Diophantine Extremality of the Patterson Measure

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Abstract

We derive universal Diophantine properties for the Patterson measure μ_G associated with a convex cocompact Kleinian group G acting on $(n+1)$-dimensional hyperbolic space. We show that μ_G is always a $\mathcal{S}$-friendly measure, for every (G, μ_G)-neglectable set $\mathcal{S}$, and deduce that if G is of non-Fuchsian type then μ_G is an absolutely friendly measure in the sense of [7]. Consequently, by a result of [2], μ_G is strongly extremal which means that the essential support of μ_G has empty intersection with the set of very well multiplicatively approximable points, a set from classical metric Diophantine approximation theory which is of n-dimensional Lebesgue measure zero but which has Hausdorff dimension equal to n.

1 Introduction and statements of results

Recall that in classical Diophantine approximation theory a point $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ is called very well multiplicatively approximable if for some $\tau > 0$ there exist infinitely many $(p_1, \ldots, p_n) \in \mathbb{Z}^n$ and $q \in \mathbb{N}$ such that

$$\left(\prod_{i=1}^{n} \left|\xi_i - \frac{p_i}{q}\right|\right)^{\frac{1}{n}} < \frac{1}{q^{1+\frac{1}{n}+\tau}}.$$

Likewise, the point ξ is called badly approximable if there exists $c > 0$ such that for all $(p_1, \ldots, p_n) \in \mathbb{Z}^n$ and $q \in \mathbb{N}$ we have

$$\sum_{i=1}^{n} \left|\xi_i - \frac{p_i}{q}\right| > \frac{c}{q^{1+\frac{1}{n}}}.$$

It is well-known that the set of very well multiplicatively approximable numbers $\mathcal{W}$ as well as the set of badly approximable numbers $\mathcal{B}$ are both of n-dimensional Lebesgue measure λ_n equal to zero but have Hausdorff dimension $\dim_H$ equal to n ([8]).

A particular outcome of this paper will be to relate these classical Diophantine sets to the limit set $L(G)$ of an arbitrary convex cocompact Kleinian group of non-Fuchsian type acting on $(n+1)$-dimensional space $\mathbb{H}^{n+1}$ such that $L(G) \subset \mathbb{R}^n$ (in here 'non-Fuchsian' refers to that $L(G)$ is not contained in a finite set of spheres or hyperplanes of codimension at least 1). Namely, a consequence of the results in this paper will be that the Patterson measure μ_G of such a Kleinian group is an absolutely friendly measure in the sense of [7]. Combining this with recent results of [4] and the well-known fact that the Patterson measure μ_G of a convex cocompact Kleinian group is Ahlfors-regular, which is an immediate consequence of Sullivan's Shadow Lemma, it follows that

$$\dim_H(\mathcal{B} \cap L(G)) = \dim_H(L(G)).$$

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Furthermore, using a result of [2], absolute friendlyness of μ_G implies that μ_G -almost every element of $L(G)$ is not very well multiplicatively approximable, that is

$$\mu_G(\mathcal{W}) = 0.$$

We remark that the latter result is closely related to work of Kleinbock and Margulis in [3] which lead to a confirmation of a conjecture of Sprindzuk [9]. This conjecture states that λ_n is strongly extremal for each non-degenerate submanifold of $\mathbb{R}^n$, where a measure ν is strongly extremal if $\nu(\mathcal{W}) = 0$ (cf. [2]). In this terminology our afore mentioned result therefore says that for every convex cocompact Kleinian group of non-Fuchsian type the Patterson measure is strongly extremal.

In order to state the results in greater detail let us first fix some notation. For convenience, throughout we shall work with the Poincaré model $\left(\mathbb{D}^{n+1}, d\right)$ of $(n+1)$-dimensional hyperbolic space. In this model the boundary at infinity of hyperbolic space is given by the n-sphere $\mathbb{S}^n := \{\xi \in \mathbb{R}^{n+1} : ||\xi|| = 1\}$, which is conformally equivalent to the one-point compactification of $\mathbb{R}^n$. A Kleinian group G is a discrete subgroup of $Con(n)$, where $Con(n)$ refers to the group of isometries of $\mathbb{D}^{n+1}$. Let $L(G)$ denote the limit set of G, that is the smallest closed G-invariant subset of $\mathbb{S}^n$. We always assume that G is non-elementary convex cocompact, which means that G has a fundamental domain with finitely many sides, does not have parabolic elements and $L(G)$ is an uncountable proper subset of $\mathbb{S}^n$. Also, by Ahlfors's Lemma we can assume without loss of generality that G does not have elliptic elements. It is well-known that in this situation $\dim_H(L(G)) = \delta > 0$, where $\delta = \delta(G) := \inf\left\{s \geq 0 : \sum_{g\in G} \exp(-sd(0, g(0))) < \infty\right\}$ refers to the exponent of convergence of G (see [1], [11]). Finally, recall that to each G we can associate a Patterson measure μ_G. This measure is always a probability measure supported on $L(G)$, and it is unique in the convex cocompact case. Furthermore, μ_G is a δ-conformal measure, which means that for each Borel set $S \subset \mathbb{S}^n$ and for all $g \in G$,

