemma 3.1), since $n_k \nearrow +\infty$ when $k \nearrow +\infty$, and since $F : Q \to Q$ is an open map, which has no critical points, proceeding, with obvious modifications, as in [DU1] (comp. [UZ1]), we obtain the following first basic result.

Theorem 3.6. *The weak-limit measure m is h-conformal for the map $F : Q \to Q$.*

Let us now introduce the main set $J_r = J_r(F)$ we will be dealing with throughout the rest of this paper.

Definition 3.7. *$J_r(f) \subset \mathbf{C}$ is the set of those points $z \in \mathbf{C}$ for which there exists an unbounded sequence $\{n_k(z)\}_{k=1}^\infty$ such that*

$$\mathrm{dist}\Big(\{f^{n_k(z)}(z)\}_{k=1}^\infty, \beta_0^\infty\Big) > 0$$

and the set $\mathrm{Re}\{f^{n_k(z)}(z)\}_{k=1}^\infty$ is bounded. The set $J_r = J_r(F) \subset Q$ is defined to be $\pi(J_r(f))$.

Given $M > 0$ we define the set $J_{r,M}$ to consist of those points $z \in Q$ for which there exists an unbounded sequence $\{n_k(z)\}_{k=1}^\infty$ such that

$$\mathrm{dist}\Big(\{f^{n_k(z)}(z)\}_{k=1}^\infty, \beta_0^\infty\Big) > 0$$

and $\{F^{n_k(z)}(z)\}_{k=1}^\infty \subset Q_M$. Obviously, $J_r = \bigcup J_{r,M}$. We shall now prove our second basic result.

Theorem 3.8. *If m is a t-conformal probability measure for $F : Q \to Q$ with $t > 1$, then $m(J_r) = 1$. Even more, there exists $M > 0$ such that $m(J_{r,M}) = 1$.*

Proof. Fix $M > 0$ and define

$$E_M = \Big\{x \in Y_M : \forall_{k \ge 0}\Big(F^k(x) \in Y_M^+ \Rightarrow F^{k+1}(x) \in Y_M\Big)\Big\}.$$

We shall show first that there exists $M > 0$ arbitrarily large such that $m(E_M) = 0$. Notice that $F(Y_M^+ \cap E_M) \subset E_M$ and therefore $Y_M^+ \cap E_M \subset F^{-1}(E_M)$. Thus the same argument as in formula (3.5) gives us the following.

$$\begin{aligned} m(E_M \cap Y_M^+) \le m(F^{-1}(E_M)) &\preceq \sum_{k=-\infty}^{+\infty} |M + 2k\pi i|^{-t} m(Y_M \cap E_M) \\ &\preceq M^{1-t} m(Y_M \cap E_M) \le \frac{1}{4} m(E_M) \end{aligned} \tag{3.15}$$

for all $M > 0$ large enough. It remains to show that there exists an arbitrarily large M such that

$$m(E_M \cap Y_M^-) \le \frac{1}{4} m(E_M).$$

This task requires a much more involved reasoning. Again, as in the proof of tightness, the difficulty is caused by the fact that the set $E_M \cap Y_M^-$ is mapped by F into a neighbourhood of 0 with small derivative. Thus, we shall need to estimate carefully the measure $m(F(E_M \cap Y_M^-))$. For every $n \ge 2$ put

$$M_n = \log 4 + \sum_{j=1}^n \log|\beta_j| + \log|\beta_{n-1}| = \log 4 + (n+1)\log|\lambda| + \sum_{j=0}^{n-1} \alpha_j + \alpha_{n-2}.$$

Similarly as (3.9) we can show that

$$m(B_n \cap G) \preceq |\beta_{n+2}|^{-\kappa} m(F^2(G)) \tag{3.16}$$

for $\kappa = \frac{t-1}{4}$, for all Borel sets $G \subset Q$ and all n (independent of G) large enough. We shall now check that for all $n \geq 2$ large enough

$$F\Big(Y_{M_n}^-\Big) \subset \tilde{B}_n \quad \text{and} \quad F^{n+1}\Big(Y_{M_n}^-\Big) \subset Y_{M_n}^+. \tag{3.17}$$

Indeed,

$$F\Big(Y_{M_n}^-\Big) \subset B\Big(\hat{0}, e^{-M_n}\Big) = B\left(\hat{0}, \frac{1}{4}\prod_{j=1}^{n} |\beta_j|^{-1}|\beta_{n-1}|^{-1}\right)$$

and applying (3.12) we see that the first inclusion in the formula (3.17) is satisfied. Also

$$F^n(\tilde{B}_n) = B_n \subset Y_{\alpha_n - |\beta_{n-1}|^{-1}} \subset Y_{M_n}^+$$

for all $n \geq 2$ large enough (again, this follows easily from Lemma 3.5). Combining this and the first inclusion in (3.17) we see that the second inclusion in (3.17) is also satisfied. Fix now $k \geq 2$ so large that (3.16) and (3.17) hold for all $n \geq k$. It follows from the second inclusion in (3.17) and from the definition of E_{M_k} that

$$F^{n+1}(E_{M_k} \cap Y_{M_n}^-) \subset E_{M_k} \cap Y_{M_n}^+. \tag{3.18}$$

For every $n \geq k$ put

$$Z_n = Y_{M_n}^- \setminus Y_{M_{n+1}}^-.$$

It then follows from the first inclusion in (3.17) and from (3.18) that

$$\begin{aligned}
m\Big(E_{M_k} \cap Z_n\Big) &\leq (\inf\{|F'(z)|^t : z \in E_{M_k} \cap Z_n\})^{-1} m\Big(F\Big(E_{M_k} \cap Z_n\Big)\Big) \\
&\leq (\inf\{|F'(z)|^t : z \in E_{M_k} \cap Z_n\})^{-1} m\Big(F\Big(E_{M_k} \cap Y_{M_n}^-\Big) \\
&\leq \exp(M_{n+1}t) m\Big(\tilde{B}_n \cap F^{-n}(E_{M_k})\Big) \\
&= \left(4\prod_{j=1}^{n+1} |\beta_j| \cdot |\beta_n|\right)^t \cdot m\Big(\tilde{B}_n \cap F^{-n}(E_{M_k})\Big)
\end{aligned}$$

Notice that the estimate (3.13) (which was established for measures m_n) is also valid for an arbitrary t-conformale measure t (with $\kappa = \frac{t-1}{4}$). Applying (3.13) with m_n replaced by m, k replaced by n, and with $G = B_n \cap E_{M_k}$, we get

$$m(\tilde{B}_n \cap F_0^{-n}(E_{M_k})) = m\Big(F_0^{-n}\Big(B_n \cap E_{M_k}\Big)\Big) \leq |\beta_{n+2}|^{-\kappa} m\Big(F^2\Big(B_n \cap E_{M_k}\Big)\Big),$$

and consequently

$$\begin{aligned}
m\Big(E_{M_k} \cap Z_n\Big) &\leq \left(4\prod_{j=1}^{n+1} |\beta_j| \cdot |\beta_n|\right)^t |\beta_{n+2}|^{-\kappa} m\Big(F^2\Big(B_n \cap E_{M_k}\Big)\Big) \\
&\leq |\beta_{n+2}|^{-u} m\Big(F^2\Big(B_n \cap E_{M_k}\Big)\Big)
\end{aligned} \tag{3.19}$$

for $u = \frac{\kappa}{2}$ and for all n large enough. The latter easily follows again from Lemma 3.5. We shall check now that $F^2(B_n \cap E_{M_k}) \subset E_{M_k}$. Indeed, by Koebe's distortion theorem and (1.1) we get

$$F(B_n) \subset B(\hat{\beta}_{n+1}, K|\lambda||\beta_{n+1}| \cdot |\beta_{n-1}|^{-1}) \subset Y_{\alpha_{n+1}-K|\lambda||\beta_{n+1}|\cdot|\beta_{n-1}|^{-1}} \subset Y^+_{M_n}$$

Therefore $F\Big(B_n \cap E_{M_k}\Big) \subset E_{M_k} \cap Y^+_{M_n}$, and consequently

$$F^2\Big(B_n \cap E_{M_k}\Big) \subset F\Big(E_{M_k} \cap Y^+_{M_n}\Big) \subset Y_{M_k}.$$

