How points escape to infinity under exponential maps

Bogusława Karpińska* and Mariusz Urbański †

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Abstract

We investigate the finer fractal structure of the set of points escaping to infinity under iteration of an arbitrary exponential map. Providing exact formulas, we show how sensitively the Hausdorff dimension depends on the rate of growth of canonical Devaney-Krych codes.

1 Introduction

We consider complex exponential maps $E : \mathbb{C} \mapsto \mathbb{C}$, $E(z) = \lambda \exp z$ where $\lambda \neq 0$ is a complex parameter. It is known (see [3]) that for these maps the Julia set is the closure of the set of points escaping to infinity under iteration of E.

Let

$$I_\infty = \{z : \ |E^n(z)| \to \infty \text{ as } n \to \infty\}.$$

This set is actually dynamically boring. However it has a very rich geometrical structure. Topologically, for a big set of parameters λ, the set I_∞ consists of uncountably many infinite "hairs". In order to contribute towards better understanding of geometry of I_∞, our aim is to show that the set I_∞ exhibits a natural finer fractal structure. Since $|E^n(z)| = |\lambda| \exp(\operatorname{Re} E^{n-1}(z))$, we have

$$I_\infty = \{z : \ \operatorname{Re}(E^n(z)) \to \infty \text{ as } n \to \infty\}.$$

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Therefore, for every $q \geq 0$, $I_\infty = \bigcup_{k\geq 0} I_\infty^{k,q}$, where

$$I_\infty^{k,q} = \{z \in I_\infty : \ \forall n \geq k \quad \mathrm{Re}(E^n(z)) > q\}.$$

In order to shorten notation, we will write I_∞^0 for $I_\infty^{0,q}$ and will always assume that $q > 0$ is large enough.
We divide the right half-plane $H = \{z : \ \mathrm{Re}\, z > 0\}$ into infinitely many strips (as in [2]):

$$P_j = \{z \in H : (2j-1)\pi \leq \mathrm{Im}\, z < (2j+1)\pi\} \quad \text{where } j \in \mathbb{Z}.$$

Every point z which remains in H under iteration of E has uniquely defined sequence of integers $s(z) = (s_0, s_1, ..)$ such that

$$s_k = j \qquad \text{iff} \qquad E^k(z) \in P_j.$$

The sequence $s(z)$ is called the *itinerary of the point* z. The necessary condition for a given sequence s to be the itinerary of a point z (remaining in H) is the following: $|s_n|$ cannot grow faster than $E^n(x)$ for some real x (because the imaginary part of $E^n(z)$ is at most equal to $|E^n(z)|$). In fact (see [2]) this is also a sufficient condition.
So, itineraries of escaping points cannot grow faster than moduli of their n-th iterations. It follows from [6] (see also [4]) that the Hausdorff dimension of the set

$$\{z \in I_\infty^0 : 2\pi|s_n(z)| \geq |E^n(z)|/2\}$$

is equal to 2.
The set of points which escape to ∞ in such way that $s_n(z)$ grows relatively slowly in comparison to $|E^n(z)|$ can have small Hausdorff dimension. It follows from [5] and [7] that for every $\epsilon \in (0,1)$

$$\mathrm{HD}(\{z \in I_\infty^0 : 2\pi|s_n(z)| \leq |E^n(z)|^\epsilon\}) \leq 1 + 2\epsilon.$$

We are interested in the structure of the set I_∞ in terms of the Hausdorff dimension of some of its significant subsets. We find a natural condition for the rate of growth of itineraries which ensures that the Hausdorff dimension of the set of points whose itineraries grow at most in this rate, is equal to arbitrarily chosen number $t \in [1,2]$.

For all integers $k \geq 0,\ l \geq k$ and every $\epsilon > 0$ define

$$D_\epsilon^{k,l} = \left\{ z \in I_\infty^{k,q} : 2\pi|s_n(z)| \leq \frac{|E^n(z)|}{(\log|E^n(z)|)^\epsilon} \ \text{ for all } \ n \geq l \right\},$$

where $q > 0$ is fixed and large enough.

The main result of our article is the following.

Theorem A. *For every* $\epsilon > 0$ *and all integers* $0 \leq k \leq l$

$$\mathrm{HD}(D_\epsilon^{k,l}) = 1 + \frac{1}{1+\epsilon}.$$

Notice that I_∞ has no compact (in $\mathbb{C}$) forward invariant subsets and for any Borel probability invariant measure μ on the Julia set, $\mu(I_\infty) = 0$. Therefore, in order to prove Theorem A we have to look for subtler geometrical methods than those rutinely used in a conformal setting. Since the set of points in I_∞^0 whose itinerary is $1, 1, 1, ..$ contains a curve (see [1]) and it belongs to $D_\epsilon^{k,l}$ for any $\epsilon > 0$ and l sufficiently big we get the following immediate consequence of Theorem A.

Corollary 1.1. *The Hausdorff dimension of the set*

$$D_\infty = \left\{ z \in I_\infty^0 : \quad \forall \epsilon > 0 \ \exists l \ \forall n \geq l \ \ 2\pi |s_n(z)| \leq \frac{|E^n(z)|}{(\log |E^n(z)|)^\epsilon} \right\}$$

is equal to 1.

2 Preliminaries

The Theorem A will turn out to be an immediate consequence of a more general result whose formulation requires some preparations.
Let $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be an increasing function such that

$$h(x) + \pi \leq \frac{x}{4e^\pi} \ \text{ for all } \ x \in \mathbb{R}_+ \text{ large enough,} \tag{1}$$

$$h(e^\pi x) \leq C h(x) \ \text{ for some constant } C$$

and

$$\lim_{x \to \infty} h(x) = +\infty.$$

Call these functions exp-adapted. If in (1), $4e^\pi$ is replaced by $2e^\pi$ the function h is then call weakly exp-adapted. Fix an exp-adapted function $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ and let

$$D(h) = D^q(h) = \{z \in I_\infty^{0,q} : 2\pi |s_n(z)| \leq h(|E^n(z)|) \ \text{ for all } n \geq 0\}.$$

We will need the following two lemmas.