$$\mu_G(g(S)) = \int_S |g'(\xi)|^\delta \, d\mu_G(\xi).$$

For the purposes of this paper the following two definitions will be crucial. For this recall that a measure ν on $\mathbb{R}^n$ is called doubling (or Federer) if there exists a constant $c > 1$ such that for each $\xi \in \mathbb{R}^n$ and $r > 0$ we have $\nu(B(\xi, 2r)) \leq c\nu(B(\xi, r))$, where $B(\xi, r)$ refers to the open n-ball centred at ξ of radius r. Also, throughout we shall use the notation $a \asymp b$ for two positive reals a, b to denote that a/b is uniformly bounded away from zero and infinity. We write $a \ll b$ if a/b is uniformly bounded away from infinity. If not stated otherwise, the bounds will always depend exclusively on the given Kleinian group G.

Definition 1 *Let $G < Con(n)$ be a Kleinian group, and let ν be a finite Borel measure on $\mathbb{S}^n$. A set $\mathcal{S}$ of Borel subsets of $\mathbb{S}^n$ is called (G, ν)-neglectable if and only if the following three conditions are satisfied.*

- *$\mathcal{S}$ is compact with respect to the Hausdorff topology in $\mathbb{S}^n$;*
- *$\nu(S) = 0$ for all $S \in \mathcal{S}$;*
- *$g(S) \in \mathcal{S}$ for all $g \in G, S \in \mathcal{S}$.*

Definition 2 *Let $\mathcal{S}$ be a given (G, ν)-neglectable set. A finite Borel measure ν is called $\mathcal{S}$-friendly if ν is a doubling measure and there exist $\sigma, r_0 > 0$ such that for all $\xi \in L(G), S \in \mathcal{S}, r \in (0, r_0)$ and $s > 0$,*

$$\frac{\nu\left(\mathcal{N}_{s\,r}(S) \cap B(\xi, r)\right)}{\nu\left(B(\xi, r)\right)} \ll s^\sigma.$$

In here, $\mathcal{N}_t(S) := \bigcup_{\xi\in S} B(\xi,t)$ *refers to the* t *-neighbourhood of* S *in* $\mathbb{S}^n$ *.*

The following theorem gives the first main result of this paper.

Theorem 1 *For every convex cocompact Kleinian group* $G < Con(n)$ *and for each* (G,μ_G) *-neglectable set* $\mathcal{S}$ *, we have that the Patterson measure* μ_G *is* $\mathcal{S}$ *-friendly.*

Subsequently, we show that Theorem 1 is applicable in the special situation of [2], [4]. For this we consider the set $\mathcal{H}(G)$ of intersections of $L(G)$ with spheres in $\mathbb{S}^n$ of codimension at least 1. That is, we let

$$\mathcal{H}(G) := \{H \cap L(G) : H \text{ a sphere in } \mathbb{S}^n \text{ of codimension at least } 1\}.$$

Our second main result is stated in the following theorem. In here, a Kleinian group G is called of non-Fuchsian type if $L(G)$ is not a subset of a finite union of elements of $\mathcal{H}(G)$.

Theorem 2 *For every convex cocompact Kleinian group* $G < Con(n)$ *which is of non-Fuchsian type we have that the Patterson measure* μ_G *is* $\mathcal{H}(G)$ *-friendly.*

In [2] and [4] it was shown that if a Ahlfors-regular measure ν in $\mathbb{S}^n$ is absolutely friendly then it is strongly extremal and $\dim_H(\mathcal{B}\cap supp(\nu)) = \dim_H(supp(\nu))$. In here, $supp(\nu)$ refers to the support of ν , and absolutely friendly means that ν is friendly with respect to the set of intersections of $L(G)$ with spheres in $\mathbb{S}^n$ of codimension 1 . Therefore, as an immediate corollary to Theorem 2 we obtain the following result.