Thus, it follows from the definition of E_{M_k} that $F^2\Big(B_n \cap E_{M_k}\Big) \subset E_{M_k}$. Using the estimate (3.19) we conclude that

$$m\Big(E_{M_k} \cap Z_n\Big) \le |\beta_{n+2}|^{-u} m\Big(\cap E_{M_k}\Big)$$

for all $k \ge 2$ large enough and all $n \ge k$. Summing these inequalities up over all $n \ge k$ and using the fact that $\bigcup_{n \ge k} Z_n = Y_{M_k}$ we obtain

$$m\Big(E_{M_k} \cap Y^-_{M_k}\Big) \le \sum_{n=k}^{\infty} |\beta_{n+2}|^{-u} m\Big(E_{M_k}\Big) \le \frac{1}{4} m\Big(E_{M_k}\Big)$$

for all $k \ge 2$ large enough. Combining this and (3.15), where $M = M_k$ (k large enough), we get $m\Big(E_{M_k}\Big) \le \frac{1}{2} m\Big(E_{M_k}\Big)$, which implies that $m\Big(E_{M_k}\Big) = 0$. Thus, by conformality of the measure m

$$m\left(\bigcup_{j=0}^{\infty} F^{-j}\Big(E_{M_k}\Big)\right) = 0.$$

In order to complete the proof, it therefore suffices to prove

Lemma 3.9.

$$Q \setminus \bigcup_{j=0}^{\infty} F^{-j}\Big(E_{M_k}\Big) \subset J_{r,M_k}.$$

Proof. Fix a point $z \in Q \setminus \bigcup_{j=0}^{\infty} F^{-j}\Big(E_{M_k}\Big)$. Then either

Case 1: There exists $j_0 = j_0(z) \ge 0$ such that $|\mathrm{Re}(F^j(z))| \le M_k$ for all $j \ge j_0$

or

Case 2: There are infinitely many $j's$ such that $\mathrm{Re}(F^j(z)) > M_k$ and $|\mathrm{Re}(F^{j+1}(z))| \le M_k$.

Consider first the Case 1, $|\mathrm{Re} f^j(z)| < M_k$ for $j > j_0$.

Since the sequence $\{\beta_n\}_{n=1}^{\infty}$ diverges to ∞, we therefore conclude that $\mathrm{dist}(\{f^n(z)\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}) > 0$. Thus $z \in J_{r,M_k}$, and we are done in this case.

Consider now the Case 2.

Fix $j \ge 1$ such that $\mathrm{Re} f^j(z) = \mathrm{Re}(F^j(z)) > M_k$ and $|\mathrm{Re} f^{j+1}(z)| = |\mathrm{Re}(F^{j+1}(z))| \le M_k$. It follows from the definition of M_k that $\alpha_0, \alpha_1, \dots, \alpha_{k-1} < M_k - 1$, and therefore

$$\pi^{-1}\Big(\{\hat{\beta}_0, \hat{\beta}_1 \dots \hat{\beta}_{k-1}\}\Big) \cap B(f^j(z), 1) = \emptyset. \tag{3.20}$$

Since $\beta_n \in Y^+_{M_{k+1}}$ for all $k \geq 2$ large enough and all $n \geq k+1$, we get also

$$f^{j+1}(z) \notin B\Big(\{\beta_n\}_{n=k+1}^{\infty}, 1\Big) \tag{3.21}$$

By Koebe's $\frac{1}{4}$-distortion theorem,

$$f(B(f^j(z),1) \supset B\Big(f^{j+1}(z), 4^{-1}|\lambda|e^{M_k}\Big).$$

On the other hand, it follows from (3.20) that $B(f^j(z),1) \cap f^{-1}\{\beta_1,\dots,\beta_n\} = \emptyset$. Thus, $f^{j+1}(z) \notin B\Big(\{\beta_n\}_{n=1}^{k}, 4^{-1}|\lambda|e^{M_k}\Big)$. Combining this and (3.21) we see that

$$f^{j+1}(z) \notin B\Big(\{\beta_n\}_{n=1}^{\infty}, \min\{1, 4^{-1}|\lambda|e^{M_k}\}\Big). \tag{3.22}$$

Obviously, we can write also

$$f^{j+1}(z) \notin B\Big(\{\beta_n\}_{n=0}^{\infty}, \min\{1, 4^{-1}|\lambda|e^{M_k}\}\Big) \tag{3.23}$$

since $|f^{j+1}(z) - \beta_0| = |f^{j+1}(z)| \geq |\lambda|e^{M_k}$. Since we are in the Case 2, $F^{j+1}(z) \in Q_{M_k}$. Thus $z \in J_{r,M_k}$ and we are done. ■

Fix now $M > 0$. For every $z \in J_{r,M}$ we fix one sequence $\{n_k(z)\}_{k=1}^{\infty}$ for which the condition in the definition of the set $J_{r,M}$ (Def. 3.7) is satisfied. Since f restricted to the ball centered at $f^{n_k(z)}(z)$ with radius $\min\{\pi, \mathrm{dist}\Big(\{f^{n_k(z)}(z)\}_{k=1}^{\infty}, \beta_0^{\infty}\Big)\}$ is univalent and since $|f'(f^{n_k(z)}(z))| = |f(f^{n_k(z)}(z))| = |\lambda|\exp(\mathrm{Re}(f^{n_k(z)}(z)) \geq |\lambda|e^{-M}$, it follows from Koebe's distortion theorem that there exists a unique holomorphic branch $f_z^{-s_k(z)} : B\Big(f^{s_k(z)}(z), 4\delta_z\Big)$ of $f^{-s_k(z)}$ sending $f^{s_k(z)}$ to z, where $s_k(z) = n_k(z) + 1$ and

$$\delta_z = \delta_z(M) = \frac{1}{16}|\lambda|e^{-M}\min\{\pi, \mathrm{dist}\Big(\{f^{n_k(z)}(z)\}_{k=1}^{\infty}, \beta_0^{\infty}\Big)\}. \tag{3.24}$$

Remark 3.10. *Note that from (3.23) one can deduce that the radius $\delta_z(M_k)$ in the above construction can be chosen to be independent of a point $z \in Q \setminus \bigcup_{j=0}^{\infty} F^{-j}(E_{M_k})$. We shall use this property in the proof of Theorem 5.2.*

Let $z \in J_{r,M}$. Then $F^{n_k(z)}(z) \in Q_M$ for all $k \geq 1$. Hence

$$|f^{s_k(z)}|(z) = |f(F^{n_k(z)})(z)| = |\lambda|\exp\Big(\mathrm{Re}(F^{n_k}(z)\Big) \leq |\lambda|e^{M}.$$

Therefore, assuming M to be large enough, we have that

$$\overline{B\Big(f^{s_k(z)}(z), 4\delta_z\Big)} \subset B(0, |\lambda|e^{2M}).$$

Definition 3.11. *Let $z \in J_r = \bigcup_M J_{r,M}$. Passing to a subsequence, we may assume that the limit $\lim_{k\to\infty} f^{s_k(z)}(z)$ exists and belongs to $B(0, |\lambda|e^{M})$ if $z \in J_{r,M}$. This limit will be denoted by $y(z)$.*

We shall prove now the following.