Lemma 2.1. *If q is sufficiently large then $\mathrm{Re}(E^n(z))$ diverges to infinity uniformly with respect to $z \in D^q(h)$.*

Proof. Fix $q > 0$ so large that

$$|\lambda| \exp(\sqrt{3}x/2) \geq 2x \quad \text{for every } x \geq q.$$

For every $z \in D^q(h)$ and every $n \geq 0$, we have

$$\begin{aligned}
\mathrm{Re}(E^n(z)) &= \sqrt{|E^n(z)|^2 - (\mathrm{Im}\, E^n(z))^2} \\
&\geq \sqrt{|E^n(z)|^2 - (h(|E^n(z)|) + \pi)^2} \\
&\geq \sqrt{|E^n(z)|^2 - |E^n(z)/2|^2} \\
&= \frac{\sqrt{3}}{2}|E^n(z)|.
\end{aligned}$$

And therefore, in view of our chioce of q, we get

$$|E^{n+1}(z)| = |\lambda| \exp(\mathrm{Re}(E^n(z))) \geq |\lambda| \exp(\sqrt{3}|E^n(z)|/2) \geq 2|E^n(z)|. \tag{2}$$

Hence

$$\mathrm{Re}(E^{n+1}(z)) \geq \sqrt{3}|E^n(z)| \geq \sqrt{3}\,\mathrm{Re}(E^n(z))$$

□

Denote by $B(x, r)$ the ball centered at x of the radius r.

Lemma 2.2. *For every $\alpha > 0$ and every $T > 0$ there exist $L > 0$ and $n_0 \geq 0$ such that for every $n \geq n_0$,*

$$|(E^{n+1})'(z)| \geq L|(E^n)'(z)|^\alpha$$

for all $z \in D^q(h) \cap B(0, T)$.

Proof. It follows from the first inequality in (2) that for every $\alpha > 0$ there exists n_0 such that

$$|E^{n+1}(z)| \geq |E^n(z)|^\alpha \tag{3}$$

for all $z \in D^q(h)$ and all $n \geq n_0$.

Let L denote the infimum of the map $z \mapsto |(E^{n_0+1})'(z)| \cdot |(E^{n_0})'(z)|^{-\alpha}$ in $B(0, T) \cap \{\mathrm{Re}\, z > 0\}$ (it exists and it is positive). We now proceed by induction. For $n = n_0$ the lemma holds (it is just the definition of L). Suppose now that it is true for some $n \geq n_0$. Using (3) we get

$$\begin{aligned}
|(E^{n+2})'(z)| &= |E'(E^{n+1})(z)| \cdot |(E^{n+1})'(z)| \\
&\geq L|E^{n+2}(z)| \cdot |(E^n)'(z)|^\alpha \\
&\geq L|E^{n+1}(z)|^\alpha \cdot |(E^n)'(z)|^\alpha \\
&= L|(E^{n+1})'(z)|^\alpha.
\end{aligned}$$

□

Let

$$H_\lambda = \{z \in \mathbb{C}: \quad \mathrm{Re}\, z > \max\{0, -|\log|\lambda|| + 2\pi\}.$$

For every $k \in \mathbb{Z}$ define

$$S_k = \left\{ z \in H_\lambda : \quad -\frac{\pi}{2} + 2k\pi - \arg\lambda \le \mathrm{Im}\, z \le \frac{\pi}{2} + 2k\pi - \arg\lambda \right\}.$$

Notice that each sufficiently high iterate of any point escaping to infinity (in I_∞) lies in $\bigcup_{k\in\mathbb{Z}} S_k$. Notice also that the whole orbit of any point in I_∞^0 is contained in this union. Consider a covering of S_k by squares B with the following properties:

(a) the lengths of sides of B are equal to π

(b) the horizontal sides are contained in S_k

(c) the right vertical side does not belong to B.

Denote by $\mathcal{B}$ the family of all elements of these (all $k \in \mathbb{Z}$) coverings. Fix $q \ge \max\{0, -|\log|\lambda|| + 2\pi\}$. Notice that if $\mathrm{Re}\, z \ge -|\log|\lambda|| + 1$, then

$$|E'(z)| \ge |\lambda| \exp(-|\log|\lambda|| + 1) = e > 1. \tag{4}$$

If $z \in I_\infty^0$ (which implies that $\{z, E^1(z), E^2(z), ...\}$, the whole orbit of z, stays in H_λ), then for every $n \ge 0$ there exists a unique square $B_n(z) \in \mathcal{B}$ such that $E^n(z) \in B_n(z)$. It follows immediately from (4) that there exists a unique holomorphic inverse branch $E_z^{-n} : B_n(z) \to H_\lambda$ of E^n sending $E^n(z)$ to z. Inequality (4) implies also that

(i) $E^k(E_z^{-n}(B_n(z))) \subset \{z \in \mathbb{C}: \ \mathrm{Re}\, z > |\log|\lambda|| + 1\}$ for every $0 \le k \le n$.

(ii) $\mathrm{diam}(E_z^{-n}(B_n(z))) \le \sqrt{2}\pi e^{-n}$

(iii) there exists a constant $K \ge 1$ independent on n and z such that for all $x, y \in B_n(z)$

$$\frac{|(E_z^{-n})'(x)|}{|(E_z^{-n})'(y)|} \le K.$$

Notice that for all $\xi, \eta \in B_{n-1}(z)$,

$$\frac{|E'(\xi)|}{|E'(\eta)|} \le e^\pi \tag{5}$$

Put

$$K_n(z) = E_z^{-n}(B_n(z)).$$

Combining (iii) with (5) we get that

$$\frac{|(E^n)'(x)|}{|(E^n)'(y)|} \leq K \quad \text{for all} \quad x, y \in K_{n-1}(z) \tag{6}$$

with an appropiately larger constant K (independent of n and z).
It follows from (ii) that if $z \in I_\infty^k$ then

$$\bigcap_{n \geq k} K_n(z) = \{z\}.$$

Observe that $E^n(K_{n-1}(z))$ is a half-annulus "centered" at the origin. Denote its inner and outer radii respectively by $r_n(z)$ and $R_n(z)$. Note that

$$\frac{R_n(z)}{r_n(z)} = e^\pi$$

and, in view of (5),

$$K^{-1}|E'(E^{n-1}(z))| \leq r_n(z) \leq R_n(z) \leq K|E'(E^{n-1}(z))| \tag{7}$$

with sufficiently large K. Consequently

$$K^{-n} \prod_{j=1}^n r_j(z) \leq |(E^n)'(z)| \leq K^n \prod_{j=1}^n r_j(z) \tag{8}$$

and there exists a constant $M \geq 1$ such that

$$M^{-n} \left(\prod_{j=1}^n r_j(z) \right)^{-1} \leq \operatorname{diam}(K_n(z)) \leq M^n \left(\prod_{j=1}^n r_j(z) \right)^{-1}. \tag{9}$$