- *For each convex cocompact Kleinian group* G *of non-Fuchsian type, the Patterson measure is strongly extremal and* $\dim_H(\mathcal{B}\cap L(G)) = \dim_H(L(G))$ *.*

Remark: We remark that recent work in [12] of the second author shows that for an irreducible finite conformal iterated function system satisfying the open set condition, the Gibbs measure associated with some Hölder family of functions is always absolutely friendly and hence in particular strongly extremal.

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2 Friendlyness of the Patterson measure

2.1 Convex cocompact geometry

We briefly recall some of the geometry which will be important in this paper. For the proofs we refer to [5], [6], [10] and [11].
The shadow map $\Pi : \mathbb{D}^{n+1} \to \mathbb{S}^n$ is given for $A \subset \mathbb{D}^{n+1}$ by $\Pi(A) := \{\xi \in \mathbb{S}^n : s_\xi \cap A \neq \emptyset\}$. In here s_ξ refers to the hyperbolic ray in $\mathbb{D}^{n+1}$ starting at the origin $0 \in \mathbb{D}^{n+1}$ and terminating at ξ . We then have the following three well-known elementary facts from hyperbolic geometry (see [10]).

Elementary Shadow Lemma *Let* $\rho > 0$ *be given. For each* $z \in \mathbb{D}^{n+1}$ *such that* $d(0,z) > \rho$ *, we have for the spherical diameter* $\operatorname{diam}(\Pi(b(z,\rho)))$ *of the shadow of the open hyperbolic ball* $b(z,\rho)$ *centred at* z *and of radius* ρ *,*

$$\operatorname{diam}(\Pi(b(z,\rho))) \asymp_\rho e^{-d(0,z)}.$$

In here, $\asymp_\rho$ *indicates that the involved constants depend on* ρ *.*

Light Cone Lemma *Let* $\rho > \log 2$ *be given. For each* $g \in Con(n)$ *non-elliptic such that* $d(0, g(0)) > \rho$ *, we have*

$$\mathbb{S}^n \setminus B\left(\Pi(g^{-1}(0)), 2\pi e^{-\rho}\right) \subset g^{-1}\left(\Pi(b(g(0),\rho))\right) \subset \mathbb{S}^n \setminus B\left(\Pi(g^{-1}(0)), e^{-\rho}\right).$$

Geometric Distortion Lemma *Let* $\rho > 0$ *be given. For each* $g \in Con(n)$ *non-elliptic such that* $d(0, g(0)) > \rho$ *, we have*

$$\left|\left(\gamma^{-1}\right)'(\xi)\right|^{-1} \asymp_\rho \operatorname{diam}\left(\Pi(b(g(0),\rho))\right) \quad \textit{for each } \xi \in \Pi(b(g(0),\rho)).$$

It is well-known that the convex core $\mathcal{C}(G)$ of a convex cocompact Kleinian group G is a compact subset of $\mathbb{D}^{n+1}$, where $\mathcal{C}(G)$ is defined as the convex hull of the intersection of a fundamental domain of G with the set of geodesics connecting different elements of $L(G)$. Throughout, let $\kappa > 0$ refer to the hyperbolic diameter of $\mathcal{C}(G)$, and let $\tau := \exp(-3\kappa)$. Also, we partition G into 'annuli' as follows

$$A_k := \{g \in G : 3\kappa k \leq d(0, g(0)) < 3\kappa(k+1)\} \text{ for } k \in \mathbb{N}.$$

Combining compactness of $\mathcal{C}(G)$ and the fact that for a convex cocompact Kleinian group the injective radius is uniformly bounded away from zero, we obtain the following.

Hedlund's Lemma *Let* $G < Con(n)$ *be a convex cocompact Kleinian group. For each* $\xi \in L(G)$ *and* $k \in \mathbb{N}$ *, there exists* $g \in A_k$ *such that* $\xi \in \Pi(b(g(0),\kappa))$ *.*

Recall that an element $\xi \in L(G)$ is called uniformly radial if $s_\xi \subset \bigcup_{g\in G} b(g(0),\rho)$, for some $\rho > 0$. Therefore, an immediate consequence of Hedlund's Lemma is that for convex cocompact groups every element of $L(G)$ is uniformly radial. Using the fact that the injective radius is uniformly bounded away from zero, a further immediate implication is that for each $k \in \mathbb{N}$ the set $\{\Pi(b(g(0),\kappa)) : g \in A_k\}$ is a covering of $L(G)$ of multiplicity uniformly bounded from above.