Lemma 3.12. *For every $M > 0$ and every $z \in J_{r,M}$*

$$\lim_{k\to\infty} |(f^{n_k(z)})'(z)| = \lim_{k\to\infty} |(f^{s_k(z)})'(z)| = +\infty.$$

Proof. The idea of this proof is the same as of the proof of Lemma 2.2. Put $s_k = s_k(z)$, $k \ge 1$. Suppose on the contrary that $\liminf_{k\to\infty} |(f^{s_k})'(z)| < +\infty$. Without loss of generality we may assume that $\tau = \lim_{k\to\infty} |(f^{s_k})'(z)| < +\infty$ and that $f^{s_k}(z) \in B(y(z), \delta_z)$ for all $k \ge 1$. Consider the family $\{f_z^{-s_k} : B(y(z), 3\delta_z) \to \mathbb{C}\}_{k\ge 1}$ of holomorphic inverse branches of f^{s_k} sending $f^{s_k}(z)$ to z. Applying $\frac{1}{4}$-Koebe's distortion theorem, we see that $f_z^{-s_k}\Big(B(y(z), 3\delta_z)\Big) \supset B(z, (8\tau)^{-1})$ for all $k \ge 1$ large enough. Thus $f^{s_k}(B(z, (8\tau)^{-1})) \subset B(y(z), 3\delta_z)$, and consequently the family of maps $f^{s_k} : B(z, (8\tau)^{-1}) \to \mathbb{C}$ is normal, which contradicts the fact that $z \in J(F)$ and shows that $\lim_{k\to\infty} |(f^{s_k})'(z)| = +\infty$. Since $|f'|$ is uniformly bounded on Q_M and since $s_k = n_k + 1$, we conclude that also $\lim_{k\to\infty} |(f^{n_k})'(z)| = +\infty$. We are done. ■

Lemma 3.13. *Let ν be an arbitrary conformal measure for F. Then for every non-empty open set $U \subset Q$, we have*

$$\limsup_{n\to\infty} \nu(F^n(U)) = 1.$$

Proof. Let $\tilde U$ be a connected component of $\pi^{-1}(U)$. Since periodic points of f are dense in $\mathbb{C}$, $\tilde U$ contains a repelling periodic point ω. Denote the period of ω by p. There then exists an open ball $W \subset \tilde U$ centered at ω such that $f^p(W) \supset W$. Since $\omega \in \mathbb{C} = J(f^p)$, $\bigcup_{n\ge 0} f^{pn}(W) \supset \mathbb{C} \setminus \{0\}$. Hence for every $n \ge 1$

$$F^{pn}(\pi(W)) = \pi(f^{pn}(W)) \supset \pi(W)$$

and

$$\bigcup_{n\ge 0} F^{pn}(\pi(W)) = \bigcup_{n\ge 0} \pi(f^{pn}(W)) = \pi\left(\bigcup_{n\ge 0} f^{pn}(W)\right) \supset \pi(\mathbb{C} \setminus \{0\}) = Q. \tag{3.25}$$

Thus

$$\lim_{n\to\infty} \nu\Big(F^{pn}(\pi(W))\Big) = \nu\left(\bigcup_{n\ge 0} F^{pn}(\pi(W))\right) = \nu(Q) = 1.$$

Since $\pi(W) \subset \pi\tilde U = U$, we are done. ■

Corollary 3.14. *If ν is a conformal measure for F and $U \subset Q$ is an arbitrary open set then $\nu(U) > 0$.*

Now, we are ready to conclude the following theorem. After the above preparation, the proof is rather standard. It follows the idea of the analogous theorem in [UZ1]. However, we present it here for the sake of completeness and since some details are different.

Theorem 3.15. *The h-conformal measure m is a unique probability t-conformal measure for $F : J(F) \to J(F)$ with $t > 1$. In addition, m is conservative and ergodic.*

Proof. For all $s, l \geq 1$ put

$$Z_{s,l} = \{z \in J_{r,s} : \delta_z(s) \geq 1/l\}$$

(see (3.24) for the definition of $\delta_z(s)$). Fix $z \in Z_{s,l}$. Recall that, by Definition 3.11, $y(z) = \lim_{k\to\infty} f^{s_k}(z)$. Without loss of generality we may assume that $|y(z) - f^{s_k(z)}(z)| < (4Kl)^{-1}$ for all $k \geq 1$. Consider the holomorphic inverse branches $F_z^{-s_k(z)} : B(\pi(y(z)), 3/l) \to Q$ sending $F^{s_k(z)}(z)$ to z. Suppose that ν is an arbitrary t-conformal measure with $t > 1$. Since, by $\frac{1}{4}$-Koebe's distortion theorem and the standard version of Koebe's distortion theorem,

$$F_z^{-s_k(z)}(B(\pi(y(z)), 3/l)) \supset F_z^{-s_k(z)}(B(F^{s_k(z)}(z), 2/l)) \supset B\left(z, \frac{1}{2l}|(F^{s_k(z)})'(z)|^{-1}\right)$$

and

$$F_z^{-s_k(z)}\left(B\left(\pi(y(z)), \frac{1}{4Kl}\right)\right) \subset F_z^{-s_k(z)}\left(B\left(F^{s_k(z)}(z), \frac{1}{2Kl}\right)\right) \subset B\left(z, \frac{1}{2l}|(F^{s_k(z)})'(z)|^{-1}\right).$$

Using the conformality of the measure ν along with the standard version of Koebe's distortion theorem, and the fact that $\inf\{\nu(B(w, (2Kl)^{-1})) : w \in B(\hat{0}, |\lambda|e^s)\} > 0$, we deduce that

$$B(\nu, l, s)^{-1} r_k(z)^t \leq \nu(B(z, r_k(z))) \leq B(\nu, l, s) r_k(z)^t, \tag{3.26}$$

where $r_k(z) = (2l)^{-1}|(F^{s_k(z)})'(z)|^{-1}$ and $B(\nu, l)$ depends only on ν l and s. Fix now E, an arbitrary bounded Borel set contained in $Z_{s,l}$. Since m is regular, for every $x \in E$ there exists a radius $r(x) > 0$ of the form $r_k(x)$ such that

$$m(\bigcup_{x\in E} B(x, r(x)) \setminus E) < \varepsilon. \tag{3.27}$$

Now by the Besicovič theorem (see [Gu]) we can choose a countable subcover $\{B(x_i, r(x_i))\}_{i=1}^\infty$, $r(x_i) \leq \varepsilon$, from the cover $\{B(x, r(x))\}_{x\in E}$ of E, of multiplicity bounded by some constant $C \geq 1$, independent of the cover. Therefore by (3.26) and (3.27), we obtain

$$\begin{aligned}
\nu(E) &\leq \sum_{i=1}^\infty \nu(B(x_i, r(x_i))) \leq B(\nu, l, s) \sum_{i=1}^\infty r(x_i)^t \\
&\leq B(\nu, l, s) B(m, l, s) \sum_{i=1}^\infty r(x_i)^{t-h} m(B(x_i, r(x_i))) \\
&\leq B(\nu, l, s) B(m, l, s) C\varepsilon^{t-h} m(\bigcup_{i=1}^\infty B(x_i, r(x_i))) \\
&\leq CB(\nu, l, s) B(m, l, s)\varepsilon^{t-h}(\epsilon + m(E)).
\end{aligned} \tag{3.28}$$

In the case when $t > h$, letting $\varepsilon \searrow 0$ we obtain $\nu(Z_{s,l}) = 0$. Since $J_r = \bigcup_{s\geq1}\bigcup_{l\geq1} Z_{s,l}$, we thus conclude that $\nu(J_r) = 0$. This contradiction shows that $t \leq h$. If $t < h$, then exchanging the role of ν and m in the above reasoning , we would get $m(J_r) = 0$. Thus $t = h$. Then

(3.28) and (3.28) with exchanged roles of measures m and ν show that the measures ν and m are equivalent.

Let us now prove that any h-conformal measure ν is ergodic. Indeed, suppose to the contrary that $F^{-1}(G) = G$ for some Borel set $G \subset J(F)$ with $0 < m(G) < 1$. But then the two conditional measures ν_G and $\nu_{J(F)\setminus G}$

$$\nu_G(B) = \frac{\nu(B \cap G)}{\nu(G)}, \quad \nu_{J(F)\setminus G}(B) = \frac{\nu(B \cap (J(F) \setminus G))}{\nu(J(f) \setminus G)}$$

would be h-conformal and mutually singular; a contradiction.

If now ν is again an arbitrary h-conformal measures, then by a simple computation based on the definition of conformal measures we see that the Radon-Nikodyn derivative $\phi = d\nu/dm$ is constant on grand orbits of F. Therefore by ergodicity of m we conclude that ϕ is constant m-almost everywhere. As both m and ν are probability measures, it implies that $\phi = 1$ a.e., hence $\nu = m$.