It is easy to see that for $z \in D(h)$ radii $R_n(z)$ and $r_n(z)$ grow superexponentialy fast:

Lemma 2.3. *If $z \in D(h)$ (and q is large enough) then for every $n \in \mathbb{N}$ the following inequality holds*

$$R_{n+1}(z) \geq r_{n+1}(z) \geq \exp\left(\frac{\sqrt{3}}{2} r_n(z) \right) = \exp\left(\frac{\sqrt{3}}{2e^\pi} R_n(z) \right).$$

Proof. Take q so large that (1) is satisfied for all $x \geq q$. Since

$$h(|E^n(z)|) \leq \frac{|E^n(z)|}{2e^\pi} \leq \frac{R_n(z)}{2e^\pi} = \frac{r_n(z)}{2} \leq r_n(z),$$

there exists $z' \in E^n(K_{n-1}(z))$ such that $|z'| = r_n(z)$ and $\operatorname{Im} z' = h(|E^n(z)|)$. Then

$$r_{n+1}(z) \geq |\lambda| \exp(\operatorname{Re}(E^n(z)) - \pi) \geq |\lambda| e^{-\pi} \exp(\operatorname{Re} z').$$

But

$$\operatorname{Re} z' \geq \sqrt{r_n(z)^2 - (h(|E^n(z)|)^2} \geq \frac{\sqrt{3}}{2} r_n(z).$$

Hence

$$r_{n+1}(z) \geq e^{-\pi}|E(z')| = |\lambda| \exp\bigl(\operatorname{Re}(z') - \pi\bigr) \geq |\lambda| e^{-\pi} \exp\left(\frac{\sqrt{3}}{2} r_n(z)\right).$$

$\square$

3 Main technical result

In this section we prove our main technical result, Proposition 3.3 and its immediate consequence, Theorem B. Let ∂_∞ be the set of those points z in I^0_∞ that $B_n(z) \cap \partial E(B_{n-1}(z)) \neq \emptyset$ for infinitely many n. As the first little lemma, we show that the set ∂_∞ can be neglected in our considerations.

Lemma 3.1. $\operatorname{HD}(\partial_\infty) \leq 1$.

Proof. Let

$$\partial_n = \bigcup_{z \in I^0_\infty} E^{-n}(A_+(r_n(z), r_n(z) + 2\pi) \cup A_+(R_n(z) - 2\pi, R_n(z)))$$

where $A_+(a, b)$ denotes the half-annulus with radii a and b. For every $k \geq 0$ the set $\bigcup_{n \geq k} \partial_n$ is a covering of ∂_∞. The half-annuli $A_+(r_n(z), r_n(z) + 2\pi) \cup A_+(R_n(z) - 2\pi, R_n(z))$ can be covered with $M_1 R_n(z)$ sets of diameters less than 1, where M_1 is a constant. Therefore, because of (6), $K_{n-1}(z) \cap \partial_n$ can be covered with no more than $M_1 R_n(z)$ sets $J_{i,n}(z)$ of diameters less than $M_2 |(E^n)'(z)|^{-1}$, where M_2 is a constant. Fix $T \geq 2q$. Since any two sets $K_{n-1}(z)$ and $K_{n-1}(z')$ are either disjoint or equal, one can find a finite set $Z_n \subset I^0_\infty$ such that $K_{n-1}(z)$ and $K_{n-1}(z')$ are disjoint for $z, z' \in Z_n$, $z \neq z'$ and

$$\partial_n \cap B_+(0, T) \subset \bigcup_{z \in Z_n} K_{n-1}(z) \subset B_+(0, 2T),$$

where $B_+(0, r)$ denotes half-disk $B(0, r) \cap \{z : \operatorname{Re} z > 0\}$. Fix now $\epsilon > 0$ and take $n \geq 1$ so large that Lemma 2.2 is satisfied for $\alpha = 2/\epsilon$ and $2T$.

Applying this lemma, using (7) and (4), we get

$$\begin{aligned}
\sum_{z\in Z_n}\sum_{J_{i,n}}(\operatorname{diam} J_{i,n}(z))^{1+\epsilon} &\leq \sum_{z\in Z_n} M_1M_2^{1+\epsilon}R_n(z)|(E^n)'(z)|^{-(1+\epsilon)}\\
&\leq KM_1M_2^{1+\epsilon}\sum_{z\in Z_n}|E'(E^{n-1}(z))|\cdot|(E^n)'(z)|^{-(1+\epsilon)}\\
&\leq KM_1M_2^{1+\epsilon}\sum_{z\in Z_n}|(E^n)'(z))|^{-\epsilon}|(E^{n-1})'(z)|^{-1}\\
&\leq KL^{-\epsilon}M_1M_2^{1+\epsilon}\sum_{z\in Z_n}|(E^{n-1})'(z)|^{-2}|(E^{n-1})'(z)|^{-1}\\
&\leq KL^{-\epsilon}M_1M_2^{1+\epsilon}e^{-(n-1)}\sum_{z\in Z_n}|(E^{n-1})'(z)|^{-2}
\end{aligned}$$

Since the Lebesgue measure of each set of the form $K_{n-1}(z)$ is proportional to $|(E^{n-1})'(z)|^{-2}$, we get that there exists a contstant $M_3 > 0$ such that the last term in the above inequality is less than or equal to $M_3e^{-(n-1)}\cdot$ area$(B_+(0,2T))$.
Hence

$$\sum_{n=k}^{\infty}\sum_{z\in Z_n}\sum_{J_{i,n}}(\operatorname{diam} J_{i,n}(z))^{1+\epsilon} \leq M_3\cdot\text{area}(B_+(0,2T))\sum_{n=k}^{\infty}e^{-(n-1)} = 2\pi T^2M_3e^2\frac{e^{-k}}{e-1}.$$

Since $\lim_{k\to\infty} e^{-k} = 0$, we therefore conclude that the $(1+\epsilon)$-dimensional Hausdorff measure of $\partial_\infty \cap B_+(0,T)$ is equal to 0 for every $\epsilon > 0$. Hence $\mathrm{HD}(\partial_\infty) \leq 1$. □

Let $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a weakly exp-adapted function and let $\delta \in [0,1]$.