A combination of the Elementary Shadow Lemma, the Geometric Distortion Lemma and the Light Cone Lemma with the δ-conformality of the Patterson measure then gives rise to the following well-known result ([11]).

Sullivan's Shadow Lemma *Let* $G < Con(n)$ *be a Kleinian group, and let* $\rho > 0$ *be fixed. For each* $g \in G$ *we have*

$$\mu_G\left(\Pi(b(g(0),\rho))\right) \asymp_\rho \left(\operatorname{diam}\left(\Pi(b(g(0),\rho))\right)\right)^\delta \asymp_\rho e^{-\delta d(0,g(0))}.$$

For a convex cocompact Kleinian group G Sullivan's Shadow Lemma then immediately gives that the Patterson measure μ_G is a doubling measure which is δ-Ahlfors regular, in the sense that $\mu_G(B(\xi,r)) \asymp r^\delta$, for all $\xi \in L(G)$ and $0 < r < 1$.

2.2 Friendlyness with respect to neglectable sets

Throughout this section let $G < Con(n)$ be a convex cocompact Kleinian group and let μ_G be the associated Patterson measure. For a set $\mathcal{S}$ of subsets of $\mathbb{S}^n$, we define the function $\Delta_\mathcal{S}$ for $t > 0$ by

$$\Delta_\mathcal{S}(t) := \sup\left\{\mu_G\left(\mathcal{N}_t(S)\right) : S \in \mathcal{S}\right\}.$$

Proposition 1 *There exists* $\tau_0 > 0$ *such that for each* G*-invariant set* $\mathcal{S}$ *of subsets of* $\mathbb{S}^n$ *we have*

$$\Delta_\mathcal{S}(s \cdot t) \ll \Delta_\mathcal{S}(s)\ \Delta_\mathcal{S}(t) \quad \textit{for all } s,t \in (0,\tau_0).$$

PROOF: Let $s,t>0$ be given such that $\tau^{m+1}\le s<\tau^m$ and $\tau^{l+1}\le t<\tau^l$, for $m,l\in\mathbb{N}$. Let $S\in\mathcal{S}$ be arbitrary, and define for $k\in\mathbb{N}$,

$$A_k(S):=\{g\in A_k : S\cap\Pi(b(g(0),\kappa))\neq\emptyset\}.$$

By the Elementary Shadow Lemma we have that if $g\in A_k(S)$ then $\operatorname{diam}(\Pi(b(g(0),\kappa)))\asymp\tau^k$. Also, if $g\in A_m(S)$ and $h\in A_{m+l}(S)$ such that $\Pi(b(g(0),\kappa))\cap\Pi(b(h(0),\kappa))\neq\emptyset$ then the Geometric Distortion Lemma implies $\operatorname{diam}\left(g^{-1}\Pi(b(h(0),\kappa))\right)\asymp\tau^l$. Using these observations together with Sullivan's Shadow Lemma, the G-invariance of $\mathcal{S}$, and the fact that $\{\Pi(b(g(0),\kappa)) : g\in A_k(S)\}$ is a covering of S of multiplicity uniformly bounded from above, we obtain

$$\begin{aligned}
\mu_G(\mathcal{N}_{st}(S)) &\asymp \sum_{h\in A_{m+l}(S)}\mu_G(\Pi(b(h(0),\kappa)))\\
&\ll \sum_{g\in A_m(S)}\sum_{f\in A_l(g^{-1}(S))}\tau^{(m+l)\delta}\\
&\ll \sum_{g\in A_m(S)}\tau^{m\delta}\sum_{f\in A_l(g^{-1}(S))}\tau^{l\delta}\\
&\ll \sum_{g\in A_m(S)}\tau^{m\delta}\Delta_{\mathcal{S}}(t)\ll\Delta_{\mathcal{S}}(s)\Delta_{\mathcal{S}}(t).
\end{aligned}$$

Since $S\in\mathcal{S}$ was chosen to be arbitrary, the assertion of the proposition follows. □

Corollary 1 *There exists $\sigma,\tau_1>0$ such that for each (G,μ_G)-neglectable set $\mathcal{S}$ we have*

$$\Delta_{\mathcal{S}}(t)\le t^\sigma \quad \textit{for all } \; t\in(0,\tau_1).$$