It remains to show that m is conservative. We shall prove first that every forward invariant ($F(E) \subset E$) subset E of $J(F)$ is either of measure 0 or 1. Indeed, suppose to the contrary that $0 < m(E) < 1$. In view of the second part of Theorem 3.8, it suffices to show that

$$m(E \cap J_{r,M}) = 0,$$

where M comes from Theorem 3.8. Let

$$Z = \left\{ z \in E \cap J_{r,M} : \lim_{r\to 0} \frac{m\Big(B(z,r) \cap E \cap J_{r,M}\Big)}{m(B(z,r))} = 1 \right\}. \tag{3.29}$$

In view of the Lebesgue density theorem (see for example Theorem 2.9.11 in [Fe]), $m(Z) = m(E)$. Since $m(E) > 0$ we find at least one point $z \in Z$. Let $\{n_k(z)\}_{k=1}^\infty$ be sequence associated to z by virtue of the definition of the set $J_{r,M}$. Let δ_z be the number defined in formula (3.24) and let $y(z)$ be defined as in Definition 3.11. Put $\eta = \delta_z/8$. Suppose that $m(B(y(z),\eta) \setminus E) = 0$. By conformality of m, $m(F(Y)) = 0$ for all Borel sets Y such that $m(Y) = 0$. Hence,

$$\begin{aligned} 0 = m\Big(F^n(B(y(z),\eta) \setminus E)\Big) &\geq m\Big(F^n(B(y(z),\eta)) \setminus F^n(E)\Big) \\ &\geq m\Big(F^n(B(y(z),\eta)) \setminus E\Big) \geq m\Big(F^n(B(y(z),\eta)\Big) - m(E) \end{aligned} \tag{3.30}$$

It therefore follows from Lemma 3.13 that $0 \geq 1 - m(E)$, which is a contradiction. Consequently $m(B(x,\eta) \setminus E) > 0$. Hence for every $j \geq 1$ large enough, $m\Big(B(F^{n_j(z)}(z), 2\eta) \setminus E\Big) \geq m\Big(B(y(z),\eta) \setminus E\Big) > 0$. Therefore, as $F^{-1}(J(F) \setminus E) \subset J(F) \setminus E$, the standard application of Koebe's Distortion Theorem shows that

$$\limsup_{r\to 0} \frac{m(B(z,r) \setminus E)}{m(B(z,r))} > 0$$

which contradicts (3.29). Thus either $m(E) = 0$ or $m(E) = 1$.

Now conservativity is straightforward. One needs to prove that for every Borel set $B \subset J(F)$ with $m(B) > 0$ one has $m(G) = 0$, where

$$G = \{x \in J(F) : \sum_{n \geq 0} \chi_B(f^n(x)) < +\infty\}.$$

Indeed, suppose that $m(G) > 0$ and for all $n \geq 0$ let

$$G_n = \{x \in J(F) : \sum_{k \geq n} \chi_B(F^n(x)) = 0\} = \{x \in J(F) : f^k(x) \notin B \text{ for all } k \geq n\}.$$

Since $G = \bigcup_{n \geq 0} G_n$, there exists $n_0 \geq 0$ such that $m(G_{n_0}) > 0$. Since all the sets G_n are forward invariant we conclude that $m(G_{n_0}) = 1$. But on the other hand all the sets $F^{-n}(B)$, $n \geq k$, are of positive measure and are disjoint from G_{n_0}. This contradiction finishes the proof of conservativity of m. ■

4. Invariant measure

In this section we show the existence and uniqueness of a probability F-invariant measure equivalent to m. We first prove the following.

Lemma 4.1. *Up to a multiplicative constant there exists a unique F-invariant, σ-finite measure μ, which is conservative, ergodic and equivalent to the h-conformal measure m.*

The idea of the proof of Lemma 4.1 is to apply a general sufficient condition for the existence of σ-finite absolutely continuous invariant measure proven in [Ma]. In order to formulate this condition suppose that X is a σ-compact metric space, m is a Borel probability measure on X, positive on open sets, and that a measurable map $T : X \to X$ is given with respect to which measure m is quasi-invariant, i.e. $m \circ T^{-1} << m$. Moreover we assume the existence of a countable partition $\alpha = \{A_n : n \geq 0\}$ of subsets of X which are all σ-compact and of positive measure m. We also assume that $m(X \setminus \bigcup_{n \geq 0} A_n) = 0$, and if additionally for all $m, n \geq 1$ there exists $k \geq 0$ such that

$$m(T^{-k}(A_m) \cap A_n) > 0, \tag{4.1}$$

then the partition α is called irreducible. Martens' result comprising Proposition 2.6 and Theorem 2.9 of [Ma] reads the following.

Theorem 4.2. *Suppose that $\alpha = \{A_n : n \geq 0\}$ is an irreducible partition for $T : X \to X$. Suppose that T is conservative and ergodic with respect to the measure m. If for every $n \geq 1$ there exists $K_n \geq 1$ such that for all $k \geq 0$ and all Borel subsets A of A_n*

$$K_n^{-1} \frac{m(A)}{m(A_n)} \leq \frac{m(T^{-k}(A))}{m(T^{-k}(A_n))} \leq K_n \frac{m(A)}{m(A_n)}, \tag{4.2}$$

then T has a σ-finite T-invariant measure μ absolutely continuous with respect to m. Additionally μ is equivalent with m, conservative and ergodic, and unique up to a multiplicative constant.

Proof of Lemma 4.1 (sketch). Since in the sequel we will not only need Lemma 4.1 but a bit more, namely the way in which the invariant measure claimed in Theorem 4.2 is produced, we shall also describe this procedure briefly. Following Martens, one considers the following sequences of measures

$$S_k m = \sum_{i=0}^{k-1} m \circ T^{-i} \quad \text{and} \quad Q_k m = \frac{S_k m}{S_k m(A_0)}.$$

It is proven in [Ma] that each weak limit μ of the sequence $Q_k m$ has the properties required in Theorem 4.2, where a sequence $\{\nu_k : k \geq 1\}$ of measures on X is said to converge weakly if for all $n \geq 1$ the measures ν_k converge weakly on all compact subsets of A_n. In fact it turns out that the sequence $Q_k m$ converges and

$$\mu(F) = \lim_{n \to \infty} Q_k m(F)$$

for every Borel set $F \subset X$. Of course $\mu(A) \leq 1 < \infty$. Making use of (4.1) and (4.2) one proves (see Lemma 2.4 in [Ma]) the following.

Lemma 4.3. *For every $n \geq 0$ we have $0 < \mu(A_n) < \infty$, even more, the Radon-Nikodym derivative $\frac{d\mu}{dm}$ is bounded on A_n.*

and

Lemma 4.4. *For all $i, j \geq 0$ there exists a costant $\kappa > 0$ such that*

$$\frac{S_n m(D)}{S_n m(E)} \leq \kappa \frac{m(D)}{m(E)}$$

for all $n \geq 1$ and all Borel sets $D \subset A_i$ and $E \subset A_j$.

Let us pass now to our map $F : Q \to Q$. The ergodicity and conservativity of the measure m is proven in Theorem 3.15. Thus, we only need to construct an irreducible partition α with property (4.2). Indeed, set $Y = J(F) \setminus \hat{\beta}_1^\infty$. For every $y \in Y$ consider a ball $B(y, r(y)) \subset Q$ such that $r(y) > 0$, $m(\partial B(y, r(y))) = 0$, and $r(y) < (1/2)\mathrm{dist}(y, \hat{\beta}_1^\infty)$. The balls $B(y, r(y))$, $y \in Y$, cover Y and, obviously, one can choose a countable cover, say $\{\tilde{A}_n : n \geq 0\}$, from them. We may additionally require that the family $\{\tilde{A}_n : n \geq 0\}$ is locally finite that is that each point $x \in Y$ has an open neighborhood intersecting only finitely many balls $\tilde{A}_n$, $n \geq 0$. We now define the family $\alpha = \{A_n : n \geq 0\}$ inductively setting

$$A_0 = \tilde{A}_0 \quad \text{and} \quad A_{n+1} = \tilde{A}_{n+1} \setminus \bigcup_{k=1}^{n} \overline{\tilde{A}_n}$$

(and throwing away empty sets). Obviously α is a disjoint family and

$$\bigcup_{n \geq 1} A_n \supset J(F) \setminus (\hat{\beta}_1^\infty \cup \bigcup_{n \geq 0} \partial \tilde{A}_n).$$

Hence, in view of the last assumption of our theorem, $m\Big(\bigcup_{n\geq 0} A_n\Big) = 1$. The distortion condition (4.2) follows now from Koebe's distortion theorem with all constants $K_n = K$, and irreducibility of partition α follows from openess of the sets A_n and Lemma 3.13. ■

For the proof of Theorem 4.6, the main result of this section, we will need the following.