Definition 3.2. We say that the sequence of real positive numbers $\{x_n\}_{n=1}^{\infty}$ satisfies the condition $\Delta(h,\delta)$ if for every $c > 0$ there exists n_0 such that for every $n \geq n_0$ the following inequality holds

$$c^n\frac{x_1^\delta..x_{n-1}^\delta}{h(x_1)..h(x_{n-1})} \leq \left(\frac{h(x_n)}{x_n}\right)^\delta. \tag{10}$$

We say that the sequence $\{x_n\}_{n=1}^{\infty}$ satisfies condition $\Delta'(h,\delta)$ if for every $c > 0$ and for every n_0 there exists $n \geq n_0$ such that

$$c^n\frac{x_1^\delta..x_{n-1}^\delta}{h(x_1)..h(x_{n-1})} > \left(\frac{h(x_n)}{x_n}\right)^\delta. \tag{11}$$

The proof of Theorem A is based on the following.

Theorem B. *Suppose that $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is an exp-adapted function. If for every $z \in D(h)$ the sequence $r_n(z)$ satisfies condition $\Delta(h, \delta)$ then*

$$\mathrm{HD}(D(h)) \geq 1 + \delta.$$

If for every $z \in D(h)$ the sequence $R_n(z)$ satisfies condition $\Delta'(h, \delta)$ then

$$\mathrm{HD}(D(h)) \leq 1 + \delta.$$

This theorem is in turn an immediate consequence of Proposition 3.3 formulated below. If $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a weakly exp-adapted function and $q \geq 0$ is large enough, then consider the set

$$D_*(h) = D_*^q(h) = \{z \in I_\infty^{0,q} : 2\pi(|s_n(z)| + 1) \leq h(|E^n(z)|) \ \text{ for all } n \geq 0\}.$$

Proposition 3.3. *Suppose that $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a weakly exp-adapted function. If for every $z \in D_*(h)$ the sequence $r_n(z)$ satisfies condition $\Delta(h, \delta)$ then $\mathrm{HD}(D_*(h)) \geq 1 + \delta$.*
If for every $z \in D_*(h)$ the sequence $R_n(z)$ satisfies condition $\Delta'(h, \delta)$ then $\mathrm{HD}(D_*(h)) \leq 1 + \delta$.

Before proving this proposition, we shall show how it implies Theorem B. Indeed, first, if h is exp-adapted, then $D_*(h) \subset D(h)$. Secondly, if h is exp-adapted, then $2h$ is weakly exp-adapted and $D(h) \subset D_*(2h)$. Now, to finish the argument it suffices to notice that if a sequence of positive reals satisfies the condition $\Delta(h, \delta)$ (respectively $\Delta'(h, \delta)$), then it also satisfies the condition $\Delta(2h, \delta)$ (respectively $\Delta'(2h, \delta)$).

We shall now pass to the proof of Proposition 3.3 which is based on a Cantor-type construction and several lemmas. Fix an arbitrary square $B_0 \in \mathcal{B}$. We shall construct inductively a sequence (actually two sequences) $\{\mathcal{B}^n\}_{n=0}^\infty$ of subfamilies of $\mathcal{B}$ as follows. Put $\mathcal{B}^0 = \{B_0\}$ and suppose that the family $\mathcal{B}^n$ has been constructed. Consider $B \in \mathcal{B}^n$. Denote by $r(E(B))$ and $R(E(B))$ the inner and outer radius (respectively) of the half-annulus $E(B)$. Select from $\mathcal{B}$ all the squares Q that are contained in the half-annulus $E(B)$ and with the property that

$$\min_{z \in Q}\{|\operatorname{Im} z|\} \leq \sup_{z \in Q}\{h(|z|)\} \tag{12}$$

(1st variant, if chosen in the 1st step kept forever)

or alternatively:

(*) All the squares $Q \in \mathcal{B}$ that are contained in the right hand-side half-annulus $A_+(2r(E(B)), \frac{1}{2}R(E(B)))$

and with the property that

$$4\pi + \max_{z\in Q}\{|\operatorname{Im} z|\} \le \inf_{z\in Q}\{h(|z|)\} \tag{13}$$

(2nd variant, if chosen in the 1st step kept forever)

These squares will be called the successors of B. The family $\mathcal{B}^{n+1}$ consists of all successors of all squares from $\mathcal{B}^n$. Notice that if $B_i \in \mathcal{B}^i$, $0 \le i \le n$, and B_{i+1} is a successor of B_i then there exists a unique holomorphic inverse branch $E_*^{-n} : B_n \to B_0$ of E^n such that $E^i(E_*^{-n}(B_n)) \subseteq B_i$ for all $i = 0, 1, \dots, n$. Let $E^{-n}(\mathcal{B}^n)$ be the family of all sets $E_*^{-n}(B_n)$, where $B_n \in \mathcal{B}^n$. If $K_{n+1} \in E^{-(n+1)}(\mathcal{B}^{n+1})$ then there exists a unique $K_n \in E^{-n}(\mathcal{B}^n)$ such that $K_{n+1} \subseteq K_n$. The set K_{n+1} will be called a child of K_n. The family of all children of K_n will be denoted $ch(K_n)$.