PROOF: Since $\mathcal{S}$ is compact and $\mu_G(S)=0$ for all $S\in\mathcal{S}$, we have that $\lim_{t\to0}\Delta_{\mathcal{S}}(t)=0$. Hence, there exists $0<\tau_1'<1$ such that $\Delta_{\mathcal{S}}(t)<1$ and such that the inequality in Proposition 1 holds, for all $t\in(0,\tau_1')$. Let us consider the function $\Phi:(-\log\tau_1',\infty)\to\mathbb{R}^+$ given by

$$\Phi(x):=-\log\Delta_{\mathcal{S}}\left(e^{-x}\right).$$

Clearly, Φ is a strictly positive function which is unbounded, and by Proposition 1 we have that there exists a constant $c>0$ such that

$$\Phi(x+y)>\Phi(x)+\Phi(y)-c \quad \text{for all} \; x,y\in(-\log\tau_1',\infty).$$

This implies that there exist $\sigma,\tau_1>0$ such that

$$\Phi(x)\ge\sigma x \quad \text{for all} \; x\in(-\log\tau_1,\infty).$$

By rewriting this in terms of $\Delta_{\mathcal{S}}$, the corollary follows.

□

Proof of Theorem 1.1 Let $\mathcal{S}$ be a given (G,μ_G)-neglectable set. Consider $S\in\mathcal{S}$, $0<r<r_0, s>0$ and $\xi\in L(G)$. Since ξ is uniformly radial, the Geometric Distortion Lemma gives that there exists $g\in G$ such that $\xi\in\Pi(b(g(0),\kappa))$ and such that $\left|\left(g^{-1}\right)'\right|\asymp 1/r$ on $\Pi(b(g(0),\kappa))$. Clearly, this also holds for $B(\xi,r)$, that is we have $\left|\left(g^{-1}\right)'\right|\asymp 1/r$ on

$B(\xi, r)$. Using the δ-conformality of μ_G, Sullivan's Shadow Lemma and Corollary 1, it then follows

$$\begin{aligned}\mu_G\left(B(\xi,r)\cap\mathcal{N}_{sr}(S)\right) &\asymp r^{\delta}\,\mu_G\left(g^{-1}\left(B(\xi,r)\cap\mathcal{N}_{sr}(S)\right)\right)\\ &\ll r^{\delta}\,\mu_G\left(\mathcal{N}_s(g^{-1}(S))\right)\ll r^{\delta}\,\Delta_{\mathcal{S}}(s)\\ &\ll \mu_G(B(\xi,r))\,s^{\sigma}.\end{aligned}$$

□

2.3 Friendlyness with respect to $\mathcal{H}(G)$

Proposition 2 *For every convex cocompact Kleinian group $G < Con(n)$ of non-Fuchsian type we have that the set $\mathcal{H}(G)$ is (G,μ_G)-neglectable.*

Proof: We already saw in section 2.1 that if G is convex cocompact then μ_G is a doubling measure. Also, we clearly have that $\mathcal{H}(G)$ is G-invariant and compact with respect to the Hausdorff topology in $\mathbb{S}^n$. Therefore, it remains to show that $\mu_G(H)=0$ for all $H\in\mathcal{H}(G)$. For this, assume by way of contradiction that there exists $H\in\mathcal{H}(G)$ such that $\mu_G(H)>0$. Then let ξ be a μ_G-density point of $H\cap L(G)$. Since G is convex cocompact, ξ is a uniformly radial limit point, and therefore $s_\xi\subset\bigcup_{g\in G} b(g(0),\kappa)$. Hence, there exists a sequence $(g_m)_{m\in\mathbb{N}}$ of elements $g_m\in G$ such that $\xi\in\Pi\left(b(g_m(0),\kappa)\right)$ and $d(0,g_m(0)) < d(0,g_{m+1}(0))$, for all m. Since $\xi\in H\cap L(G)$ is a μ_G-density point, it follows

$$\lim_{m\to\infty}\frac{\mu_G\left(\Pi\left(b(g_m(0),2\kappa)\right)\cap H\right)}{\mu_G\left(\Pi\left(b(g_m(0),2\kappa)\right)\right)}=1.$$

Now let $\epsilon>0$ be fixed. We then have, for each m sufficiently large,

$$\mu_G\left(\Pi\left(b(g_m(0),2\kappa)\right)\setminus H\right)<\epsilon\mu_G\left(\Pi\left(b(g_m(0),2\kappa)\right)\right).$$