Lemma 4.5. *There exists $R \in (0, \pi/2)$ such that for every $t > 0$ there exists a constant $C(t) > 0$ such that*

$$m(B(\hat{\beta}_n, r)) \leq C(t) r^t |(F^n)'(0)|^{h-t}$$

for all $n \geq 0$ and all $r \in [0, 1)$.

Proof. Combining (3.12) and (3.13), we see that for every $k \geq 2$

$$m(B(\hat{0}, r_k)) \preceq |\beta_{k+2}|^{-\kappa},$$

where κ is the fixed positive number, introduced in (3.9) and $r_k = (4|\beta_1||\beta_2|\cdot\ldots\cdot|\beta_k|)^{-1}|\beta_{k-1}|^{-1}$. Consider an arbitrary radius $r \in (0, r_2]$. Then $r_{k+1} < r \leq r_k$ for some $k \geq 2$. Hence, using our super-growing condition (1.1) and Lemma 3.5, we get easily

$$m(B(\hat{0}, r)) \leq m(B(\hat{0}, r_k)) \preceq |\beta_{k+2}|^{-\kappa} \leq C_0(t) r_{k+1}^t \leq C_0(t) r^t \tag{4.3}$$

for some constant $C_0(t)$ and we are done with the case $n = 0$. Since $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} |\beta_n| = +\infty$, the set $\hat{\beta}_n$ has no accumulation point in Q and there exists $R > 0$ such that holomorphic inverse branches $F_0^{-n} : B(\hat{\beta}_n, 2R) \to Q$ of F^n sending $\hat{\beta}_n$ to $\hat{0}$ are well-defined for all $n \geq 1$. Hence, using Koebe's distortion theorem and h-conformality of the measure m, we get for every $r \in (0, R)$ that

$$m\Big(B(\hat{0}, K|(F^n)'(0)|^{-1} r)\Big) \geq m\Big(F_0^{-n}(B(\hat{\beta}_n, r))\Big) \geq K^{-h} |(F^n)'(0)|^{-h} m(B(\hat{\beta}_n, r)).$$

Hence

$$\begin{aligned}
&m(B(\hat{\beta}_n, r)) \leq K^h |(F^n)'(0)|^h m\Big(B(\hat{0}, K|(F^n)'(0)|^{-1} r)\Big) \\
&\preceq K^h C_0(t) |(F^n)'(0)|^h \Big(Kr|(F^n)'(0)|^{-1}\Big)^t = K^{h+t} C_0(t) r^t |(F^n)'(0)|^{h-t} = C(t) r^t |(F^n)'(0)|^{h-t}.
\end{aligned}$$

where $C(t) = K^{h+t} C_0(t)$. ■

Our next, technically most involved, goal is to prove the following main result of this section.

Theorem 4.6. *The σ-finite, F-invariant measure μ equivalent to the h-conformal measure m, produced in Lemma 4.1, is finite.*

Proof. Let $\{A_n\}_{n\geq 0}$ be the irreducible partition constructed just before Lemma 4.5. We may assume without loss of generality that $A_0 = B(z', \xi) \subset B(\hat{0}, R)$ for some $z' \in Q$ and some $\xi \in (0, R)$. Fix $r \in (0, R]$. Decreasing $r > 0$ if necessary, we may assume that

$$\pi^{-1}(z') \cap \bigcup_{k=0}^{\infty} B(\beta_k, r) = \emptyset.$$

We shall write $z' + 2\pi ij$ to denote the unique point in $\pi^{-1}(z')$ with imaginary part in the interval $[2\pi j, 2\pi(j+1))$. Let

$$D_{n,j} := B(2\pi ij, |4\beta_n|)$$

Take now $n \geq 1$ so large that for every $j \in \boldsymbol{Z}$

$$D_{n,j} \cap \{\beta_k : k \geq n+1\} = \emptyset.$$

The last property is guaranteed by the super-growing condition (see (1.1) and (1.2)). Fix a point

$$z'' \in B(2\pi ij, 2|\beta_n|) \setminus \bigcup_{k=0}^{n} B(\beta_k, r).$$

It the follows from Lemma 3, p. 152 in [Ha] that there exists a simply connected open set $D'_{n,j} \subset D_{n,j}$ such that $z' + 2\pi ij, z'' \in D'_{n,j}$, $\beta_1, \beta_2, \ldots, \beta_n \notin D'_{n,j}$ and for all $n \geq 1$ large enough

$$\begin{aligned}\rho_h(z' + 2\pi ij, z''; D'_{n,j}) &\leq \rho_h(z' + 2\pi ij, z''; D_{n,j}) + \frac{1}{2}A\left(n + \log\frac{1}{r}\right) \leq A + \frac{1}{2}A\left(n + \log\frac{1}{r}\right)\\ &\leq A\left(n + \log\frac{1}{r}\right),\end{aligned}$$

where $A > 0$ is some unversal constants and ρ_h is the hyperbolic metric in respective domain $D'_{n,j}$ or $D_{n,j}$. Consequently, it follows from Koebe's distortion theorem that if $H : D'_{n,j} \to \mathbb{C}$ is a univalent holomorphic function, then

$$\frac{|H'(z' + 2\pi ij)|}{|H'(z'')|} \preceq \exp\left(6A\left(n + \log\frac{1}{r}\right)\right) \preceq e^{6An}\left(\frac{1}{r}\right)^{6A}. \tag{4.4}$$

Consider now an arbitrary geometric disk $S \subset Q$ with radius $\leq r/2$ and the center $w \notin B(\hat{\beta}_0^\infty, r)$. In particular $2S \cap \hat{\beta}_0^\infty = \emptyset$. Let $n = n(S)$ be the least integer such that $S \subset Q_{\alpha_{n(S)}+1}$. Notice that all holomorphic branches of all backward iterates of F are well defined on $2S$. Fix one such branch $F_\nu^{-k} : 2S \to Q$. Then $F_\nu^{-k} = \pi \circ f_\nu^{-k} \circ \pi_\nu^{-1}$, where π_ν^{-1} is an appropriate holomorphic inverse branch of π defined on $2S$, and $f_\nu^{-k} : \pi_\nu^{-1}(2S) \to \mathbb{C}$ is an appropriate holomorphic inverse branch of f^k. Write $\pi_\nu^{-1}(w) = w + 2\pi ij$ for some $j \in \boldsymbol{Z}$ and note that $\pi_\nu^{-1}(w) \in \pi_\nu^{-1}(2S) \subset B\Big(2\pi ij, 2|\beta_{n(S)}|\Big)$ and $\pi_\nu^{-1}(w) \notin B(\beta_0^\infty, r)$. Apply now considerations leading to (4.4) with $z'' = \pi_\nu^{-1}(w)$. Since $D'_{n,j}$ is simply connected, since $D'_{n,j} \cap \beta_0^\infty = \emptyset$ and since $\pi_\nu^{-1}(w)\in\pi_\nu^{-1}(2S)\cap D'_{n,j}$, there exists a unique holomorphic continuation $f_{\nu'}^{-k}:D'_{n,j}\to\mathbb{C}$ of the branch $f_\nu^{-k}:\pi_\nu^{-1}(2S)\cap D'_{n,j}$. So,

$$f_{\nu'}^{-k}|_{\pi_\nu^{-1}(2S)\cap D'_{n,j}}=f_\nu^{-k}|_{\pi_\nu^{-1}(2S)\cap D'_{n,j}}.$$