For every $n \ge 0$ define X_n to be the union of closures of all elements of $E^{-n}(\mathcal{B}^n)$. Clearly $X_{n+1} \subseteq X_n$. Now we construct the sequence $\{\mu_n\}_{n=0}^{\infty}$ of Borel probability measures on the sets X_n as follows. Let μ_0 be the normalized Lebesgue measure on $X_0 = \overline{B_0}$. Suppose now that the measure μ_n on X_n has been defined. The measure μ_{n+1} on X_{n+1} is defined on each $K_{n+1} \in E^{-(n+1)}(\mathcal{B}^{n+1})$ as follows

$$\mu_{n+1}|_{K_{n+1}} = \frac{\operatorname{area}(K_{n+1})}{\sum_{K\in ch(K_n)} \operatorname{area}(K)} \cdot \mu_n|_{K_n}, \tag{14}$$

where K_n is the unique element of $E^{-n}(\mathcal{B}^n)$ containing K_{n+1}. Notice that $\mu_{n+1}(K_n \cap X_{n+1}) = \mu_n(K_n)$, and therefore

$$\mu_k(K_n) = \mu_n(K_n).$$

for all $k \ge n$ and every $K_n \in E^{-n}(\mathcal{B}^n)$. Thus, there exists a unique Borel measure μ on the set $X_\infty = \bigcap_{n\ge 0} X_n$ (denoted respectively by X_∞^1 and X_∞^2 if we want to indicate that it comes from the first or second variant of our construction) such that

$$\mu(K_n) = \mu_n(K_n) \qquad \text{for every} \quad K_n \in E^{-n}(\mathcal{B}^n). \tag{15}$$

It follows from (14), (15), (9) and (6) that

$$\begin{aligned} \mu(K_n(z)) &\asymp \operatorname{area}(K_n(z)) \cdot \prod_{i=1}^{n} \frac{\operatorname{area}(K_{i-1}(z))}{\sum_{K\in ch(K_{i-1}(z))} \operatorname{area}(K)} \\ &= c(z)^{2n} \left(\prod_{j=1}^{n} r_j(z) \right)^{-2} \prod_{i=1}^{n} \frac{\operatorname{area}(K_{i-1}(z))}{\sum_{K\in ch(K_{i-1}(z))} \operatorname{area}(K)} \end{aligned}$$

with some $c(z) \in [M^{-1}, M]$. But by (6) and by the inductive step of our construction,

$$\frac{\operatorname{area}(K_{i-1}(z))}{\sum_{K\in ch(K_{i-1}(z))} \operatorname{area}(K)} \asymp \frac{\operatorname{area} E^i(K_{i-1}(z))}{\sum_{K\in ch(K_{i-1}(z))} \operatorname{area}(E^i(K))} \asymp \frac{r_i^2(z)}{h(r_i(z))r_i(z)}.$$

Hence, there exists a constant $C \geq 1$ such that with some $c_1(z), c_2(z) \in [(CM)^{-1}, CM]$, we have that

$$\begin{aligned} \mu(K_n(z)) &= c_1^n(z)\left(\prod_{j=1}^n r_j(z)\right)^{-2} \prod_{i=1}^n \frac{r_i^2(z)}{r_i(z)h(r_i(z))} \qquad (16) \\ &= c_1^n(z)\left(\prod_{j=1}^n r_j(z)\right)^{-(1+\delta)} \prod_{i=1}^n \frac{r_i^\delta(z)}{h(r_i(z))} \\ &= c_2^n(z)\operatorname{diam}(K_n(z))^{1+\delta} \prod_{i=1}^n \frac{r_i^\delta(z)}{h(r_i(z))}. \end{aligned}$$

Let ∂_∞^0 be the set of those points z in I_∞^0 that $B_n(z)$ never intersects the boundary of the half annulus $E(B_{n-1}(z))$. Of crucial importance is the following straightforward.

Observation 3.4. $B_0 \cap D_*(h) \cap \partial_\infty^0 \subseteq X_\infty^1$ and $D_*(h) \supseteq X_\infty^2$.

Proof. Fix $z \in B_0 \cap D_*(h) \cap \partial_\infty^0$. Our aim is to show that $B_n(z) \in \mathcal{B}^n$ (first variant construction) for all $n \geq 0$. For $n = 0$ this is obvious and we proceed by induction. Suppose that $B_n(z) \in \mathcal{B}^n$ for some n. Since $z \in D_*(h)$ we obtain

$$\min_{w\in B_{n+1}(z)}\{|\operatorname{Im} w|\} \leq 2\pi(|s_{n+1}(z)|+1) \leq h(|E^{n+1}(z)|) \leq \sup_{w\in B_{n+1}(z)} h(|w|),$$

and since $B_{n+1}(z)$ does not intersect the boundary of $E(B_n(z))$ (as $z \in \partial_\infty^0$), we get that $B_{n+1}(z) \subseteq E(B_n(z))$. Thus $B_{n+1}(z) \in \mathcal{B}^{n+1}$ and, in consequence, $z \in X_\infty^1$.
Assume now that $z \in X_\infty^2$. Then

$$\begin{aligned} 2\pi(|s_{n+1}(z)|+1) = 2\pi(|s_{n+1}(z)|-1) + 4\pi &\leq \max_{w\in B_{n+1}(z)}\{|\operatorname{Im} w|\} + 4\pi \\ &\leq \inf_{w\in B_{n+1}(z)}\{h(|w|)\} \leq h(|E^{n+1}(z)|). \end{aligned}$$

So, $z \in D_*(h)$. $\square$

We shall now prove the second part of Theorem B.

Lemma 3.5. *If for every $z \in D_*(h)$ the sequence $r_n(z)$ satisfies condition $\Delta'(h,\delta)$ then* $\mathrm{HD}(D_*(h)) \le 1+\delta$.

Proof. For every $n \ge 0$ let ∂_∞^n be the set of all points $z \in I_\infty^0$ such that $B_{j+1}(z) \cap \partial E(B_j(z)) = \emptyset$ for all $j \ge n$. Notice that $\partial_\infty = \bigcup_{n\ge 0} I_\infty^0 \setminus \partial_\infty^n$. By Lemma 3.1 it is enough to show that $\mathrm{HD}(D_*(h) \cap \partial_\infty^n) \le 1+\delta$ for every $n \ge 1$. And since $E^n(D_*(h) \cap \partial_\infty^n) \subset D_*(h) \cap \partial_\infty^0$, it is in fact sufficient to demonstrate that $\mathrm{HD}(D_*(h) \cap \partial_\infty^0) \le 1+\delta$. Obviously it suffices to prove that $\mathrm{HD}(D_*(h) \cap \partial_\infty^0) \le 1+\delta$. And eventually, by Observation 3.4, it is sufficient to show that