Using the δ-conformality of μ_G, Sullivan's Shadow Lemma, and the fact that $\left|\left(g_m^{-1}\right)'\right|$ is comparable to $\exp(d(0,g_m(0))$ on $\Pi(b(g_m(0),2\kappa))$, we have

$$\begin{aligned}\mu_G\left(g_m^{-1}\left(\Pi\left(b(g_m(0),2\kappa)\right)\setminus H\right)\right) &\asymp e^{\delta d(0,g_m(0))}\mu_G\left(\Pi\left(b(g_m(0),2\kappa)\right)\setminus H\right)\\ &< \epsilon e^{\delta d(0,g_m(0))}\mu_G\left(\Pi\left(b(g_m(0),2\kappa)\right)\right)\\ &\ll \epsilon.\end{aligned}$$

We then make the following three observations. First, by compactness of $L(G)$ there exists $\eta\in L(G)$ such that on a subsequence (m_k) we have

$$\lim_{k\to\infty} g_{m_k}^{-1}(\xi)=\eta.$$

Secondly, since $\mathcal{H}(G)$ is G-invariant and compact, there exists $H_0\in\mathcal{H}(G)$ such that on a subsequence (m_k') of (m_k) we have, where convergence is with respect to the Hausdorff topology,

$$\lim_{k\to\infty} g_{m_k'}^{-1}(H)=H_0.$$

Third, we clearly have there exists a constant $R>0$ such that the boundaries of $g_{m_k'}^{-1}\Pi(b(g_{m_k'}(0),\kappa))$ and $g_{m_k'}^{-1}\Pi(b(g_{m_k'}(0),2\kappa))$ are uniformly separated by $2R$. Therefore, since $\lim g_{m_k'}^{-1}(\xi)=\eta$ and $g_{m_k'}^{-1}(\xi)\in g_{m_k'}^{-1}\Pi(b(g_{m_k'}(0),\kappa))$, it follows for all $k\in\mathbb{N}$,

$$B(\eta,R)\subset g_{m_k'}^{-1}\Pi(b(g_{m_k'}(0),2\kappa).$$

Combining this observation with the measure estimate above, we obtain $\mu_G(B(\eta,R)\setminus H_0)\ll\epsilon$. Since this estimate holds for all ϵ, it follows

$$\mu_G(B(\eta,R)\setminus H_0)=0.$$

Now, if $L(G)\cap(B(\eta,R)\setminus H_0)\neq\emptyset$ then $L(G)\cap(B(\eta,R)\setminus H_0)$ is a nonempty open subset of $L(G)$, and hence $\mu_G(B(\eta,R)\setminus H_0)>0$. Clearly, this gives a contradiction, and therefore

$$B(\eta,R)\cap L(G)\subset H_0.$$

Then, since $L(G)=\bigcup_{g\in G} g\,(B(\eta,R)\cap L(G))$, compactness of $L(G)$ gives that there exist $h_1,\ldots,h_l\in G$ such that

$$L(G)\subset\bigcup_{i=1}^{l} h_i\,(H_0)\,.$$

This contradicts the assumtion that G is of non-Fuchsian type, and hence the assertion follows. $\square$

Proof of Theorem 1.2: Combine Theorem 1 and Proposition 2. $\square$

Remark: We remark that in the proof of Proposition 2, in order to deduce that $\mu_G(H)=0$ for all $H\in\mathcal{H}(G)$, we did not use the fact that the elements of $\mathcal{H}(G)$ are intersections of spheres with $L(G)$. Hence, a straight forward adaptation of the proof shows that if $\mathcal{S}$ is any G-invariant set of subsets of $\mathbb{S}^n$ which is closed with respect to the Hausdorff topology, then $\mu_G(S)=0$ for all $H\in\mathcal{S}$, unless finitely many elements of $\mathcal{S}$ contain $L(G)$. Consequently, Theorem 2 admits the following generalization.

- *Let $G<Con(n)$ be a convex cocompact Kleinian group, and let $\mathcal{S}$ be a G-invariant set of subsets of $\mathbb{S}^n$, which is compact with respect to the Hausdorff topology, and which has the property that $L(G)$ is not contained in a finite union of elements of $\mathcal{S}$. We then have that $\mathcal{S}$ is (G,μ_G)-neglectable, and hence Theorem 1 implies that μ_G is a $\mathcal{S}$-friendly measure.*

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