In particular $f_{\nu'}^{-k}(\pi_\nu^{-1}(w))=f_\nu^{-k}(\pi_\nu^{-1}(w))$. Since $(2A_0+2\pi ij)\cap\beta_0^\infty=\emptyset$ and since $z'+2\pi ij\in(A_0+2\pi ij)\cap D'_{n,j}$, there exists a unique holomorphic inverse branch $f_{\nu''}^{-k}:2A_0+2\pi ij\to\mathbb{C}$ such that $f_{\nu''}^{-k}|_{(A_0+2\pi ij)\cap D'_{n,j}}=f_{\nu'}^{-k}|_{(A_0+2\pi ij)\cap D'_{n,j}}$. In particular $f_{\nu''}^{-k}(z'+2\pi ij)=f_{\nu'}^{-k}(z'+2\pi ij)$. Let $F_{\nu''}^{-k}:2A_0\to Q$ be defined by the formula $F_{\nu''}^{-k}=\pi\circ f_{\nu''}^{-k}\circ\pi_j^{-1}$, where $\pi_j^{-1}(z)=z+2\pi ij$. Notice that the mapping $F_\nu^{-k}\mapsto F_{\nu''}^{-k}$ is a bijection between the set of all holomorphic inverse branches of F^k defined on $2S$ and those defined on $2A_0$. Applying (4.4) to the map $f_{\nu'}^{-k}:D'_{n,j}\to\mathbb{C}$, and using Koebe's distortion theorem to the maps $f_{\nu'}^{-k}:\pi^{-1}(2S)\to\mathbb{C}$ and $f_{\nu''}^{-k}:2A_0+2\pi ij\to\mathbb{C}$, we obtain for all $x\in S$ and all $y\in A_0$ that

$$\frac{|(F_\nu^{-k})'(x)|}{|(F_{\nu''}^{-k})'(y)|}\le K^2\frac{|(F_\nu^{-k})'(w)|}{|(F_{\nu''}^{-k}(z')|}=K^2\frac{|(f_{\nu'}^{-k})'(\pi^{-1}(w)|}{|(f_{\nu'}^{-k}(z'+2\pi ij)|}\preceq e^{6An(S)}\left(\frac{1}{r}\right)^{6A}.$$

Since $m(F_\nu^{-k}(S))\le\left(\sup_S\{|(F_\nu^{-k})'|\}\right)^h m(S)$ and since $m(F_{\nu''}^{-k}(A_0))\ge\left(\inf_{A_0})\{|(F_\nu^{-k})'|\}\right)^h m(A_0)$, we therefore get

$$\frac{m(F_\nu^{-k}(S))}{m(F_{\nu''}^{-k}(A_0))}\le\left(\frac{\sup_S\{|(F_\nu^{-k})'|\}}{\inf_{A_0}\{|(F_{\nu''}^{-k})'|\}}\right)^h\frac{m(S)}{m(A_0)}\preceq e^{6An(S)}\left(\frac{1}{r}\right)^{6Ah}m(S).$$

Summing over all branches ν, we thus get

$$m(F^{-k}(S))\preceq\exp(6Ahn(S))\left(\frac{1}{r}\right)^{6Ah}m(F^{-k}(A_0))m(S).$$

It immediately implies that

$$\mu(S)\le\exp(6Ahn(S))\left(\frac{1}{r}\right)^{6Ah}m(S). \tag{4.5}$$

Assume now $R\in(0,1)$ to be so small that all the balls $B(\hat\beta_n,2R)$, $n\ge0$, are mutually disjoint. Our goal now is to estimate the measure μ of the neighbourhood of $\hat\beta_n$. To do so, we divide each ball $B(\hat\beta_n,R)$ into geometric annuli

$$B(\hat\beta_n,\frac{R}{2^k})\setminus B(\hat\beta_n,\frac{R}{2^{k+1}}).$$

Obviously, each annulus can be covered by a bounded (independent of k and n) number of balls $B_{n,k}$ with radius equal to $R2^{-(k+1)}$. So, consider an arbitrary ball $B_{n,k}\subset B(\hat\beta_n,\frac{R}{2^k})$, $k\ge0$, with radius equal to $R2^{-(k+1)}$ and the center at the distance from $\hat\beta_n$ exactly equal to $2^{-k}R$. It then follows from (4.5) and Lemma 4.5 that

$$\mu(B_k)\le C(t)\cdot\exp(6Ahn)\left(\frac{2^k}{R}\right)^{6Ah}\left(\frac{R}{2^k}\right)^t|(f^n)'(0)|^{h-t}\preceq C(t)|(f^n)'(0)|^{-u}2^{-uk}$$

with an arbitrarily large u assuming t to be large enough. Notice now that there exists an integer $L \geq 1$ so large that appropriately choosing for each $k \geq 0$, L balls of the form $B_{n,k}$, we will cover the punctured disk $B(\hat{\beta}_n, R) \setminus \{\hat{\beta}_n\}$. Hence

$$\sum_{n\geq 0} \mu(B(\hat{\beta}_n, R)) \preceq L \sum_{n\geq 0}\sum_{k=0}^{\infty} |(f^n)'(0)|^{-u} 2^{-uk} \preceq \sum_{n\geq 0} |(f^n)'(0)|^{-u} < \infty. \tag{4.6}$$

Obviously, there exists an integer $T \geq 1$ such that each set

$$W_n = \{z \in Q : \alpha_n \leq \mathrm{Re} z \leq \alpha_{n+1}\} \setminus \bigcup_{k=0}^{\infty} B(\hat{\beta}_j, R),\ n \geq 0,$$

can be covered by no more than $T(\alpha_{n+1} - \alpha_n)$ balls with radii $R/2$ and centers lying in W_n. Applying then (4.5) and (3.6) (each of these balls is contained in $Y^+_{\alpha_n}$), we obtain

$$\mu(W_n) \preceq T(\alpha_{n+1} - \alpha_n) \exp\Big(6Ah(n+1)\Big) \left(\frac{2}{R}\right)^{6Ah} \exp\Big(\frac{1}{2}(1-h)\alpha_n\Big) \leq Ce^{-u\alpha_n} \tag{4.7}$$

for some constant C and for some positive u. The last inequality follows from super-growing condition (1.1). Since the measure μ is F-invariant and since there exists $N > 0$ so large that $F(Y^-_N) \subset B(\hat{0}, R)$, we conclude from (4.6) that $\mu(Y^-_1) < \infty$. Combining this along with (4.6) and (4.7), we deduce that

$$\mu\left(Y^-_1 \cup \bigcup_{n=0}^{\infty} \Big(B(\hat{\beta}_n, R) \cup W_n\Big)\right) < \infty.$$

Since it is obvious that the complement of this set has a finite measure μ, we are done. ■

5. Hausdorff and Packing Measures

Let H^h and P^h be be respectively the h-dimensional Hausdorff and packing measures (see [TT], comp. [PU] for example, for its definition and some basic properties). The results of this section provide in a sense a complete description of the geometrical structure of the sets $J_r(F)$ and $J_r(f)$ and also they exhibit the geometrical meaning of the h-conformal measure m. The short proof of the first result improves on the argument from the proof of Proposition 4.9 from [UZ1].

Proposition 5.1. *We have $\mathrm{P}^h(J_r(f)) = \mathrm{P}^h(J_r(F)) = \infty$. In fact $\mathrm{P}^h(G) = \infty$ for every open nonempty subset of $J_r(f)$.*

Proof. Since $m(J_r(F) \cap Y^-_M)) > 0$ for every $M \in I\!R$, it follows from Birkhoff's ergodic theorem, Theorem 4.6 and Theorem 3.15 that there exists a set $E \subset J_r(F)$ such that $m(E) = 1$ and $\liminf_{n\to\infty} \mathrm{Re}F^n(z) = -\infty$. for every $z \in E$. Fix $z \in E$. The above formula means that there exists an unbounded increasing sequence $\{n_k\}_{k=1}^{\infty}$, depending on z, such that

$$\lim_{k\to\infty} \mathrm{Re}(F^{n_k}(z)) = -\infty. \tag{5.1}$$

Since $\lim_{n\to\infty}\alpha_n=+\infty$, the balls $B(F^{n_k}(z),1)$ are for all k large enough, say $k\ge q$, disjoint from the set $\hat\beta_0^\infty$. Fix $k\ge q$ and consider the ball $B(z,K^{-1}|(F^{n_k})'(z)|^{-1})$. Then

$$B(z,K^{-1}|(F^{n_k})'|(z)|^{-1})\subset F_z^{-n_k}(B(F^{n_k}(z),1)),$$

where F_z^{-n} is the holomorphic inverse branch of F^n, defined on $B(F^n(z),1)$ and mapping $F^n(z)$ to z. Applying Koebe's distortion theorem and conformality of the measure m, we obtain

$$\begin{aligned} m(B(z,K^{-1}|(F^{n_k})'(z)|^{-1}) &\le K^h|(F^{n_k})'(z)|^{-h}m(B(F^{n_k}(z),1))\\ &\le K^{2h}(K^{-1}|(F^{n_k})'(z)|^{-1})^h m(Y^-_{\mathrm{Re}F^{n_k}(z)-1}) \end{aligned}$$