$$\mathrm{HD}(X_\infty^1) \le 1+\delta.$$

Let μ be the measure on X_∞^1 constructed above. Fix $n \ge 2$. Cover X_∞^1 by countably many mutually disjoint sets $K_{n-1}(z_j)$ such that $z_j \in X_\infty^1$ for all j. Fix $z_j = z$. Consider an arbitrary set $F \subseteq E(B_{n-1}(z)) = E^n(K_{n-1}(z))$. Then in view of (8)

$$\mathrm{diam}(E_z^{-n}(F)) \le K^n \left(\prod_{k=1}^n r_k(z)\right)^{-1} \cdot \mathrm{diam}\, F \tag{17}$$

where $E_z^{-n} : E(B_{n-1}(z)) \to K_{n-1}(z)$ is the unique holomorphic inverse branch of E^n defined on $E(B_{n-1}(z))$ and sending $E^n(z)$ to z. Now consider the covering $\mathcal{G}_z^n$ of $E(B_{n-1}(z)) \cap \{w : |\,\mathrm{Im}\, w| \le h(R_n(z))\}$ by squares G with the following properties

- the length of each edge of G is equal to $h(R_n(z))$
- one of the horizontal edges of G is contained in the real axis
- at least two of the edges of G are contained in $E(B_{n-1}(z))$.

By (16) we therefore get

$$\mu(K_n(z)) = c_1^n(z) \left(\prod_{j=1}^n r_j(z)\right)^{-(1+\delta)} \prod_{i=1}^n \frac{r_i^\delta(z)}{h(r_i(z))}. \tag{18}$$

Let $\tilde{G} = G \cap E(B_{n-1}(z))$ and put $\tilde{\mathcal{G}}_z^n = \{G \cap E(B_{n-1}(z)) : G \in \mathcal{G}_z^n\}$. Assuming that n is big enough (which implies that $h(R_n(z))$ is as large as we wish) we see that there exists a universal constant $\kappa \in (0,1)$ such that G contains at least $\kappa h^2(R_n(z)) \ge \kappa h^2(r_n(z))$ squares from $\mathcal{B}^n$. Since

$\operatorname{diam} G \leq \sqrt{2}h(R_n(z)) \leq C\sqrt{2}h(r_n(z))$, using (18) and (17) we therefore get

$$\begin{aligned}
&\mu(E_z^{-n}(\tilde{G})) \geq \\
&\geq \kappa h^2(r_n(z))c_1^n(z)\left(\prod_{i=1}^n r_i(z)\right)^{-(1+\delta)} \prod_{i=1}^n \frac{r_i^\delta(z)}{h(r_i(z))} \\
&= \kappa(C\sqrt{2})^{-(1+\delta)}c_1^n(z)\left(\frac{C\sqrt{2}h(r_n(z))}{\prod_{i=1}^n r_i(z)}\right)^{1+\delta}\left(\frac{r_n(z)}{h(r_n(z))}\right)^\delta \cdot \prod_{i=1}^{n-1}\frac{r_i^\delta(z)}{h(r_i(z))} \\
&\geq \frac{\kappa(CK^{1+\delta})^{-n}}{(C\sqrt{2})^{1+\delta}}(\operatorname{diam}(E^{-n}(\tilde{G})))^{1+\delta}\left(\frac{r_n(z)}{h(r_n(z))}\right)^\delta \cdot \prod_{i=1}^{n-1}\frac{r_i^\delta(z)}{h(r_i(z))}.
\end{aligned}$$

Using (11) and the assumptions of our lemma, we thus obtain

$$\mu(E^{-n}(\tilde{G})) \geq (\operatorname{diam}(E^{-n}(\tilde{G})))^{1+\delta}. \tag{19}$$

The squares G need not be disjoint but it is possible to choose the covering $\mathcal{G}$ so that its multiplicity does not exceed 2. Since the union of all squares $\tilde{G} \in \tilde{\mathcal{G}}_z^n$ contains all the successors of $E^{n-1}\big(K_{n-1}(z)\big)$, the set $E_z^{-n}\left(\bigcup_{\tilde{G}\in\tilde{\mathcal{G}}_z^n}\tilde{G}\right)$ covers all the children of $K_{n-1}(z)$, and consequently, covers $K_{n-1}(z) \cap X_\infty^1$. Hence

$$\bigcup_j \bigcup_{\tilde{G}\in\tilde{\mathcal{G}}_{z_j}^n} E_{z_j}^{-n}(\tilde{G}) \supset X_\infty^1.$$

By (19)

$$\begin{aligned}
\sum_j \sum_{\tilde{G}\in\tilde{\mathcal{G}}_{z_j}^n} (\operatorname{diam} E_{z_j}^{-n}(\tilde{G}))^{1+\delta} &\leq \sum_j \sum_{\tilde{G}\in\tilde{\mathcal{G}}_{z_j}^n} \mu(E_{z_j}^{-n}(\tilde{G})) \\
&\leq \sum_j \mu\big(K_{n-1}(z_j)\big) \leq 2\mu(B_0).
\end{aligned}$$

Since the diameters of the sets $E_{z_j}^{-n}(\tilde{G})$, $j \geq 1$, $\tilde{G} \in \tilde{\mathcal{G}}_{z_j}^n$, converge uniformly to 0 as $n \to \infty$, we conclude that the Hausdorff measure of the set X_∞^1 is less or equal than 2. Hence $\mathrm{HD}(X_\infty^1) \leq 1+\delta$ and the proof is complete. $\square$

Lemma 3.6. *If for every $z \in D_*(h)$ the sequence $r_n(z)$ satisfies condition $\Delta(h,\delta)$ then $\mathrm{HD}(D_*(h)) \geq 1+\delta$.*

Proof. By Observation 3.4 that it is sufficient to show that

$$\mathrm{HD}(X_\infty^2) \geq 1+\delta.$$

It follows from (10) and (16) that for all $Q \in E^{-n}(\mathcal{B}^n)$ and all $n \geq 1$ large enough,

$$\begin{aligned}\mu(Q) &\leq c_2^n(z) \prod_{i=1}^{n} \frac{r_i^\delta(z)}{h(r_i(z))} (\operatorname{diam} Q)^{1+\delta} \\ &\leq c_2^n(z) \prod_{i=1}^{n-1} \frac{r_i^\delta(z)}{h(r_i(z))} \frac{r_n^\delta(z)}{h^\delta(r_n(z))} (\operatorname{diam} Q)^{1+\delta} \\ &\leq (\operatorname{diam} Q)^{1+\delta}\end{aligned} \tag{20}$$

Fix a constant $D \geq 2$. Take an arbitrary point $z \in X_\infty^2$. Our aim is to show that if r is small then $\mu(B(z,r)) \leq \mathrm{const} \cdot r^{1+\delta}$. Take the least $n \geq 1$ such that

$$\operatorname{diam} K_n(z) \leq r. \tag{21}$$

Consider an arbitrary $Q \in E^{-n}(\mathcal{B}^n)$ such that $Q \cap B(z,r) \neq \emptyset$. Using (6), (*) and (21), we obtain that if $r > 0$ is sufficiently small (so that $n \geq 1$ is large enough), then

$$K^{-1} \operatorname{diam} K_n(z) \leq \operatorname{diam} Q \leq K \operatorname{diam} K_n(z) \leq Kr. \tag{22}$$

Since $\delta \in [0,1]$, applying this, (20) and (21), we get that

$$\mu(Q) \leq (\operatorname{diam} Q)^{1+\delta} \leq K^{1+\delta} (\operatorname{diam} K_n(z))^{1+\delta} \leq K^{1+\delta} r^{1+\delta}. \tag{23}$$

Denote the family of all sets $Q \in E^{-n}(\mathcal{B}^n)$ intersecting $B(z,r)$ by $\mathcal{F}(z,r)$. We shall consider several cases.

Case 1: $r \leq D \operatorname{diam} K_n(z)$
Since $z \in X_\infty^2$, we get

$$\begin{aligned} K_{n-1}(z) &= E_z^{-n}(A_+(r_n(z), R_n(z))) \supseteq E_z^{-n}(B(E^n(z), r_n(z))) \\ &\supseteq B(z, K^{-1}|(E^n)'(z)|^{-1} r_n(z)) \\ &\supseteq B(z, (\sqrt{2}\pi K^2)^{-1} r_n(z) \operatorname{diam} K_n(z)) \\ &\supseteq B(z, D(K+1) \cdot \operatorname{diam} K_n(z)) \supseteq B(z, (K+1)r) \end{aligned} \tag{24}$$

if n is sufficiently large for $r_n(z)$ to be bigger than $\sqrt{2}\pi K^2(K+1)D$ (which happens if $r > 0$ is small enough). By (22) and (24), if $Q \in \mathcal{F}(z,r)$, then $Q \subset K_{n-1}(z)$. Since E^n is injective on $K_{n-1}(z)$ and since $\operatorname{diam}\bigl(E^n(B(z,r))\bigr)$ is less than $2Kr|(E^n)'(z)|$, the number of squares $E^n(Q)$, $Q \in \mathcal{F}(z,r)$ is bounded above by $4K^2\pi^{-1}r^2|(E^n)'(z)|^2$. Since

$$r \leq D \operatorname{diam} K_n(z) \leq DK\sqrt{2}\pi|(E^n)'(z)|^{-1},$$

we obtain that

$$\sharp\{E^n(Q):\ Q\in\mathcal{F}(z,r)\}\le 8K^4M^2\pi.$$

But $\sharp\{E^n(Q):\ Q\in\mathcal{F}(z,r)\}=\sharp\mathcal{F}(z,r)$. So using (23) we get that

$$\mu(B(z,r))\le\sum_{Q\in\mathcal{F}(z,r)}\mu(Q)\le\sum_{Q\in\mathcal{F}(z,r)}K^{1+\delta}r^{1+\delta}\le 8\pi K^{5+\delta}M^2r^{1+\delta}.$$

Case 2: $D\,\mathrm{diam}\,K_n(z)\le r\le D^{-1}\,\mathrm{diam}\,K_{n-1}(z)$
Since $z\in X^2_\infty$, using (7), we get with $D\ge 2$ large enough that

$$\begin{aligned}K_{n-1}(z)&\supseteq B(z,K^{-1}r_n(z)|(E^n)'(z)|^{-1})\supseteq B(z,K^{-2}|(E^{n-1})'(z)|^{-1})\\&\supseteq B(z,(\sqrt{2}\pi K^3)^{-1}\,\mathrm{diam}(K_{n-1}(z)))\supseteq B(z,(K+1)r).\end{aligned}\tag{25}$$

By (22) and (25), if $Q\in\mathcal{F}(z,r)$, then $Q\subseteq K_{n-1}(z)$. Hence applying (18), we get that

$$\begin{aligned}\mu(B(z,r))&\le\sum_{Q\in\mathcal{F}(z,r)}\mu(Q)\\&\le\sum_{Q\in\mathcal{F}(z,r)}c_1^n(z)\left(\prod_{i=1}^n r_i(z)\right)^{-(1+\delta)}\prod_{i=1}^n\frac{r_i^\delta(z)}{h(r_i(z))}\\&=\sharp\mathcal{F}(z,r)\cdot c_1^n(z)\prod_{i=1}^n\frac{r_i^{-1}(z)}{h(r_i(z))}.\end{aligned}\tag{26}$$

Now we consider two subcases.

Case 2a: $\mathrm{diam}\big(E^n(B(z,r))\big)\le 2h(r_n(z))$.
Since $\sharp\mathcal{F}(z,r)\preceq\mathrm{area}\big(E^n(B(z,r))\big)$, using (8), we can continue (26) as follows.

$$\begin{aligned}\mu(B(z,r))&\preceq c_1^n(z)\,\mathrm{area}(E^n(B(z,r))\prod_{i=1}^n\frac{(r_i(z))^{-1}}{h(r_i(z))}\\&\le\pi K^nc_1^n(z)r^2\prod_{i=1}^n r_i^2(z)\prod_{i=1}^n\frac{(r_i(z))^{-1}}{h(r_i(z))}\\&\le K^nC^nr^2\prod_{i=1}^n\frac{r_i(z)}{h(r_i(z))}\\&=K^nC^nr^{1+\delta}r^{1-\delta}\prod_{i=1}^n\frac{r_i(z)}{h(r_i(z))}.\end{aligned}$$

But $\operatorname{diam}\big(E^n(B(z,r))\big) \le 2h(r_n(z))$, so using (8) again, we get that

$$r \cdot \prod_{i=1}^{n} r_i(z) \le K^n h(r_n(z)).$$

Therefore applying (10) with $c = K^{2-\delta}C$, we finally obtain

$$\begin{aligned}
\mu(B(z,r)) &\le K^n C^n r^{1+\delta} \left(K^n \frac{h(r_n(z))}{r_1(z)..r_n(z)} \right)^{1-\delta} \cdot \prod_{i=1}^{n} \frac{r_i(z)}{h(r_i(z))} \\
&= r^{1+\delta} (CK^{2-\delta})^n \cdot \prod_{i=1}^{n-1} \frac{r_i(z)^\delta}{h(r_i(z))} \cdot \left(\frac{r_n(z)}{h(r_n(z))} \right)^\delta \\
&\le r^{1+\delta},
\end{aligned}$$

and we are done in this case.