Since by (5.1), $\lim_{k\to\infty}m(Y^-_{\mathrm{Re}F^{n_k}(z)-1})=0$, we see that

$$\liminf_{r\to 0}\frac{m(B(z,r))}{r^h}=0.$$

Since $m(G\cap J_r(F))>0$ for every non-empty open subset of $J_r(F)$, this implies (see an appropriate Converse Frostman's Type Theorem in [DU2] or [PU]) that $\mathrm{P}^h(G)=\infty$. Since $J_r(f)=\pi^{-1}((J_r(F))$ and π is a local isometry, we are therefore done. ■

Theorem 5.2. *The h-dimensional Hausdorff measure restricted to the set $J_r(F)$ is positive, finite, and absolutely continuous with respect to the h-conformal measure m.*

Proof. We shall show first that $\mathrm{H}^h|_{J_r(F)}$ is absolutely continuous with respect to m and finite. Fix $z\in J_{r,M}(F)$. This implies (see the considerations after Remark 3.10) that there exists an increasing unbounded sequence $\{s_k\}_{k=1}^\infty$ and a positive number $\delta_z(M)$ such that $f^{s_k(z)}(z)\to y(z)$, $|y(z)|<|\lambda|e^M$ and a holomorphic inverse branch of f^{-s_k} sending $f^{s_k}(z)$ to z is well defined on the ball $B(f^{n_k}(z),\delta_z(M))$. Denote by $\pi_k^{-1}:B(F^{s_k}(z),\pi)\to\mathbb{C}$ the holomorphic inverse branch of the projection $\pi:Q\to\mathbb{C}$, which sends $F^{s_k}(z)$ to $f^{s_k}(z)$. Then the composition $F_z^{-s_k}=\pi\circ f_z^{-s_k}\circ\pi_k^{-1}:B(F^{s_k}(z),\delta_z(M))\to\mathbb{C}$ is well-defined and forms a holomorphic inverse branch of F^{s_k} sending $F^{s_k}(z)$ to z. In this way, taking an appropriate component of preimage $F^{-s_k}(B(F^{s_k}(z)),\delta_z(M))$ we get a neighbourhood of z, contained in a ball of radius $K\frac{1}{|(F^{s_k})'(z)|}\delta_z(M)$. This allows us to construct, for every point $z\in J_{r,M}$ a sequence of balls of radii $r_k(z)$ converging to 0 (see Lemma 3.12) for which we can estimate, using conformality of the measure m:

$$m(B(z,r_k(z))\succeq r_k^h\cdot\inf_{\mathrm{Re}w\le|\lambda|\exp(M)}m(B(w,\delta_z(M)).$$

This is enough to conclude (see an appropriate Converse Frostman's Type Theorem in [DU2] or [PU]) that for every M the measure $\mathrm{H}^h_{|J_{r,M}}$ is absolutely continuous with respect to m and, consequently, $\mathrm{H}^h_{|J_r(F)}$ is absolutely continuous with respect to m. Now, there exists $M=M_k$ so that $m(J_{r,M})=1$ (see Theorem 3.8). Moreover, there exists a positive number δ_M such that for m almost every point $z\in J_{r,M}$ it holds $\delta_z(M)\ge\delta_M$ (see Remark 3.10). So, there exists a set $H \subset J_{r,M}$ such that $m(H) = 1$ and for for each point $z \in H$ one can find a sequence of radii $r_k(z) \to 0$ for which a uniform estimate holds:

$$m(B(z, r_k(z))) \succeq r_k^h \cdot \inf_{\mathrm{Re} w \le |\lambda| \exp(M)} m(B(w, \delta_M)) \ge C r_k^h$$

where the constant C does not depend on z. This implies in a standard way that the h-dimensional measure of the set H is finite. Since $m(J_{r,M} \setminus H) = 0$, we conclude that $\mathrm{H}^h(J_{r,M} \setminus H) = 0$ (we already know that H_h is absolutely continuous with respect to m). Consequently, $\mathrm{H}^h(J_{r,M}) = \mathrm{H}^h(H) < \infty$. Finally, we know from Theorem 3.8 that $m\left(J_r(F) \setminus J_{r,M}(F)\right) = 0$. Therefore (again, by absolute continuity)

$$\mathrm{H}^h\left(J_r(F) \setminus J_{r,M}(F)\right) = 0.$$

This proves that the h-dimensional Hausdorff measure of $J_r(F)$ is finite.

We shall prove now that $\mathrm{H}^h(J_r(F)) > 0$. Since β_n tends to infinity fast (see (1.1)), there exists $1 > \theta > 0$ such that $|\beta_j - \beta_i| > \theta$ for $i \neq j$. So, fix $z \in J_r$ and $r \in (0, \pi)$. Since, by Lemma 3.12, $\limsup_{n\to\infty} |(f^n)'(z)| = +\infty$, there exists a least $n = n(z, r) \ge 0$ such that

$$r|(f^{(n+1)})'(z)| \ge \theta(32K)^{-1}.$$

Thus

$$r|(f^n)'(z)| < \theta(32K)^{-1}. \tag{5.2}$$

Suppose that the holomorphic inverse branch of f^n defined on $B(f^n(z), 32r|(f^n)'(z)|)$ and sending $f^n(z)$ to z, does not exist. Then $n \ge 1$, and let $1 \le k \le n$ be the largest integer such that the holomorphic inverse branch of $f^{n-(k-1)}$ defined on $B(f^n(z), 32r|(f^n)'(z)|)$ and sending $f^n(z)$ to $f^{k-1}(z)$ does not exist. This implies that $0 \in f_k^{-(n-k)}(B(f^n(z), 32r|(f^n)'(z)|))$, where $f_k^{-(n-k)} : B(f^n(z), 32r|(f^n)'(z)|) \to \mathbb{C}$ is the holomorphic inverse branch of f^{n-k} sending $f^n(z)$ to $f^k(z)$. Thus, $\beta_k \in B(f^n(z), 32r|(f^n)'(z)|$. But $32r|(f^n)'(z)| < \frac{\theta}{K} < \frac{\theta}{2}$ and, by the definition of θ, there are no other images of 0 even in the ball $B(f^n(z), 64r|(f^n)'(z)|)$. So, we can use Koebe distortion theorem for the map $f_k^{-(n-k)}$ as follows.

$$0 \in f_k^{-(n-k)}(B(f^n(z), 32r|(f^n)'(z)|)) \subset B(f^k(z), 32Kr|(f^k)'(z)|),$$

and therefore $|f^k(z)| < 32Kr|(f^k)'(z)|$. On the other hand, since $r|(f^{k-1})'(z)| < (32K)^{-1}$, we conclude that

$$32Kr|(f^k)'(z)| = 32Kr|(f^{k-1})'(z)| \cdot |f'(f^{k-1}(z))| = 32Kr|(f^{k-1})'(z)| \cdot |f^k(z)| < \theta|f^k(z)| < |f^k(z)|.$$

This contradiction shows that the holomorphic inverse branch $f_z^{-n} : B(f^n(z), 32r|(f^n)'(z)|) \to \mathbb{C}$ of f^n sending $f^n(z)$ to z, is well-defined. Now, the map f restricted to $B(f^n(z), 32r|(f^n)'(z)|)$ is 1-to-1, and by Koebe's $\frac{1}{4}$-Theorem

$$f(B(f^n(z), 32r|(f^n)'(z)|)) \supset B\Big(f^{n+1}(z), 8r|(f^{n+1})'(z)|\Big).$$