Case 2b: $\operatorname{diam}\big(E^n(B(z,r))\big) > 2h(r_n(z))$.
In this case we estimate the cardinality of $\mathcal{F}(z,r)$ in a different way. Using first (13) and then (8), we get

$$\begin{aligned}
\sharp\mathcal{F}(z,r) &\preceq \operatorname{area}(E^n(B(z,r)) \cap \{w : \ |\operatorname{Im} w| \le h(R_n(z))\}) \\
&\le 2 \operatorname{diam} E^n(B(z,r)) h(R_n(z)) \\
&\le 2Ch(r_n(z)) \cdot 2rK^n \cdot \prod_{i=1}^{n} r_i(z).
\end{aligned}$$

Hence, we can continue (26) as follows.

$$\begin{aligned}
\mu(B(z,r)) &\preceq 4C(Kc_1(z))^n r \frac{h(r_n(z))}{h(r_1(z))...h(r_n(z))} \\
&\le 4C(CKM)^n r^{1+\delta} r^{-\delta} \prod_{i=1}^{n-1} (h(r_i(z)))^{-1}.
\end{aligned}$$

Since $\operatorname{diam}\big(E^n(B(z,r))\big) > 2h(r_n(z))$, we get from (8) that

$$r \cdot r_1(z)..r_n(z) \ge K^{-n} h(r_n(z)).$$

Thus

$$\mu(B(z,r)) \le 4CK(CMK^{1+\delta})^n r^{1+\delta} \cdot \prod_{i=1}^{n-1} \frac{(r_i(z))^\delta}{h(r_i(z))} \cdot \left(\frac{r_n(z)}{h(r_n(z))} \right)^\delta.$$

Using (10) we obtain

$$\mu(B(z,r)) \le r^{1+\delta}.$$

Case 3: $r > D^{-1} \operatorname{diam} K_{n-1}(z)$.
This means that $\operatorname{diam} K_{n-1}(z) < Dr$. But by our choice of n, $\operatorname{diam} K_{n-1}(z) > r = D^{-1}(Dr)$. This implies that $\operatorname{diam} K_{n-2}(z) > Dr$ if $r > 0$ is small enough so that $n \ge 1$ is large enough. So, $n-1$ is the number ascribed to the radius Dr as in the begining of the proof and the *Case 1* holds. Therefore

$$\mu(B(z,r)) \le \mu(B(z,Dr)) \le (Dr)^{1+\delta}.$$

□

4 Conclusions

It is obvious that for every $\epsilon > 0$ the function

$$h_\epsilon(x) = \frac{x}{(\log x)^\epsilon}$$

is exp-adapted. In order to prove the Theorem A, it is therefore sufficient to apply Theorem B and make use of the following observation.

Proposition 4.1. *If $z \in D(h_\epsilon) = D_\epsilon$, then the sequence $r_n(z)$ satisfies condition $\Delta(h_\epsilon, \delta)$ for every $\delta < \frac{1}{1+\epsilon}$ and the sequence $R_n(z)$ satisfies $\Delta'(h_\epsilon, \delta)$ for every $\delta > \frac{1}{1+\epsilon}$.*

Proof. Since $R_n(z) = e^\pi r_n(z)$ we need only to check that $r_n(z)$ satisfies $\Delta(h_\epsilon, \delta)$ if $\delta < \frac{1}{1+\epsilon}$ and $\Delta'(h_\epsilon, \delta)$ if $\delta > \frac{1}{1+\epsilon}$.
The inequality in $\Delta(h_\epsilon, \delta)$ is equivalent to the following

$$c^n (\log r_n(z))^{\epsilon\delta} \le \frac{(r_1(z) ... r_{n-1}(z))^{1-\delta}}{(\log r_1(z) ... \log r_{n-1}(z))^\epsilon}.$$

Since $r_{n-1}(z) \asymp \log r_n(z)$, it follows from lemma 2.3 that the above inequality holds (for sufficiently big n) if $\epsilon\delta < 1 - \delta$.
Similarily, if $\epsilon\delta > 1 - \delta$ then for n big enough

$$c^n (\log r_n(z))^{\epsilon\delta} > \frac{(r_1(z) .. r_{n-1}(z))^{1-\delta}}{(\log r_1(z) .. \log r_{n-1}(z))^\epsilon}.$$

which is our claim. □

In this way we have proved that

$$\mathrm{HD}(D_\epsilon) \ge 1+\delta \quad \text{for every } 0<\delta<\frac{1}{1+\epsilon}$$

and

$$\mathrm{HD}(D_\epsilon) \le 1+\delta \quad \text{for every } \ \delta>\frac{1}{1+\epsilon}.$$

Passing with δ to the limit $\frac{1}{1+\epsilon}$, we obtain

$$\mathrm{HD}(D_\epsilon) = 1+\frac{1}{1+\epsilon}.$$

Since $\mathrm{HD}\left(\bigcup_{n=0}^\infty E^{-n}(D_\epsilon)\right) = \mathrm{HD}(D_\epsilon)$, Theorem A is thus proved.

Bogusława Karpińska
Faculty of Mathematics and Information Sciences
Warsaw University of Technology
Plac Politechniki 1
Warsaw 00-661
Poland
E-mail: bkarpin@impan.gov.pl

Mariusz Urbański
Department of Mathematics
University of North Texas
P.O. Box 311430
Denton, TX 76203-1430
USA
E-mail:urbanski@unt.edu
Web: http://www.math.unt.edu/∼urbanski

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