Hence there exists a unique holomorphic inverse branch $f_z^{-(n+1)} : B\Big(f^{n+1}(z), 8r|(f^{n+1})'(z)|\Big) \to \mathbb{C}$ of f^{n+1} mapping $f^{n+1}(z)$ to z. Applying Koebe's $\frac{1}{4}$-Theorem again, we see that

$$f_z^{-(n+1)}\Big(B\Big(f^{n+1}(z), 4r|(f^{n+1})'(z)|\Big)\Big) \supset B(z,r). \tag{5.3}$$

Since the ball $B\Big(f^{n+1}(z), 4r|(f^{n+1})'(z)|\Big)$ intersects at most $\frac{1}{2\pi}4r|(f^{n+1})'(z)|+1 \le r|(f^{n+1})'(z)|$ horizontal strips of the form $2\pi ij + (\mathbb{R} \times [0,2\pi))$, $j \in \mathbb{Z}$, using (5.3) Koebe's Distortion Theorem, h-conformality of the measure m and, at the end, (5.2), we get

$$\begin{aligned} r^{-h}(m(B(z,r))) &\le r^{-h}K^h|(f^{n+1})'(z)|^{-h}(r|(f^{n+1})'(z)|)m\Big(\pi\Big(B\Big(f^{n+1}(z), 4r|(f^{n+1})'(z)|\Big)\Big)\Big) \\ &\le r^{-h}K^h|(f^{n+1})'(z)|^{-h}(r|(f^{n+1})'(z)|) \\ &= K^h(r|(f^{n+1})'(z)|)^{1-h} \le K^h(\theta)^{1-h}(32K)^{h-1}, \end{aligned}$$

We are done by applying an appropriate Converse Frostman's Type Theorem in [DU2] or [PU]. ■

Theorem 5.3. *We have* $\mathrm{HD}(J_r(f)) = \mathrm{HD}(J_r(F)) = h \in (1,2)$.

Proof. It follows immediately from Theorem 5.2 that $\mathrm{HD}(J_r(f)) = \mathrm{HD}(J_r(F)) = h$. We know already that $h > 1$. In order to prove that $h < 2$, let us recall that it follows from Proposition 5.1 that for every open set Z such that $Z \cap J_r(f) \ne \emptyset$ we have $P^h(J_r(f) \cap Z) = \infty$. In particular, if Z is a ball, assuming $h = 2$ we get $\infty = P^2(Z \cap J_r(f)) \le P^2(Z)$, thus $P^2(Z) = \infty$, which is a contradiction. ■

Finally, let us notice that the following result of Lyubich ([Lyu]) and Rees ([Re]) can be deduced as a corollary:

Corollary 5.4. *If λ is a super-growing parameter then for Lebesgue almost every point $z \in \mathbb{C}$ $\omega(z) = \beta_0^\infty \cup \{\infty\}$.*

Proof. Since $\mathrm{HD}(J_r(f)) < 2$, the complement of J_r is a set of full measure. Fix a point $z \notin J_r(f)$. By the definition of $J_r(f)$, this implies that $\omega(z) \subset \{\infty\} \cup \beta_0^\infty$. We only have to check that, actually, the equality holds for almost every point. So, assume that $\omega(z) = \infty$. The set of such points has Lebesgue measure 0; actually, this is true for a large class of maps, see e.g. [McM] or [EL]. Next, assume that $\omega(z) = \{\infty\} \cup \beta_k^\infty$ for some $k > 0$. Thus, there exists an infinite sequence of integers s_i such that $f^{s_i}(z) \to \beta_k$. Then, denoting $n_i = s_i - 1$, we see that $\mathrm{Re} f^{n_i}(z) \to \mathrm{Re}\beta_k$ and, moreover, $\mathrm{dist}(f^{n_i}(z), \beta_k^\infty) > 0$. Consequently, $z \in J_r(f)$, a contradiction. We are done. ■

Remark 5.5. *Actually, the result in Lyubich's paper is stated for $\lambda = 1$, but his proof extends in a straightforward way to all maps f_λ with super-growing parameter λ. So, the statement of the corollary is not new.*

It is natural to ask about the dependence of the dimension $\mathrm{HD}(J_r(f_\lambda))$ on λ. Below, we sketch a proof of one partial result: Denote by $M_{c,N}$ the set of parameters for which the condition (1.1) holds with a constant c and for all $n > N$. Let $\lambda \in M_{c,N}$. Using the arguments of [UZ1], one can easily check that there exists a neighbourhood U of λ in $\mathbb{C}$ and a constant $s > 1$ such that $\mathrm{HD}(J_M(f_\mu)) > s$ for every $\mu \in U$. This, in turn implies that $h_\mu = HD(J_r(f_\mu)) > s$. Following the way of proof of Lemma 3.4 we can show that the family of conformal measures $m_\mu, \mu \in M_{c,N} \cap U$ is tight. If $\lambda_n \in M_{c,N}$ and $\lambda_n \to \lambda$ then one can choose a subsequence n_k such that $m_{\lambda_{n_k}}$ converges weakly to some measure m . One can also assume that $h_{\lambda_{n_k}} \to h \geq s$. Then one can check that this limit measure is h-conformal for the limit map f_λ. It follows now from the uniqueness of conformal measure that $h = h_\lambda$. Thus, the following theorem is true (we omit details of the proof).

Theorem 5.6. *The function* $\lambda \mapsto \mathrm{HD}(J_r(f_\lambda))$ *is continuous in the set* $M_{c,N}$. *In particular, it is continuous in* $\{\lambda \in \mathbb{R}, \lambda > \frac{1}{e}\}$.

References

[DU1] M. Denker, M. Urbański, On Sullivan's conformal measures for rational maps of the Riemann sphere, Nonlinearity 4 (1991), 365-384.

[DU2] M. Denker, M. Urbański, Geometric measures for parabolic rational maps, Ergod. Th. and Dynam. Sys. 12 (1992), 53-66.

[EL] A.Eremenko, M. Lyubich, Dynamical properties of some classes of entire functions, Ann. Inst.Fourier, Grenoble 42 (1992), 989-1020

[Fe] H. Federer, Geometric Measure Theory, Springer 1969.

[Gu] M. Guzmán: Differentiation of integrals in $\mathbb{R}^n$. Lect. Notes in Math. 481, Springer Verlag.

[Ha] W. Hayman, The maximum modulus and valency of functions meromorphic in the unit circle, Acta Math. 86 (1951), 89-257.

[Lyu] M. Lyubich, The measurable dynamics of the exponential map, Siberian Journ. Math, 28 (1987), 111-127

[Ma] M. Martens, The existence of σ-finite invariant measures, Applications to real one-dimensional dynamics, Preprint.

[McM] C. McMullen, Area and Hausdorff dimension of Julia sets of entire functions, Trans. Amer. Math. Soc., 300 (1987), 325-342

[PU] F. Przytycki, M. Urbański, Fractals in the Plane - the Ergodic Theory Methods, available on the web:http://www.math.unt.edu/~urbanski, to appear in Cambridge Univ. Press.

[Re] M. Rees, The exponential map is not recurrent, Math. Z. 191 (1986), 593-598

[TT] S.J. Taylor, C. Tricot, Packing measure, and its evaluation for a Brownian path, Trans. A.M.S. 288 (1985), 679 - 699.

[UZ1] M. Urbański, A. Zdunik, The finer geometry and dynamics of exponential family, Michigan Math. J. 51 (2003) 227 - 250.

[UZ2] M. Urbański, A. Zdunik, Real analyticity of Hausdorff dimension of finer Julia set of exponential family, Preprint 2002, to appear Erg. Th. Dynam. Sys.

[We] Qiu Weiyuan, Hausdorff dimension of the M-set of $\lambda \exp(z)$, Acta Math. Sinica 10 (1994), 362 - 386

Mariusz Urbański; Department of Mathematics, University of North Texas, P.O. Box 311430, Denton TX 76203-1430, USA
urbanski@unt.edu, http://www.math.unt.edu/~urbanski

Anna Zdunik, Institute of Mathematics, Warsaw University, ul. Banacha 2, 02-097 Warszawa, Poland.
A.Zdunik@mimuw.edu.pl